INT Program INT-18-3

Probing Nucleons and Nuclei in High Energy Collisions

October 1 - November 16, 2018

Relating TMDs & Collinear PDFs in the CSS formalism

“putting weighted TMDs on better footing”

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October 11, 2018

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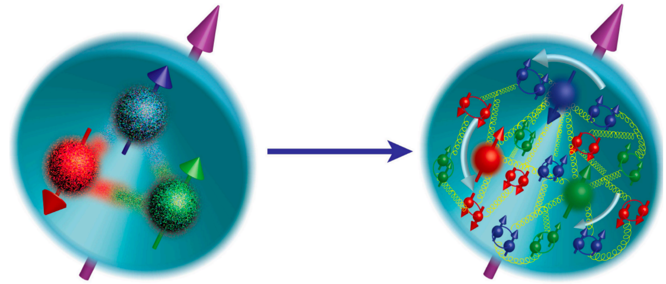
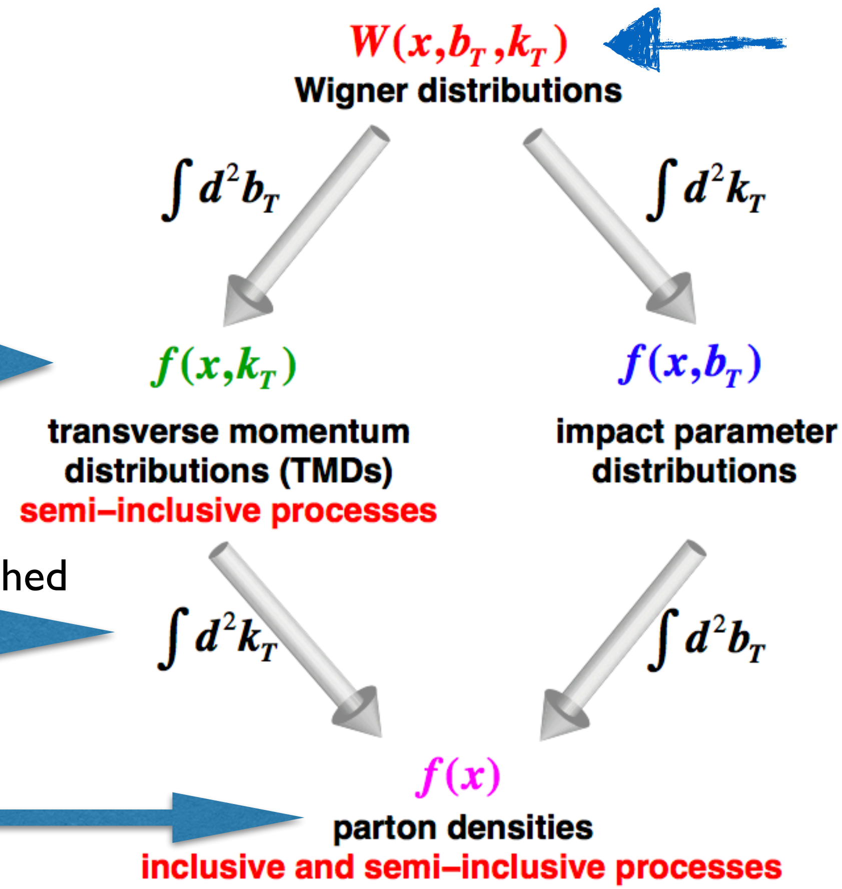


EIC White Paper

Last week ...



TMD to collinear

nb CSS TMD factorisation carried out coordinate space: can we can shed some light through CSS ?

Must consider UV and IR Divergences and TMD evolution, CS NPB 1982… CSS 1985
Ji Ma Yuan PRD 2004/5, Collins 2011 Cambridge Press, Aybat Rogers 2011 PRD ….

From a talk of Davison Soper

Transverse momentum

- The partons in a proton carry momentum components transverse to the beam direction.

- Thus there are transverse momentum dependent (TMD) parton distributions

$$f_{a/A}(x, \boldsymbol{k}_\perp, Q^2)$$

- If you are going into the woods, you have to be careful: there are some subtle issues in the definitions of these.

- On an intuitive level

$$f_{a/A}(x, Q^2) \sim \int d\boldsymbol{k}_\perp \; f_{a/A}(x, \boldsymbol{k}_\perp, Q^2)$$

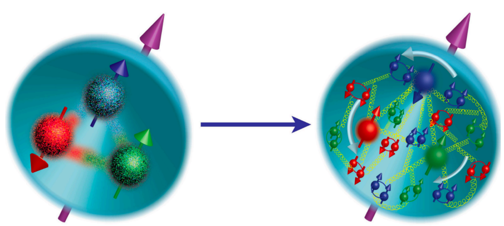
Overview comments

- Report on relating TMD factorization & collinear factorisation in studying nucleon structure in Collins-Soper-Sterman formalism

- Using enhanced version of CSS framework, able to obtain at @ “LO” the well-known relation between the unpolarized FT-TMD & $f_1(x,\mu)$, & the Sivers function, and the (collinear twist-3) Qiu-Sterman function $T_F(x,\mu) \sim f_{1T}^{\perp(1)}(x,\mu)$
  nb … power counting remains open question

  - **Phys.Rev. D (2016) Collins, Gamberg, Prokudin, Sato, Rogers, Wang**
  - **Phys. Lett B (2018) Gamberg , Metz, Pitonyak, Prokudin**

- Relies on a modification of the so called *W+Y* construction used to “match” the cross section as a function of q_T point-by-point, from small $q_T \sim m$ (hadronic scale), to large $q_T \sim Q$



Overview comments

- We modify the "*standard matching prescription*" traditionally used in CSS formalism relating low & high q_T behavior cross section @ moderate Q in particular where studies of TMDs are relevant

