Multiparticle electroweak physics on the lattice: introduction and application to kaon decays

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Outline

My original talk is "Electromagnetism on $K \to \pi\pi$"

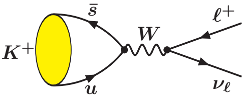
suggested by S. Sharpe $\longrightarrow$

"Multiparticle electroweak physics on the lattice: introduction and application to kaon decays"

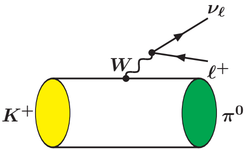
- Theoretical issues relevant for multiparticle matrix elements in finite volume
- Applications on the lattice
  - Multiparticle appears in the final state
    - Timelike pion form factor
    - $\pi\gamma \to \rho$ amplitude
    - $B \to K^*\ell\nu$ decay amplitude [Talk by Luka Leskovec]
    - Form factor for resonance [Talk by Alessandro Baroni]
    - $K \to \pi\pi$ decay (emerged into the last topic)
  - Multiparticle appears in the intermediate state
    - Use rare kaon decay as an axample
- Electromagnetism on $K \to \pi\pi$

A mission of lattice calculations is to evaluate the hadronic effects

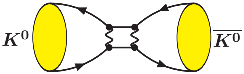
- Lattice QCD is powerful for "standard" hadronic matrix elements with



$K_{\ell 2} \to f_K$



$K_{\ell 3} \to f_+(0)$



K^0-$\overline{K^0}$ mixing $\to B_K$

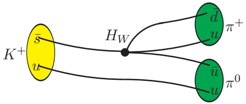
  - single local operator insertion
  - only single stable hadron or vacuum in the initial/final state
  - spatial momenta carried by particles need to be small compared to $1/a$

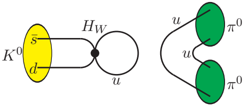
	N_f	FLAG average	Frac. Err.
f_K/f_π	$2+1+1$	1.1933(29)	0.25%
$f_+(0)$	$2+1+1$	0.9706(27)	0.28%
$\hat{B}_K$	$2+1$	0.7625(97)	1.27%

- Recent progress in lattice calculations $\Rightarrow$ go beyond "standard" quantities

Examples of beyond "standard" electroweak quantities

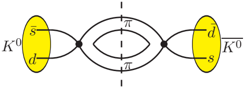
- $K \to \pi\pi$ decays and direct CP violation

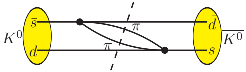




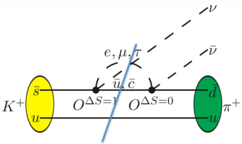
Final state involves multiparticle

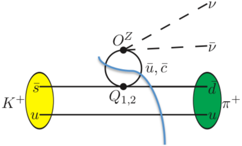
- Long-distance contributions to flavor changing processes
  - K^0-$\overline{K^0}$ mixing





  - Rare kaon decays: $K \to \pi\nu\bar{\nu}$ or $K \to \pi\ell^+\ell^-$





Intermediate state involves multiparticle

To go beyond "standard" quantities, multiparticle plays important role

Theoretical development

Lellouch-Lüscher formalism

Enclose a $\pi\pi$ scattering system in a finite box with size L

- The energy level is quantized following Lüscher formalism

$$\phi(E, L) + \delta(E) = n\pi$$

- Without weak interaction, K and $\pi\pi$ form a two-state system

$$H = \begin{pmatrix} E_{\pi\pi} & 0 \\ 0 & M_K \end{pmatrix}$$

  One can tune the volume to make $E_{\pi\pi} = M_K$

- Turning on the weak interaction, degenerate perturbation theory yields

$$E_{\pi\pi} = M_K \pm |A_L|, \quad |A_L| = |\langle \pi\pi | H_W | K \rangle_L|$$

- Lüscher formalism still holds for QCD + Weak

$$\underbrace{\phi(E + \Delta E) + \delta(E + \Delta E)}_{\text{1) } \Delta E \text{ causes the shift in } \phi + \delta} \quad + \quad \underbrace{\Delta\delta}_{\Downarrow} \quad = \quad n\pi$$

2) H_W causes the shift in the functional form of δ

Lellouch-Lüscher formalism

Include K and weak interaction into $\pi\pi$ scattering

- The $\pi\pi$ scattering amplitude in the infinite volume is shifted by

$$\Delta T(\pi\pi \to K \to \pi\pi) = \langle \pi\pi, \mathrm{out}|H_W|K\rangle \frac{1}{q^2 - M_K^2} \langle K|H_W|\pi\pi, \mathrm{in}\rangle$$

  - $\langle \pi\pi, \mathrm{out}|H_W|K\rangle\langle K|H_W|\pi\pi, \mathrm{in}\rangle \quad \Rightarrow \quad |A_\infty|^2$
  - $q^2 - M_K^2 \quad \Rightarrow \quad \pm 2M_K \Delta E$

- The scattering amplitude is related to δ through $T = 16\pi E \frac{e^{2i\delta}-1}{2ik}$

$$\Delta\delta = \frac{k}{16\pi E}\Delta T = \mp \frac{k}{32\pi E^2 \Delta E}|A_\infty|^2$$

- Putting $\Delta\delta$ into the quantization condition yields

$$\frac{d(\phi+\delta)}{dE}|A_L|^2 = \frac{k}{32\pi E^2}|A_\infty|^2$$

The derivation beatifully uses K-$\pi\pi$ two-state system and degenerate PT

- A new derivation based Kim, Scharajda, Sharpe's method ['04]

Lattice QCD to treat with multiparticle intermediate state

- In infinite volume we need to compute the amplitude

$$\mathcal{A} = \int_{-\infty}^{\infty} dt \, \langle \bar{K}^0 | T[H_W(t) H_W(0)] | K^0 \rangle$$

and to determine the K_L-K_S mass difference

$$\Delta M_K = M_{K_L} - M_{K_S} = 2 \mathcal{PV} \sum\!\!\!\!\!\!\int_\alpha \frac{\langle \bar{K}^0 | H_W | \alpha \rangle \langle \alpha | H_W | K^0 \rangle}{M_K - E_\alpha}$$

- On the lattice with size L, we can obtain

$$\Delta M_K(L) = 2 \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle_L \, {}_L\langle n | H_W | K^0 \rangle}{M_K - E_n}$$

  - $|n\rangle$ could be given by multi-hadron state $|\pi\pi\rangle$
  - Significant FV effects, especially when $E_n = E_{\pi\pi} \to M_K$

