Lorentz Invariance and QCD Equation of Motion Relations for GPDs and GTMDs

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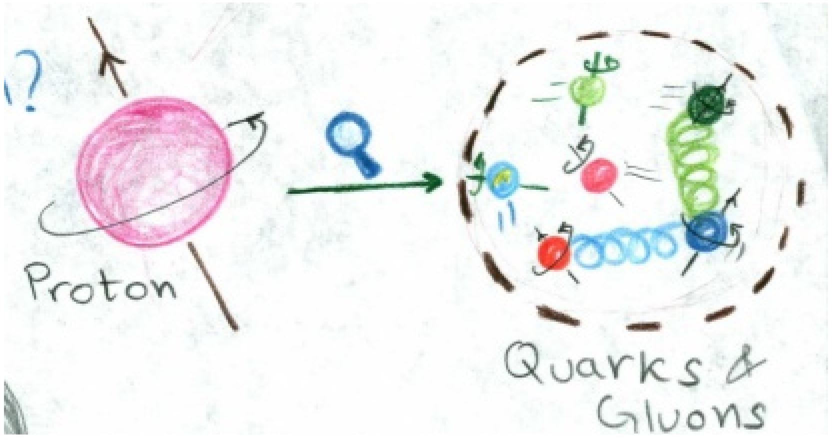


Probing Nucleons and Nuclei in High
Energy Collisions – Week 3
INT, Seattle

People Involved

- Simonetta Liuti, University of Virginia
- Michael Engelhardt, New Mexico State University
- Aurore Courtoy, UNAM Mexico

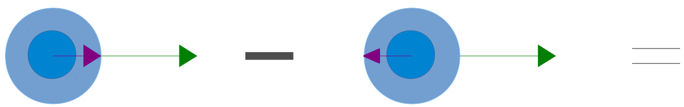
AR, Engelhardt and Liuti arxiv:1709.05770

AR, Courtoy, Engelhardt and Liuti PRD 94 (2016)

Outline

- Spin Crisis !
- Orbital Angular Momentum
  - GTMD definition
  - GPD definition Ji
- What's the connection? Lorentz Invariance Relations
- Equation of Motion
- Quark Gluon Structure of Twist Three GPDs
- Conclusions

Proton Spin Crisis



$g_1^P(x)$ Quark Spin Contribution

$$\frac{1}{2M}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-i\lambda x}\left\langle P,S\left|\bar{\psi}\left(\frac{\lambda n}{2}\right)\gamma_\mu\gamma_5\psi\left(-\frac{\lambda n}{2}\right)\right|P,S\right\rangle = \Lambda g_1(x)p_\mu + g_T(x)S_{\perp_\mu}$$

Measured by EMC experiment in 1980s to be small, present values about 30% of total !!

Proton Spin Crisis

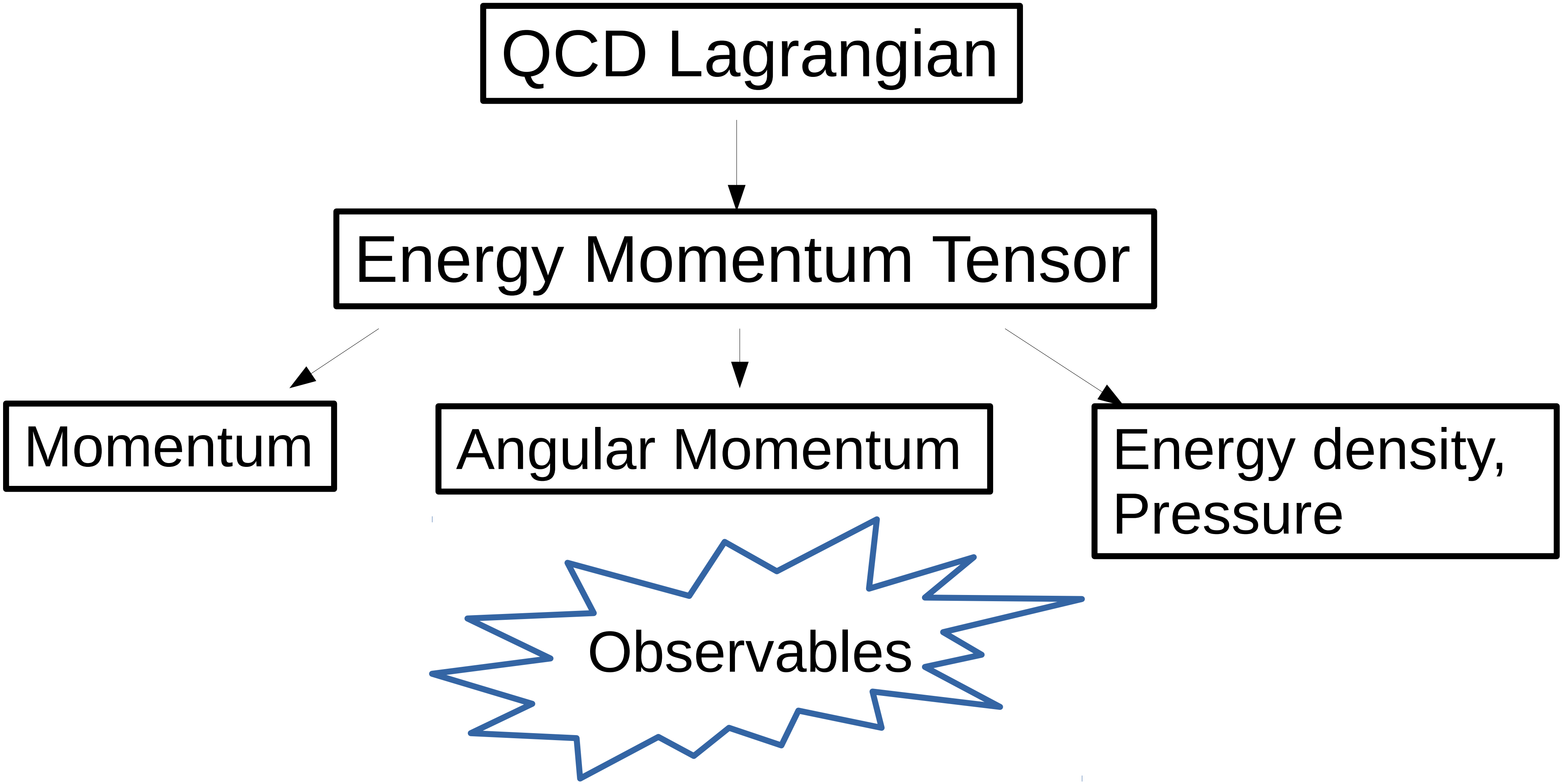
$= \; g_1^P(x)$ Quark Spin Contribution

$$\frac{1}{2M}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-i\lambda x}\left\langle P,S\left|\bar{\psi}\left(\frac{\lambda n}{2}\right)\gamma_\mu\gamma_5\psi\left(-\frac{\lambda n}{2}\right)\right|P,S\right\rangle = \Lambda g_1(x)p_\mu + g_T(x)S_{\perp_\mu}$$

Measured by EMC experiment in 1980s to be small, present values about 30% of total !!

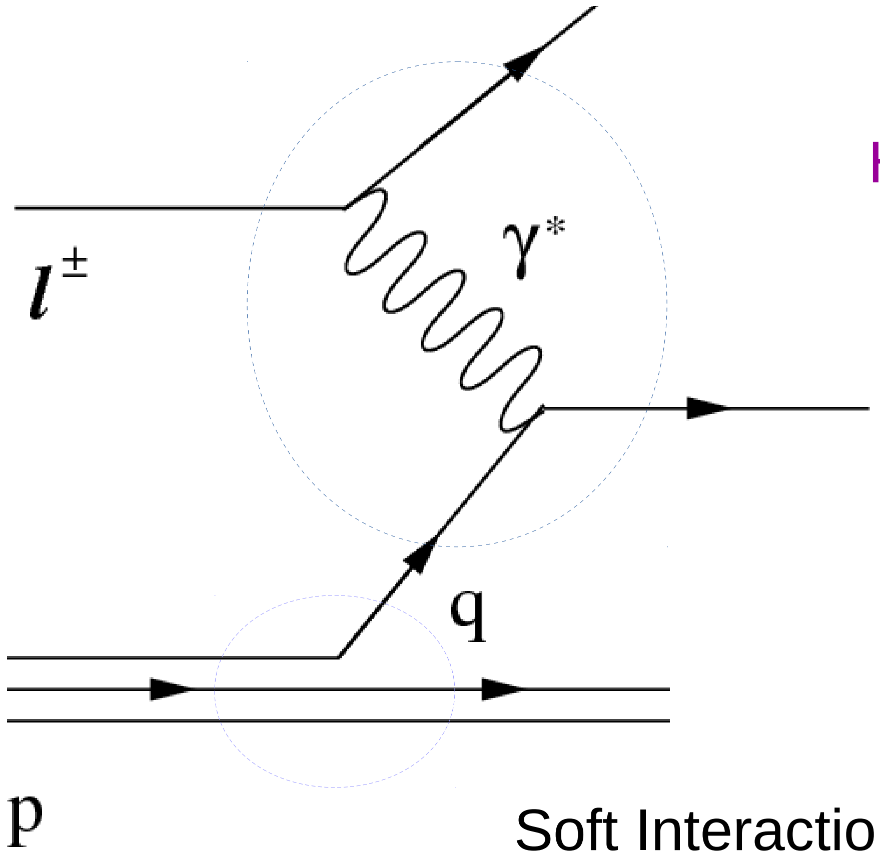
What are other sources ?
Partonic Orbital Angular Momentum

QCD Energy Momentum Tensor



Deeply Virtual Compton Scattering, moments of GPDs etc.

Hard and Soft Parts



Hard Interaction

Deep Inelastic Scattering

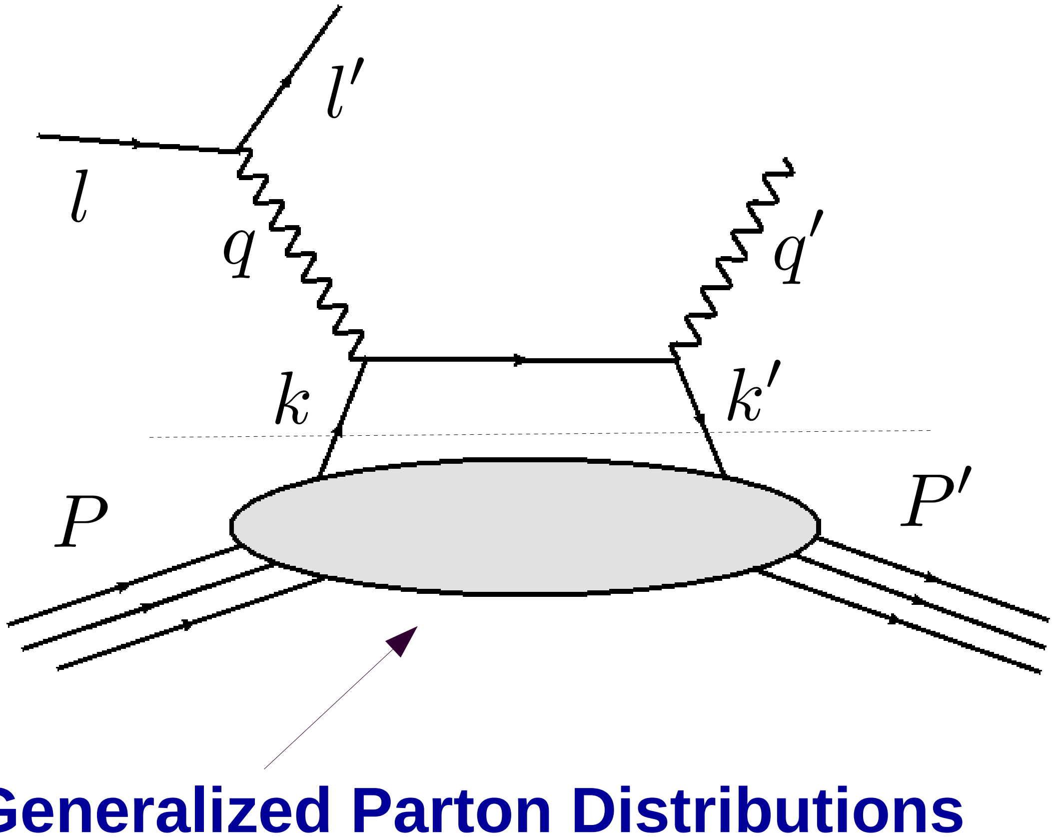
Soft Interaction

$$\int \frac{dz^-}{2\pi} e^{ik^+z^-} \langle p, S \mid \bar{\psi}(-z/2)\gamma^+\psi(z/2) \mid p, S\rangle_{z^+=z_T=0} = f_1(x)$$