Matching studies in CSS related approaches

…

NPB Collins & Soper(1982), & Sterman 1985

NPB (1991) Arnold, Kauffman

PRD (1998) Nadolsky Stump Yuan

PRL (2001) Qiu, Zhang

PRD (2003) Berger, Qiu

NPB (2006) Bozzi, Catani, DeFlorian, Grazzini …

NPB (2006) Y. Koike, J. Nagashima, W. Vogelsang

arXiv (2014) Sun, Isacson, Yuan-CP, Yuan-F

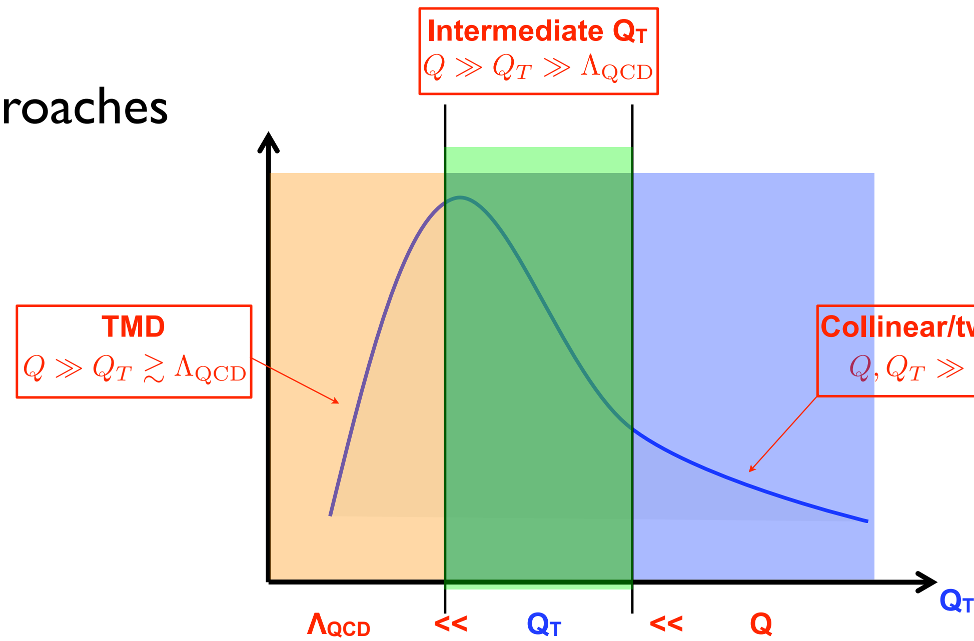
JHEP (2015) Boglione, Hernandez, Melis Prokudin

PRD (2016) Collins, Gamberg, Prokudin, Rogers, Sato, Wang

PLB (2018) Gamberg , Metz, Pitonyak, Prokudin

PLB (2018) Echevarria, Kasemets, Lansberg, Pisano, Signori

….



Nobuo's talk …

Y-term & Matching

$$\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\mathrm{SIDIS}}(\alpha_s(\mu),\mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T}\ \tilde{F}_{j/H_1}(x,b_T;\mu,\zeta_1)\,\tilde{D}_{H_2/j'}(z,b_T;\mu,\zeta_2) \ + \ Y_{\mathrm{SIDIS}}$$

$$d\sigma(m \lesssim q_T \lesssim Q, Q) = W(q_T,Q) + FO(q_T,Q) + ??\ O\left(\frac{m}{Q}\right)^c d\sigma(q_T,Q)\ ??$$

<u>If we do we double count</u>

We subtract out the double counting such that the cross section is matched (SIDIS,DY, $e^+ e^-$) in the “overlap region”:Designed s.t. valid to leading order in *m/Q* uniformly in q_T (*<u>see role of “approximations” in TMD factorization</u>*)

$$Y(q_T,Q) = FO(q_T,Q) - ASY(q_T,Q)$$

$$d\sigma(m \lesssim q_T \lesssim Q, Q) = W(q_T,Q) + Y(q_T,Q) + O\left(\frac{m}{Q}\right)^c d\sigma(q_T,Q)$$

JCC Cambridge Press 2011, Collins arXiv: 1212.5974, Catani et al. NPB 06, 15, Collins, Gamberg, Prokudin, Rogers, Sato, Wang PRD 2016 ……

Moments of TMDs and collinear pdfs

Naive connection of moments of TMDs and collinear pdfs based on matrix elements and a Parton Model picture "factorization"

$$\int d^2\vec{k}_T \, \frac{\vec{k}_T^2}{2M^2} \, \underbrace{f_{1T}^{\perp}(x, k_T)}_{\text{TMD}} = \underbrace{f_{1T}^{\perp(1)}(x)}_{\text{kinematical CT3}} = \underbrace{-\frac{T_F(x,x)}{2M}}_{\text{dynamical CT3}}$$

Qiu & Sterman 1991

Boer, Mulder, Pijlman (2003); Meissner (2009); ...

$$\int d^2\vec{p}_T \, \frac{\vec{p}_T^2}{2z^2 M_h^2} \, \underbrace{H_1^{\perp}(z, p_T)}_{\text{TMD}} = \underbrace{H_1^{\perp(1)}(z)}_{\text{kinematical CT3}}$$

Yuan and Zhou (2009)

Must confront central issues of collinear and TMD evolution and factorisation considering polarised TMDs

Consider the less exotic case

"Parton Model"

TMD CT2

$$\int d^2\vec{k}_T \quad \boldsymbol{f_1(x, k_T)} \quad = \quad \boldsymbol{f_1(x)}$$

$$\vdots$$

TMD CT2

$$\int d^2\vec{p}_T \quad \boldsymbol{D_1(z, p_T)} \quad = \quad \boldsymbol{D_1(z)}$$

$$\vdots$$

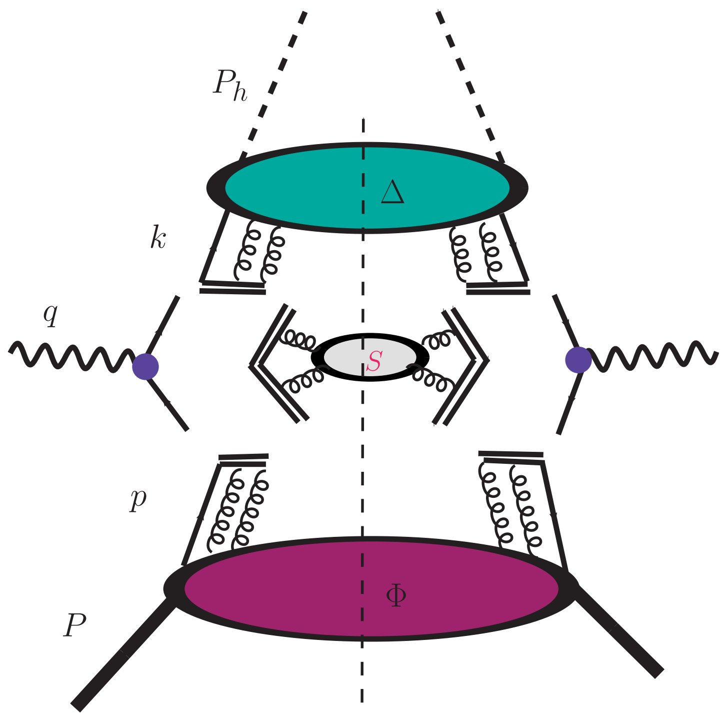
Ignore UV divergences and effects from soft-gluon radiation

Some comments on the subject

- Collins QCD Book Ch. 9 &13
- Ji Ma Yuan PRD 2005
- Aybat Rogers PRD 2011
- Aybat Collins Qiu Rogers PRD 2012
- Vogelsang INT talk 2/27/14
- Collins, Gamberg, Prokudin, Sato, Wang PRD 2016
- Gamberg, Metz, Pitonyak, Prokudin PLB 2018

Review of TMD factorization

- ★ Collins Soper (81), Collins, Soper, Sterman (85), Boer (01) (09) (13), Ji,Ma,Yuan (04), Collins-Cambridge University Press (11), Aybat Rogers PRD (11), Abyat, Collins, Qiu, Rogers (11), Aybat, Prokudin, Rogers (11), Bacchetta, Prokudin (13), Sun, Yuan (13),Echevarria, Idilbi, Scimemi JHEP 2012, Collins Rogers 2015 ....