- Finite volume correction

[Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510]

$$\Delta M_K(L) - \Delta M_K(\infty) = 2 \cot(\phi + \delta) \frac{d(\phi + \delta)}{dE}\bigg|_{E = M_K} \langle \bar{K}^0 | H_W | \pi\pi, M_K \rangle_L \, {}_L\langle \pi\pi, M_K | H_W | K^0 \rangle$$

Put three processes together

- $\pi\pi$ scattering, Lüscher's quantization condition

$$\phi(q) + \delta(k) = n\pi, \quad n \in \mathbb{Z}, \quad q = \frac{k}{2\pi/L}, \quad E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$$

- $K \to \pi\pi$ transition, Lellouch-Lüscher formula

$$|\langle 0|\sigma(0)|\pi\pi, E\rangle_\infty|^2 = \frac{2\pi E^2}{k^2}\left(\frac{d(\phi+\delta)}{dk}\right)|\langle 0|\sigma(0)|\pi\pi, E\rangle_L|^2$$

- K_L-K_S mass difference

$$\Delta M_K(L) - \Delta M_K(\infty) = 2\cot(\phi+\delta)\frac{d(\phi+\delta)}{dE}\bigg|_{E=M_K}\langle\bar{K}^0|H_W|\pi\pi, M_K\rangle_L\,{}_L\langle\pi\pi, M_K|H_W|K^0\rangle$$

Similar structure $\Rightarrow$ Uniform treatment of the FV effects in the three processes

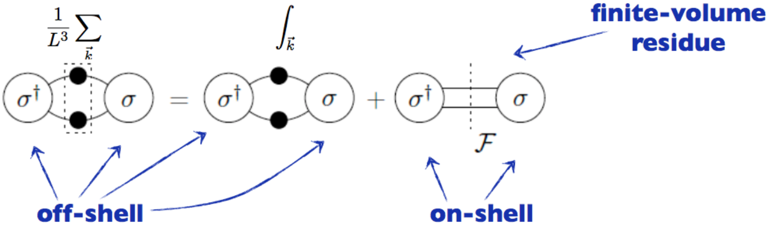
I borrow two instructive pages from Steve's talk

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

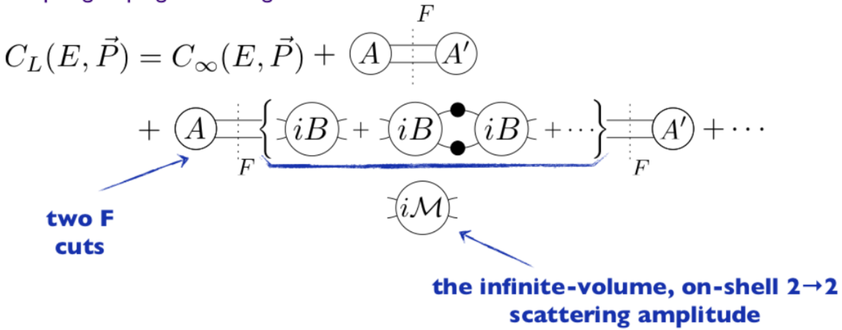
$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*\prime})g^*(\hat{q}^{*\prime})$$

- Diagrammatically



- And keep regrouping according to number of "F cuts"



Finite volume correction to correlator

$$\Delta C(P) = C_L(P) - C_\infty(P) = A\mathcal{F}\sum_n (i\mathcal{M}\mathcal{F})^n A' = A\frac{1}{\mathcal{F}^{-1} - i\mathcal{M}}A' = \frac{k}{8\pi P_0}\frac{e^{i(\phi+\delta)}}{\sin(\phi+\delta)}|A|^2$$

- Product 1: Lüscher quantization condition

  $C_L(P)$ has poles at $P_0 = E_n$, which match the poles in $\Delta C(P)$

$$\sin(\phi+\delta)\big|_{P_0=E_n} = 0 \quad \Rightarrow \quad (\phi+\delta)\big|_{P_0=E_n} = n\pi$$

Product 2: Lellouch-Lüscher formalism

Agadjanov, Bernard, Meissner, Rusetsky, NPB 886 (2014) 1199
Briceno, Hansen, Walker-Loud, PRD 91 (2015) 034501
Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510

- Let the poles in $\Delta C(P)$ and $C_L(P)$ have the same residues

$$\lim_{P_0\to E_n}(P_0-E_n)\Delta C(P)=\lim_{P_0\to E_n}(P_0-E_n)C_L(P)$$

- Spectral representation for $C_L(P)$ is given by

$$C_L(P)=\sum_n \frac{2E_n|\langle 0|\sigma(0)|\pi\pi,E_n\rangle_L|^2}{P_0^2-E_n^2}$$

- The expression for ΔC is given by

$$\Delta C(P)=\frac{k}{8\pi P_0}\frac{e^{i(\phi+\delta)}}{\sin(\phi+\delta)}|A|^2$$

- Picking up the residues leads to the Lellouch-Lüscher formula

$$|A|^2=\left.\frac{d(\phi+\delta)}{dk}\right|_{P_0=E_n}\frac{2\pi E_n^2}{k_n^2}|\langle 0|\sigma(0)|E_n\rangle_L|^2$$

Product 3: FV correction to long-distance observables

Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510

For the $K_L - K_S$ system, we need to evaluate the principal value of the branch-cut integral, which can be achieved by taking the real part of KSS formula

$$\mathrm{Re}\,\Delta C(P) = C_L(P) - \mathcal{PV}[C_\infty(P)] = \frac{k}{8\pi P_0}\mathrm{ctg}\,(\phi+\delta)|A|^2$$

Using the spectral representation for $C_L(P)$, we can rewrite

$$\mathcal{PV}[C_\infty(P)] = \sum_n \frac{2E_n|\langle 0|\sigma(0)|E_n\rangle_L|^2}{P_0^2 - E_n^2} - \frac{k}{8\pi P_0}\mathrm{ctg}\,(\phi+\delta)|A|^2$$

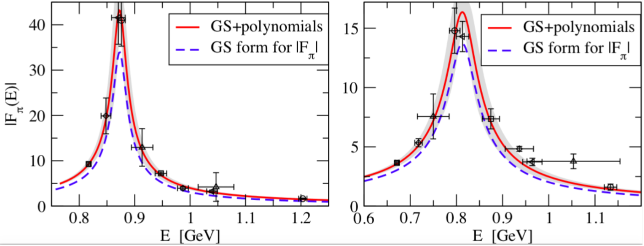
This gives FV correction for ΔM_K (if we replace $|\langle 0|\sigma(0)|E_n\rangle_L|$ by $|\langle K|H_W|E_n\rangle_L|$)

