Status of hadronic tensor calculation on the lattice

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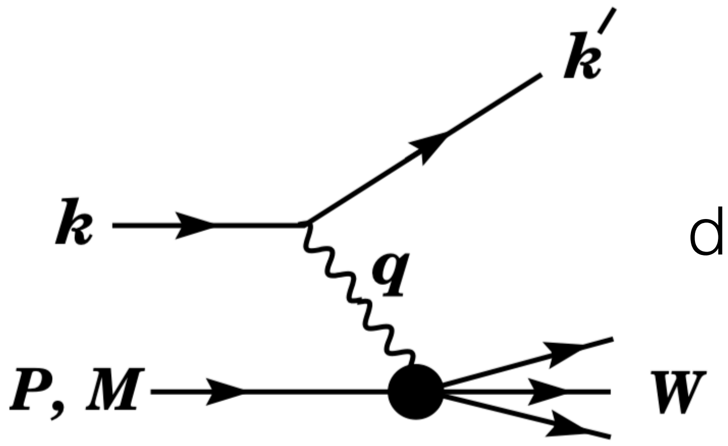
χQCD collaboration

10/15/2018 INT 18-3@Seattle

Hadronic tensor



deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS)

to leading order perturbation $$\frac{d^2\sigma}{dxdy} = \frac{2\pi y \alpha^2}{Q^4} \sum_j \eta_j L_j^{\mu\nu} W_{\mu\nu}^j$$

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4 z e^{iq\cdot z} \left\langle P, S \left| \left[J_\mu^\dagger(z) J_\nu(0) \right] \right| P, S \right\rangle$$

for unpolarized cases $$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P \cdot q} F_2(x, Q^2)$$

extract PDFs $$F_i = \sum_a C_i^a \otimes f_a$$

Hadronic tensor is scale independent! No need to do renormalization.

Structure functions are frame independent! No need of large external momentum.

Hadronic tensor on the lattice

four-point function with two vector currents

$$C_4 = \sum_{\boldsymbol{x_f}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x_f}} \sum_{\boldsymbol{x_2 x_1}} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})} \left\langle \chi_N(\boldsymbol{x_f}, t_f) J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

nucleon two-point function

$$C_2 = \sum_{\boldsymbol{x_f}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x_f}} \left\langle \chi_N(\boldsymbol{x_f}, t_f) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

Euclidean hadronic tensor

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \frac{E_p}{m_N} \frac{\mathrm{Tr}[\Gamma_e C_4]}{\mathrm{Tr}[\Gamma_e C_2]} \rightarrow \sum_{\boldsymbol{x_2 x_1}} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})} \langle P, S \,|\, J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \,|\, P, S \rangle$$

$$= \sum_n A_n e^{-(E_n - E_p)\tau}, \tau \equiv t_1 - t_2$$

K.F. Liu, PRD 62, 074501 (2000)

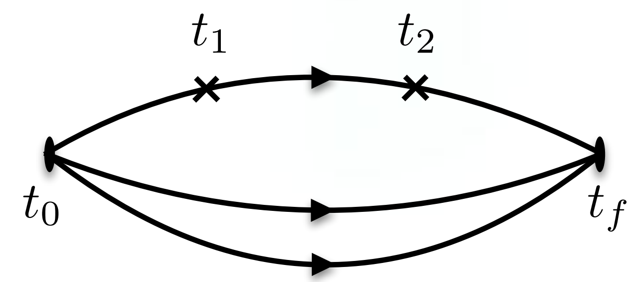
K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

exponential behavior w.r.t. the time difference between the two currents

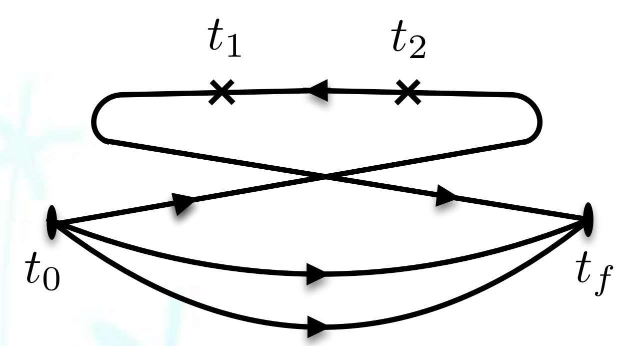
Contractions

$$C_4 = \sum_{\mathbf{x}_f} e^{-ip\cdot x_f} \sum_{\mathbf{x}_2 \mathbf{x}_1} e^{-iq\cdot(\mathbf{x}_2 - \mathbf{x}_1)} \left\langle \chi_N(\mathbf{x}_f, t_f) J_\mu(\mathbf{x}_2, t_2) J_\nu(\mathbf{x}_1, t_1) \bar{\chi}_N(\mathbf{0}, t_0) \right\rangle$$

1. valence and connected-sea parton



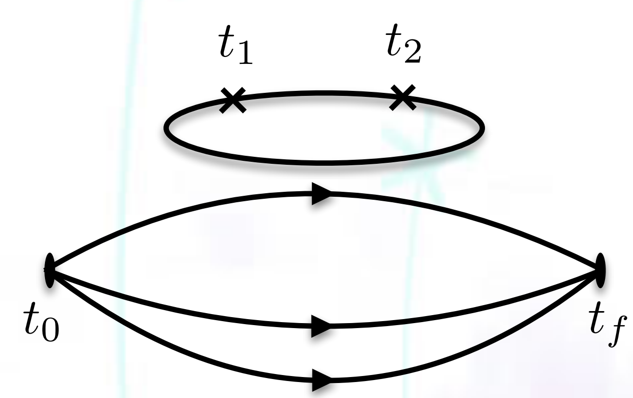
2. connected-sea anti-parton



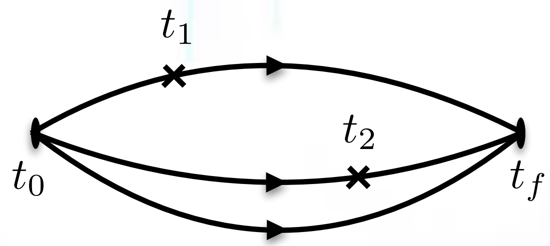
responsible for the Gottfried sum rule violation!

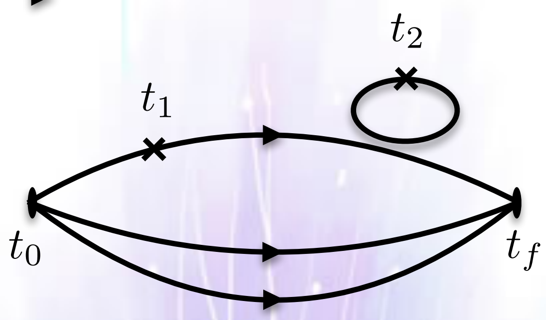
K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

3. disconnected-sea parton and anti-parton



4. pure higher-twist ones





...

Back to the Minkowski space

Euclidean hadronic tensor

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) \sim \sum_n A_n e^{-\nu_n \tau}, \nu_n \equiv E_n - E_p$$

In an integral form

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \int d\nu W_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\nu) e^{-\nu\tau}$$

Laplace transform

Formally, an inverse Laplace transform will do

$$W_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\nu) = \frac{1}{i}\int_{c-i\infty}^{c+i\infty} d\tau e^{\nu\tau} \tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau)$$

Practically, need to solve the inverse problem of the Laplace transform

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \int d\nu W_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\nu) e^{-\nu\tau}$$

several (O(10)) discrete data points

spectral function with infinite nu

lack of information, an ill-defined problem

More about the inverse problem

A general form

$$c(\tau) = \int k(\tau, \nu)\omega(\nu)d\nu$$

In our case, the Laplace kernel

$$k(\tau, \nu) = e^{-\nu\tau}$$

$\omega(\nu)$ is a continuum function, but numerically, we need to discretize it.