TMD Factorization & Evolution

$$\frac{d\sigma}{d\boldsymbol{q}_{\rm T}^2 dQ^2 \ldots} = W(q_{\rm T}, Q) + Y(q_{\rm T}, Q) + O\left(\frac{m}{Q}\right)^{\rm c} \frac{d\sigma}{d\boldsymbol{q}_{\rm T}^2 dQ^2 \ldots}$$

$$W_{UU}(q_{\rm T}, Q) = \sum_{jj'} H_{jj'}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T \cdot \boldsymbol{q}_T} \; \tilde{f}_{j/H_1}(x, b_T; \mu, \zeta_1) \, \tilde{D}_{H_2/j'}(z, b_T; \mu, \zeta_2)$$

In full QCD, the auxiliary parameters μ and ζ are exactly arbitrary and this is reflected in the the Collins-Soper (CS) equations for the TMD PDF, and the renormalization group (RG) equations

TMD factorization & evolution from b-space rep of SIDIS cross section

interpret as a multipole expansion in terms of $b_T\,[\mathrm{GeV}^{-1}]$ conjugate $\boldsymbol{P}_{h\perp}$

Boer, Gamberg, Musch, Prokudin, **JHEP (2011)**

$\tilde{W}_{UU}(x,z,b,Q^2)$

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\,dy\,d\phi_S\,dz_h\,d\phi_h\,|\boldsymbol{P}_{h\perp}|d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B\,y\,Q^2}\,\frac{y^2}{(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\int\frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|\Big\{J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,T}+\varepsilon\,J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,L} \\
&+\quad\sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos\phi_h}+\varepsilon\cos(2\phi_h)\,J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LU}^{\sin\phi_h} \\
&+\quad S_\parallel\Big[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin\phi_h}+\varepsilon\sin(2\phi_h)\,J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin 2\phi_h}\Big] \\
&+S_\parallel\lambda_e\Big[\sqrt{1-\varepsilon^2}\,J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}^{\cos\phi_h}\Big] \\
&+|\boldsymbol{S}_\perp|\Big[\sin(\phi_h-\phi_S)\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\left(\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)}+\varepsilon\,\mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)}\right) \\
&\qquad+\ \varepsilon\ \sin(\phi_h+\phi_S)\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\qquad+\ \varepsilon\ \sin(3\phi_h-\phi_S)\,J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)}
\end{aligned}
$$

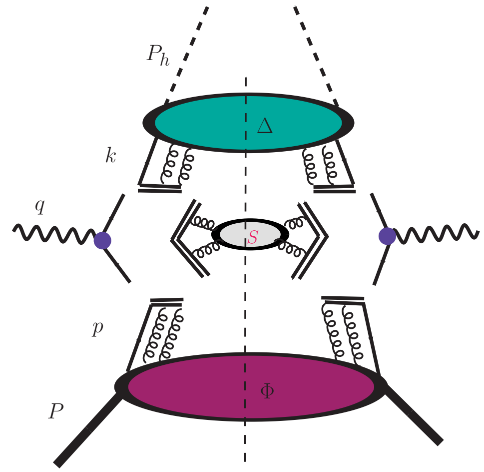
$$\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} = -\mathcal{P}[\tilde{f}_{1T}^{\perp(1)}\,\tilde{D}_1]$$

$$\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} = -\mathcal{P}[\tilde{h}_1\,\tilde{H}_1^{\perp(1)}]$$

$$\ldots \qquad + \qquad Y$$

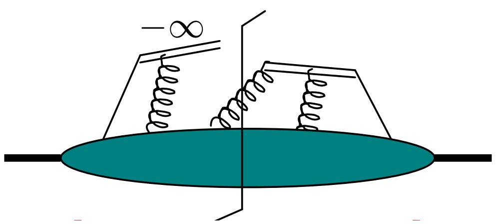
Transverse Momentum Dependent Evolution

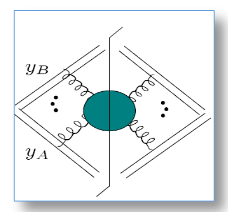
TMD factorization/evolution CSS in b space emerge from region analysis & Ward Identities



- ✦ *Collins Soper, NPB 1982*
- ✦ *Collins Soper Sterman NPB 1985*
- ✦ *Ji Ma Yuan PRD 2005*
- ✦ *Aybat Rogers PRD 2011*
- ✦ *Aybat Collins Qiu Rogers PRD 2012*
- ✦ *Collins 2011 Cambridge Press*

- •TMDs w/Gauge links: color invariant
- •TMD PDFs & Soft factor have rapidity/LC divergences
- • Rapidity regulator introduced to regulate these divergences



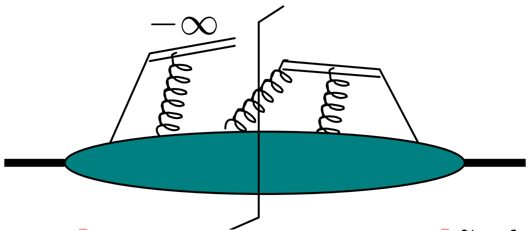


$$\tilde{f}^{\rm sub}_{j/H}(x, b_T; \mu, y_n) = \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} \underbrace{\tilde{f}^{\rm unsub}_{j/H}(x, b_T; \mu, y_P - y_B)} \sqrt{\frac{\tilde{S}(b_T; y_A, y_n)}{\tilde{S}(b_T; y_A, y_B)\tilde{S}(b_T; y_n, y_B)}} \times UV_{renorm}$$

$$\Updownarrow$$

$$\tilde{f}^{\rm unsub}_{j/H}(x, b_T; \mu, y_P - y_B) = \int \frac{db^-}{2\pi} e^{-ixP^+b^-} \langle P|\bar{\psi}(0)\gamma^+ \mathcal{U}_{[0,b]}\psi(b)|P\rangle|_{b^+=0}$$

Evolution follows from their independence of rapidity scale



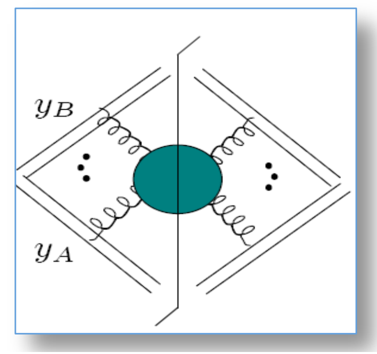
Collins Cambridge press 2011, Aybat & Rogers 2011 PRD

$$\tilde{F}_H^{\rm sub}(x, b_T; \mu, y_n) = \lim_{\substack{y_A \to \infty \\ y_B \to -\infty}} \tilde{F}_H^{\rm unsub}(x, b_T; \mu, y_P - y_B) \sqrt{\frac{\tilde{S}(b_T; y_A, y_n)}{\tilde{S}(b_T; y_A, y_B)\tilde{S}(b_T; y_n, y_B)}}$$

From operator definition get

Collins-Soper Equation:

$$-\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$



$$\tilde{K}(b_T; \mu) = \frac{1}{2} \frac{\partial}{\partial y_n} \ln \frac{\tilde{S}(b_T; y_n, -\infty)}{\tilde{S}(b_T; +\infty, y_n)}$$

Soft factor further "repartitioned"
This is done to

1) cancel LC divergences in "unsubtracted" TMDs
2) separate "right & left" movers i.e. full factorization
3) remove double counting of momentum regions