Application: electroweak physics with multiparticle in the final state

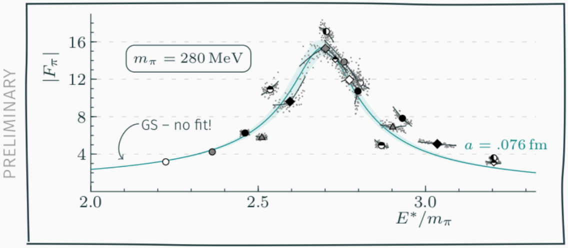
Vector channel, $\sigma \to J_\mu$, $\langle \pi\pi, E | J_\mu | 0 \rangle \to F_\pi(E)$ [H. Meyer, PRL107 (2011)]

$$|F_\pi(s)|^2 = \frac{\gamma}{g(\gamma)^2} \frac{3\pi s}{2k^5} \left(q \frac{\partial \phi(q)}{\partial q} + k \frac{\partial \delta(k)}{\partial k} \right) |\langle \pi\pi, E | J_\mu | 0 \rangle_V|^2$$

JLQCD @ m_π = 380 and 290 MeV, XF, Aoki, Hashimoto, Kaneko, PRD91 (2015)

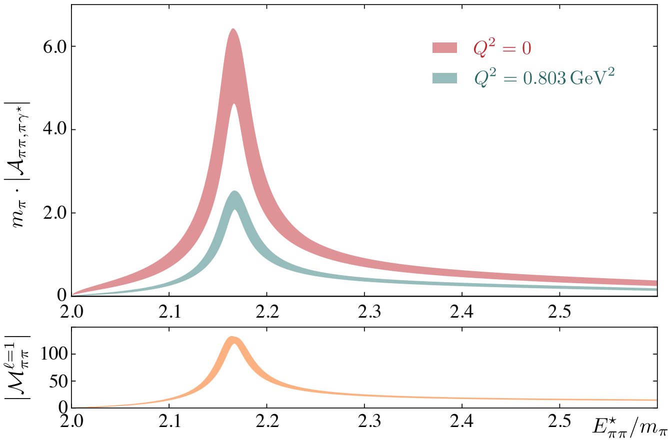


Ben Horz's talk on Monday



HadSpec calculation performed @ m_π = 400 MeV

$$\mathcal{H}^{\mu}_{\pi\pi,\pi\gamma^\star} = \langle \pi, P_\pi | \mathcal{J}^\mu(0) | \pi\pi, P_{\pi\pi}, \ell = 1 \rangle, \quad Q^2 = -(P_\pi - P_{\pi\pi})^2, \quad E^{\star 2}_{\pi\pi} = P^2_{\pi\pi}$$



Briceno, Dudek, Edwards, Shultz,Thomas,Wilson [PRL115, 2015; PRD93, 2016]

- Work also from [Leskovec, Meinel, et. al., arXiv:1611.00282]
- Study $\pi\gamma^* \to \pi\pi$ in a dispersive approach [Martin Hoferichter's talk today]

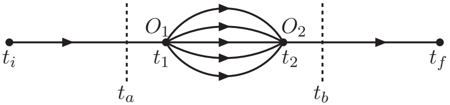
Another three topics related to final-state multiparticle

- $B \to K^* \ell \nu$ decay amplitude [Talk by Luka Leskovec today]

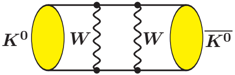
- Form factor for resonance [Talk by Alessandro Baroni today]

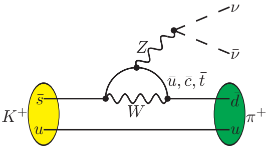
- $K \to \pi\pi$ decay (emerged into the topic of electromagnetism)

Application: electroweak physics with multiparticle in the intermediate state

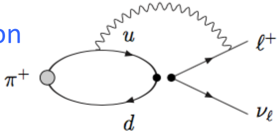


2^{nd} order electroweak interaction

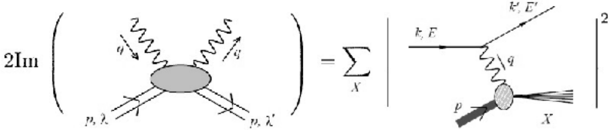




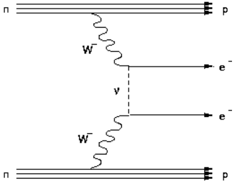
Electromagnetic correction



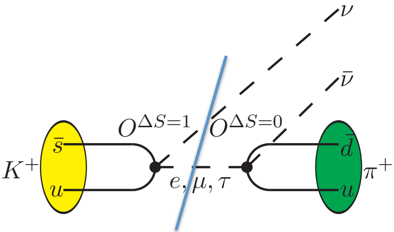
Inclusive decay and deep inelastic scattering



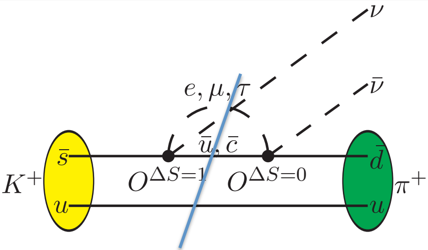
Neutrinoless double beta decay



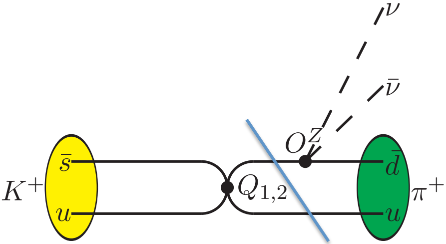
Use $K^+ \to \pi^+\nu\bar{\nu}$ as an example



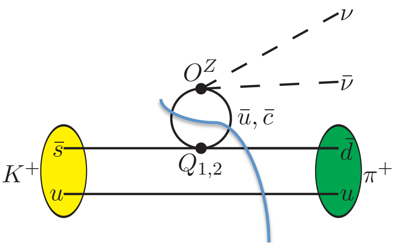
$K^+ \to \ell^+\nu \quad \& \quad \ell^+ \to \pi^+\bar{\nu}$



$K^+ \to \pi^0\ell^+\nu \quad \& \quad \pi^0\ell^+ \to \pi^+\bar{\nu}$

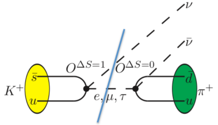
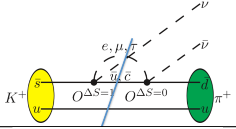
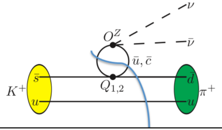