$$a^\pm = \frac{a^0 \pm a^3}{\sqrt{2}}$$

Exclusive Processes

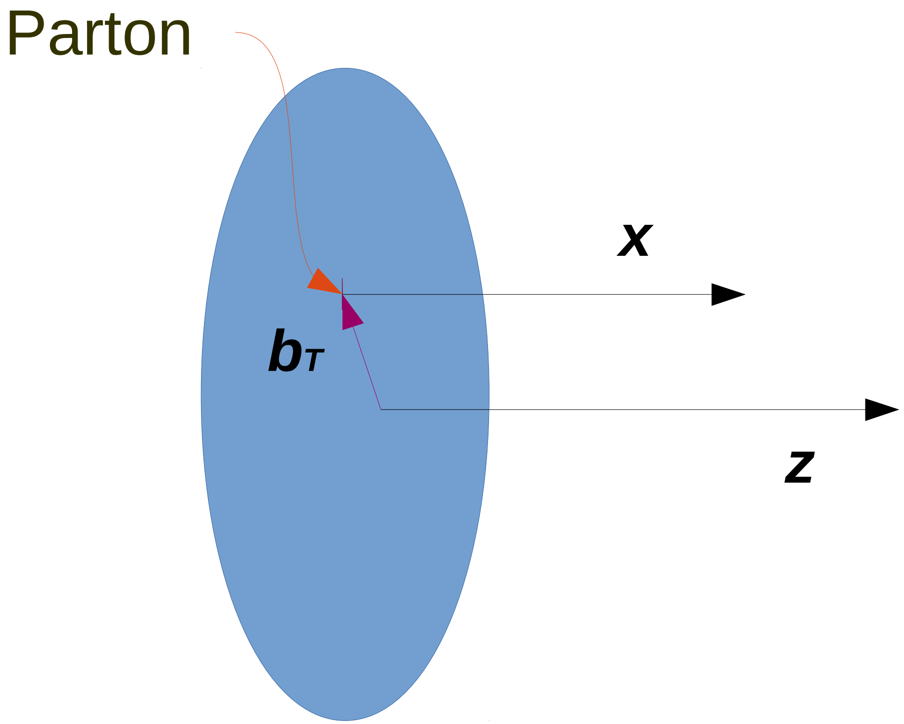
- Need a high energy photon to probe the partons
- The proton needs to remain intact to access spatial distribution
- **Deeply Virtual Compton Scattering**



$$\int \frac{dz_-}{2\pi} e^{ixP^+z^-} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\gamma^+\psi(z/2) \mid p, \Lambda\rangle_{z^+=z_T=0} = \overline{U}(P', S') \left[\gamma^+ H + \frac{i\sigma^{+\Delta}}{2M} E\right] U(P, S)$$

Generalized Parton Distributions

- GPDs are the Fourier transform of the spatial distribution of partons in protons and neutrons.



GPD based definition of Angular Momentum

$$J^i_{q,g} = \frac{1}{2}\epsilon^{ijk}\int d^3x\,\left(T^{0k}_{q,g}x^j - T^{0j}_{q,g}x^k\right)$$

$$\vec{J}_q = \int d^3x\psi^\dagger\left[\vec{\gamma}\gamma_5 + \vec{x}\times i\vec{D}\right]\psi \qquad \vec{J}_g = \int d^3x\left(\vec{x}\times\left(\vec{E}\times\vec{B}\right)\right)$$

$$J_q = \frac{1}{2}\int_{-1}^{1} dx x(H_q(x,0,0) + E_q(x,0,0))$$

Xiangdong Ji, PRL 78.610,1997

To access OAM, we take the difference between total angular momentum and spin

$$\mathcal{L}_q = J_q - \frac{1}{2}\Delta\Sigma$$

OAM Total Spin

Direct description of OAM

- The moment in x of the GPD G_2 shown to be OAM

$$\int dx x G_2 = \int dx x (H+E) - \int dx \tilde{H}$$

Kiptily and Polyakov, Eur Phys J C 37 (2004)

Hatta and Yoshida, JHEP (1210), 2012

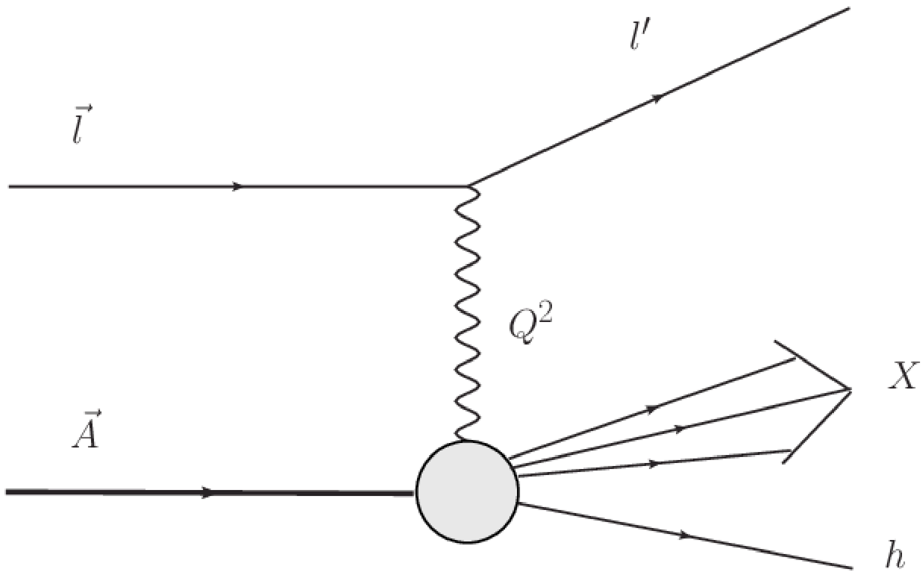
$$G_2 \equiv \tilde{E}_{2T} + H + E$$

$$\mathcal{F}_{\perp\mu}(x,\xi,\Delta) = \overline{U}(P',S')\Big\{ \underline{(H+E)}\,\gamma_\mu^\perp + G_1 \frac{\Delta_\mu^\perp}{2M} + \underline{G_2}\gamma_\mu^\perp + G_3 \Delta_\mu^\perp \hat{n} + G_4 i\epsilon_{\mu\nu}^\perp \Delta_\perp^\nu \hat{n}\gamma_5 \Big\} U(P,S)$$

$$F_{\Lambda,\Lambda'}^{[\gamma^i]}(x,\xi,\Delta) = \frac{1}{2(P^+)^2}\bar{U}\left[i\sigma^{+i} H_{2T} + \frac{\gamma^+ \Delta_T^i}{2M} E_{2T} + \frac{P^+ \Delta_T^i}{M^2} \tilde{H}_{2T} - \frac{P^+ \gamma^i}{M} \underline{\tilde{E}_{2T}} \right] U$$

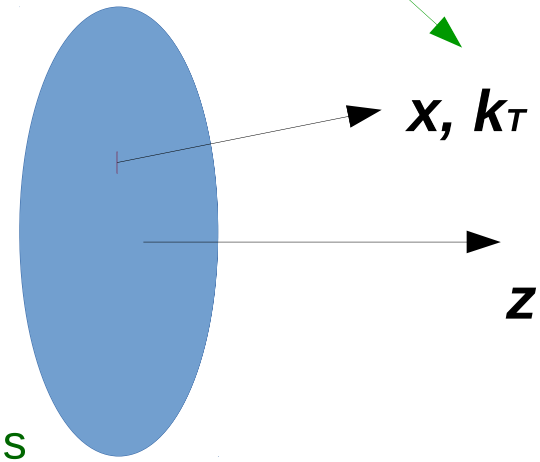
Meissner Metz and Schlegel, JHEP 0908 (2009)

Intrinsic Transverse Momentum



The partonic transverse momentum is correlated with the momentum of the observed hadron

Semi inclusive Deep Inelastic Scattering

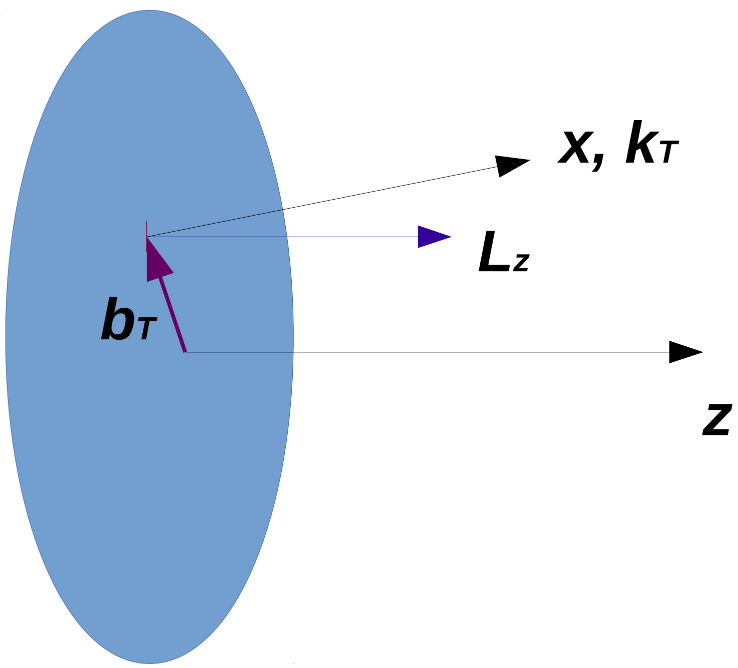


However, the target does not remain intact, no access to the spatial distribution of partons

Transverse Momentum Distributions

Partonic Orbital Angular Momentum II

- Consider measuring both the intrinsic transverse momentum and the spatial distribution of partons
- $L_{q,z} = \mathbf{b_T} \times \mathbf{k_T}$



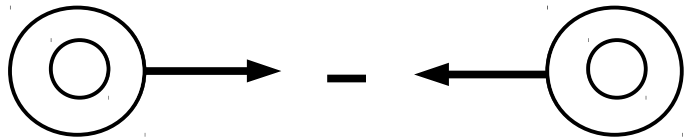
$$W_{\Lambda,\Lambda'}^{[\gamma^+]} = \frac{1}{2M}\bar{U}(p',\Lambda')[F_{11} + \frac{i\sigma^{i+}k_T^i}{\bar{p}_+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{\bar{p}_+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{M^2}F_{14}]U(p,\Lambda)$$

Generalized Transverse Momentum Distributions (related by Fourier transform to Wigner Distributions)

Meissner Metz and Schlegel,
JHEP 0908 (2009)

GTMDs that describe OAM

- How does F_{14} connect to OAM ?