Along with .... Renormalization group Equations

$$\frac{d\tilde{K}}{d\ln\mu} = -\gamma_K(g(\mu))$$

$$\frac{d\ln\tilde{F}(x, b_T; \mu, \zeta)}{d\ln\mu} = -\gamma_F(g(\mu); \zeta/\mu^2)$$

RGE:
get anomalous
for *F* & *K*

Solve Collins Soper & RGE eqs. to obtain "evolved TMDs"

Ted's Talk

One TMD PDF: Solution to Evolution

Ex: Cutoff Prescription:

$$\mathbf{b}_*(\mathbf{b}_\mathrm{T}) \equiv \frac{\mathbf{b}_\mathrm{T}}{\sqrt{1 + b_T^2/b_\mathrm{max}^2}}$$

$$\mu_b \equiv C_1/|\mathbf{b}_*(b_T)|$$

$$\tilde{F}_{f/P}(x, \mathbf{b}_\mathrm{T}; Q, Q^2) =$$

Collinear PDFs

$$\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P}(\hat{x}, \mu_b) \times$$

$$\times \exp\left\{ \ln \frac{Q}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{Q} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{Q}{\mu'} \gamma_K(g(\mu')) \right] \right\} \times$$

$$\times \exp\left\{ -\underline{g_{f/P}(x, b_T; b_\mathrm{max})} - \underline{g_K(b_T; b_\mathrm{max})} \ln \frac{Q}{Q_0} \right\}$$

Nonperturbative parts large b_T

Unpolarized and Sivers evolve in same way

Recall the correlator in *b*-space Bessel Transform

$$\tilde{\Phi}^{[\gamma^+]}(x,\boldsymbol{b}_T) \;=\; \tilde{f}_1(x,\boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma}\, b_{T\rho} S_{T\sigma}\, M\,\tilde{f}_{1T}^{\perp(1)}(x,\boldsymbol{b}_T^2)$$

Boer Gamberg Musch Prokudin JHEP 2011

See lattice studies of Engelhardt et al , Musch 2009-2018

It obeys Collins Soper Equation, thus unpolarised and Sivers evolve in the same manner

$$\frac{\partial\,\tilde{\phi}^{i}_{f/P}(x,\mathbf{b}_{\mathrm{T}};\mu,\zeta_F)\epsilon_{ij}S_T^j}{\partial\ln\sqrt{\zeta_F}} = \tilde{K}(b_T;\mu)\tilde{\phi}^{i}_{f/P}(x,\mathbf{b}_{\mathrm{T}};\mu,\zeta_F)\epsilon_{ij}S_T^j.$$

Aybat Rogers Collins Qiu PRD 2012
also see Kang Yuan Xiao PRL 2011

TMD Evolution-Solution for unpolarised & Sivers

- TMD/CSS Evolution/Factorization carried out in *b*-space "Bessel transforms"

Boer Gamberg Musch Prokudin 2011 JHEP
Collins Aybat Rogers Qiu 2012 PRD

$$\tilde{\Phi}^{[\gamma^+]}(x, \boldsymbol{b}_T) \;=\; \tilde{f}_1(x, \boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M \tilde{f}_{1T}^{\perp(1)}(x, \boldsymbol{b}_T^2)$$

Correlator obeys CSS equation so,

$$\tilde{\boldsymbol{f}}_{\boldsymbol{1}}(\boldsymbol{x}, \boldsymbol{b_T}; \boldsymbol{Q^2}, \boldsymbol{\mu_Q}) \;\sim\; \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{f_1(\hat{x}; \mu_{b_*})}\right)$$
$$\times\ \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

Collins (2011); ...

Qiu & Sterman PRL 1991

$$\tilde{\boldsymbol{f}}_{\boldsymbol{1T}}^{\perp(1)}(\boldsymbol{x}, \boldsymbol{b_T}; \boldsymbol{Q^2}, \boldsymbol{\mu_Q}) \;\sim\; \left(\tilde{C}^{f_{1T}^{\perp}}(\hat{x}_1, \hat{x}_2, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{T_F(\hat{x}_1, \hat{x}_2; \mu_{b_*})}\right)$$
$$\times\ \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_{1T}^{\perp}}(b_T, Q)\right]$$

Aybat, Collins, Qiu, Rogers (2012); Echevarria, Idilbi, Kang, Vitev (2014); ...

Putting Solution of CSS Eqn. Together
W term

$$\tilde{W}_{\rm UU}(b_T, Q) = \tilde{W}^{\rm OPE}_{\rm UU}(b_*(b_T), Q)\tilde{W}^{\rm NP}_{\rm UU}(b_T, Q)$$

$$= \sum_j H_j(\mu_Q, Q)\, \tilde{f}_1^j(x, b_*(b_T); \mu^2_{b_*}, \mu_{b_*})\, \tilde{D}_1^{h/j}(z, b_*(b_T); \mu^2_{b_*}, \mu_{b_*})$$

$$\times \exp\left\{\tilde{K}(b_*(b_T); \mu_{b_*}) \ln\left(\frac{Q^2}{\mu^2_{b_*}}\right) + \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu'); 1) - \ln\left(\frac{Q^2}{\mu'^2}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

$$\times \exp\left\{-g_{\rm pdf}(x, b_T; Q_0, b_{max}) - g_{\rm ff}(z, b_T; Q_0, b_{max}) - g_K(b_T; b_{max}) \ln\left(\frac{Q^2}{Q_0^2}\right)\right\},$$

$$\tilde{W}^{\rm siv}_{\rm UT}(b_T, Q) = \tilde{W}^{\rm siv,OPE}_{\rm UT}(b_*(b_T), Q)\tilde{W}^{\rm siv,NP}_{\rm UT}(b_T, Q)$$

$$= \sum_j H_j(\mu_Q, Q)\, \tilde{f}_{1T}^{\perp(1)j}(x, b_*(b_T); \mu^2_{b_*}, \mu_{b_*})\, \tilde{D}_1^{h/j}(z, b_*(b_T); \mu^2_{b_*}, \mu_{b_*})$$

$$\times \exp\left\{\tilde{K}(b_*(b_T); \bar{\mu}) \ln\left(\frac{Q^2}{\mu^2_{b_*}}\right) + \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu'); 1) - \ln\left(\frac{Q^2}{\mu'^2}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

$$\times \exp\left\{-g_{\rm siv}(x, b_T; Q_0, b_{max}) - g_{\rm ff}(z, b_T; Q_0, b_{max}) - g_K(b_T; b_{max}) \ln\left(\frac{Q^2}{Q_0^2}\right)\right\},$$

Matching of the small and large $b_{\rm T}$ behaviour of solution to CSS $b_{\rm max}$

$$b_*(b_T) \equiv \sqrt{\frac{b_T^2}{1 + b_T^2/b_{max}^2}}, \qquad \mu_{b_*} \equiv \frac{C_1}{b_*(b_T)},$$

Re-factorization collinear pdfs OPE

$$\tilde{f}_1^j(x, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}) = \sum_{j'} \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{j/j'}^{\rm pdf}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*}))\, f_1^{j'}(\hat{x}; \mu_{b_*}) + O((m\, b_*(b_T))^p)\,,$$

$$\tilde{D}_1^{h/j}(z, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}) = \sum_{i'} \int_z^1 \frac{d\hat{z}}{\hat{z}^3} \tilde{C}_{i'/j}^{\rm ff}(z/\hat{z}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*}))\, D_1^{h/i'}(\hat{z}; \mu_{b_*}) + O((m\, b_*(b_T))^p)\,,$$