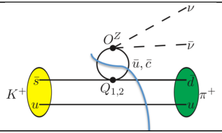
$K^+ \xrightarrow{H_W} \pi^+ \quad \& \quad \pi^+ \xrightarrow{V_\mu} \pi^+$



$K^+ \xrightarrow{H_W} \pi^+\pi^0 \quad \& \quad \pi^+\pi^0 \xrightarrow{A_\mu} \pi^+$

Branching ratios and decay widths for $K \to \{n\}$ decays

$K \to \{n\}$	Branching ratio	Relevant diagrams
$K^+ \to \mu^+\nu_\mu$	$6.355(11) \times 10^{-1}$	
$K^+ \to 2\pi\mu^+\nu_\mu$	$4.254(32) \times 10^{-5}$	
$K^+ \to \pi^0 e^+\nu_e$	$3.353(34) \times 10^{-2}$	
$K^+ \to 3\pi e^+\nu_e$	$< 3.5 \times 10^{-6}$	
$K^+ \to \pi^+\pi^0$	$2.066(8) \times 10^{-1}$	
$K^+ \to 3\pi$	$7.35(5) \times 10^{-2}$	

$K \to \{n\}$	Decay width [eV]	Relevant diagrams
$K_S \to 2\pi$	$7.343(13) \times 10^{-6}$	
$K_L \to 3\pi$	$4.125(30) \times 10^{-9}$	

FV effects from $\pi\pi$ intermediate state

Calculation has been performed at m_π = 420 MeV and 170 MeV

For $m_\pi = 170$ MeV, $K \to \pi\pi$ is possible

- Matrix elements for $K \to \pi\pi$ and $\pi\pi \to \pi$

$a^4 \cdot \langle \pi\pi \vert Q_{1,2} \vert K \rangle$	$a \cdot \langle \pi \vert A_\mu \vert \pi\pi \rangle$	expectation
$-i \cdot 9.653(25) \times 10^{-5}$	$i \cdot 3.1910(60)$	$i \cdot 3.2149(62)$

Expectation value is given by $\langle \pi | A_\mu | \pi\pi \rangle \approx \langle \pi | \pi \rangle \cdot \langle 0 | A_\mu | \pi \rangle$.

- Parameters relevant for finite-size correction

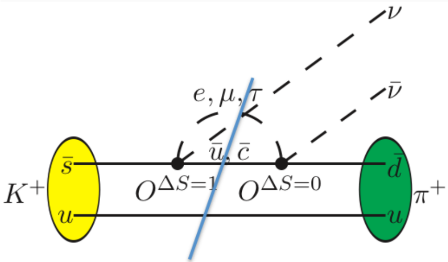
ϕ	$\delta = k \cdot a_{\pi\pi}$	$d\phi/dE$	$d\delta/dE = a_{\pi\pi} \cdot \frac{m_K}{4k}$	$\cot(\phi + \delta)$
1.449(2)	−0.0678(16)	13.858(8)	−0.367(9)	0.192(3)

- Results

$F_0(s)$	$F_0^{(\pi\pi)}(s)$	$\Delta_{FV} F_0(s)$
$2.05(12) \cdot 10^{-2}$	$-1.536(5) \cdot 10^{-3}$	$4.28(7) \cdot 10^{-4}$

$I = 2$ $\pi\pi$-state contributes 7.5%. The finite-size correction is about 2.1%.

FV effects from $\pi\ell$ intermediate state



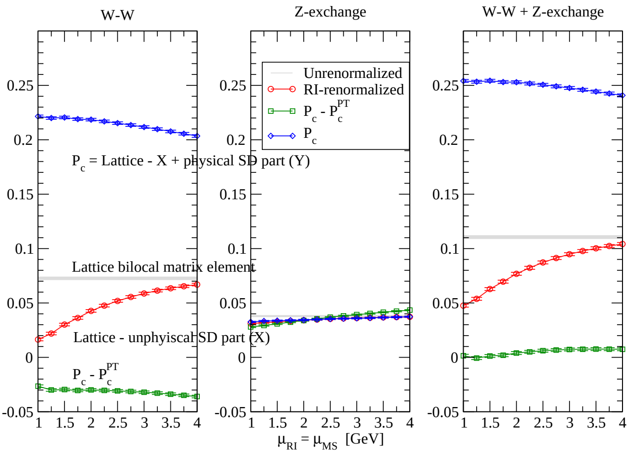
$$A_{\rm FV}^{\pi^0\ell^+} = \left(\frac{1}{L^3}\sum_{\vec{k}}\int\frac{dk_0}{2\pi} - \mathcal{P}\int\frac{d^4k}{(2\pi)^4}\right)\left\{A_\alpha^{K^+\to\pi^0}(p_K,k)\frac{1}{k^2+m_\pi^2}A_\beta^{\pi^0\to\pi^+}(k,p_\pi)\right\}$$
$$\times\left\{\bar{u}(p_\nu)\gamma^\alpha(1-\gamma_5)\frac{i(\not{P}-\not{k})+m_{\bar{\ell}}}{(P-k)^2+m_{\bar{\ell}}^2}\gamma^\beta(1-\gamma_5)v(p_{\bar{\nu}})\right\},$$

Lattice results

Published results @ $m_\pi = 420$ MeV, $m_c = 860$ MeV

[Bai, Christ, XF, Lawson, Portelli, Schrajda, PRL 118 (2017) 252001]

$$P_c = 0.2529(\pm 13)_{\mathrm{stat}}(\pm 32)_{\mathrm{scale}}(-45)_{\mathrm{FV}}$$



Lattice QCD is now capable of first-principles calculation of rare kaon decay

- The remaining task is to control various systematic effects

$K \to \pi\pi$ decays and direct CP violation

Direct and indirect CP violation

- The experimentally detected states $K_{L/S}$ are not CP eigenstates

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1+\bar{\epsilon}^2}}\left(|K^0_{\mp}\rangle + \bar{\epsilon}|K^0_{\pm}\rangle\right)$$

- $K_L \to 2\pi$ $(CP = +)$
  - $K^0_+ \to 2\pi$ (indirect CP violation, ϵ or ϵ_K)
  - $K^0_- \to 2\pi$ (direct CP violation, ϵ')
- Experimental measurement

$$\frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \equiv \eta_{+-} \equiv \epsilon + \epsilon'$$

$$\frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \equiv \eta_{00} \equiv \epsilon - 2\epsilon'$$