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

If the number of nu is less than the number of tau, chi square fitting

If the number of nu is equal to the number of tau, linear equations

If the number of nu is larger than the number of tau, no unique solution

plug in Bayesian prior information

Solving the inverse problem

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

✦ **Backus-Gilbert (BG)**

G. Backus and F. Gilbert, Geophysical Journal International 16, 169 (1968)

If the kernels can span a complete function basis

$$\sum_{\tau} c(\tau, \nu_0)k(\tau, \nu) \sim \delta(\nu - \nu_0)$$

$$\sum_{\tau} c(\tau, \nu_0)c(\tau) \sim \int \delta(\nu - \nu_0)\omega(\nu)d\nu = \omega(\nu_0)$$

The actual incompleteness of the kernels leads to bad resolution.

✦ **Maximum Entropy (ME)**

M. Asakawa et al., Prog. Part. Nucl. Phys. 46, 459 (2001)

$$P[\omega | D, \alpha, m] \propto \frac{1}{Z_S Z_L} e^{Q(\omega)} \qquad Q = \alpha S - L$$

$$S = \sum_{\nu} \left[\omega(\nu) - m(\nu) - \omega(\nu) \log \left(\frac{\omega(\nu)}{m(\nu)} \right) \right] \Delta\nu$$

Maximum search is using SVD in a reduced parameter space ($O(10^1)$).

Hyper parameter alpha is averaged over based on assumptions.

Solving the inverse problem

$$c(\tau) = \sum_{\nu} k(\tau,\nu)\omega(\nu)\Delta\nu$$

✦ **Bayesian Reconstruction (BR)**

Y. Burnier and A. Rothkopf, PRL 72, 1790 (1994)

$$P[\omega\,|\,D,\alpha,m] \propto e^{Q'(\omega)} \quad Q' = \alpha S - L - \gamma(L - N_\tau)^2$$

No over fitting

$$S = \sum_{\nu}\left[1 - \frac{\omega(\nu)}{m(\nu)} + \log\left(\frac{\omega(\nu)}{m(\nu)}\right)\right]\Delta\nu$$

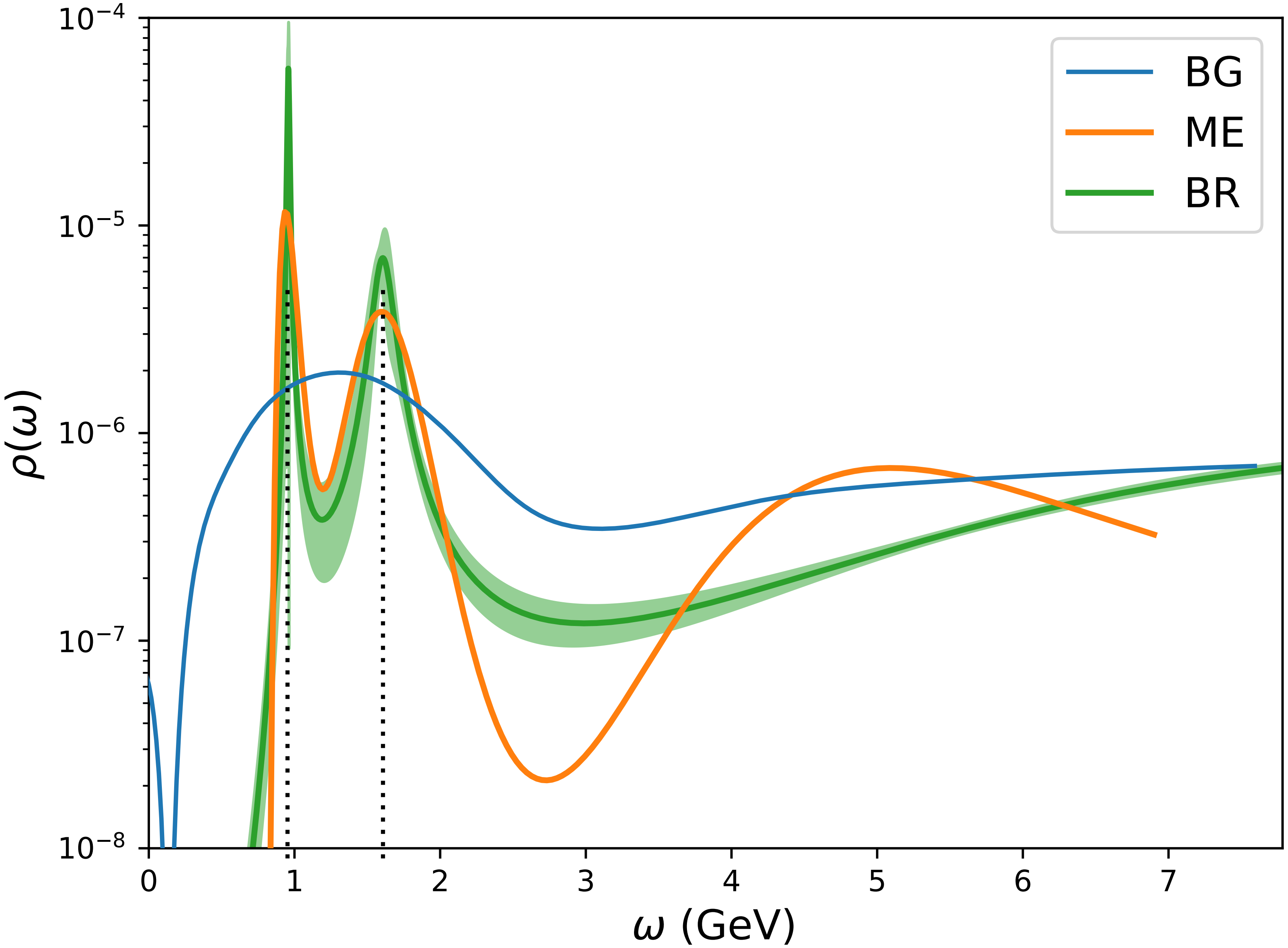
Hyper parameter alpha is integrated over.

$$P[\omega\,|\,D,m] = \frac{P[D\,|\,\omega,I]}{P[D\,|\,m]}\int d\alpha P[\alpha\,|\,D,m]$$

Maximum search is in the entire parameter space ($O(10^3)$).

High precision architecture is needed (e.g., 512-bit floating point number).