★ Collins-Cambridge Univ Press 2011, Aybat Rogers PRD 2011, Collins Rogers PRD 2015 ....

$$\tilde{f}_{1T}^{\perp(1)j}(x, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}) = -\frac{1}{2M_P} \sum_{j'} \int_x^1 \frac{d\hat{x}_1}{\hat{x}_1} \frac{d\hat{x}_2}{\hat{x}_2} \tilde{C}_{j/j'}^{\rm siv}(\hat{x}_1, \hat{x}_2, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) T_F^{j'}(\hat{x}_1, \hat{x}_2; \mu_{b_*}) + O((m\, b_*(b_T))^{p'})\,,$$

★ Kang, Xiao, Yuan PRL 2011

★ Abyat, Collins, Qiu, Rogers PRD 2012

TMD Evolution-Solution for unpolarised

With $\mu_b = C_1/b_*$ as hard scale, the b dependence of TMDs is calculated in perturbation theory and related to their collinear parton distribution (PDFs), fragmentation functions (FFs), or multiparton correlation functions, thru an OPE

$$\boldsymbol{\tilde{f}_1(x, b_T; Q^2, \mu_Q)} \quad \sim \quad \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{f_1(\hat{x}; \mu_{b_*})}\right)$$

Collins (2011); ...

$$\times \ \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

Collins 2011 QCD Aybat Rogers PRD 2011

CSS Modification Unpolarized FT-TMD

$$\tilde{f}_1(x, b_T; Q^2, \mu_Q) \sim \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes f_1(\hat{x}; \mu_{b_*})\right)$$

$$\times \ \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

perturbative Sudakov factor

$$\underbrace{-\ln(Q/\mu_{b_*})\tilde{K}(b_*, \mu_{b_*}) - \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu'); 1) - \gamma_K(\alpha_s(\mu'))\ln(Q/\mu')\right]}_{\text{same for unpol. and pol.}}$$

non-perturbative Sudakov factor

$$g_{f_1}(x, b_T) + g_K(b_T)\ln(Q/Q_0)$$

different for each TMD

universal

$$b_*(b_T) \equiv \sqrt{\frac{b_T^2}{1 + b_T^2/b_{\max}^2}} \qquad \mu_{b_*} = C_1/b_*(b_T)$$

Note: $b_*(0) = 0$ and $(\mu_{b_*})_{b_* \to 0} = \infty$ ➡ problematic large logarithms in S_{pert}

(Bozzi, Catani, de Florian, Grazzini (2006); Collins, Gamberg, Prokudin, Rogers, Sato, Wang (2016))

Parsi Petronzio NPB 1979, Altarelli et al. NPB 1984 CSS NPB250, Bozzi Catani, de Florian Grazzini NPB 2006

$bQ << 1$ contributions to the W term

- Addressed "q_T resummation" Parsi Petronzio NPB 1979, Altarelli et al. NPB 1984 CSS NPB250, Bozzi Catani, de Florian Grazzini NPB 2006

- & "TMD CSS analysis" Collins, Gamberg, Prokudin, Rogers, Sato, Wang PRD 2016 studying the Fourier transform of the W term in the $W+Y$ matching in q_T of the SIDIS cross section from coordinate b-space to q_T momentum space

- Regulate the large logs(Q^2b^2) at small b in the FT they Bozzi et al. , replace logs(Q^2b^2) with logs(Q^2b^2+1) cutting off the $b << 1/Q$ contribution

- Also Kulesza,Sterman,Vogelsang PRD 2002 in threshold resummation studies

We place another boundary condition on now small b_T

"Improved CSS" (Unpolarized) (Collins, Gamberg, Prokudin, Rogers, Sato, Wang (2016))*

Place a lower cut-off on b_T: $b_T \to b_c(b_T)$ where $b_c(b_T) = \sqrt{b_T^2 + \left(\frac{b_0}{C_5 Q}\right)^2} = \sqrt{b_T^2 + b'^2_{min}}$,

$$\Rightarrow \mu_{b_*} \to \bar{\mu} \equiv \frac{C_1}{b_*(b_c(b_T))} \quad \text{so } \mu_{b_*} \text{ is cut off at} \quad \mu_c \approx C_1 Q$$

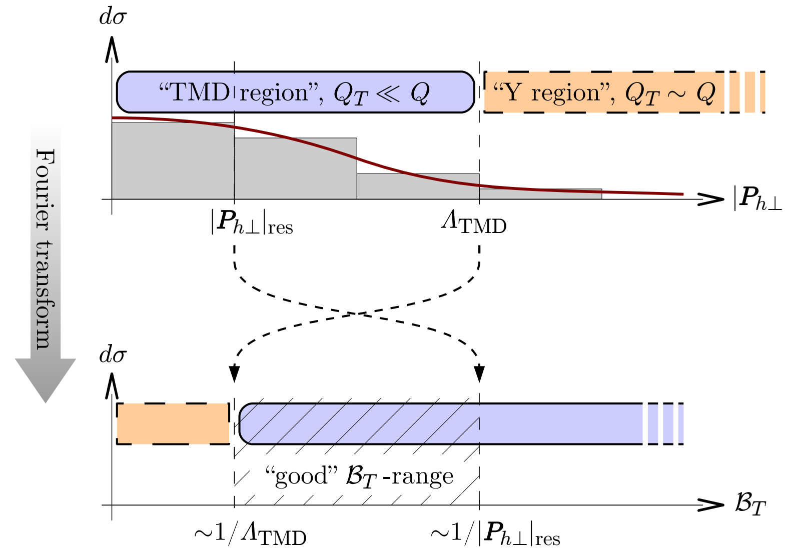
Modification to CSS W Term

B.C. Introduce small b-cuttoff

Similar to Catani et al. NPB 2006 & *"Bessel Weighting" ppr. Boer LG Musch Prokudin JHEP 2011*

$$b_c(b_T) = \sqrt{b_T^2 + b_0^2/\left(C_5 Q\right)^2} \implies b_c(0) \sim 1/Q$$

Regulate unphysical divergences from in *W* term



$$\tilde{W}_{New}(q_T, Q; \eta, C_5) = \Xi\left(\frac{q_T}{Q}, \eta\right) \int \frac{d^2 b_T}{(2\pi)^2} e^{i q_T \cdot b_T} \tilde{W}^{OPE}\left(b_*(b_c(b_T)), Q\right) \tilde{W}_{NP}(b_c(b_T)), Q; b_{max})$$

Generalized B.C.

$$b_*(b_c(b_{\rm T})) \longrightarrow \begin{cases} b_{\rm min} & b_{\rm T} \ll b_{\rm min} \\ b_{\rm T} & b_{\rm min} \ll b_{\rm T} \ll b_{\rm max} \\ b_{\rm max} & b_{\rm T} \gg b_{\rm max}\,. \end{cases}$$