- Using $|\eta_{+-}|$ and $|\eta_{00}|$ as input, PDG quotes

$$|\epsilon| \approx \frac{1}{3}\left(2|\eta_{+-}| + |\eta_{00}|\right) = 2.228(11)\times 10^{-3}, \quad \mathrm{Re}[\epsilon'/\epsilon] \approx \frac{1}{3}\left(1 - \frac{|\eta_{00}|}{|\eta_{+-}|}\right) = 1.66(23)\times 10^{-3}$$

ϵ' is 1000 times smaller than the indirect CP violation ϵ

Thus direct CP violation ϵ' is very sensitive to New Physics

$K \to \pi\pi$ decays and CP violation

- Theoretically, Kaon decays into the isospin $I = 2$ and 0 $\pi\pi$ states

$$\Delta I = 3/2 \text{ transition:} \quad \langle \pi\pi(I=2)|H_W|K^0\rangle = A_2 e^{i\delta_2}$$
$$\Delta I = 1/2 \text{ transition:} \quad \langle \pi\pi(I=0)|H_W|K^0\rangle = A_0 e^{i\delta_0}$$

- If CP symmetry were protected $\Rightarrow$ A_2 and A_0 are real amplitudes

- ϵ and ϵ' depend on the $K \to \pi\pi(I)$ amplitudes A_I

$$\epsilon = \bar{\epsilon} + i\left(\frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]}\right)$$
$$\epsilon' = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\mathrm{Re}[A_2]}{\mathrm{Re}[A_0]}\left(\frac{\mathrm{Im}[A_2]}{\mathrm{Re}[A_2]} - \frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]}\right)$$

The target for lattice QCD is to calculate both amplitude A_2 and A_0

Results for $\mathrm{Re}[A_0]$, $\mathrm{Im}[A_0]$ and $\mathrm{Re}[\epsilon'/\epsilon]$

- Lattice results for A_2 @ $m_\pi = 140$ MeV [RBC-UKQCD, PRD91 (2015)]

$$\mathrm{Re}[A_2] = 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}$$
$$\mathrm{Im}[A_2] = -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}$$

- Lattice results for A_0 @ $m_\pi = 140$ MeV [RBC-UKQCD, PRL115 (2015)]

$$\mathrm{Re}[A_0] = 4.66(1.00)_{\mathrm{stat}}(1.26)_{\mathrm{syst}} \times 10^{-7}\ \mathrm{GeV}$$
$$\mathrm{Im}[A_0] = -1.90(1.23)_{\mathrm{stat}}(1.08)_{\mathrm{syst}} \times 10^{-11}\ \mathrm{GeV}$$

- Experimental measurement

$$\mathrm{Re}[A_2] = 1.479(3) \times 10^{-8}\ \mathrm{GeV}$$
$$\mathrm{Re}[A_0] = 3.3201(18) \times 10^{-7}\ \mathrm{GeV}$$
$$\mathrm{Im}[A_2]\ \&\ \mathrm{Im}[A_0]\ \text{are unknown}$$

- Determine the direct CP violation $\mathrm{Re}[\epsilon'/\epsilon]$

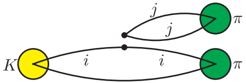
$$\mathrm{Re}[\epsilon'/\epsilon] = 0.14(52)_{\mathrm{stat}}(46)_{\mathrm{syst}} \times 10^{-3} \quad \text{Lattice}$$
$$\mathrm{Re}[\epsilon'/\epsilon] = 1.66(23) \times 10^{-3} \quad \text{Experiment}$$

2.1 σ deviation $\Rightarrow$ require more accurate lattice results

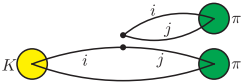
Resolve the puzzle of $\Delta I = 1/2$ rule

$\Delta I = 1/2$ rule: $A_0 = 22.5 \times A_2$ $\Rightarrow$ a > 50 year puzzle

- Wilson coefficient only contributes a factor of ~ 2
- $\mathrm{Re}[A_2]$ and $\mathrm{Re}[A_0]$ are dominated by diagrams C_1 and C_2



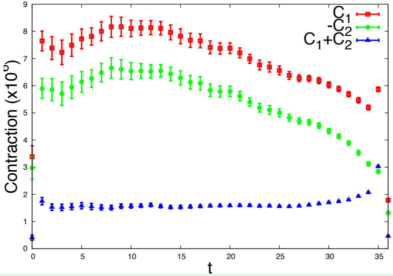
C_1: color diagonal



C_2: color mixed

Color counting in LO PT $\Rightarrow$ $C_2 = C_1/3$; Non-PT effects $\Rightarrow$ $C_2 \approx -0.7 C_1$

- $\mathrm{Re}[A_2] \propto C_1 + C_2$, while $\mathrm{Re}[A_0] \propto 2C_1 - C_2$ $\Rightarrow$ another factor of ~ 10
  - Such cancellation is first observed in an earlier calculation [RBC-UKQCD, PRL110 (2013) 152001]
  - It is further confirmed in the latest calculation of A_2 [RBC-UKQCD, PRD91 (2015) 074502]



Puzzle of $\Delta I = 1/2$ rule is resolved from first principles

Electromagnetism on $K \to \pi\pi$

[N. Christ & XF, arXiv:1711.09339]

How large the EM corrections to $K \to \pi\pi$

Direct CP violation in $K \to \pi\pi$

$$\epsilon' = \frac{1}{3}(\eta_{+-} - \eta_{00}) = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}} \frac{\mathrm{Re}\, A_2}{\mathrm{Re}\, A_0} \left(\frac{\mathrm{Im}\, A_2}{\mathrm{Re}\, A_2} - \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0} \right)$$

- Turn on EM interaction, $A_I \to A_I^\gamma$, $\delta_I \to \delta_I^\gamma$, $I = 0, 2$

Though $A_2^\gamma - A_2$ is an $O(\alpha_e)$ effect, its size could be enhanced by a factor of 22 due to the mixing with A_0 and $\Delta I = 1/2$ rule

- ChPT+Large-N_c: Cirigliano et al, hep-ph/0008290, hep-ph/0310351
  –"the isospin violating correction for ϵ' is below 15%"

Technical issues on including electromagnetism

- Lellouch-Lüscher's formalism relies on a short-range interaction
  $\Rightarrow$ long-range EM requires the change in the FV formalism
- EM interaction mixes $I = 0$ and $I = 2$ $\pi\pi$ scattering
  $\Rightarrow$ $K \to \pi\pi$ decay becomes a coupled-channel problem
- Possible photon radiation
  $\Rightarrow$ coupled channels further mixed with 3-particle channel $(\pi\pi\gamma)$

