Rapidity factorization and evolution of TMD s

I. Balitsky

JLAB & ODU

INT 8 Nov 2018

Outline

1. TMD factorization for particle production in hadron collisions:
   - Rapidity factorization for particle production.
   - Approximate solution of classical YM equations.
   - Power corrections to TMD factorization for Higgs production
   - Power corrections to TMD factorization for Z-boson production
2. Evolution of gluon TMDs:
   - Method of calculation: shock-wave approach + light-cone expansion.
   - One loop: real corrections and virtual corrections.
   - One-loop evolution of gluon TMD
   - DGLAP, Sudakov and BK limits of TMD evolution equation
   - Conclusions
   - In works: conformal properties of TMD factorization

TMD factorization

Factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta)\mathcal{D}_{f/B}(x_B, b_\perp, \eta)\sigma(ff \to H)$$

$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

where η is the rapidity, $\mathcal{D}_{f/A}(x, z_\perp, \eta)$ is the TMD density of a parton f in hadron A, and $\sigma(ff \to H)$ is the cross section of production of particle H of invariant mass $m_H^2 = Q^2$ in the scattering of two partons.

TMD factorization

Factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta) \mathcal{D}_{f/B}(x_B, b_\perp, \eta) \sigma(ff \to H)$$

$$+ \text{ power corrections } + \text{ ``Y} - \text{terms}''$$

where η is the rapidity, $\mathcal{D}_{f/A}(x, z_\perp, \eta)$ is the TMD density of a parton f in hadron A, and $\sigma(ff \to H)$ is the cross section of production of particle H of invariant mass $m_H^2 = Q^2$ in the scattering of two partons.

To understand how this formula works one needs to find one-loop evolution and first power corrections

A typical factorization formula for production of a particle with a small transverse momentum in hadron-hadron collisions:

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta)\mathcal{D}_{f/B}(x_B, b_\perp, \eta)\sigma(ff \to H)$$
$$+ \text{ power corrections } + \text{ Y} - \text{terms}$$

When we increase transverse momentum $q_\perp^2$ of the produced particle:

- At first the leading power TMD analysis with (nonperturbative) TMDs applies,
- then at some point power corrections kick in,
- and finally at $q_\perp^2 \sim Q^2$ they are transformed into so-called Y-term making smooth transition to collinear factorization formulas.

In the first part of this talk I try to answer the question about the first transition, namely at what $q_\perp^2$ power corrections become significant.

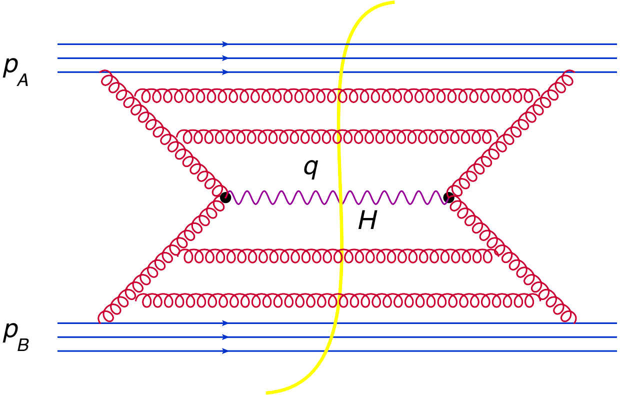
Higgs production by gluon fusion in pp scattering

Suppose we produce a scalar particle (Higgs) in a gluon-gluon fusion.

For simplicity, assume the vertex is local:

$$\mathcal{L}_\Phi = g_\Phi \int dz\, \Phi(z) F^2(z), \qquad F^2 \equiv F^a_{\mu\nu} F^{\mu\nu}_a$$

.



$s >> Q^2 >> q_\perp^2$

$q^2 = Q^2 = M_H^2$

"Hadronic tensor"

$$W(p_A,p_B,q) \stackrel{\rm def}{=} \sum_X \int d^4x\, e^{-iqx}\langle p_A,p_B|F^2(x)|X\rangle\langle X|F^2(0)|p_A,p_B\rangle$$
$$= \int d^4x\, e^{-iqx}\langle p_A,p_B|F^2(x)F^2(0)|p_A,p_B\rangle$$