Unpolarized and Sivers W term

$$\tilde{W}_{\mathrm{UU}}(b_c(b_T),Q) = \sum_j H_j(\mu_Q,Q)\,\tilde{f}_1^{j}(x,b_c(b_T);Q^2,\mu_Q)\,\tilde{D}_1^{h/j}(z,b_c(b_T);Q^2,\mu_Q)\,,$$

$$\tilde{W}_{\mathrm{UT}}^{\mathrm{siv}}(b_c(b_T),Q) = \sum_j H_j(\mu_Q,Q)\,\tilde{f}_{1T}^{\perp(1)j}(x,b_c(b_T);Q^2,\mu_Q)\,\tilde{D}_1^{h/j}(z,b_c(b_T);Q^2,\mu_Q)\,.$$

✦ Phys. Lett B (2018) Gamberg , Metz, Pitonyak, Prokudin

Unpolarized FT TMD

$$\tilde{f}_1^{j}(x,b_c(b_T);Q^2,\mu_Q) = \sum_{j'}\int_x^1 \frac{d\hat{x}}{\hat{x}}\,\tilde{C}^{\mathrm{pdf}}_{j/j'}(x/\hat{x},b_*(b_c(b_T));\bar{\mu}^2,\bar{\mu},\alpha_s(\bar{\mu}))\,f_1^{j'}(\hat{x};\bar{\mu})$$
$$\times\exp\left\{\tilde{K}(b_*(b_c(b_T));\bar{\mu})\ln\left(\frac{Q}{\bar{\mu}}\right)+\int_{\bar{\mu}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1)-\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$
$$\times\exp\left\{-g_{\mathrm{pdf}}(x,b_c(b_T);Q_0,b_{max})-g_K(b_c(b_T);b_{max})\ln\left(\frac{Q}{Q_0}\right)\right\}\,,$$

Sivers FT TMD or 1st Bessel moment

$$\tilde{f}_{1T}^{\perp(1)j}(x,b_c(b_T);Q^2,\mu_Q) = -\frac{1}{2M_P}\sum_{j'}\int_x^1\frac{d\hat{x}_1}{\hat{x}_1}\frac{d\hat{x}_2}{\hat{x}_2}\,\tilde{C}^{\mathrm{siv}}_{j/j'}(\hat{x}_1,\hat{x}_2,b_*(b_c(b_T));\bar{\mu}^2,\bar{\mu},\alpha_s(\bar{\mu}))\,T_F^{j'}(\hat{x}_1,\hat{x}_2;\bar{\mu})$$
$$\times\exp\left\{\tilde{K}(b_*(b_c(b_T));\bar{\mu})\ln\left(\frac{Q}{\bar{\mu}}\right)+\int_{\bar{\mu}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1)-\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

Modified FT TMDs enhanced CSS

"Improved CSS" (Unpolarized) (Collins, Gamberg, Prokudin, Rogers, Sato, Wang (2016))

Place a lower cut-off on b_T: $b_T \to b_c(b_T)$ where $b_c(b_T) = \sqrt{b_T^2 + b_0^2/(C_5 Q)^2}$

$$\Rightarrow \quad \mu_{b_*} \to \bar{\mu} \equiv \frac{C_1}{b_*(b_c(b_T))} \quad \text{so } \mu_{b_*} \text{ is cut off at } \mu_c \approx \frac{C_1 C_5 Q}{b_0}$$

$$\tilde{f}_1(x, b_c(b_T); Q^2, \mu_Q) \sim \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_c(b_T)); \bar{\mu}^2, \bar{\mu}, \alpha_s(\bar{\mu})) \otimes f_1(\hat{x}; \bar{\mu})\right)$$
$$\times \exp\left[-S_{pert}(b_*(b_c(b_T)); \bar{\mu}, Q, \mu_Q) - S_{NP}^{f_1}(b_c(b_T), Q)\right]$$

"Improved CSS" (Polarized) (Gamberg, Metz, DP, Prokudin, **Phys. Lett B (2018)**)

$$\tilde{\Phi}^{[\gamma^+]}(x, \vec{b}_T; Q^2, \mu_Q) = \tilde{f}_1(x, b_T; Q^2, \mu_Q) - iM\epsilon^{ij} b_T^i S_T^j \tilde{f}_{1T}^{\perp(1)}(x, b_T; Q^2, \mu_Q)$$

$b_T \to b_c(b_T)$

NO $b_T \to b_c(b_T)$ replacement – kinematic factor NOT associated with the scale evolution

$b_T \to b_c(b_T)$

Relationship between moments of regularised TMDs and collinear pdfs LO result-done in b-space w/ the OPE - small b region

Relies on the small b limit with $b_{\min}$ cutoff $\quad b_{min} \propto \frac{1}{Q}$

$$\int d^2\boldsymbol{k}_\mathrm{T}\, f_1^j(x, k_T; Q^2, \mu_Q; C_5) = \tilde{f}_1^j(x, b'_{min}; Q^2, \mu_Q) = f_1^j(x; \mu_c) + O(\alpha_s(Q))$$

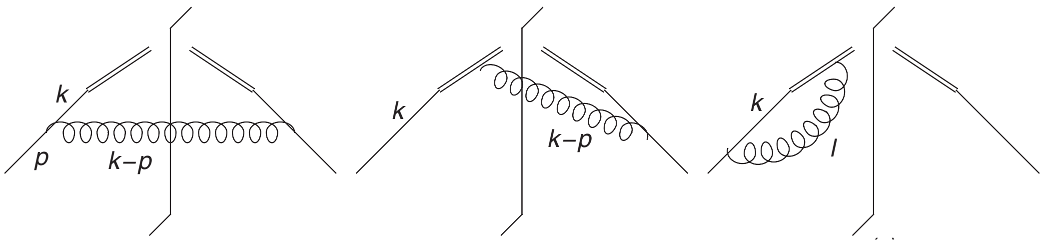
$$z^2 \int d^2\boldsymbol{p}_\mathrm{T}\, D_1^j(z, p_T; Q^2, \mu_Q; C_5) = z^2 \tilde{D}_1^{h/j}(z, b'_{min}; Q^2, \mu_Q) = D_1^{h/j}(z; \mu_c) + O(\alpha_s(Q))\,,$$

$$\int d^2\boldsymbol{k}_\mathrm{T} \frac{k_T^2}{2M_P^2} f_{1T}^{\perp j}(x, k_T; Q^2, \mu_Q; C_5) = \tilde{f}_{1T}^{\perp(1)j}(x, b'_{min}; Q^2, \mu_Q) = \frac{-1}{2M_P} T_F^j(x, x; \mu_c) + O(\alpha_s(Q))$$

$$\int d^2\boldsymbol{p}_\mathrm{T} \frac{p_T^2}{z^2 2M_h^2} H_1^{\perp j}(z, p_T; Q^2, \mu_Q; C_5) = \tilde{H}_1^{\perp(1)j}(z, b'_{min}; Q^2, \mu_Q) = \tilde{H}_1^{\perp(1)j}(z, \mu_c) + O(\alpha_s(Q))$$