Include EM interaction in the Coulomb gauge

$$\mathcal{L}_{\rm int} = \underbrace{\sum_{q=u,d,s} e_q \vec{A}(x)\cdot \bar{q}\vec{\gamma}q(x)}_{\text{Transverse radiation}} - \underbrace{\sum_{q,q'=u,d,s} \int \frac{d^3\vec{x}'}{4\pi} \frac{\rho_q(\vec{x}',t)\rho_{q'}(\vec{x},t)}{|\vec{x}'-\vec{x}|}}_{\text{Coulomb potential}}$$

- Adding transverse photon to $\pi\pi$ $\Rightarrow$ three-particle problem
- At current stage, focus on Coulomb potential only

Mixing of isospin states

Focus on Coulomb potential, no $\pi\pi\gamma$ state

However, $I = 2$ and $I = 0$ $\pi\pi$ states still mix with each other

- No EM: relation between charged $c = +-, 00$ and isopsin $s = 0, 2$ $\pi\pi$ states

$$|(\pi\pi)_c\rangle^{\text{out}} = \sum_{s=0,2} \Omega_{cs} |(\pi\pi)_s\rangle^{\text{out}}, \quad \Omega_{cs} = \begin{pmatrix} \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- With EM:

$$|(\pi\pi)_c^\gamma\rangle^{\text{out}} = \sum_{s=0,2} \Omega_{cs}^\gamma |(\pi\pi)_s^\gamma\rangle^{\text{out}}, \quad \Omega_{cs}^\gamma = \begin{pmatrix} \cos\theta^\gamma & \sin\theta^\gamma \\ -\sin\theta^\gamma & \cos\theta^\gamma \end{pmatrix}$$

Define ${}^{\text{out}}\langle(\pi\pi)_s^\gamma|H_W|K^0\rangle = e^{i\delta_s^\gamma} A_s^\gamma$

$$\epsilon' = \frac{1}{3}(\eta_{+-} - \eta_{00}) = \frac{\sin 2\theta}{\sin 2\theta^\gamma} \frac{ie^{i(\delta_2^\gamma - \delta_0^\gamma)}}{\sqrt{2}} \frac{\text{Re}\, A_2^\gamma}{\text{Re}\, A_0^\gamma} \left(\frac{\text{Im}\, A_2^\gamma}{\text{Re}\, A_2^\gamma} - \frac{\text{Im}\, A_0^\gamma}{\text{Re}\, A_0^\gamma} \right)$$

$\frac{\sin 2\theta}{\sin 2\theta^\gamma}$ is a small correction $\Rightarrow$ focus on A_s^γ and δ_s^γ

Determination of A_s^γ and δ_s^γ from lattice QCD

Turn off EM and calculate correlators with $I = 0, 2$ operators

$$
\begin{aligned}
C_{II'}(t) &= \langle \phi_{\pi\pi,I}(t)\phi^\dagger_{\pi\pi,I'}(0)\rangle \\
&= \sum_{s=0,2} \langle 0|\phi_{\pi\pi,I}|(\pi\pi)_s\rangle e^{-E_s t}\langle(\pi\pi)_s|\phi^\dagger_{\pi\pi,I'}|0\rangle \delta_{s,I}\delta_{s,I'} \\
&= (UMU^\dagger)_{II'}
\end{aligned}
$$

where

$$
U = \begin{pmatrix} \langle 0|\phi_{\pi\pi,0}|(\pi\pi)_0\rangle & 0 \\ 0 & \langle 0|\phi_{\pi\pi,2}|(\pi\pi)_2\rangle \end{pmatrix}, \quad M = \begin{pmatrix} e^{-E_0 t} & \\ & e^{-E_2 t} \end{pmatrix}
$$

Turn on EM and calculate correlators with the same operators

$$
\begin{aligned}
C^\gamma_{II'}(t) &= \langle \phi_{\pi\pi,I}(t)\phi^\dagger_{\pi\pi,I'}(0)\rangle^\gamma \\
&= \sum_{s=0,2} {}^\gamma\langle 0|\phi_{\pi\pi,I}|(\pi\pi)^\gamma_s\rangle e^{-E^\gamma_s t}\langle(\pi\pi)^\gamma_s|\phi^\dagger_{\pi\pi,I'}|0\rangle^\gamma \\
&= (U^\gamma M^\gamma U^{\gamma\dagger})_{II'}
\end{aligned}
$$

where

$$
U^\gamma = \begin{pmatrix} {}^\gamma\langle 0|\phi_{\pi\pi,0}|(\pi\pi)^\gamma_0\rangle & {}^\gamma\langle 0|\phi_{\pi\pi,0}|(\pi\pi)^\gamma_2\rangle \\ {}^\gamma\langle 0|\phi_{\pi\pi,2}|(\pi\pi)^\gamma_0\rangle & {}^\gamma\langle 0|\phi_{\pi\pi,2}|(\pi\pi)_2\rangle \end{pmatrix}, \quad M^\gamma = \begin{pmatrix} e^{-E^\gamma_0 t} & \\ & e^{-E^\gamma_2 t} \end{pmatrix}
$$

Determination of A_s^γ and δ_s^γ from lattice QCD

- Use the coefficient matrix to construct a ratio $U^{-1}U^\gamma = 1 + \begin{pmatrix} N_{00}^{(1)} & N_{02}^{(1)} \\ N_{20}^{(1)} & N_{22}^{(1)} \end{pmatrix}$
- Build a ratio for the 2×2 correlation matrix: $R(t) = C^{-\frac{1}{2}}(t)C^\gamma(t)C^{-\frac{1}{2}}(t)$
- Time dependence of $R(t)$ yields

$$R(t) = \begin{pmatrix} 1 + 2N_{00}^{(1)} + E_0^{(1)}t & N_{20}^{(1)}e^{(E_2-E_0)t/2} + N_{02}^{(1)}e^{(E_0-E_2)t/2} \\ N_{20}^{(1)}e^{(E_2-E_0)t/2} + N_{02}^{(1)}e^{(E_0-E_2)t/2} & 1 + 2N_{22}^{(1)} + E_2^{(1)}t \end{pmatrix}$$

  - $E_s^{(1)} = E_s^\gamma - E_s$ can be used to determine δ_s^γ, $s = 0, 2$
  - $N_{ll'}^{(1)}$ can be used to construct U^γ and compute $A_s^\gamma = \langle(\pi\pi)_s^\gamma|H_W|K^0\rangle$

Need to modify Lüscher quantization condition and Lellouch-Lüscher relation to include EM effects