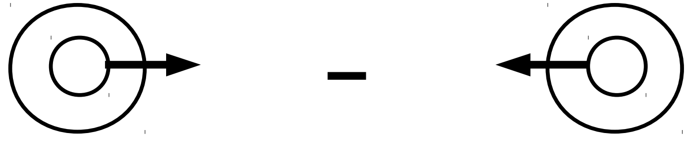
Unpolarized quark in a longitudinally polarized proton

$$\mathcal{W}(x,\mathbf{k}_T,\mathbf{b}) = \int \frac{d^2\Delta_T}{(2\pi)^2} e^{ib.\Delta_T} \left[W^{\gamma^+}_{++} - W^{\gamma^+}_{--} \right]$$

$$L = \int dx \int d^2k_T \int d^2\mathbf{b} (\mathbf{b}\times\mathbf{k}_T) \mathcal{W}(x,\mathbf{k}_T,\mathbf{b}) = -\int dx \int d^2k_T \frac{k_T^2}{M^2} F_{14}$$

Lorce et al PRD84, (2011)
Hatta Phys. Lett. B708 (2011)

- Another GTMD relevant to OAM



G_{11} describes a longitudinally polarized quark in an unpolarized proton. Measures spin orbit correlation.

The Two Definitions

- Weighted average of b_T X k_T

$$L_z = -\int dx \int d^2k_T \frac{k_T^2}{M^2} F_{14}$$

- Difference of total angular momentum and spin

$$\mathcal{L}_q = J_q - \frac{1}{2}\Delta\Sigma$$

$$\frac{1}{2}\int_{-1}^{1} dx x(H_q + E_q) \qquad \frac{1}{2}\int_{-1}^{1} dx \tilde{H}_q$$

The Two Definitions

- Weighted average of b_T X k_T

GTMD

$$L_z = -\int dx \int d^2 k_T \frac{k_T^2}{M^2} F_{14} \rightarrow F_{14}^{(1)}$$

- Difference of total angular momentum and spin

GPD

$$\mathcal{L}_q = J_q - \frac{1}{2}\Delta\Sigma$$

$$\frac{1}{2}\int_{-1}^{1} dx\, x(H_q + E_q) \qquad\qquad \frac{1}{2}\int_{-1}^{1} dx\, \tilde{H}_q$$

Is there a connection ?

- We find that

$$F_{14}^{(1)}(x) = \int_x^1 dy \left(\tilde{E}_{2T}(y) + H(y) + E(y)\right)$$

- This is a form of Lorentz Invariant Relation (LIR)
- This is a distribution of OAM in x
- Derived for a straight gauge link

Derivation of Generalized LIRs

To derive these we look at the parameterization of the quark quark correlator function at different levels

$$\int \frac{d^4 z}{2\pi} e^{ik.z} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle$$

Generalized Parton Correlation Functions (GPCFS)

Integrate over k^-

Meissner Metz and Schlegel, JHEP 0908 (2009)

$$\int \frac{dz_- d^2 z_T}{2\pi} e^{ixP^+ z^- - k_T . z_T} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle_{z^+=0}$$

GTMDs

Integrate over k_T

$$\int \frac{dz_-}{2\pi} e^{ixP^+ z^-} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle_{z^+=z_T=0}$$

GPDs

Generalized Lorentz Invariance Relations

- Parametrization of the quark quark correlator at different levels
- LIRs occur because the number of GPCFs is less than the number of GTMDs.

$$\mathcal{W}^{[\gamma^\mu]}_{\Lambda\Lambda'} = \frac{\bar{U}U}{M}(P^\mu A_1^F + k^\mu A_2^F + \Delta^\mu A_3^F) + i\frac{\bar{U}\sigma^{\mu k}U}{M}A_5^F + i\frac{\bar{U}\sigma^{\mu\Delta}U}{M}A_6^F + i\frac{\bar{U}\sigma^{k\Delta}U}{M^3}(P^\mu A_8^F + k^\mu A_9^F + \Delta^\mu A_{17}^F)$$

Integrate over k^-

Explicit k_T coefficient

$$W^{[\gamma^+]}_{\Lambda,\Lambda'} = \frac{1}{2M}\bar{U}(p',\Lambda')\left[F_{11} + \frac{i\sigma^{i+}k_T^i}{\bar{p}_+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{\bar{p}_+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{M^2}F_{14}\right]U(p,\Lambda)$$

Integrate over k_T

$$F^{[\gamma^i]}_{\Lambda,\Lambda'} = \frac{1}{2(P^+)^2}\bar{U}\left[i\sigma^{+i}H_{2T} + \frac{\gamma^+\Delta_T^i}{2M}E_{2T} + \frac{P^+\Delta_T^i}{M^2}\tilde{H}_{2T} - \frac{P^+\gamma^i}{M}\tilde{E}_{2T}\right]U$$

$$\int \frac{d^4z}{2\pi}e^{ik.z}\langle p',\Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p,\Lambda\rangle$$

Generalized Lorentz Invariance Relations

- As the quark quark correlator is non-local, the parametrization depends on choice of gauge link
- At the completely unintegrated level, we have no knowledge of the light-cone direction for a straight gauge link, hence fewer functions occur at this level for this case as compared to staple gauge link case

Gauge link

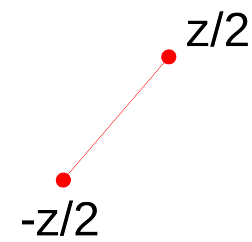
$$\int \frac{d^4z}{2\pi} e^{ik.z} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\mathcal{U}\Gamma\psi(z/2) \mid p, \Lambda\rangle$$

Non local operator

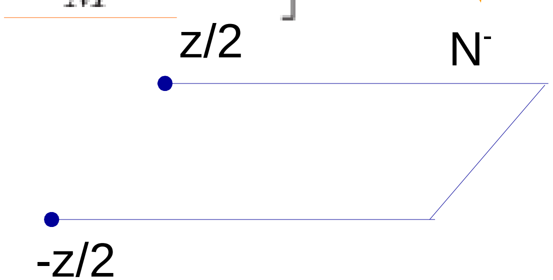
Generalized Lorentz Invariance Relations

$$\int \frac{d^4z}{2\pi} e^{ik.z} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\mathcal{U}\Gamma\psi(z/2) \mid p, \Lambda \rangle$$

$$\mathcal{W}^{[\gamma^\mu]}_{\Lambda\Lambda'} = \frac{\bar{U}U}{M}(P^\mu A_1^F + k^\mu A_2^F + \Delta^\mu A_3^F) + i\frac{\bar{U}\sigma^{\mu k}U}{M}A_5^F + i\frac{\bar{U}\sigma^{\mu\Delta}U}{M}A_6^F + i\frac{\bar{U}\sigma^{k\Delta}U}{M^3}(P^\mu A_8^F + k^\mu A_9^F + \Delta^\mu A_{17}^F)$$



$$\begin{aligned} W^{[\gamma^\mu]}_{\lambda\lambda'}&(P,k,\Delta,N;\eta) \\ = \bar{u}(p',\lambda')\Bigg[&\frac{P^\mu}{M}A_1^F + \frac{k^\mu}{M}A_2^F + \frac{\Delta^\mu}{M}A_3^F + \frac{N^\mu}{M}A_4^F + \frac{i\sigma^{\mu k}}{M}A_5^F + \frac{i\sigma^{\mu\Delta}}{M}A_6^F + \frac{i\sigma^{\mu N}}{M}A_7^F \\ &+ \frac{P^\mu\, i\sigma^{k\Delta}}{M^3}A_8^F + \frac{k^\mu\, i\sigma^{k\Delta}}{M^3}A_9^F + \frac{N^\mu\, i\sigma^{k\Delta}}{M^3}A_{10}^F + \frac{P^\mu\, i\sigma^{kN}}{M^3}A_{11}^F + \frac{k^\mu\, i\sigma^{kN}}{M^3}A_{12}^F \\ &+ \frac{N^\mu\, i\sigma^{kN}}{M^3}A_{13}^F + \frac{P^\mu\, i\sigma^{\Delta N}}{M^3}A_{14}^F + \frac{\Delta^\mu\, i\sigma^{\Delta N}}{M^3}A_{15}^F + \frac{N^\mu\, i\sigma^{\Delta N}}{M^3}A_{16}^F\Bigg]u(p,\lambda)\,, \end{aligned} \quad (2.19)$$



Generalized Lorentz Invariance Relations

- The same set of As describe the whole vector sector.

γ^+

$$F_{14}^{(1)} = \int d\sigma d\sigma' d\tau \frac{M^3}{2} J \left[A_8^F + x A_9^F\right] \qquad J = \sqrt{x\sigma - \tau - \frac{x^2 P^2}{M^2} - \frac{\Delta_T^2 \sigma'^2}{M^2}}$$

$$H + E = \int d\sigma d\sigma' d\tau \frac{M^3}{J} \sigma' A_5^F + A_6^F + \left(\frac{\sigma}{2} - \frac{x P^2}{M^2}\right)\left(A_8^F + x A_9^F\right)$$

γ_T^i

$$\tilde{E}_{2T} = \int d\sigma d\sigma' d\tau \frac{M^3}{J} \left[\left(x\sigma - \tau - \frac{x^2 P^2}{M^2} - \frac{\Delta_T^2 \sigma'^2}{M^2}\right) A_9^F - \sigma' A_5^F - A_6^F\right]$$

$$\sigma \equiv \frac{2k.P}{M^2}, \qquad \tau \equiv \frac{k^2}{M^2}, \qquad \sigma' \equiv \frac{k.\Delta}{\Delta^2} = \frac{k_T.\Delta_T}{\Delta_T^2}$$

$$-\frac{dF_{14}^{(1)}}{dx} = \tilde{E}_{2T} + H + E$$

$$F_{14}^{(1)}(x) = \int_x^1 dy \left(\tilde{E}_{2T}(y) + H(y) + E(y)\right)$$

Distribution of OAM in x !

k_T^2 moment of a twist two function

Twist three function

An analogy

- The proton electromagnetic current is parameterized by the Dirac and Pauli form factors

$$J^\mu = e\overline{U}(P', S')\left[\gamma^\mu F_1 + \frac{i\sigma^{\mu\Delta}}{2M}F_2\right]U(P, S)$$

- We know that the vector GPDs should integrate to some combination of the same form factors irrespective of twist

$$\int dx H(x, 0, t) = F_1(t)$$

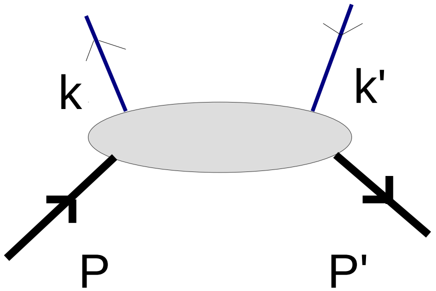
Higher Twist

$$\int \frac{dz_-}{2\pi} e^{ixP^+z^-} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle_{z^+=z_T=0}$$

$$\gamma^+, \gamma^+\gamma^5, \sigma^{i+}\gamma^5$$

Leading twist – twist 2

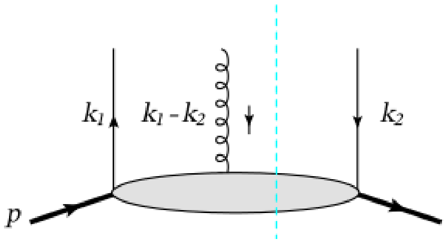
- Involve only good components
- Simple interpretation in terms of parton densities



$$\gamma^i, \gamma^i\gamma^5, \sigma^{ij}\gamma^5, 1, \gamma^5, \sigma^{+-}\gamma^5$$