Consistency w/ NLO Corrections

At small *b* can calculate coefficient function
CS NPB 1982, JCC 2011 Cambridge press, Aybat Roger PRD 2011, Bacchetta & Prokudin NPB 2013



plus soft diagrams

$$\tilde{C}^{[1]}_{j/f}(x, \mathbf{b}_T) = \tilde{F}^{[1]}_{j/f}(x, \mathbf{b}_T) - f^{[1]}_{j/f}(x)$$

$$\begin{aligned}\tilde{C}^{\mathrm{PDF}}_{f/j}(x, b_{\mathrm{T}}; \zeta_{\mathrm{PDF}}, \mu, \alpha_s(\mu)) =& \delta_{fj}\delta(1-x) + \delta_{fj} 2C_{\mathrm{F}} \Bigg\{ 2\ln\left(\frac{2e^{-\gamma_{\mathrm{E}}}}{\mu b_{\mathrm{T}}}\right)\left[\left(\frac{2}{1-x}\right)_+ - 1 - x\right] + 1 - x \\ & - \delta(1-x)\left[\frac{1}{2}\left[\ln\left(\frac{b_{\mathrm{T}}\mu}{2e^{-\gamma_{\mathrm{E}}}}\right)^2\right]^2 + \ln\left(\frac{b_{\mathrm{T}}\mu}{2e^{-\gamma_{\mathrm{E}}}}\right)^2 \ln\left(\frac{\zeta_{\mathrm{PDF}}}{\mu^2}\right)\right]\Bigg\}\left(\frac{\alpha_s(\mu)}{4\pi}\right) \\ & + \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right)\end{aligned}$$

Note the coefficient function is IR safe the collinear divergence is subtracted; we get NLO correction from replacing

Get NLO result (LO) Splitting function in limit

$$b_c(b_T) = \sqrt{b_T^2 + b_0^2/\left(C_5 Q\right)^2} \implies b_c(0) \sim 1/Q$$

$$f_{j/P}^{\overline{ms}}\left(x, Q^2; \mu_F^2\right) = \int \frac{dz}{z} f_{j/P}^0\left(\frac{x}{z}\right)\left(\delta(1-z) + \frac{\alpha_s}{2\pi} P_{q/a}(z)\left(\ln\frac{Q^2}{\mu_F^2} + \text{finite terms}\right)\right)$$

$$f_{j/P}^0\left(x\right) = \delta\left(1-x\right)$$

Vogelsang INT talk 2/27/14 gets result in joint resummation; again collinear divergence is subtracted in similar manner

Investigating Sivers and Collins at NLO

- Scimemi & Vladimirov EPC 2018

Agreement between TMD and Collinear results

- Relies on further modifications of W+Y construction see
- Collins, Gamberg, Prokudin, Sato, Rogers, Wang PRD 2016

$$\frac{d\sigma}{dxdyd\phi_S\,dz} \equiv 2z^2 \int d^2\boldsymbol{q}_{\mathrm{T}}\,\Gamma(\boldsymbol{q}_{\mathrm{T}}, Q, S) = 2z^2\,\tilde{W}^{\mathrm{OPE}}_{\mathrm{UU}}(b'_{min}, Q)_{\mathrm{LO}} + O(\alpha_s(Q)) + O((m/Q)^p)$$

$$= \frac{2\alpha_{em}^2}{yQ^2}(1 - y + y^2/2)\sum_j e_j^2\, f_1^j(x;\mu_c)\, D_1^{h/j}(z;\mu_c) + O(\alpha_s(Q)) + O((m/Q)^p)$$

- Gamberg , Metz, Pitonyak, Prokudin PLB 2018

$$\frac{d\langle P_{h\perp}\,\Delta\sigma(S_T)\rangle}{dxdydz} = -4\pi z^3 M_P\,\tilde{W}^{\mathrm{siv,OPE}}_{\mathrm{UT}}(b'_{min}, Q)_{\mathrm{LO}} + O(\alpha_s(Q)) + O((m/Q)^{p'})$$

$$= \frac{2\pi\, z\,\alpha_{em}^2}{yQ^2}(1 - y + y^2/2)\sum_j e_j^2\, T_F^j(x, x;\mu_c)\, D_1^{h/j}(z;\mu_c) + O(\alpha_s(Q)) + O((m/Q)^{p'})$$

Agrees with collinear twist-3 result at leading order

Z.-B.Kang,Vitev, Xing,PRD(2013)

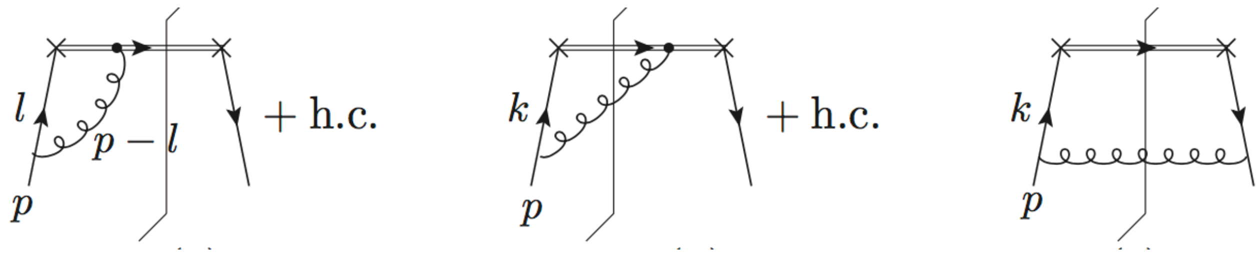
Comments

- With our method, the redefined *W* term allowed us to construct a relationship between TMD-factorization formulas and standard collinear factorization formulas, with errors relating the two being suppressed by powers of 1/Q
- Importantly, the exact definitions of the TMD pdfs and ffs are unmodified from the usual ones of factorization derivations. We preserve transverse-coordinate space version of the W term, but only modify the way in which it is used
- **We have a new now applied to transverse polarized phenomena**
- We are able to recover the well-known relations between TMD and collinear quantities expected from the leading order parton model picture operator definition
- We recover the LO collinear twist 3 result from a weighted q_T integral of the differential cross section and derive the well known relation between the TMD Sivers function and the collinear twist 3 Qiu Sterman function from iCSS approach

Extras

QCD corrections Collinear

Collinear factorisation CS 82 based on integration over transverse momentum also see JCC 2011 Cambridge Press



Independent of hadron

$$f_{j/p}(\xi;\mu) = \sum_i \int \frac{dz}{z} Z_{ji}(z,\alpha_s(\mu))\, f_{0,i/p}(\xi/z) = \overbrace{Z_{ji}}^{\text{Independent of hadron}} \otimes f_{0,i/p}$$

Collinear / DGLAP, Evolution with Scale μ

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} f_{j/P}(x;\mu) = 2\int P_{jj'}(x') \otimes f_{j'/P}(x/x';\mu)$$

Enhanced CSS definitions of TMDs

$$f_1^j(x,k_T;Q^2,\mu_Q;C_5) \equiv \int \frac{db_T}{2\pi}\, b_T J_0(k_T b_T) \tilde{f}_1^j(x,b_c(b_T);Q^2,\mu_Q)\,,$$

$$D_1^j(z,p_T;Q^2,\mu_Q;C_5) \equiv \int \frac{db_T}{2\pi}\, b_T J_0(p_T b_T)\, \tilde{D}_1^{h/j}(z,b_c(b_T);Q^2,\mu_Q)\,,$$

$$\frac{k_T^2}{2M_P^2}\, f_{1T}^{\perp j}(x,k_T;Q^2,\mu_Q;C_5) \equiv k_T \int \frac{db_T}{4\pi} b_T^2 J_1(k_T b_T)\, \tilde{f}_{1T}^{\perp(1)j}(x,b_c(b_T);Q^2,\mu_Q)\,.$$

Transverse spin case

- So it is the derivative of Sivers function or first moment evolves

$$\frac{\partial \ln \tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F)}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(b_T; \mu)$$

Matching and $W + Y$ *to collinear Factorization*

$$\int \mathrm{d}^2\boldsymbol{q}_\mathrm{T}\, \frac{\mathrm{d}\sigma}{\mathrm{d}^2\boldsymbol{q}_\mathrm{T}\ldots} = \int \mathrm{d}^2\boldsymbol{q}_\mathrm{T}\, W + \int \mathrm{d}^2\boldsymbol{q}_\mathrm{T}\, Y$$

A second/third issue is the problem of matching the TMD factorized cross section integrated over q_T to the collinear factorization formalism.