Coulomb potential with periodic boundary condition

Encode long-range EM interaction in the finite box – QED$_L$

- Coulomb potential in periodic box $V_L(\mathbf{r}) = \sum_n V(\mathbf{r} + \mathbf{n}L)$
  - $\forall \mathbf{n}$, $V(\mathbf{r} + \mathbf{n}L)$ have impact on small-$\mathbf{r}$ region and cause divergence
- Modify $V_L(\mathbf{r}) \to \hat{V}_L(\mathbf{r}) = V_L(\mathbf{r}) - \frac{1}{L^3}\int d^3\mathbf{r}\, V(\mathbf{r})$ to remove the divergence
  - This is equivalent to remove zero mode: $\hat{V}_L(\mathbf{r}) = \frac{4\pi\alpha_e}{L^3}\sum_{\mathbf{p}\neq\mathbf{0}} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2}$
- We get Lüscher quantization $\phi_c(E) + \delta(E) = n\pi$ with $\eta = \frac{\alpha_e \mu}{k} = \frac{\alpha_e}{v}$

$$\cot\phi_c(E) = (1+\pi\eta)\frac{1}{\pi}\frac{1}{kL}\sum_{\mathbf{n}}\frac{1}{-\mathbf{n}^2 + (\frac{kL}{2\pi})^2}$$

$$+\lim_{r\to 0} 8\pi\eta \left\{ \sum_{\mathbf{n}\neq\mathbf{m}} \frac{e^{i\mathbf{n}\cdot\mathbf{r}\frac{2\pi}{L}}}{\pi(2\pi)^4}\frac{1}{\mathbf{n}^2 - (\frac{kL}{2\pi})^2}\frac{1}{(\mathbf{n}-\mathbf{m})^2}\frac{1}{\mathbf{m}^2 - (\frac{kL}{2\pi})^2} - \frac{1}{4\pi}\ln(1/kr) + \frac{1}{4\pi} \right\}$$

(See also formula for scattering length [Bean & Savage, 1407.4846])

- However, $\hat{V}_L$ introduces $O(1/L)$ FV effects

$$\delta V(\mathbf{r}) \equiv \hat{V}_L(\mathbf{r}) - V(\mathbf{r}) = \left(\frac{1}{L^3}\sum_{\mathbf{p}\neq\mathbf{0}} - \int \frac{d^3\mathbf{p}}{(2\pi)^3}\right)\frac{4\pi\alpha_e}{p^2}e^{i\mathbf{p}\cdot\mathbf{r}}, \quad \lim_{\mathbf{r}\to 0}\delta V(\mathbf{r}) = -\kappa\frac{\alpha_e}{L} \approx -2.8\frac{\alpha_e}{L}$$

Coulomb potential with truncated range $R_T \leq L/2$

Not unique way to build periodic potential

$$V_L^{(T)}(\mathbf{r}) = \sum_{\mathbf{n}} V^{(T)}(\mathbf{r} + \mathbf{n}L), \quad V^{(T)}(\mathbf{r}) = \begin{cases} \alpha_e/r, & \text{for } r < R_T \\ 0, & \text{for } r > R_T \end{cases}$$

Lüscher's quantization condition holds for $V_s(\mathbf{r}) + V^{(T)}(\mathbf{r})$

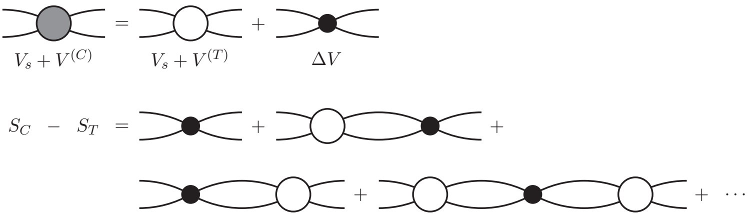
$$\phi(q) + \delta_T(k) = n\pi, \quad q = \frac{kL}{2\pi}$$

- Unfortunately $\delta_T(k)$ is unphysical

Relate the truncated scattering phase to the physical one

$$S_C = \langle E', -, C | E, +, C \rangle = 2\pi\delta(E - E') e^{2i\delta_C}$$
$$S_T = \langle E', -, T | E, +, T \rangle = 2\pi\delta(E - E') e^{2i\delta_T}$$

Skeleton expansion in Potential Theory



Truncation effects in scattering phase

The relation for scattering amplitude

$$S_C = S_T - i\,2\pi\delta(E - E')\,\langle E, -, T|\Delta V|E, +, T\rangle$$

The relation for scattering phase

$$\delta_C = \delta_T - \frac{1}{2}\langle E, +, T|\Delta V|E, +, T\rangle$$

- $\Delta V(r)$ is non-zero only for $r > R_T$
- For $\psi(r) = \langle r|E, +, T\rangle$, the functional form is known for $r > R_T$

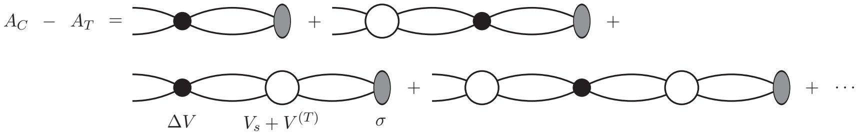
$$\psi(r) = \sqrt{\frac{\mu}{\pi k}}\frac{\sin(kr + \delta_T)}{r}, \quad \text{for S-wave}$$

- Correction to scattering phase can be evaluated

$$\langle E, +, T|\Delta V|E, +, T\rangle = \int_{R_T}^{R_\infty} d^3\mathbf{r}\left[\sqrt{\frac{\mu}{\pi k}}\frac{\sin(kr + \delta_T)}{r}\right]^2 \frac{\alpha_e}{r}$$

$\sigma \to \pi\pi$ decay amplitude

$$A_C = \langle E, -, C|\sigma\rangle, \quad A_T = \langle E, -, T|\sigma\rangle$$



The relation for decay amplitude

$$A_C - A_T = \langle E, -, T|\Delta V\, G_{TS}^{(+)}|\sigma\rangle = \langle E, -, T|\Delta V\, G_0^{(+)} \left(1 + V_{TS}\, G_{TS}^{(+)}\right)|\sigma\rangle$$

- ΔV is non-zero at $r > R_T$; $V_{TS} = V_s + V^{(T)}$ is non-zero at $r < R_T$
- The free Green function $\langle \mathbf{r}|G_0^{(+)}|\mathbf{r}'\rangle$ for $r > R_T$ and $r' < R_T$ is given by

$$\langle \mathbf{r}|G_0^{(+)}|\mathbf{r}'\rangle = \int \frac{dE'}{2\pi}\langle \mathbf{r}|E'\rangle \frac{1}{E - E' + i\varepsilon}\langle E'|\mathbf{r}'\rangle \quad \xrightarrow{r>r'} \quad -\frac{1}{2}\sqrt{\frac{\mu}{\pi k}}\frac{e^{ikr}}{r}\langle E|\mathbf{r}'\rangle$$