Tests on nucleon two-point functions



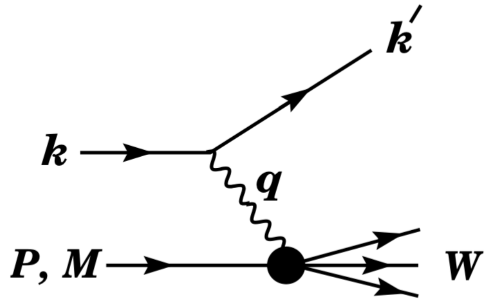
48I, $48^3 \times 96$

$a \sim 0.11$ fm

$m_\pi \sim 135$ MeV

- overlap on domain wall at the physical pion mass
- expecting peaks at ~1 GeV and ~1.5 GeV
- bad resolution of BG
- BR is shaper and more stable than ME

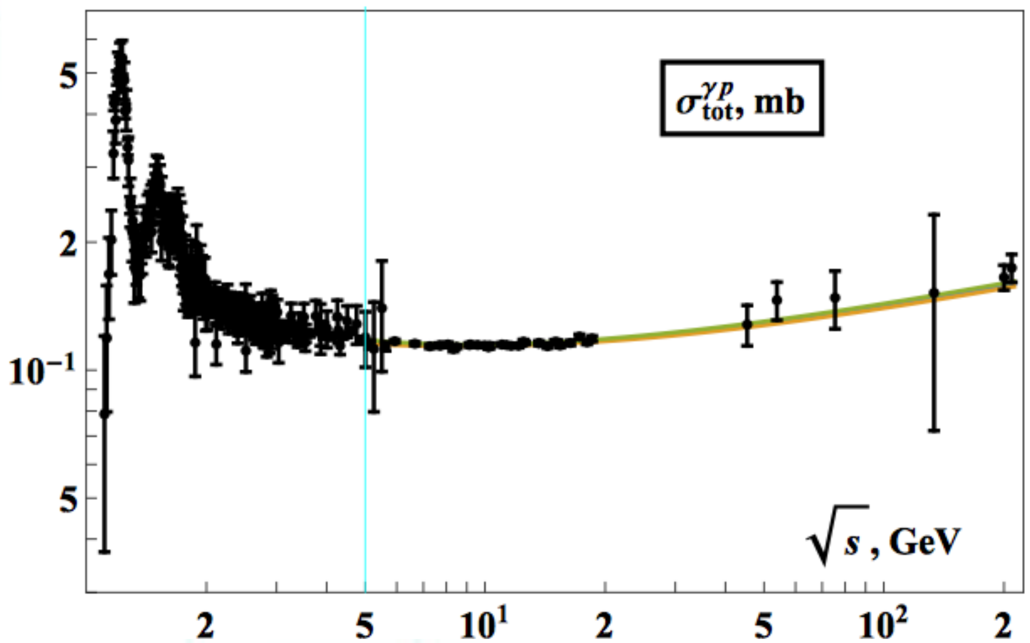
Sketch the structure function

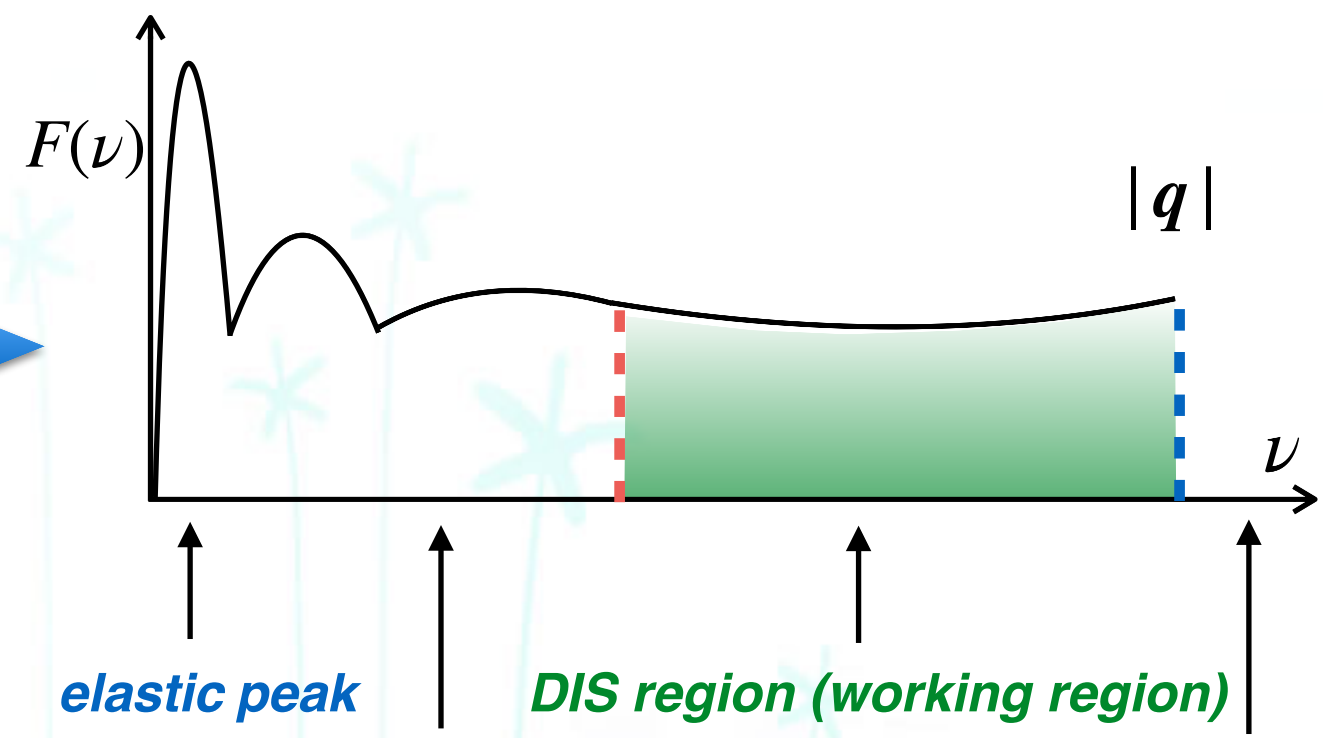


$$Q^2 = -q^2 = \boldsymbol{q}^2 - \nu^2 \qquad x = \frac{Q^2}{2m\nu}$$

fixed by momentum setup

comes from solving the inverse problem





quasi-elastic contribution

$Q^2 < 0$

$\nu > (E_{n=0} - E_p) + \Delta E$ (away from the elastic peak)

$\nu < |\boldsymbol{q}|$ (physical x and Q^2)

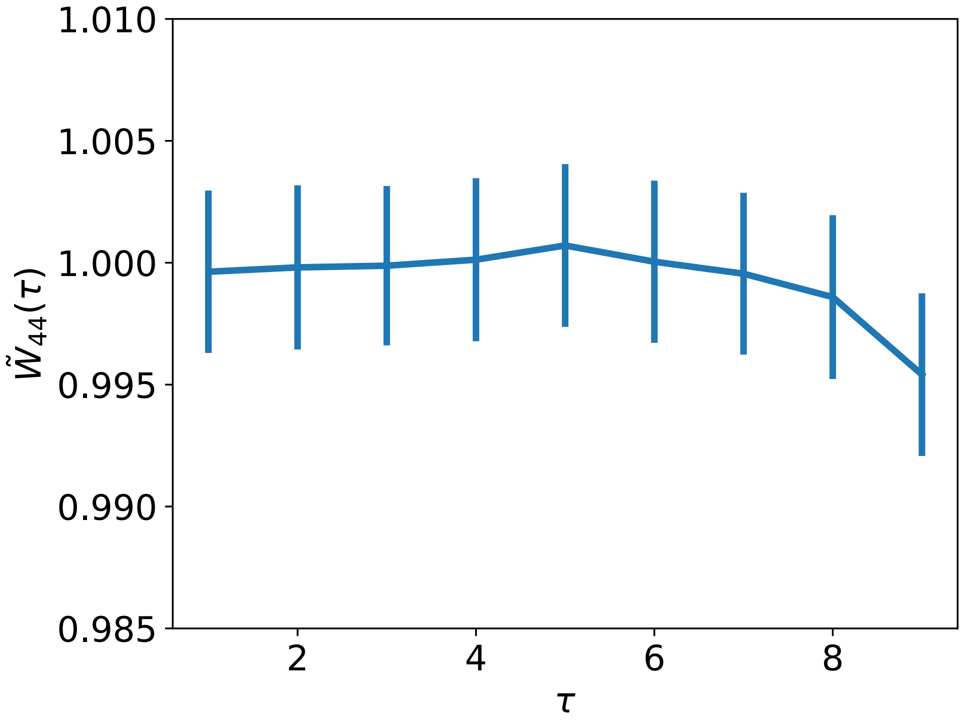
The quasi-elastic region and the region between quasi-elastic and deep inelastic scattering are also of great importance.