Higher twist – twist 3

- Involve one good and one bad component
- The bad component represents a quark gluon composite

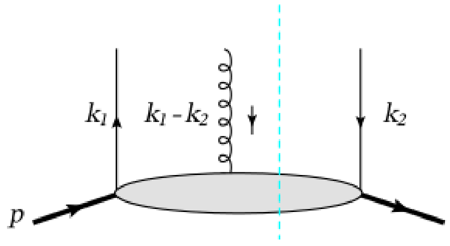


Collinear Picture : Transverse Quark Current, Higher Twist

$$\bar{\psi}(-z/2)\gamma^+\psi(z/2)$$ → Leading order quark current

Number density interpretation allowed

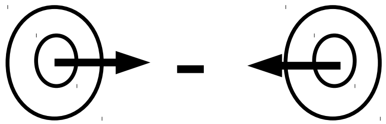
$$\bar{\psi}(-z/2)\gamma_T^i\psi(z/2)$$ → Transverse quark current, implicitly involves quark gluon interactions



Number density interpretation problematic

Through LIRs explore the connection between quark gluon interactions and intrinsic transverse momentum

Generalized Lorentz Invariance Relations



Axial Vector

$$\frac{dG_{11}^{e(1)}}{dx} = -\left(2\tilde{H}'_{2T} + E'_{2T}\right) - \tilde{H}$$

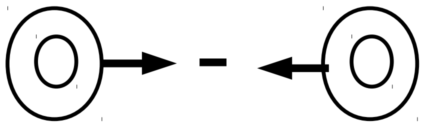
$$\frac{dG_{12}^{e(1)}}{dx} = H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T} - \left(1 + \frac{\Delta_T^2}{2M^2}\right)\tilde{H}$$

Twist two

Twist three

Vector

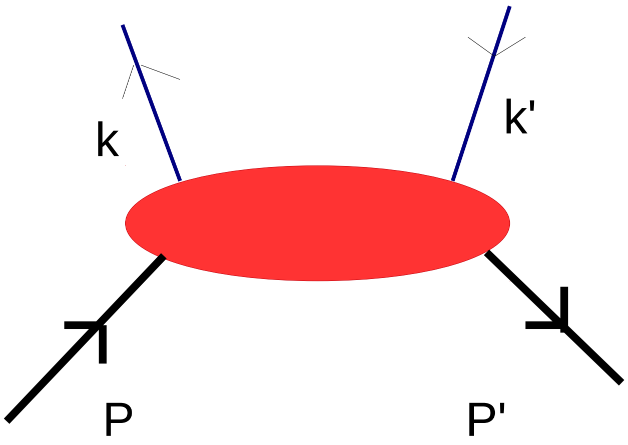
$$\frac{dF_{14}^{e(1)}}{dx} = \tilde{E}_{2T} + H + E$$



Intrinsic transverse momentum

Quark gluon interactions

Intrinsic Momentum vs Momentum Transfer Δ

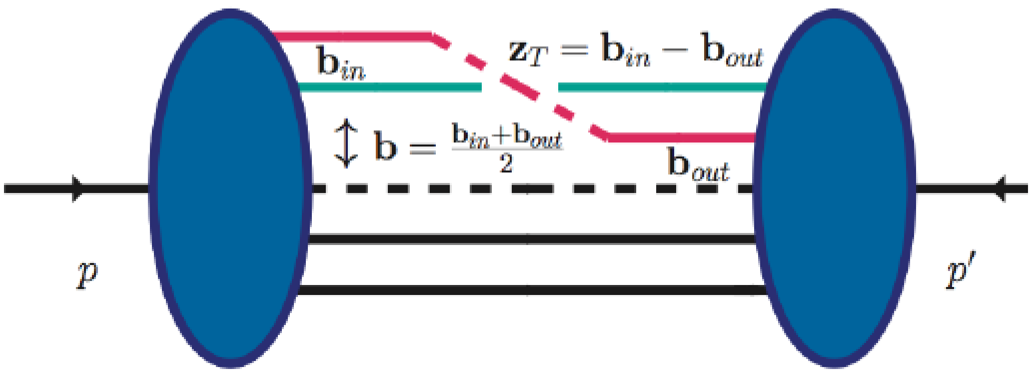


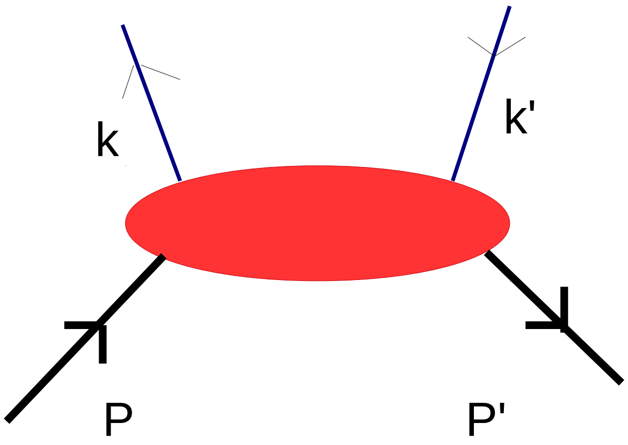
Courtoy et al PhysLett B731, 2013

Burkardt,Phys Rev D62, 2000

$$\int \frac{dz_-}{2\pi} e^{ixP^+z^-} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda \rangle_{z^+=z_T=0}$$

Intrinsic Momentum vs Momentum Transfer Δ





$$k \longleftrightarrow z$$

$$\Delta \longleftrightarrow b$$

Courtoy et al PhysLett B731, 2013

Burkardt,Phys Rev D62, 2000

$$\int \frac{dz_-}{2\pi} e^{ixP^+z^-} \langle p', \Lambda' \mid \bar{\psi}(-z/2)\Gamma\psi(z/2) \mid p, \Lambda\rangle_{z^+=z_T=0}$$

Equations of Motion Relations

How do we obtain these ?

$$
\begin{aligned}
(i\not{D} - m)\psi(z_{out}) &= (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0, \\
\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) &= \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) = 0
\end{aligned}
$$

Equations of Motion Relations

How do we obtain these ?

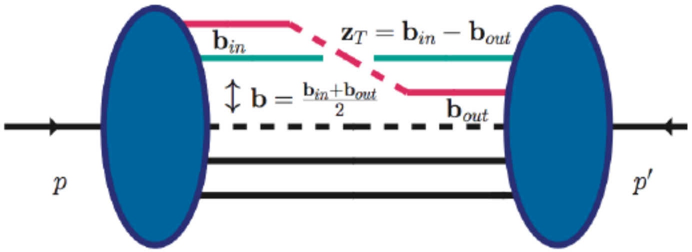
$$\begin{aligned}\mathcal{U} i\sigma^{i+}\gamma_5(i\not{D} - m)\psi(z_{out}) &= \mathcal{U} i\sigma^{i+}\gamma_5(i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0, \\ \bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m)i\sigma^{i+}\gamma_5\mathcal{U} &= \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m)i\sigma^{i+}\gamma_5\mathcal{U} = 0\end{aligned}$$

Equations of Motion Relations

How do we obtain these ?

$$\mathcal{U} i\sigma^{i+}\gamma_5 (i\not{D} - m)\psi(z_{out}) = \mathcal{U} i\sigma^{i+}\gamma_5 (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0,$$
$$\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = 0$$

$$b = \frac{z_{in} + z_{out}}{2}, \qquad z = z_{in} - z_{out}$$



$$\int db^- d^2 b_T e^{-ib\cdot\Delta} \int dz^- d^2 z_T e^{-ik\cdot z} \langle p', \Lambda' | \bar{\psi} \left[(i\overleftarrow{D} + m) i\sigma^{i+}\gamma^5 \pm i\sigma^{i+}\gamma^5 (i\overrightarrow{D} - m) \right] \psi | p, \Lambda \rangle = 0$$

Equations of Motion R

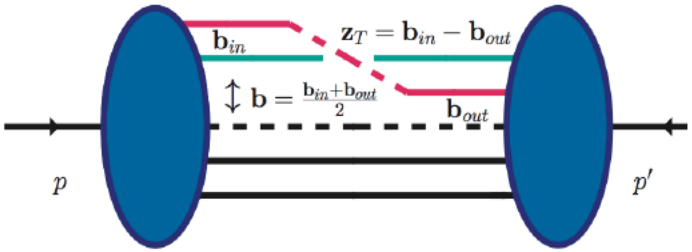
Crucial for understanding qgq contribution to GPDs!!

How do we obtain these ?

$$\mathcal{U} i\sigma^{i+}\gamma_5 (i\not{D} - m)\psi(z_{out}) = \mathcal{U} i\sigma^{i+}\gamma_5 (i\not{\partial} + g\not{A} - m)\psi(z_{out}) = 0,$$

$$\bar{\psi}(z_{in})(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = \bar{\psi}(z_{in})(i\overleftarrow{\not{\partial}} - g\not{A} + m) i\sigma^{i+}\gamma_5 \mathcal{U} = 0$$

$$b = \frac{z_{in} + z_{out}}{2}, \qquad z = z_{in} - z_{out}$$



$$\int db^- d^2 b_T e^{-ib\cdot\Delta} \int dz^- d^2 z_T e^{-ik\cdot z} \langle p', \Lambda' | \bar{\psi} \left[(i\overleftarrow{\not{D}} + m) i\sigma^{i+}\gamma^5 \pm i\sigma^{i+}\gamma^5 (i\overrightarrow{\not{D}} - m) \right] \psi | p, \Lambda \rangle = 0$$

EoM relations for Orbital Angular Momentum

$$x\tilde{E}_{2T} = -\tilde{H} + 2\int d^2k_T \frac{k_T^2 sin^2\phi}{M^2} F_{14} + \frac{\Delta^i}{\Delta_T^2}\int d^2k_T (\mathcal{M}_{++}^{i,S} - \mathcal{M}_{--}^{i,S})$$

Twist 3 **Twist 2** **Genuine Twist 3**

EoM relations for Orbital Angular Momentum

$$x\tilde{E}_{2T} = -\tilde{H} + 2\int d^2k_T \frac{k_T^2 sin^2\phi}{M^2} F_{14} + \frac{\Delta^i}{\Delta_T^2}\int d^2k_T(\mathcal{M}^{i,S}_{++} - \mathcal{M}^{i,S}_{--})$$

Twist 3 **Twist 2** **Genuine Twist 3 (explicit gluon)**

$$\mathcal{M}^{i,S}_{\Lambda'\Lambda} = \frac{i}{4}\int \frac{dz^- d^2z_T}{(2\pi)^3} e^{ixP^+z^- - ik_T\cdot z_T} \langle p', \Lambda' \mid \overline{\psi}\left(-\frac{z}{2}\right)\left[(\overrightarrow{\partial\!\!\!/} - ig A\!\!\!/)\mathcal{U}\Gamma\Big|_{-z/2} + \Gamma\mathcal{U}(\overleftarrow{\partial\!\!\!/} + igA\!\!\!/)\Big|_{z/2}\right]\psi\left(\frac{z}{2}\right) \mid p, \Lambda\rangle_{z^+=0}$$

$$\int dx \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = i\epsilon^{ij} g v^- \frac{1}{2P^+}\int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0)\psi(0)|p, \Lambda\rangle$$

Use LIRs and Equation of Motion Relations to derive Wandzura Wilczek Relations

- The equations of motion connect k_T dependent quantities with collinear objects.
- These k_T dependent quantities are also connected to collinear objects by LIRs. This is independent of equation of motion relations.
- Use the LIR to eliminate the k_T dependent quantities in equation of motion relations. This results in the Wandzura Wilczek relations for twist 3 GPDs.