Double functional integral for W

$$W(p_A,p_B,q) = \sum_X \int d^4x\, e^{-iqx}\langle p_A,p_B|F^2(x)|X\rangle\langle X|F^2(0)|p_A,p_B\rangle$$
$$= \lim_{\substack{t_f\to\infty \\ t_i\to-\infty}} \int d^4x\, e^{-iqx} \int^{\tilde{A}(t_f)=A(t_f)} D\tilde{A}_\mu DA_\mu \int^{\tilde{\psi}(t_f)=\psi(t_f)} D\tilde{\bar{\psi}} D\tilde{\psi} D\bar{\psi} D\psi\ \Psi^*_{p_A}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i))$$
$$\times\ \Psi^*_{p_B}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i)) e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi})} e^{iS_{\rm QCD}(A,\psi)} \tilde{F}^2(x)F^2(y)\Psi_{p_A}(\vec{A}(t_i),\psi(t_i))\Psi_{p_B}(\vec{A}(t_i),\psi(t_i))$$

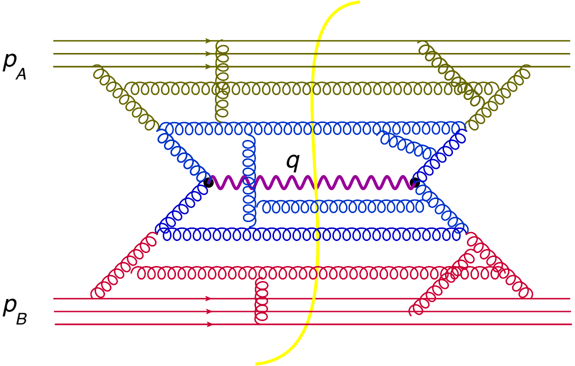
"Left" A,ψ fields correspond to the amplitude $\langle X|F^2(0)|p_A,p_B\rangle$,
"right" fields $\tilde{A},\tilde{\psi}$ correspond to amplitude $\langle p_A,p_B|F^2(x)|X\rangle$
The boundary conditions $\tilde{A}(t_f)=A(t_f)$ and $\tilde{\psi}(t_f)=\psi(t_f)$ reflect the sum over intermediate states X.

Rapidity factorization for particle production

Sudakov variables:

$$p = \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A, \;\; p_2 \simeq p_B, \;\; p_1^2 = p_2^2 = 0$$

$$x_* \equiv p_2 \cdot x = \sqrt{\frac{s}{2}}x^+, \qquad x_\bullet \equiv p_1 \cdot x = \sqrt{\frac{s}{2}}x^-$$



"Projectile" fields: $|\beta| < b$

"Central" fields

"Target" fields: $|\alpha| < a$

We integrate over “central” fields in the background of projectile and target fields.

Rapidity factorization for particle production

After integration over C fields

$$
\begin{aligned}
&W(p_A,p_B,q)\\
&= \int d^4x\, e^{-iqx}\int^{\tilde{A}(t_f)=A(t_f)} D\tilde{A}_\mu DA_\mu \int^{\tilde{\psi}_a(t_f)=\psi_a(t_f)} D\bar{\psi}_a D\psi_a D\tilde{\bar{\psi}}_a D\tilde{\psi}_a\\
&\quad\times e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi}_a)} e^{iS_{\rm QCD}(A,\psi_a)}\Psi^*_{p_A}(\vec{\tilde{A}}(t_i),\tilde{\psi}_a(t_i))\Psi_{p_A}(\vec{A}(t_i),\psi(t_i))\\
&\quad\times \int^{\tilde{B}(t_f)=B(t_f)} D\tilde{B}_\mu DB_\mu \int^{\tilde{\psi}_b(t_f)=\psi_b(t_f)} D\bar{\psi}_b D\psi_b D\tilde{\bar{\psi}}_b D\tilde{\psi}_b\\
&\quad\times e^{-iS_{\rm QCD}(\tilde{B},\tilde{\psi}_b)} e^{iS_{\rm QCD}(B,\psi_b)}\Psi^*_{p_B}(\vec{\tilde{B}}(t_i),\tilde{\psi}_b(t_i))\Psi_{p_B}(\vec{B}(t_i),\psi_b(t_i))\\
&\quad\times e^{S_{\rm eff}(U,V,\tilde{U},\tilde{V})}\mathcal{O}(q,x,y;A,\psi_a,\tilde{A},\tilde{\psi}_a;B,\psi_b,\tilde{B},\tilde{\psi}_b)
\end{aligned}
$$

$\mathcal{O}$ - sum of the *connected* diagrams for $F^2(x)F^2(0)$ in the background fields

$S_{\rm eff}$ - effective action (sum of disconnected diagrams = $e^{S_{\rm eff}}$).