Experimental program of neutrino-nucleus scattering for neutrino energy < 5 GeV needs reliable nu N cross-section.

Check of the elastic case

normalized vector current $J_4 = \bar{\psi}\gamma_4\psi$

$$\tilde{W}_{44}(\boldsymbol{p}=0,\, \boldsymbol{q}=0,\, \tau) \overset{\tau\to\infty}{=} \langle N|J_4|N\rangle\langle N|J_4|N\rangle$$
$$= F_1^2(q^2=0) = g_V^2 = 1$$

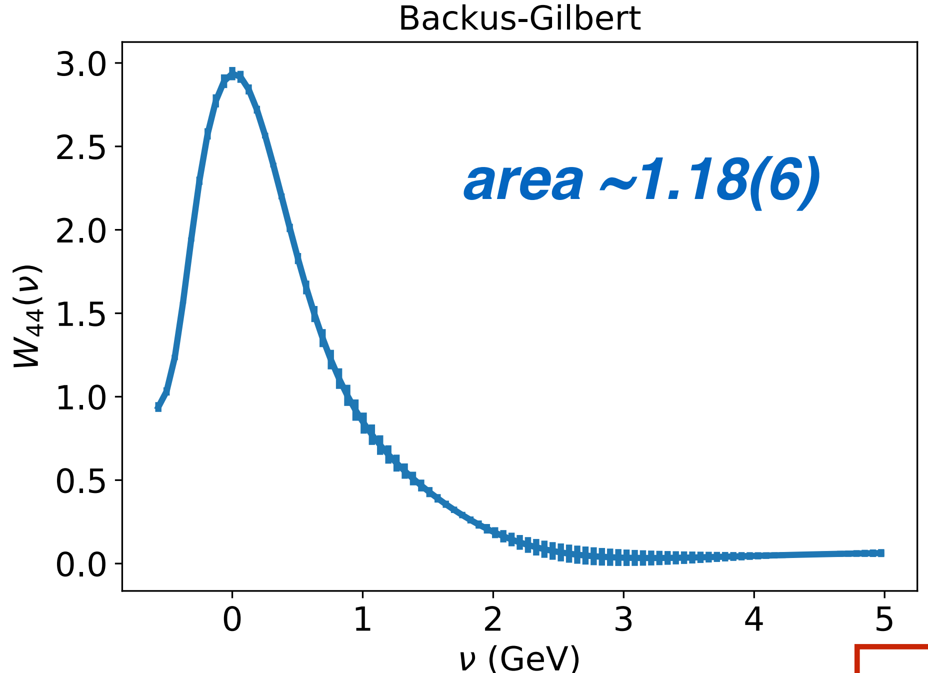


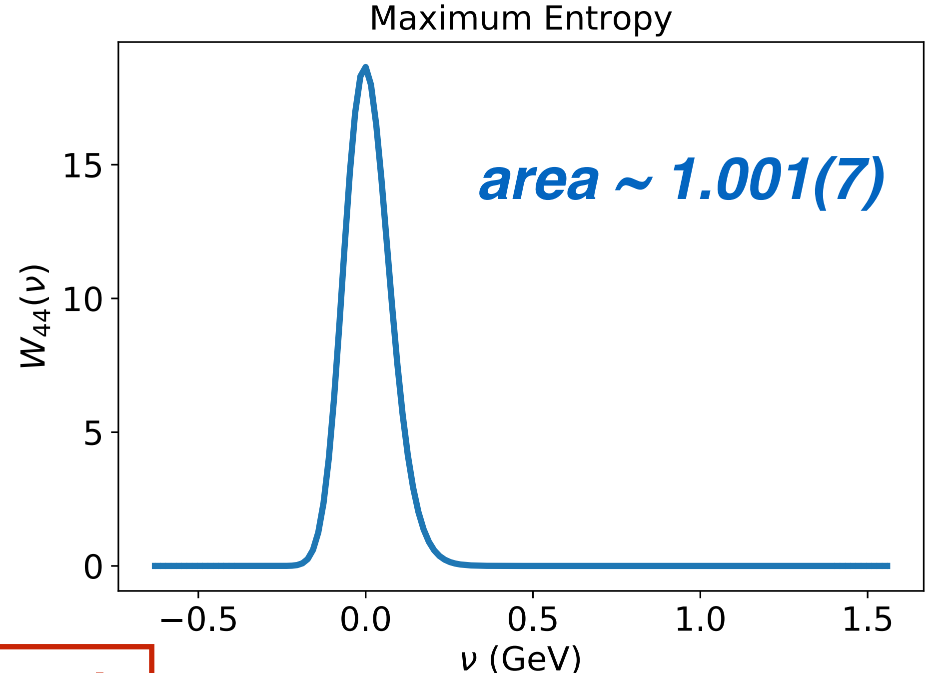
inverse

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \int d\nu W_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\nu)e^{-\nu\tau}$$

$$W_{44}(q^2,\,\nu) = \delta(q^2+2m_N\nu)\frac{2m_N}{1-q^2/4m_N{}^2}\left(G_E^2(q^2) - \frac{q^2}{4M_N^2}G_M^2(q^2)\right)$$
$$\overset{q^2=0}{=} \delta\nu G_E^2(q^2=0) = \delta\nu g_V^2 = \delta\nu$$

delta function at zero





note, different x scale

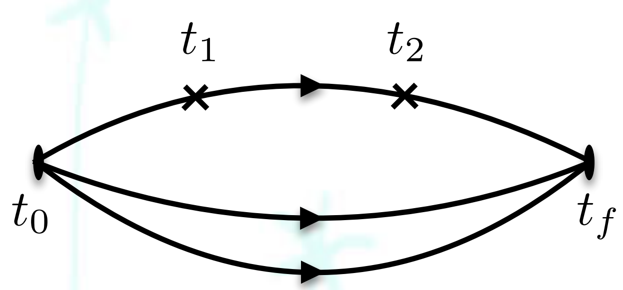
Lattice setups

clover anisotropic lattice, $24^3 \times 128,\ a_t \sim 0.035\ \text{fm},\ m_\pi \sim 380\ \text{MeV},\ \frac{2\pi}{L} \sim 0.42\ \text{GeV}$

H.-W. Lin et al., PRD 79, 034502 (2009)

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P \cdot q} F_2(x, Q^2)$$

$\mu = \nu = 1$ and $p_1 = q_1 = 0 \quad W_{11}(\nu) = F_1(x, Q^2)$



two sequential-sources for each 4-point function
554 configurations, 16 source positions

The *x*-range can be reached on this lattice is roughy [0.05, 0.3] by combining different kinematic setups.