Wandzura Wilczek Relations

$$\tilde{E}_{2T} = -\int_x^1 \frac{dy}{y}(H+E) + \left[\frac{\tilde{H}}{x} - \int_x^1 \frac{dy}{y^2}\tilde{H}\right] + \left[\frac{1}{x}\mathcal{M}_{F_{14}} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{F_{14}}\right]$$

Twist three vector GPD

Twist two

Axial vector GPD contributes to a vector GPD

Genuine Tw 3

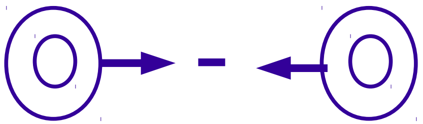
$$g_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(x) + \bar{g}_2(x)$$

Twist three PDF

Twist two

Genuine Tw 3

Moments of twist three GPDs -Quark gluon structure



$$\int dx \widetilde{E}_{2T} = -\int dx (H+E) \Rightarrow \int dx \left(\widetilde{E}_{2T} + H + E\right) = 0$$

$$\int dx x \widetilde{E}_{2T} = -\frac{1}{2}\int dx x (H+E) - \frac{1}{2}\int dx \widetilde{H}$$

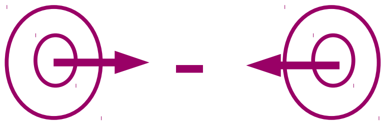
OAM

$$\int dx x^2 \widetilde{E}_{2T} = -\frac{1}{3}\int dx x^2 (H+E) - \frac{2}{3}\int dx x \widetilde{H} - \frac{2}{3}\int dx x \mathcal{M}_{F_{14}}\Big|_{v=0}$$

Genuine Twist Three

$$\int dx\, x \int d^2 k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = \frac{ig}{4(P^+)^2} \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ \gamma^5 F^{+i}(0) \psi(0) | p, \Lambda \rangle$$

Moments of twist three GPDs -Quark gluon structure



$$\int dx \left(E'_{2T} + 2\widetilde{H}'_{2T} \right) = -\int dx \widetilde{H} \qquad \Rightarrow \int dx \left(E'_{2T} + 2\widetilde{H}'_{2T} + \widetilde{H} \right) = 0$$

$$\int dx\,\underline{x} \left(E'_{2T} + 2\widetilde{H}'_{2T} \right) = -\frac{1}{2} \int dx x \widetilde{H} - \frac{1}{2} \int dx H + \boxed{\frac{m}{2M} \int dx (E_T + 2\widetilde{H}_T)}$$

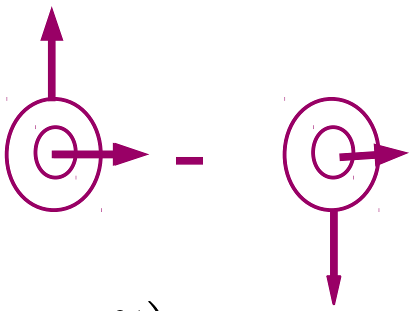
mass term

$$\int dx\,\underline{x}^2 \left(E'_{2T} + 2\widetilde{H}'_{2T} \right) = -\frac{1}{3} \int dx x^2 \widetilde{H} - \frac{2}{3} \int dx x H + \boxed{\frac{2m}{3M} \int dx x (E_T + 2\widetilde{H}_T)} - \boxed{\frac{2}{3} \int dx x \mathcal{M}_{G_{11}} \Big|_{v=0}}$$

Genuine Twist Three d_2

$$\int dx\, x \int d^2 k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = \frac{g}{4(P^+)^2} \epsilon^{ij} \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ F^{+j}(0) \psi(0) | p, \Lambda \rangle$$

Moments of twist three GPDs -Quark gluon structure

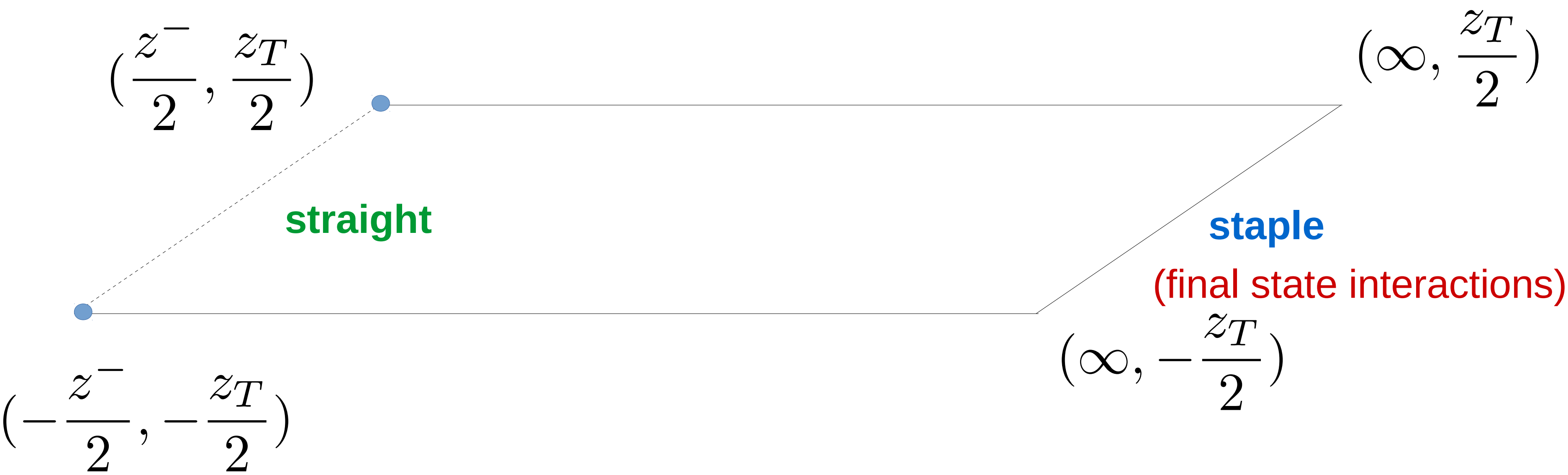


$$\int dx \left(H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T}\right) = \left(1+\frac{\Delta_T^2}{4M^2}\right)\int dx\, \widetilde{H} \overset{\Delta_T\to 0}{\Longrightarrow} \int dx \left(H'_{2T} - \widetilde{H}\right)$$

$$\equiv \int dx\, g_2 = 0$$

Off forward extension of Burkhardt Cottingham

Staple gauge link



$$\frac{dF_{14}^{e(1)}}{dx} = \tilde{E}_{2T} + H + E + \underline{\mathcal{A}_{F_{14}}}$$

LIR violating term

$$\begin{aligned}\mathcal{A}_{F_{14}}(x) &\equiv v^- \frac{(2P^+)^2}{M^2} \int d^2k_T \int dk^- \left[\frac{k_T\cdot\Delta_T}{\Delta_T^2}(A_{11}+xA_{12})+A_{14}\right. \\ &+ \left.\frac{k_T^2\Delta_T^2-(k_T\cdot\Delta_T)^2}{\Delta_T^2}\left(\frac{\partial A_8}{\partial(k\cdot v)}+x\frac{\partial A_9}{\partial(k\cdot v)}\right)\right] \\ &= \frac{dF_{14}^{(1)}}{dx} - \left.\frac{dF_{14}^{(1)}}{dx}\right|_{v=0} \qquad (1)\end{aligned}$$

$$F_{14}^{(1)} - \left.F_{14}^{(1)}\right|_{v=0} = \mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0} \; (1)$$

$$\mathcal{A}_{F_{14}}(x) = \frac{d}{dx}\left(\mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0}\right) (1)$$

$$-\int dx \left.\left(F_{14}^{(1)} - \left.F_{14}^{(1)}\right|_{v=0}\right)\right|_{\Delta_T=0} = \qquad (1)$$

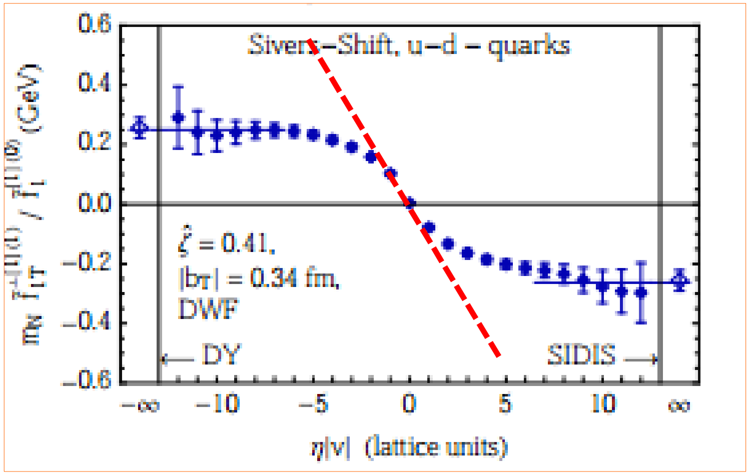
$$-\frac{\partial}{\partial\Delta^i} i\epsilon^{ij} g v^- \frac{1}{2P^+}\int_0^1 ds\, \langle p',+|\bar{\psi}(0)\gamma^+ U(0,sv)F^{+j}(sv)U(sv,0)\psi(0)|p,+\rangle\Big|_{\Delta_T=0},$$

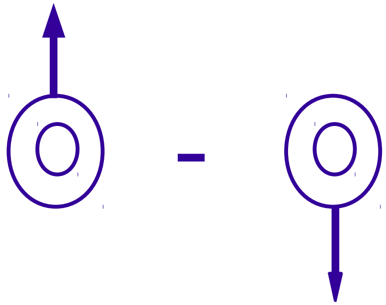
Calculating the force from Lattice data – Sivers function

$$\frac{d}{dv^-}\int dx F_{12}^{(1)}\Big|_{v^-=0} = \frac{d}{dv^-}\int dx \mathcal{M}_{F_{12}}\Big|_{v^-=0} = i(2P^+)\int dx\, x\frac{\Delta_i}{\Delta_T^2}\left(\mathcal{M}_{++}^{i,A} - \mathcal{M}_{--}^{i,A}\right) = \mathcal{M}_{G_{12}}^{n=3}$$

$$\frac{d}{dv^-}\int dx G_{12}^{(1)}\Big|_{v^-=0} = \frac{d}{dv^-}\int dx \mathcal{M}_{G_{12}}\Big|_{v^-=0} = i(2P^+)\int dx\, x\frac{\Delta_i}{\Delta_T^2}\left(\mathcal{M}_{++}^{i,S} + \mathcal{M}_{--}^{i,S}\right) = \mathcal{M}_{F_{12}}^{n=3}$$