Approximations for projectile and target fields

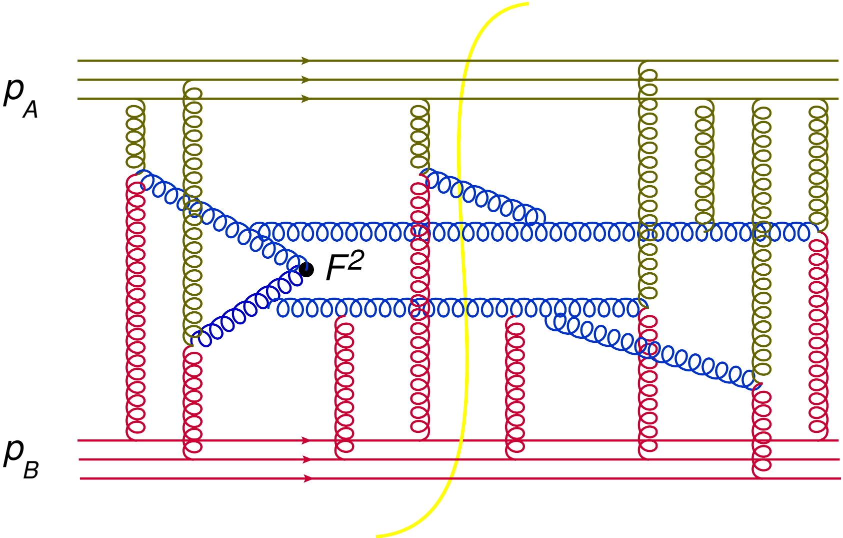
At the tree level $\beta = 0$ for $A, \tilde{A}$ fields and $\alpha = 0$ for $B, \tilde{B}$ fields $\Leftrightarrow$
$A = A(x_\bullet, x_\perp),\ \tilde{A} = \tilde{A}(x_\bullet, x_\perp)$ and $B = B(x_*, x_\perp),\ \tilde{B} = \tilde{B}(x_\bullet, x_\perp)$.

NB: because of boundary conditions $\tilde{A}(t_f) = A(t_f)$ and $\tilde{\psi}(t_f) = \psi(t_f)$ for the purpose of calculating the integral over central fields one can set

$$A(x_\bullet, x_\perp) = \tilde{A}(x_\bullet, x_\perp), \quad \psi_a(x_\bullet, x_\perp) = \tilde{\psi}_a(x_\bullet, x_\perp)$$
$$\text{and}$$
$$B(x_*, x_\perp) = \tilde{B}(x_*, x_\perp), \quad \psi_b(x_*, x_\perp) = \tilde{\psi}_b(x_*, x_\perp).$$

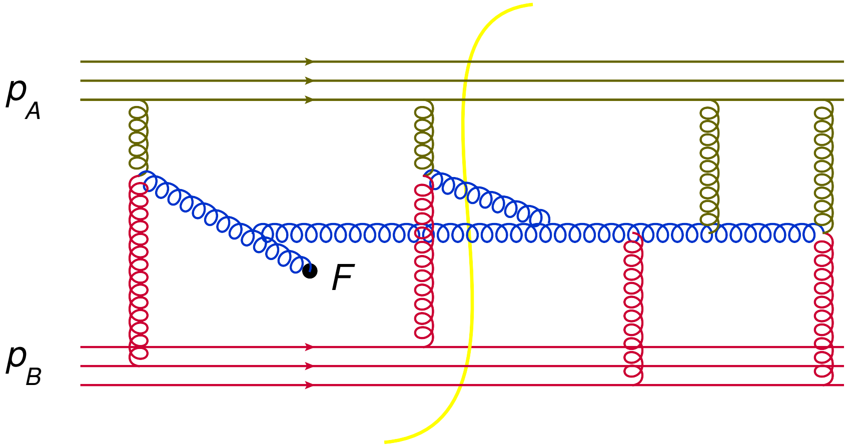
The fields A, ψ and $\tilde{A}, \tilde{\psi}$ do not depend on $x_* \Rightarrow$
if they coincide at $x_* = \infty \Rightarrow$ they coincide everywhere.
Similarly,
B, ψ_b and $\tilde{B}, \tilde{\psi}_b$ do not depend on $x_\bullet \Rightarrow$
if they coincide at $x_\bullet = \infty$ they should be equal.