This calculation:

$\boldsymbol{p}$	$\boldsymbol{q}$	E_p	$E_{n=0}$	$\lvert\boldsymbol{q}\rvert$	ν	Q^2	x
(0,3,3)	(0,-6,-6)	2.15	2.15	3.57	[2.96, 3.68]	[4, 2]	[0.16, 0.07]

More on the setups

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \sum_n A_n e^{-(E_n - E_p)\tau}$$

energy of the intermediate state n | **external nucleon energy**

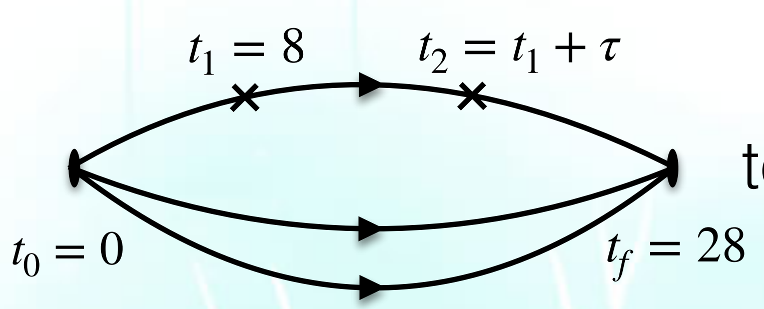
$\boldsymbol{p}$=(033), $\boldsymbol{q}$=(0-6-6) $\quad \boldsymbol{p}+\boldsymbol{q} = -\boldsymbol{p}$

$$E_0 = (m_N^2 + |\boldsymbol{p}+\boldsymbol{q}|^2) = E_p$$

the lowest energy of intermediate states

for small τ, higher intermediate states contribute, exponentially decay

for large τ, lowest intermediate state (elastic contribution) dominates, constant

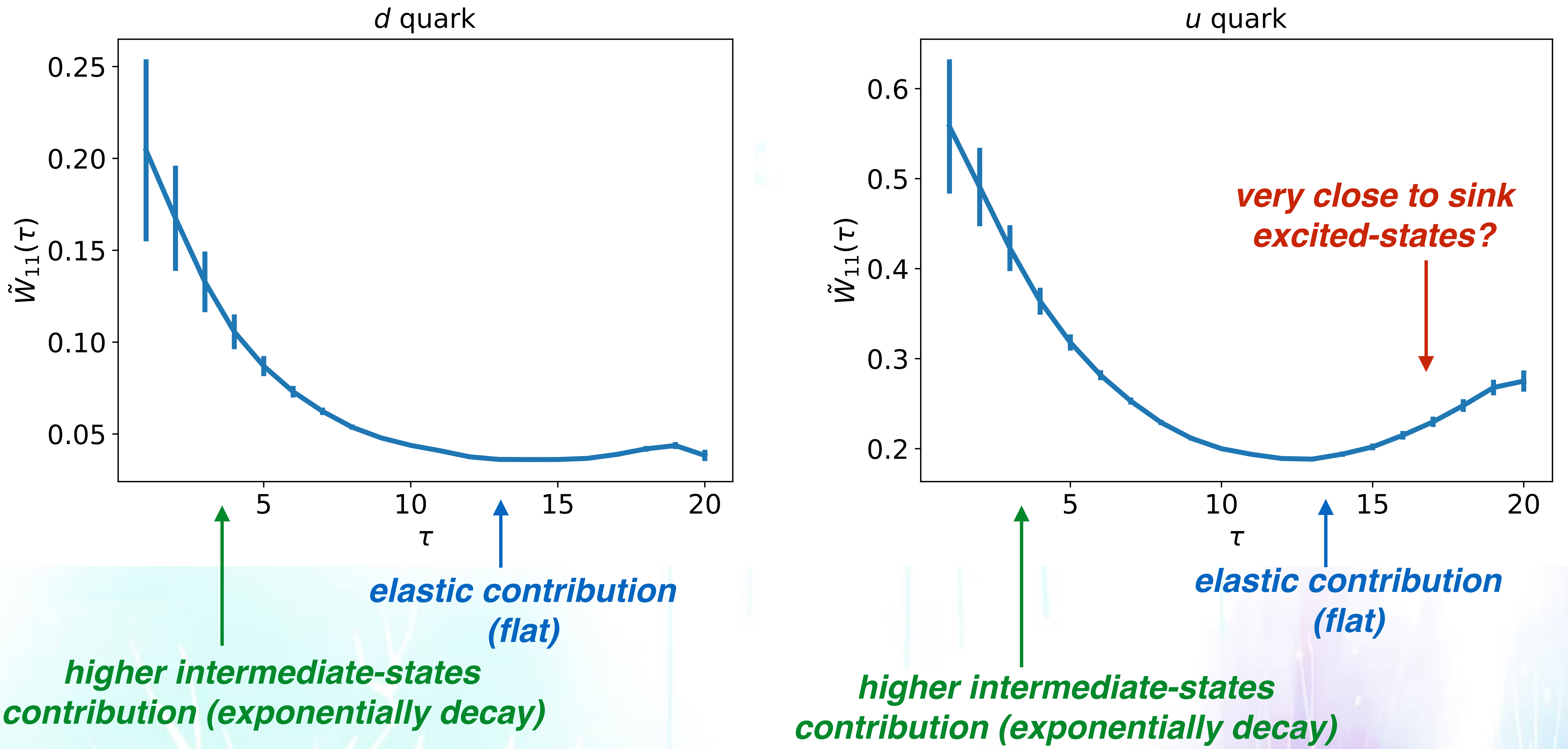


to avoid the contact point and sink excited stats $\tau \in [1,\ 12]$

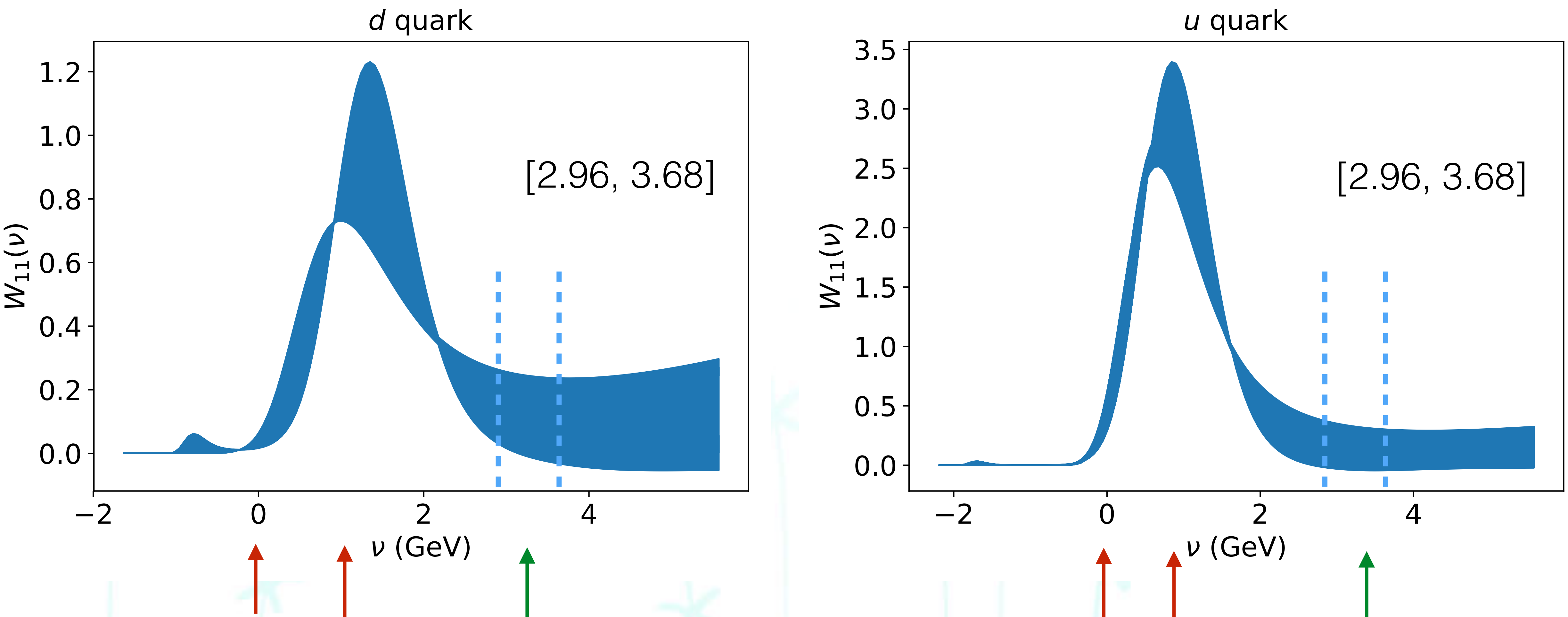
Euclidean hadronic tensor

for small τ, higher intermediate states contribute, exponentially decay

for large τ, lowest intermediate state (elastic contribution) dominates, constant