Truncation effect appears as an overall factor, and cancels in the ratio

$$A_C - A_T = \int d^3\mathbf{r}\, \psi(r)\frac{\alpha}{r}\left(-\frac{1}{2}\sqrt{\frac{\mu}{\pi k}}\frac{e^{ikr}}{r}\right) A_T$$

Solve the coupled-channel problem

Generalize Lüscher quantization condition to coupled-channel scattering

[He, XF, Liu, hep-lat/0504019; Hansen & Sharpe, 1204.0826]

$$\sin(\delta_0 + \phi(q_{+-}))\sin(\delta_2 + \phi(q_{00})) + \sin^2\theta^\gamma \sin(\delta_0 - \delta_2)\sin(\phi(q_{+-}) - \phi(q_{00})) = 0$$

where $q_c = \frac{k_c L}{2\pi}$, $c = +-, 00$, $k_{+-} = \sqrt{E^2/4 - m_{\pi^+}^2}$, $k_{00} = \sqrt{E^2/4 - m_{\pi^0}^2}$, $E \approx m_K$

Quantization condition involves three unknown quantities: δ_0, δ_2, θ^γ

- Solution 1: tune three volumes to make $E_0(L_1) = E_1(L_2) = E_2(L_3) = E$?
- Solution 2: Introduce functional form for E-dependence of δ_0, δ_2, θ'

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where $q_c=\frac{k_c L}{2\pi}$, $c=+-,00$, $k_{+-}=\sqrt{E^2/4-m_{\pi^+}^2}$, $k_{00}=\sqrt{E^2/4-m_{\pi^0}^2}$, $E\approx m_K$

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- Solution 1: tune three volumes to make $E_0(L_1)=E_1(L_2)=E_2(L_3)=E$?
- Solution 2: Introduce functional form for E-dependence of δ_0, δ_2, θ'

If mixing is caused by EM effect, the situation is simpler

- Note that $\sin(\phi(q_{+-})-\phi(q_{00}))$ is an $O(\alpha_e)$ quantity

$$\sin(\phi(q_{+-})-\phi(q_{00}))=\frac{d\phi}{dq}(q_{+-}^{(1)}-q_{00}^{(1)})+O(\alpha_e^2)$$

  where $q_{+-}^{(1)}$ and $q_{00}^{(1)}$ originate from $O(\alpha_e)$ correction to m_{π^+} and m_{π^0}

- Tune the volume to have $E_{I=0}^{(0)}=M_K$

$$\sin(\delta_0+\phi(q_{+-}))=\delta_0^{(1)}+\frac{d\phi}{dq}q_{+-}^{(1)}+\frac{d(\delta_0^{(0)}+\phi)}{dE}E_{I=0}^{(1)}+O(\alpha_e^2)$$

$$\sin(\delta_2+\phi(q_{00}))=\sin(\delta_2^{(0)}-\delta_0^{(0)})+O(\alpha_e)$$

Coupled-channel Lüscher quantization condition is simplified

- Tune the volume to have $E^{(0)}_{I=0} = M_K$

$$\delta_0^{(1)} + \left(\cos^2\theta\, q_{+-}^{(1)} + \sin^2\theta\, q_{00}^{(1)}\right)\frac{\partial\phi}{\partial q} + E^{(1)}_{I=0}\frac{\partial}{\partial E}\left(\delta_0^{(0)} + \phi\right) = 0$$

- Tune the volume to have $E^{(0)}_{I=2} = M_K$

$$\delta_2^{(1)} + \left(\cos^2\theta\, q_{00}^{(1)} + \sin^2\theta\, q_{+-}^{(1)}\right)\frac{\partial\phi}{\partial q} + E^{(1)}_{I=2}\frac{\partial}{\partial E}\left(\delta_2^{(0)} + \phi\right) = 0$$

The relations for $E^{(1)}_{I=0} \to \delta_0^{(1)}$ and $E^{(1)}_{I=2} \to \delta_2^{(1)}$ are established

Extract infinite-volume amplitude

Coupled-channel Lellouch-Lüscher relation [Hansen & Sharpe, 1204.0826]

- Tune the volume to have $E_{I=2} = M_K \quad \Rightarrow \quad \sin(\delta_2 + \phi) = O(\alpha_e)$

$$\left(2\frac{d\Delta}{dE}\right)^{\frac{1}{2}} |A^\gamma_{2,L}| = \underbrace{\sqrt{\frac{\sin(\delta_0+\phi_{00})\sin(\delta_0+\phi_{+-})}{\sin(\delta_0-\delta_2)}}}_{=\sqrt{\sin(\delta_0-\delta_2)}+O(\alpha_e)} |A^\gamma_2| - s \underbrace{\frac{\sin 2\theta \sin(\phi_{00}-\phi_{+-})}{2\sqrt{\sin(\delta_0-\delta_2)}}}_{=O(\alpha_e)} |A^\gamma_0|$$

where $\phi_c = \phi(q_c)$, $c = \pm, 00$ and $s = \mathrm{sgn}(A^\gamma_0/A^\gamma_2)$

- On the LHS, $|A^\gamma_{2,L}|$ is the amplitude calculated in the finite volume

$$\Delta = \sin(\delta_0+\phi_{+-})\sin(\delta_2+\phi_{00}) + \sin^2\theta' \sin(\delta_0-\delta_2)\sin(\phi_{+-}-\phi_{00})$$

$\Rightarrow \quad \Delta = 0$ is the coupled-channel quantization condition

$$\Rightarrow \quad \frac{d\Delta}{dE} = \frac{d(\delta_2+\phi)}{dE}\sin(\delta_0-\delta_2) + O(\alpha_e)$$

  - If turn off EM $\quad \Rightarrow \quad$ single-channel Lellouch-Lüscher relation

$$\left(2\frac{d(\delta_2+\phi)}{dE}\right)^{\frac{1}{2}} |A^\gamma_{2,L}| = |A^\gamma_2|$$

Conclusion

EW physics is a place where mulitparticle can play important role

$\Rightarrow$ Precision flavor physics requires to control the FV effects

A lot of interesting things we can do

$\Rightarrow$ Multiparticles can play role in both final and intermediate state

As the precision increases, our research frontier expand

$\Rightarrow$ Adding electromagnetism is new frontier for lattice QCD