LHS, In QCD the cross section integrated over all q_T; it is of the form of factors of collinear parton densities and/or fragmentation functions at scale Q convoluted with hard scattering that is expanded in powers of $\alpha_S(Q)$

RHS

1) Integral

$$\int d^2\boldsymbol{q}_\mathrm{T}\, W(\boldsymbol{q}_\mathrm{T}, Q, S) = \tilde{W}_\mathrm{UU}(b_T \to 0, Q)$$
$$\sim b_T^a \times (\log \text{ corrections}) = 0\,,$$
$$a = 8C_F/\beta_0, \quad \beta_0 = 11 - 2n_f/3$$

2) Using collinear factorization the *Y* term "starts" at NLO $\alpha_s^{[1]}$

b-Dependence driven by perturbative part of ev. Kernel

$$\exp\left[\int_{\mu_b*}^{\mu_Q}\frac{d\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1)-2\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right]$$

$$\begin{aligned}\tilde{W}(b_T\to 0,Q)\sim\exp\left[\frac{C_F}{\pi\beta_0}\int_{\ln\mu_b^2}^{\ln\mu_Q^2}\ln\mu'^2\right]&=\exp\left[-\frac{C_F}{\pi\beta_0}\ln\left(\frac{\mu_b^2}{\mu_Q^2}\right)\right]\\&=\exp\left[-\frac{C_F}{\pi\beta_0}\ln\left(\frac{C_1^2}{b_T^2\mu_Q^2}\right)\right]\\&=b_T^a\quad\text{where, }a=2C_F/(\pi\beta_0)>0\\&\to 0\end{aligned}$$

Must regulate the large logs in b_TQ

Nobuo’s talk

TMD Evolution-Solution for unpolarised

With $\mu_b = C_1/b_*$ as hard scale, the b dependence of TMDs is calculated in perturbation theory and related to their collinear parton distribution (PDFs), fragmentation functions (FFs), or multiparton correlation functions, thru an OPE

$$\boldsymbol{\tilde{f}_1(x, b_T; Q^2, \mu_Q)} \quad \sim \quad \left(\tilde{C}^{f_1}(x/\hat{x}, b_*(b_T); \mu_{b_*}^2, \mu_{b_*}, \alpha_s(\mu_{b_*})) \otimes \boldsymbol{f_1(\hat{x}; \mu_{b_*})}\right)$$

Collins (2011); ...

$$\times \; \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_1}(b_T, Q)\right]$$

Also relation to Parton Model?

Turn off α_s *don't get back parton model*

$$\tilde{f}(x, b_\mathrm{T}; \zeta, \mu) \to f_{j/P}(x) \exp\left\{\left(g_{j/P}(x, b_\mathrm{T}) + g_k(b_\mathrm{T}) \ln \frac{Q}{Q_0}\right)\right\}$$

$$= f_{j/P}(x) \exp\left\{\left(g_1 + g_2 \ln \frac{Q}{Q_0}\right) \frac{b_\mathrm{T}^2}{2}\right\}$$

Collins 2011 QCD Aybat Rogers PRD 2011

Collinear limit

- *Collins, Soper, Sterman NPB 1985*
- *Ji Ma Yuan, PRD 2005*
- *Collins 2011*

Consequence is that physical interpretation of integrated TMDs as collinear pdfs is at odds with parton model intuition in original version of CSS

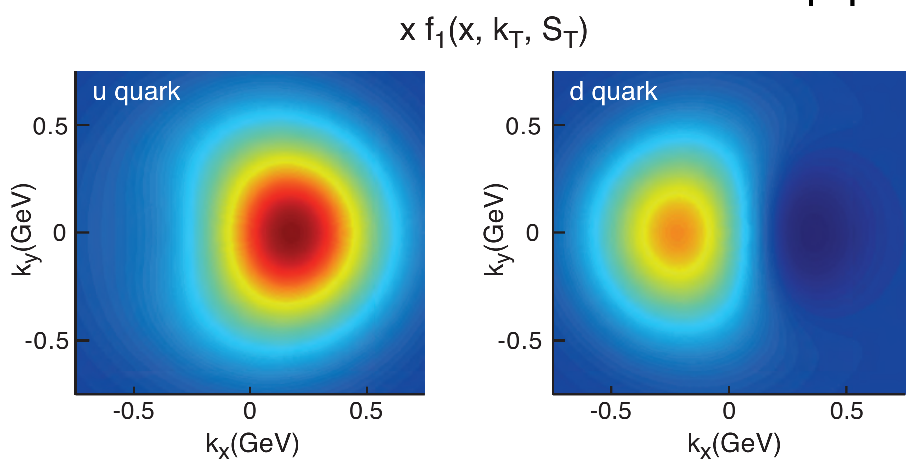
TMDs lose their physical interpretation in the "Original CSS" formalism!

$$\langle k_T^i(x)\rangle_{UT} = \int d^2k_T\, k_T^i \left(-\frac{\vec{k}_T \times \vec{S}_T}{M} f_{1T}^{\perp}(x,k_T)\right)$$

avg. TM of unpolarized quarks in a transversely polarized spin-1/2 target

Boer Mulders Teryaev PRD 1998
Burkhardt 2004,2013 PRD
Metz et al. 2013 PRD
And others …

Prokudin 2015 EIC White paper



Consistent Definition

The FT transform of the e.g. Sivers asympt. reduces to first moment of Sivers TMD

Boer, Gamberg, Musch, Prokudin, **JHEP (2011)**

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) \equiv \frac{2}{M^2} \frac{\partial}{\partial b_T^2} \tilde{f}_{1T}^{\perp}(x, b_T)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) = \frac{2\pi}{M^2} \int_0^\infty dk_T \, \frac{k_T^2}{b_T} \, J_1(k_T \, b_T) \; f_{1T}^{\perp}(x, k_T)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{\perp(1)}(x, b_T) = \frac{2}{M^2} 2\pi \int_0^\infty dk_T \, \frac{k_T^2}{2 b_T} \, \frac{k_T \, b_T}{2} f_{1T}^{\perp}(x, k_T)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{\perp(1)}(x, 0) = f_{1T}^{\perp(1)}(x)$$

Boer Mulders PRD 1998

This informs us how to study the collinear limit of transversely polarized cross section