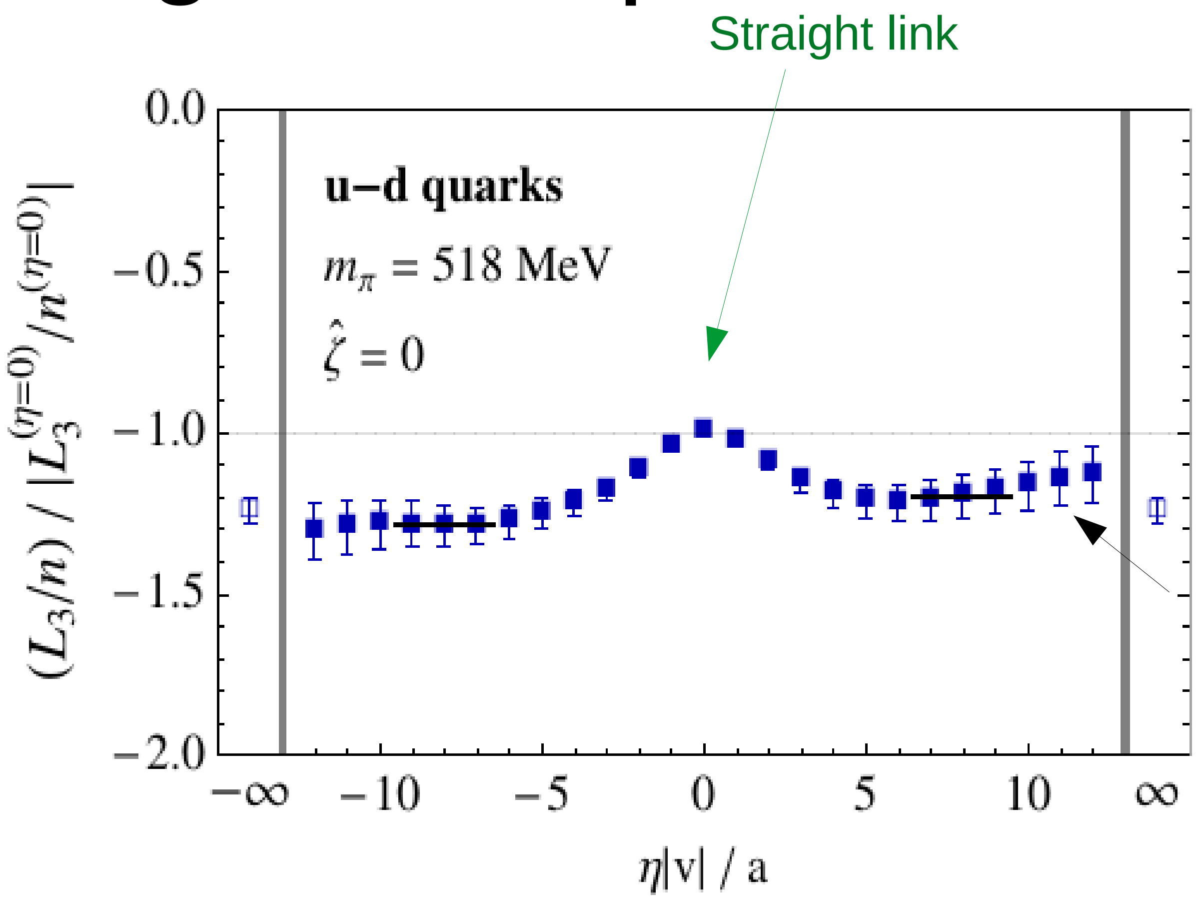
The derivative with respect to the gauge link direction gives the force!





Transversely polarized proton

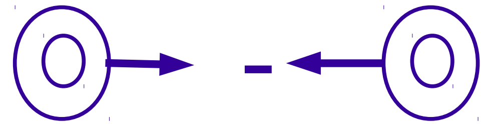
Calculating the torque from Lattice



F_{14}

Michael Engelhardt

Phys. Rev. D95 (2017)



Longitudinally polarized proton

$$\mathcal{L}_{JM} - \mathcal{L}_{Ji} = \mathcal{T}$$

Torque!

Conclusions

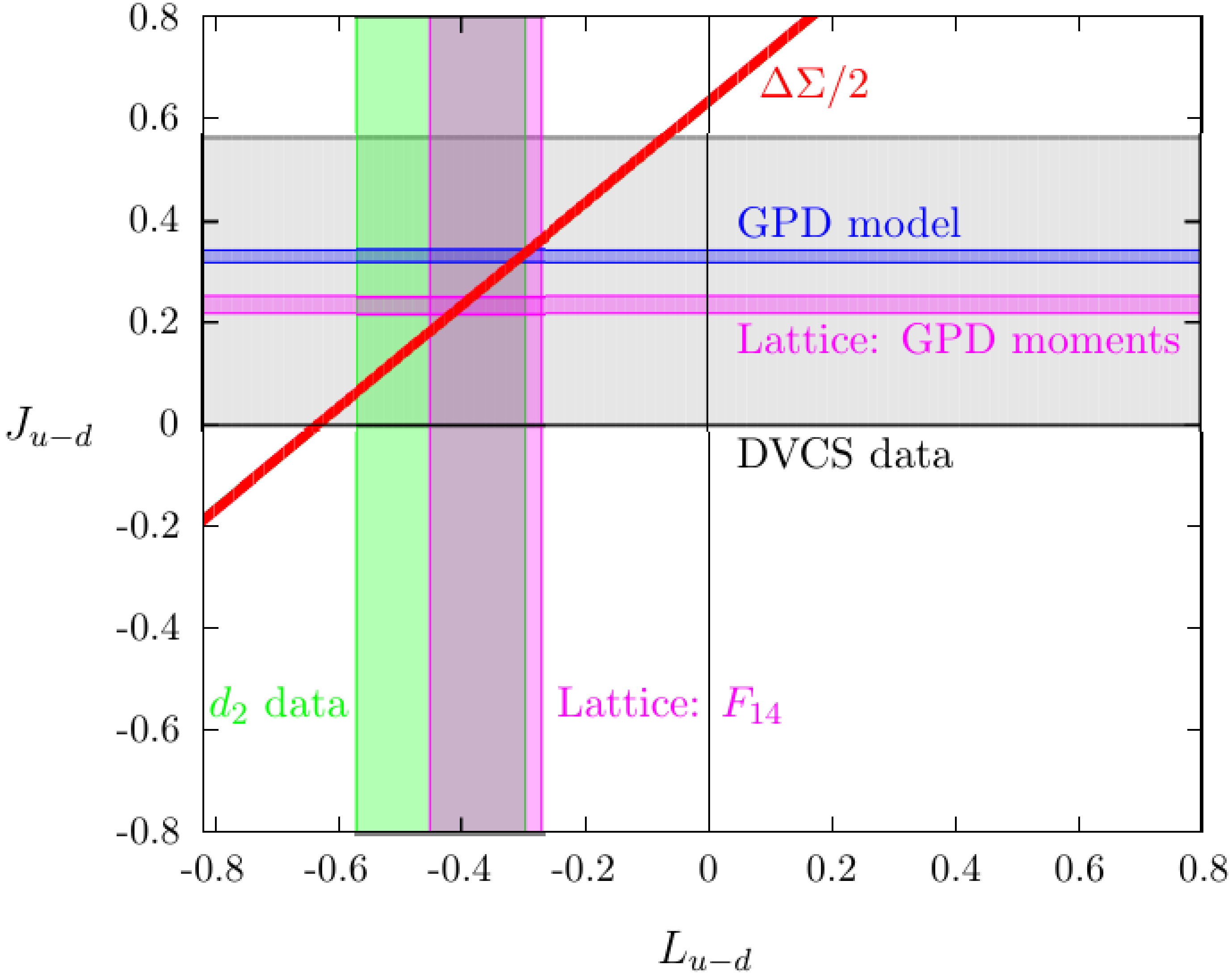
- Way of deriving the Wandzura Wilczek relations. Allows us to **write out precisely quark gluon contribution to twist 3. Study x dependence.**

- This also provides a way to measure effects that were solely associated with GTMDs by measuring the associated GPD.

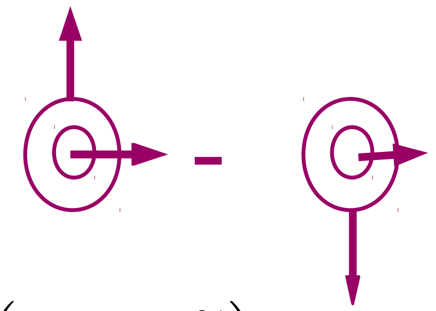
- Quark gluon quark interactions are at the heart of twist three effects.

Thank you!

Where do we stand experimentally?

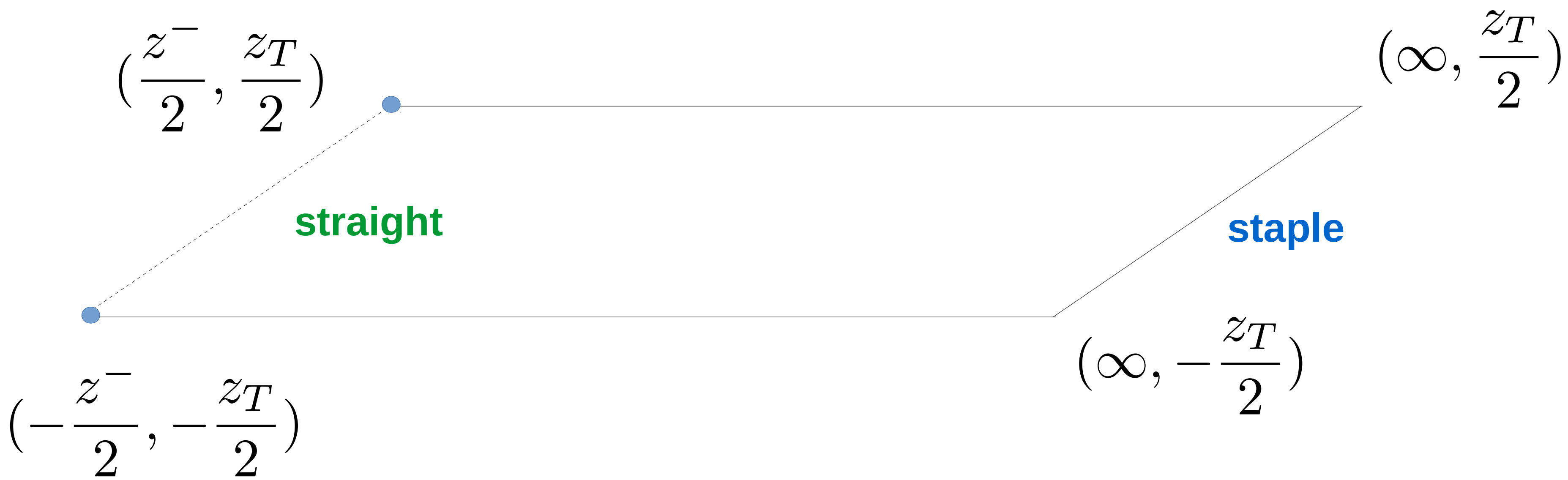


Moments of twist three GPDs -Quark gluon structure



$$
\begin{aligned}
\int dx \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) &= \left(1 + \frac{\Delta_T^2}{4M^2} \right) \int dx \widetilde{H} \quad \overset{\Delta_T \to 0}{\Rightarrow} \int dx \left(H'_{2T} - \widetilde{H} \right) \\
&\equiv \int dx \, g_2 = 0 \\
\int dx \, x \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) &= \frac{1}{2} \left(1 + \frac{\Delta_T^2}{4M^2} \right) \int dx x \widetilde{H} + \frac{\Delta_T^2}{8M^2} \int dx (H + E) \\
&+ \frac{m}{2M} \int dx \left(H_T - \frac{\Delta_T^2}{4M^2} E_T \right) \\
\int dx \, x^2 \left(H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \right) &= \frac{1}{3} \left(1 + \frac{\Delta_T^2}{4M^2} \right) \int dx x^2 \widetilde{H} + \frac{\Delta_T^2}{6M^2} \int dx x (H + E) \\
&+ \frac{2m}{3M} \int dx x \left(H_T - \frac{\Delta_T^2}{4M^2} E_T \right) + \frac{2}{3} \int dx \, x \, \mathcal{M}_{G_{12}} \Big|_{v=0} \; .
\end{aligned}
$$