Minkowski hadronic tensor (after ME)



elastic contribution *DIS region (working region)*

quasi-elastic contribution

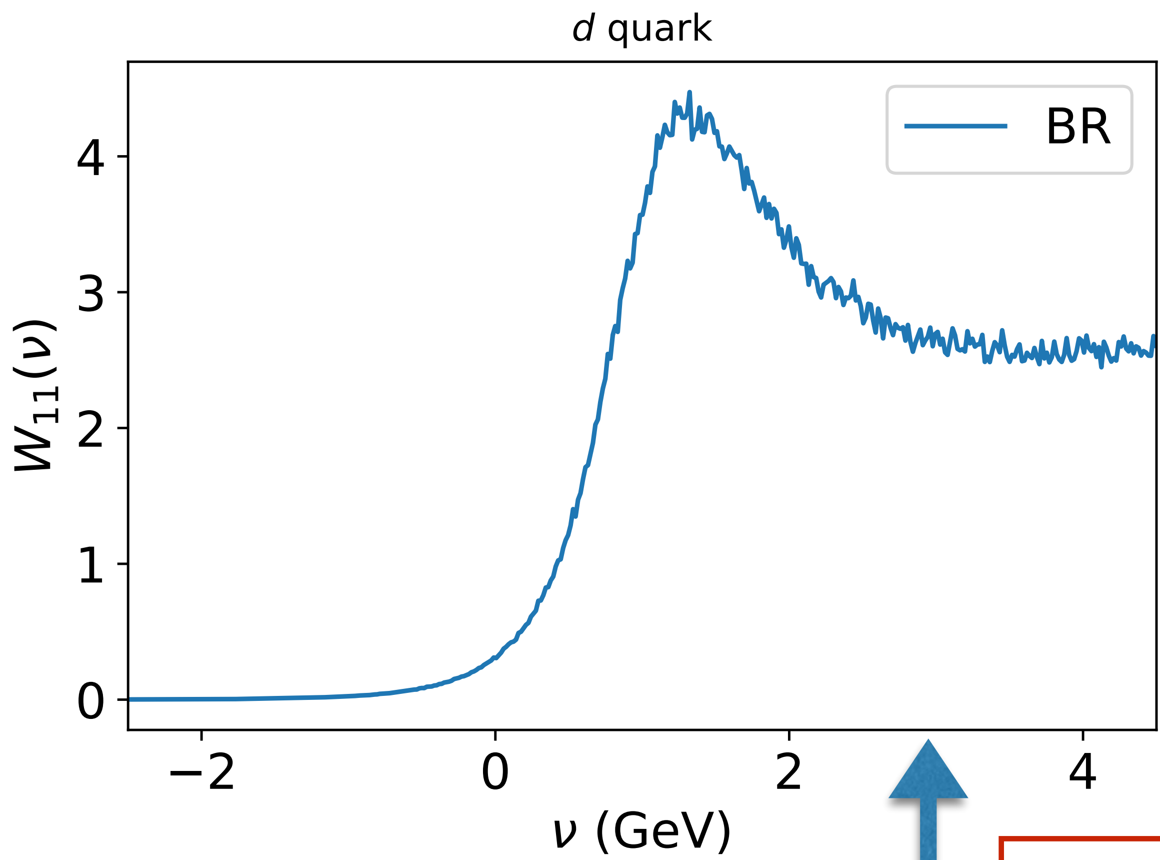
elastic contribution *DIS region (working region)*

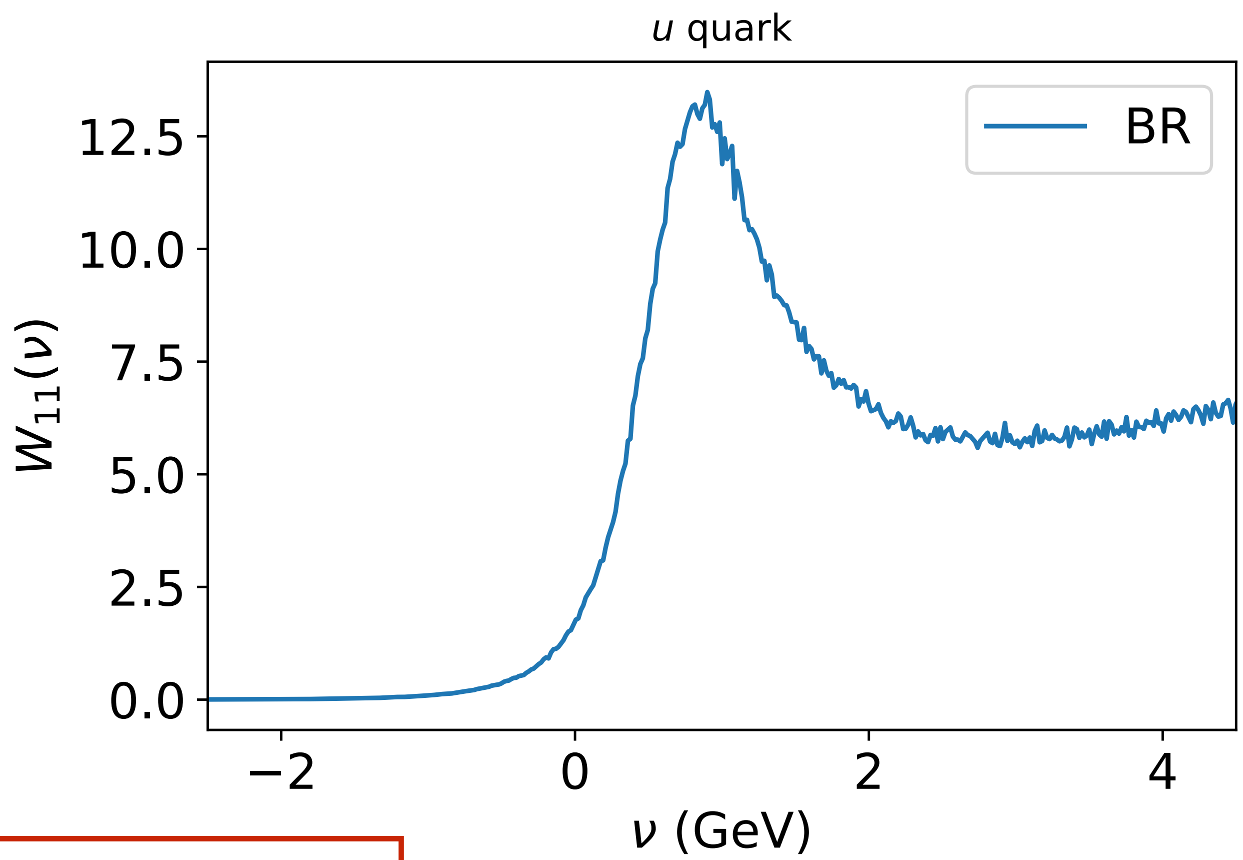
quasi-elastic contribution

- ✦ **Elastic contribution is suppressed by the large momentum transfer.** $G^2(0) \propto \frac{1}{\left(1 + \frac{Q_{\mathrm{el}}^2}{\Lambda^2}\right)^4}$