Quark gluon quark contributions



$$(\overrightarrow{\partial} - igA)\mathcal{U}\Big|_{-z/2} =$$

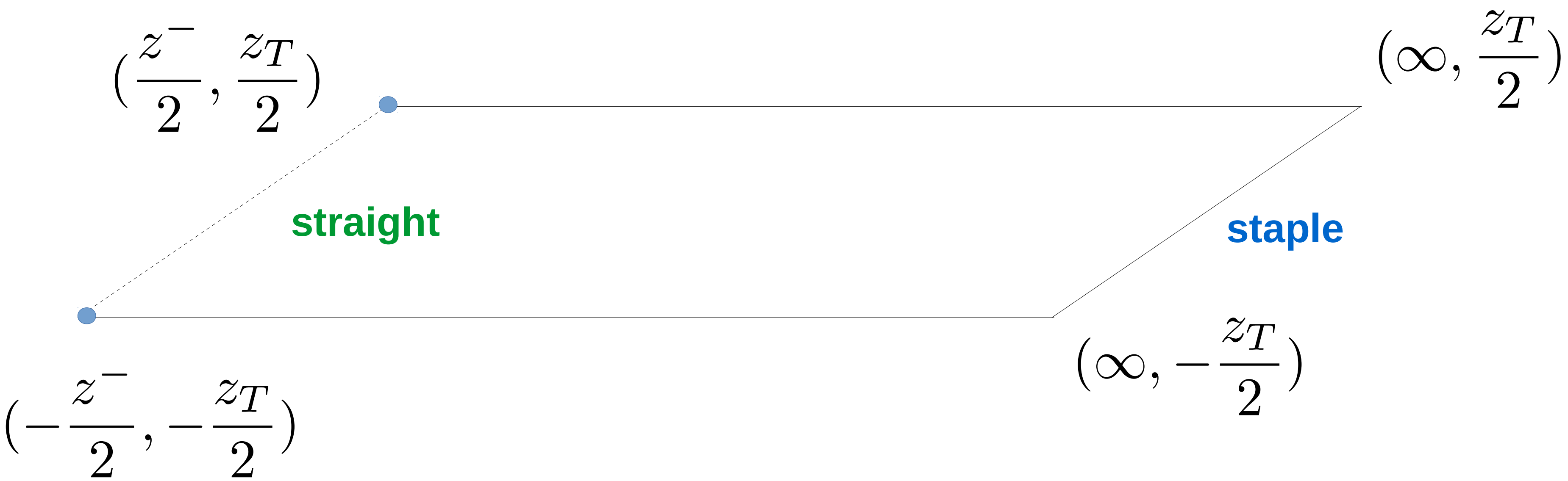
$$igz^- \int_0^1 ds\,(1-s)$$

$$\cdot U(-z/2, -z/2 + v + sz)\gamma_\mu F^{+\mu}(-z/2 + v + sz)U(-z/2 + v + sz, z/2)$$

$$+igv^- \int_0^1 ds\,U(-z/2, -z/2 + sv)\gamma_\mu F^{+\mu}(-z/2 + sv)U(-z/2 + sv, z/2)$$

staple arm

Quark gluon quark contributions



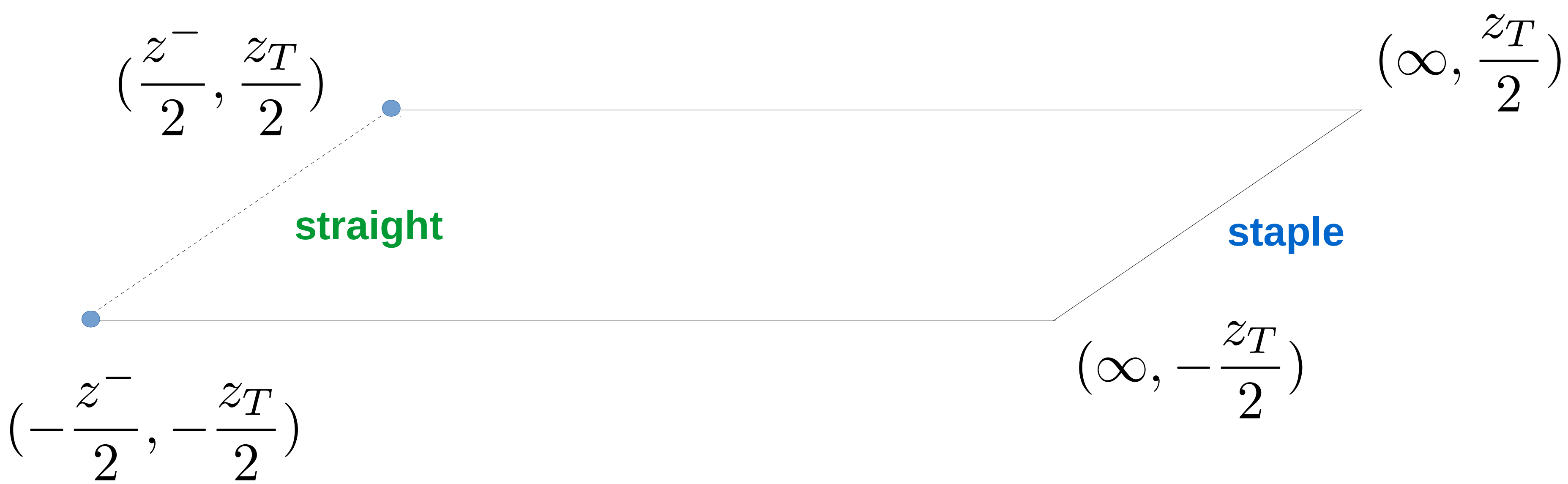
$$\int dx \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} \quad = $$

$$i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0) \psi(0) | p, \Lambda\rangle$$

$$\int dx \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} \quad = $$

$$-g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ \gamma^5 U(0, sv) F^{+i}(sv) U(sv, 0) \psi(0) | p, \Lambda\rangle$$

Quark gluon quark contributions

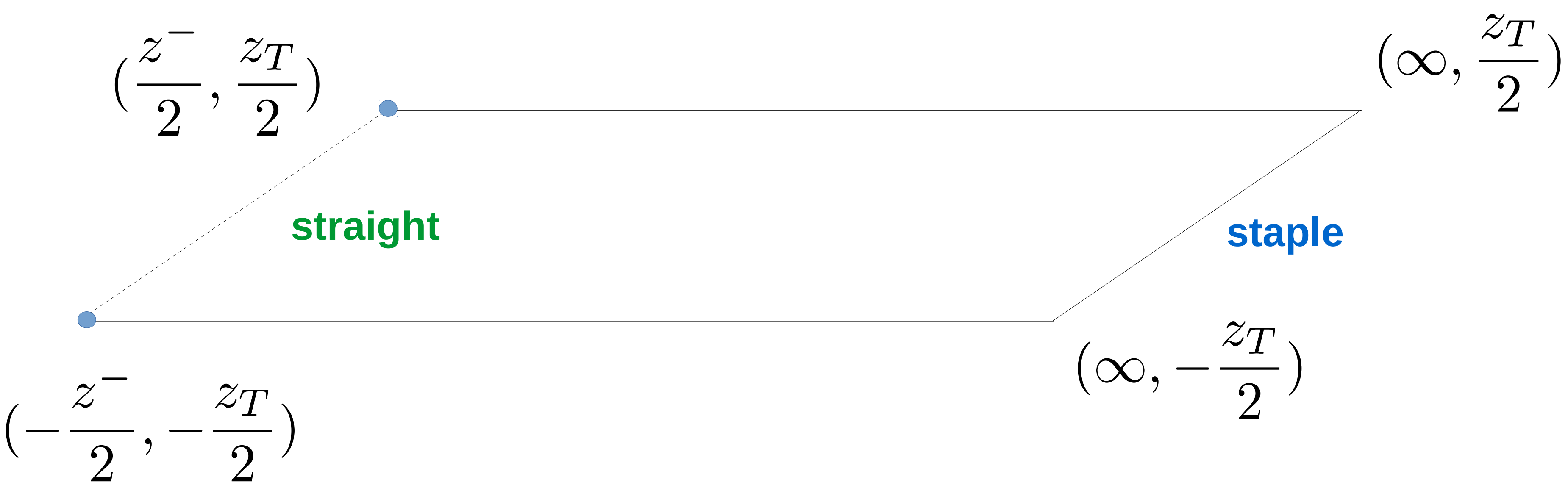


Zero for straight gauge link

$$\int dx \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0)\psi(0) | p, \Lambda\rangle$$

$$\int dx \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = -g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+\gamma^5 U(0, sv) F^{+i}(sv) U(sv, 0)\psi(0) | p, \Lambda\rangle$$

Quark gluon quark contributions



$$\int dx\, x \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = \frac{ig}{4(P^+)^2} \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ \gamma^5 F^{+i}(0) \psi(0) | p, \Lambda \rangle$$

$$\int dx\, x \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = \frac{g}{4(P^+)^2} \epsilon^{ij} \langle p', \Lambda' | \bar{\psi}(0) \gamma^+ F^{+j}(0) \psi(0) | p, \Lambda \rangle$$

Equations of Motion Relations

Starting with the equation of motion and its conjugate we arrive at the following

$$-\frac{\Delta^+}{2} W_{\Lambda\Lambda'}^{[\gamma^i\gamma^5]} + ik^+\epsilon^{ij} W_{\Lambda\Lambda'}^{[\gamma^j]} = -\frac{\Delta^i}{2} W_{\Lambda\Lambda'}^{[\gamma^+\gamma^5]} + i\epsilon^{ij} k_T^j W_{\Lambda\Lambda'}^{[\gamma^+]} - \mathcal{M}_{\Lambda\Lambda'}^{i,S}$$

$$-k^+ W_{\Lambda\Lambda'}^{[\gamma^i\gamma^5]} + \frac{i\Delta^+}{2}\epsilon^{ij} W_{\Lambda\Lambda'}^{[\gamma^j]} + k^i W_{\Lambda\Lambda'}^{[\gamma^+\gamma^5]} = i\epsilon^{ij}\frac{\Delta^j}{2} W_{\Lambda\Lambda'}^{[\gamma^+]} - mW_{\Lambda\Lambda'}^{[i\sigma^{i+}\gamma^5]} - i\mathcal{M}_{\Lambda\Lambda'}^{i,A}$$

- Each W is a correlator that can be parameterized using GTMDs / GPDs.

$$W_{\Lambda\Lambda'}^{\Gamma} = \int \frac{dz^- \, d^2\mathbf{z}_T}{(2\pi)^3} e^{ixP^+z^- - i\bar{\mathbf{k}}_T\cdot\mathbf{z}_T} \langle p', \Lambda' \mid \bar{\psi}\left(-\frac{z}{2}\right)\mathcal{U}\Gamma\psi\left(\frac{z}{2}\right) \mid p, \Lambda\rangle\Big|_{z^+=0}$$

Generalized Lorentz Invariance Relations

- Parametrization of the quark quark correlator at different levels
- LIRs occur because the number of GPCFs is less than the number of GTMDs.

$$\mathcal{W}_{\Lambda\Lambda'}^{[\gamma^\mu]} = \frac{\bar{U}U}{M}(P^\mu A_1^F + k^\mu A_2^F + \Delta^\mu A_3^F) + i\frac{\bar{U}\sigma^{\mu k}U}{M}A_5^F + i\frac{\bar{U}\sigma^{\mu\Delta}U}{M}A_6^F + i\frac{\bar{U}\sigma^{k\Delta}U}{M^3}(P^\mu A_8^F + k^\mu A_9^F + \Delta^\mu A_{17}^F)$$