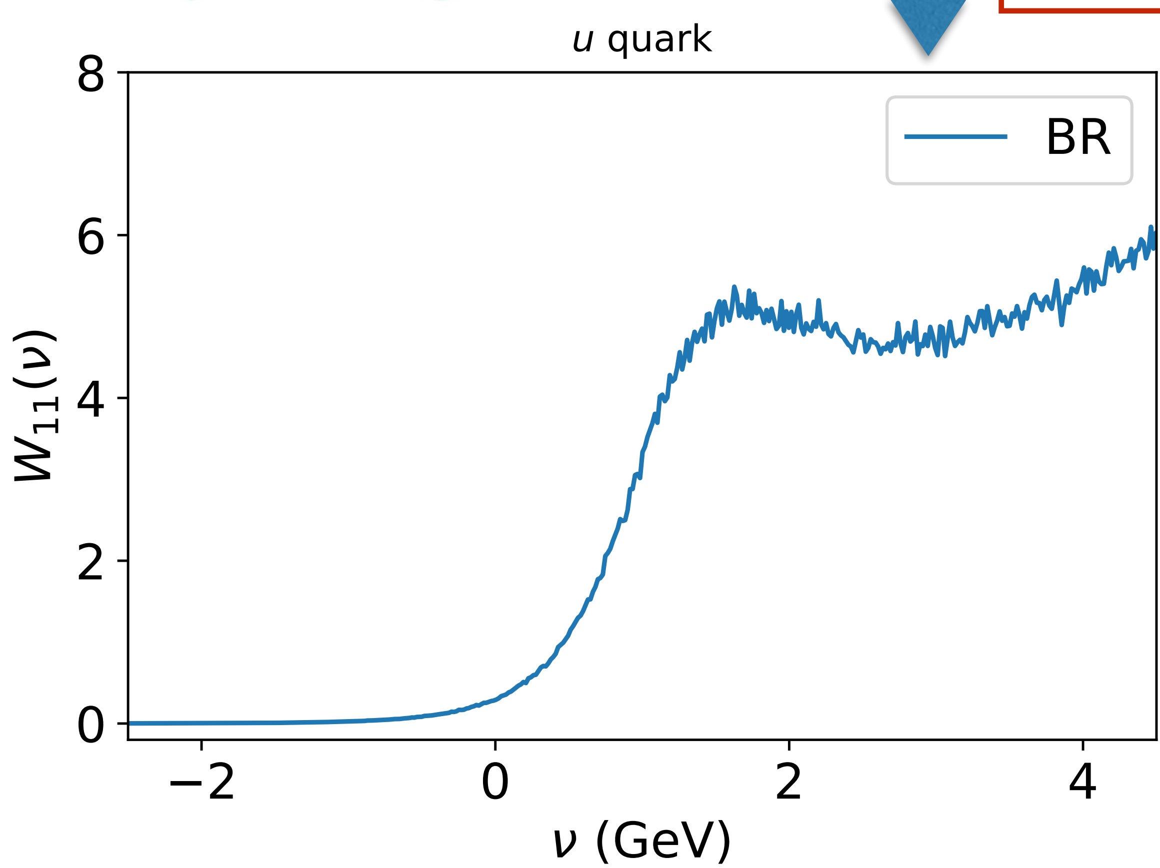
  $Q^2 \sim 13 \text{ GeV}^2,\ G^2(0) \sim 10^{-5}$
- ✦ **Quasi-elastic contribution is large and relatively stable.**
- ✦ **Large error in the DIS working region reflects the effect of different default models, no enough constraint from the data**

How about BR





different default models

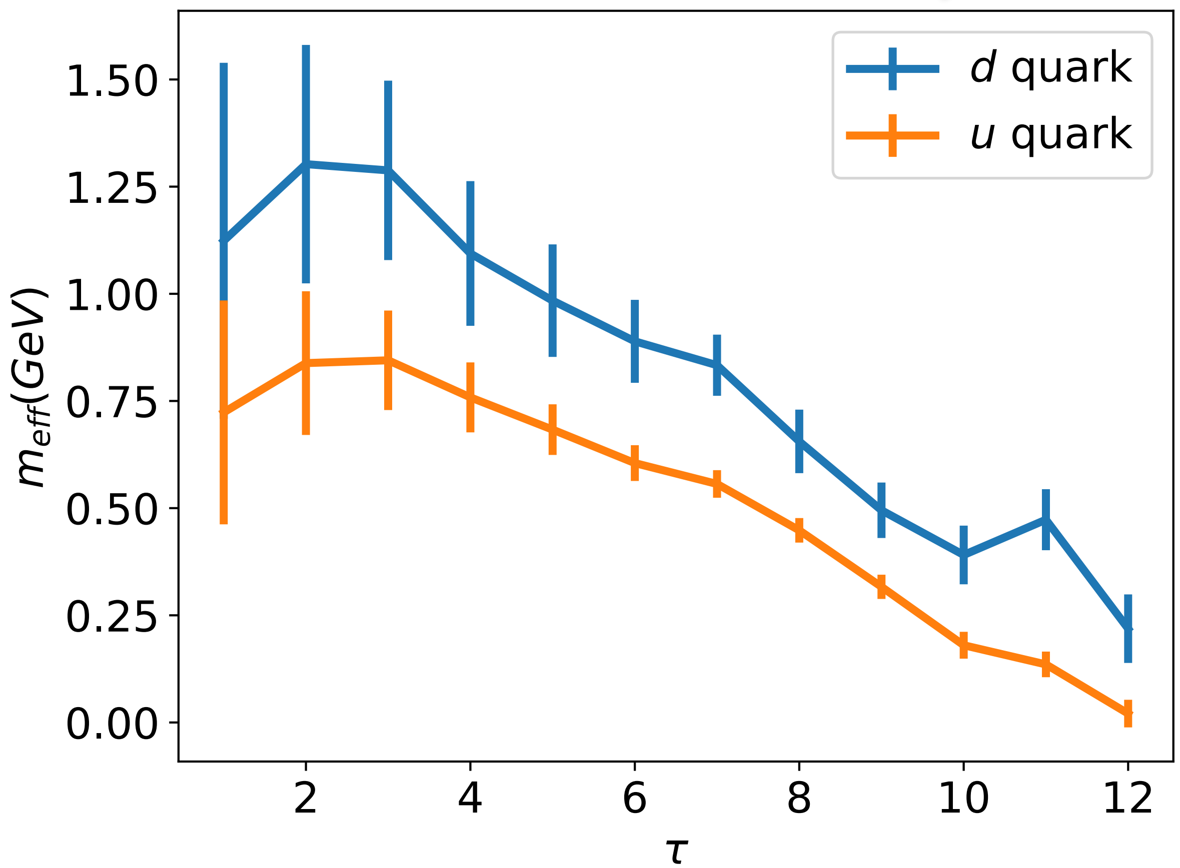


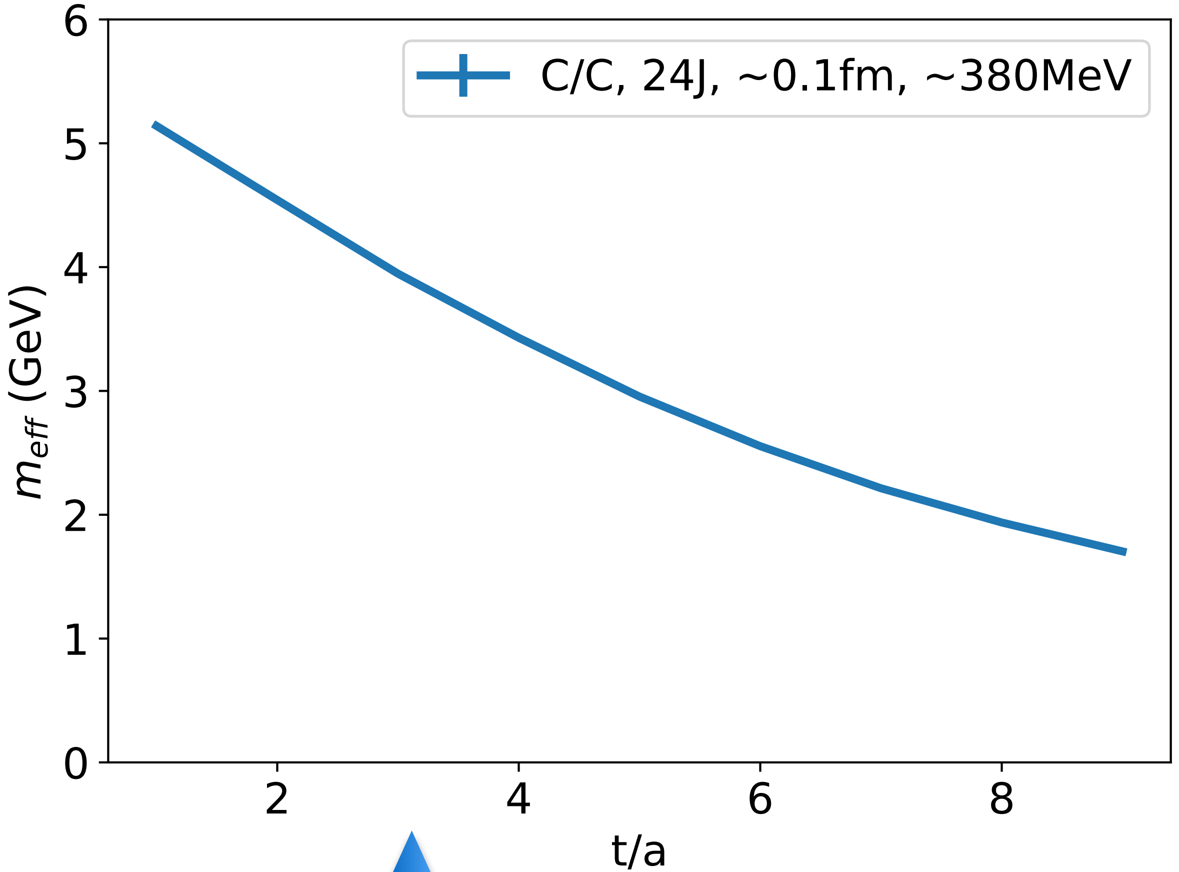
not enough information for high energy transfer

Check the effective mass

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \sum_n A_n e^{-(E_n - E_p)\tau} \qquad E_p \sim 2.15 \text{ GeV}$$

one can check the effective mass of the Euclidean hadronic tensor



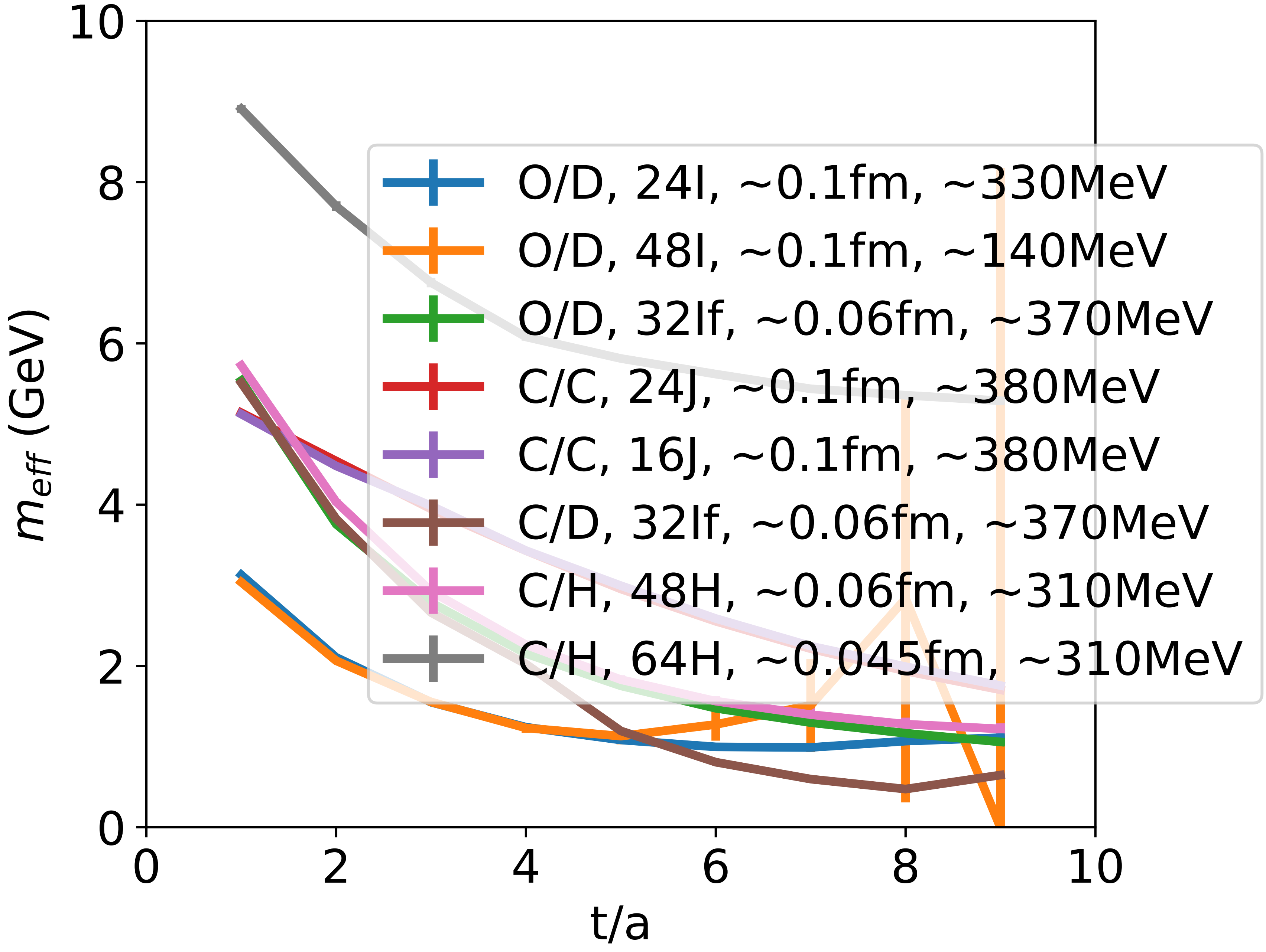


one can also check the effective mass of rho meson with point source

1.25 + 2.15 ~ 3.4, not the same as the two-point function case

But they should be connected.

Learn more from two-point functions



- ✦ **It seems how heavy we can reach is mainly connected to the lattice spacing.**
- ✦ **Other factors are not significant.**
- ✦ **It seems that the a~0.045 fm lattice can be a much better choice.**

Summary and outlook

- We tried to calculate the hadronic tensor on the lattice, which should be very helpful to understand more about the nucleon structure.
- We are beginning to have some preliminary results from this approach.
- We find that the lattice spacing plays an important role to reach highly excited states. More detailed investigation will be done for other factors.
- We can have reasonable results for the elastic (quasi-elastic) contributions.
- We are planning to work on lattices with smaller lattice spacings to have better results.

Thank you for your attention!