Explicit k_T coefficient

$$W_{\Lambda,\Lambda'}^{[\gamma^+]} = \frac{1}{2M}\bar{U}(p',\Lambda')[F_{11} + \frac{i\sigma^{i+}k_T^i}{\bar{p}_+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{\bar{p}_+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{M^2}F_{14}]U(p,\Lambda)$$

$$F_{\Lambda,\Lambda'}^{[\gamma^i]} = \frac{1}{2(P^+)^2}\bar{U}\left[i\sigma^{+i}H_{2T} + \frac{\gamma^+\Delta_T^i}{2M}E_{2T} + \frac{P^+\Delta_T^i}{M^2}\tilde{H}_{2T} - \frac{P^+\gamma^i}{M}\tilde{E}_{2T}\right]U$$

Generalized Lorentz Invariance Relations

- The As are a function of the following scalar variables :

$$\sigma \equiv \frac{2k.P}{M^2}, \qquad \tau \equiv \frac{k^2}{M^2}, \qquad \sigma' \equiv \frac{k.\Delta}{\Delta^2} = \frac{k_T.\Delta_T}{\Delta_T^2}$$

For $\Delta^+ = 0$

$$\begin{aligned}\int dk^- A(k^2, k.P, k.\Delta \ldots) &\rightarrow \frac{M^2}{2P^+}\int d\sigma A \\ &\rightarrow \frac{M^2}{2P^+}\int d\sigma' d\sigma d\tau \delta\left(\frac{k_T^2}{M^2} - x\sigma + \tau + \frac{x^2P^2}{M^2}\right)\delta\left(\sigma' - \frac{k_T.\Delta_T}{\Delta_T^2}\right) A(\sigma, \tau, \sigma')\end{aligned}$$

Generalized Lorentz Invariance Relations

$$F_{14}^{(1)} = \int d\sigma d\sigma' d\tau \frac{M^3}{2} J \left[A_8^F + x A_9^F\right]$$

$$J = \sqrt{x\sigma - \tau - \frac{x^2 P^2}{M^2} - \frac{\Delta_T^2 \sigma'^2}{M^2}}$$

$$\tilde{E}_{2T} = \int d\sigma d\sigma' d\tau \frac{M^3}{J} \left[\left(x\sigma - \tau - \frac{x^2 P^2}{M^2} - \frac{\Delta_T^2 \sigma'^2}{M^2}\right) A_9^F - \sigma' A_5^F - A_6^F\right]$$

$$H + E = \int d\sigma d\sigma' d\tau \frac{M^3}{J} \sigma' A_5^F + A_6^F + \left(\frac{\sigma}{2} - \frac{x P^2}{M^2}\right)\left(A_8^F + x A_9^F\right)$$

$$-\frac{dF_{14}^{(1)}}{dx} = \tilde{E}_{2T} + H + E$$

$$F_{14}^{(1)}(x) = \int_x^1 dy \left(\tilde{E}_{2T}(y) + H(y) + E(y)\right)$$

Distribution of OAM in x !

k_T^2 moment of a twist two function

Twist three function

EoM relations for Orbital Angular Momentum

$$x\tilde{E}_{2T} = -\tilde{H} + 2\int d^2k_T \frac{k_T^2 sin^2\phi}{M^2} F_{14} + \frac{\Delta^i}{\Delta_T^2}\int d^2k_T (\mathcal{M}^{i,S}_{++} - \mathcal{M}^{i,S}_{--})$$

$$x\left(E'_{2T} + 2\tilde{H}'_{2T}\right) = -H + \frac{m}{M}(E_T + 2\tilde{H}_T) - 2\int d^2k_T \frac{k_T^2 sin^2\phi}{M^2} G_{11} - \mathcal{M}_{G_{11}}$$

$$\mathcal{M}_{G_{11}} = \frac{2i\epsilon^{im}\Delta^m}{\Delta_T^2}(\mathcal{M}^{i,A}_{++} + \mathcal{M}^{i,A}_{--})$$

EoM relations for Transversely Polarized Proton

$$
\begin{aligned}
0 \quad = \quad & \frac{\Delta_T^2}{4M^2} E + \frac{1}{2} G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2} G_{11}^{(1)} - x \left(H'_{2T} + \frac{\Delta_T^2}{2M^2} \widetilde{H}'_{2T} \right) + \frac{m}{M} \left(H_T + \frac{\Delta_T^2}{2M^2} \widetilde{H}_T \right) \\
& - \frac{i\epsilon^{ij} \Delta^j}{2M\Delta_T^2} \int d^2 k_T \left((\Delta^1 + i\Delta^2) \mathcal{M}_{+-}^{i,A} + (-\Delta^1 + i\Delta^2) \mathcal{M}_{-+}^{i,A} \right)
\end{aligned}
$$

Axial vector

$$
\begin{aligned}
- \quad & x \left(F_{23} + \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{M^2 \Delta_T^2} F_{24} \right) + \frac{1}{2M^2} \left(\Delta_T^2 G_{13} + k_T \cdot \Delta_T G_{12} \right) \\
+ \quad & \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{M^2 \Delta_T^2} F_{12} + \frac{\Delta^i}{2M\Delta_T^2} \left((\Delta^1 - i\Delta^2) \mathcal{M}_{-+}^{i,S} + (\Delta^1 + i\Delta^2) \mathcal{M}_{+-}^{i,S} \right) = 0.
\end{aligned}
$$

Vector

EoM relations for Transversely Polarized Proton

Twist 3

$$0 = \frac{\Delta_T^2}{4M^2}E + \frac{1}{2}G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2}G_{11}^{(1)} - x\left(H'_{2T} + \frac{\Delta_T^2}{2M^2}\widetilde{H}'_{2T}\right) + \frac{m}{M}\left(H_T + \frac{\Delta_T^2}{2M^2}\widetilde{H}_T\right)$$
$$- \frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2}\int d^2k_T\left((\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,A} + (-\Delta^1 + i\Delta^2)\mathcal{M}_{-+}^{i,A}\right)$$

d_2 (in the forward limit)

Axial vector

H_{2T}

Twist 3

$$- \quad x\left(F_{23} + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{24}\right) + \frac{1}{2M^2}\left(\Delta_T^2 G_{13} + k_T\cdot\Delta_T G_{12}\right)$$
$$+ \quad \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{12} + \frac{\Delta^i}{2M\Delta_T^2}\left((\Delta^1 - i\Delta^2)\mathcal{M}_{-+}^{i,S} + (\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,S}\right) = 0.$$

Vector

EoM relations for Transversely Polarized Proton

Twist 3

$$0 = \frac{\Delta_T^2}{4M^2}E + \frac{1}{2}G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2}G_{11}^{(1)} - x\left(H'_{2T} + \frac{\Delta_T^2}{2M^2}\widetilde{H}'_{2T}\right) + \frac{m}{M}\left(H_T + \frac{\Delta_T^2}{2M^2}\widetilde{H}_T\right)$$
$$- \frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2}\int d^2k_T\left((\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,A} + (-\Delta^1 + i\Delta^2)\mathcal{M}_{-+}^{i,A}\right)$$

d_2 (in the forward limit)

Axial vector

H_{2T} Twist 3

$$- \quad x\left(F_{23} + \frac{k_T^2\Delta_T^2 - (k_T \cdot \Delta_T)^2}{M^2\Delta_T^2}F_{24}\right) + \frac{1}{2M^2}\left(\Delta_T^2 G_{13} + k_T \cdot \Delta_T G_{12}\right)$$
$$+ \quad \frac{k_T^2\Delta_T^2 - (k_T \cdot \Delta_T)^2}{M^2\Delta_T^2}F_{12} + \frac{\Delta^i}{2M\Delta_T^2}\left((\Delta^1 - i\Delta^2)\mathcal{M}_{-+}^{i,S} + (\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,S}\right) = 0.$$

$$f_{1T}^{\perp(1)} = -F_{12}^{o(1)} = \mathcal{M}_{F_{12}}|_{\Delta_T=0}$$

Vector

LIR violating term

$$\begin{aligned}
\mathcal{A}_{F_{14}}(x) &\equiv v^- \frac{(2P^+)^2}{M^2} \int d^2k_T \int dk^- \left[\frac{k_T \cdot \Delta_T}{\Delta_T^2} (A_{11} + xA_{12}) + A_{14} \right. \\
&+ \left. \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{\Delta_T^2} \left(\frac{\partial A_8}{\partial (k \cdot v)} + x \frac{\partial A_9}{\partial (k \cdot v)} \right) \right] \\
&= \frac{dF_{14}^{(1)}}{dx} - \left. \frac{dF_{14}^{(1)}}{dx} \right|_{v=0} \qquad (1)
\end{aligned}$$

$$F_{14}^{(1)} - \left. F_{14}^{(1)} \right|_{v=0} = \mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0} \, (1)$$

$$\mathcal{A}_{F_{14}}(x) = \frac{d}{dx} \left(\mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0} \right) (1)$$

$$- \int dx \, \left(F_{14}^{(1)} - \left. F_{14}^{(1)} \right|_{v=0} \right) \Big|_{\Delta_T = 0} = \qquad (1)$$

$$- \frac{\partial}{\partial \Delta^i} i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds \, \langle p', + | \bar{\psi}(0) \gamma^+ U(0, sv) F^{+j}(sv) U(sv, 0) \psi(0) | p, + \rangle \Big|_{\Delta_T = 0},$$

Wandzura Wilczek Relations

$$2\tilde{H}'_{2T} + E'_{2T} = -\int_x^1 \frac{dy}{y}\tilde{H} + \left[\frac{H}{x} - \int_x^1 \frac{dy}{y^2} H\right]$$

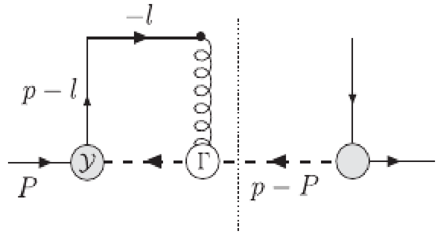
$$+ \quad \frac{m}{M}\left[\frac{1}{x}(2\tilde{H}_T + E_T) - \int_x^1 \frac{dy}{y^2}(2\tilde{H}_T + E_T)\right] + \frac{\mathcal{M}_{G_{11}}}{x} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{G_{11}}$$

Wandzura Wilczek Relations

$$
\begin{aligned}
H'_{2T} - \frac{\Delta_T^2}{4M^2} E'_{2T} \quad = \quad & \left(1 + \frac{\Delta_T^2}{4M^2}\right) \int_x^1 \frac{dy}{y} \widetilde{H} + \frac{m}{M} \left[\frac{1}{x} \left(H_T - \frac{\Delta_T^2}{4M^2} E_T \right) \right. \\
& \left. - \int_x^1 \frac{dy}{y^2} \left(H_T - \frac{\Delta_T^2}{4M^2} E_T \right) \right] + \frac{\Delta_T^2}{4M^2} \left[\frac{1}{x} (H + E) - \int_x^1 \frac{dy}{y^2} (H + E) \right] \\
& + \left[\frac{\mathcal{M}_{G_{12}}}{x} - \int_x^1 \frac{dy}{y^2} \mathcal{M}_{G_{12}} \right] - \int_x^1 \frac{dy}{y} \mathcal{A}_{G_{12}} . \qquad (1)
\end{aligned}
$$

Including Final State Interactions

- Ji → Straight Gauge link
- Jaffe Manohar → Staple Link



- The difference is the torque

$$\mathcal{L}_q^{JM} - L_q^{Ji} = \int \frac{d^2 z_T dz^-}{\left(2\pi\right)^3} \left\langle P',\Lambda' \middle| \bar{\psi}(z)\gamma^+(-g)\int_{z^-}^{\infty} dy^- U\left[z_1 G^{+1}(y^-) - z_2 G^{+2}(y^-)\right] U\psi(z) \middle| P,\Lambda \right\rangle \Big|_{z^+=0}$$

Burkardt (2013)