$F^2_{\mu\nu}(C)$ in the tree approximation



$F_{\mu\nu}(C)$ = sum of tree diagrams in external A and B fields

$F_{\mu\nu}(C)$ in the tree approximation



$F_{\mu\nu}(C)$ = sum of tree diagrams in external $\tilde{A}, A$ and $\tilde{B}, B$ fields with sources $\tilde{J}_\mu = D^\mu F_{\mu\nu}(\tilde{A} + \tilde{B})$ and $J_\mu = D^\mu F_{\mu\nu}(A + B)$

$F_{\mu\nu}(C)$ in the tree approximation

Since $\tilde{A} = A$ and $\tilde{B} = B$ the sources and background fields are the same to the left and to the right of the cut

$\Rightarrow$

$F_{\mu\nu}(C)$ is a sum of diagrams with *retarded* Green functions

(F. Gelis, R. Venugopalan)

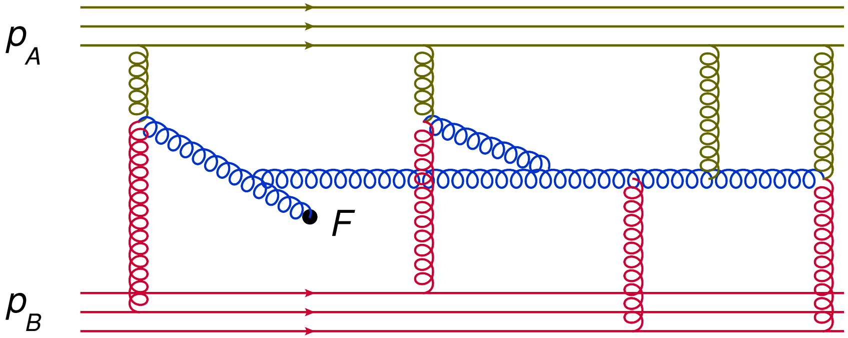
$F_{\mu\nu}(C)$ in the tree approximation

Since $\tilde{A} = A$ and $\tilde{B} = B$ the sources and background fields are the same to the left and to the right of the cut

$\Rightarrow$

$F_{\mu\nu}(C)$ is a sum of diagrams with *retarded* Green functions

(F. Gelis, R. Venugopalan)



Classical solution

The sum of diagrams with retarded Green functions $\Leftrightarrow$ solution of classical YM equations

$$D^\nu F^a_{\mu\nu} \;=\; \sum_f g\bar\psi^f t^a\gamma_\mu\psi^f, \quad (\not{P}+m_f)\psi^f \;=\; 0$$

with boundary conditions

$$A_\mu(x) \stackrel{x_*\to-\infty}{=} \bar A_\mu(x_\bullet,x_\perp), \quad \psi(x) \stackrel{x_*\to-\infty}{=} \psi_a(x_\bullet,x_\perp)$$
$$A_\mu(x) \stackrel{x_\bullet\to-\infty}{=} \bar B_\mu(x_*,x_\perp), \quad \psi(x) \stackrel{x_\bullet\to-\infty}{=} \psi_b(x_*,x_\perp)$$

following from $C_\mu, \psi_c \stackrel{t\to-\infty}{\to} 0$.
The projectile and target fields satisfy YM equations

$$D^\nu F^a_{\mu\nu} \;=\; \sum_f g\bar\psi^f_a t^a\gamma_\mu\psi^f_a, \quad (\not{P}+m_f)\psi^f_a \;=\; 0$$
$$D^\nu F^a_{\mu\nu} \;=\; \sum_f g\bar\psi^f_b t^a\gamma_\mu\psi^f_b, \quad (\not{P}+m_f)\psi^f_b \;=\; 0$$

Method of solution: start with $\bar A_\mu + \bar B_\mu$ and correct by computing Feynman diagrams (with retarded propagators) with a source $J_\nu = D^\mu F^{\mu\nu}(U+V)$

Classical solution in $A_*^{\rm projectile}=A_\bullet^{\rm target}=0$ gauge

Convenient gauge: $A_*=0$ for the projectile and $A_\bullet=0$ for the target.

$$U_i(x_\bullet,x_\perp)~\sim~m_\perp,\quad U_\bullet(x_\bullet,x_\perp)~\sim m_\perp^2,\quad U_*~=~0$$
$$V_i(x_*,x_\perp)~\sim~m_\perp^2,\quad V_*(x_*,x_\perp)~\sim m_\perp^2,\quad V_\bullet~=~0$$

and we have to solve

$$D^\nu F^a_{\mu\nu}~=~\sum_f g\bar\psi^f t^a\gamma_\mu\psi^f,\quad (\not{P}+m_f)\psi^f~=~0$$

with boundary conditions

$$A_\mu(x)\stackrel{x_*\to-\infty}{=}U_\mu(x_\bullet,x_\perp),\quad \psi(x)\stackrel{x_*\to-\infty}{=}\Sigma_a(x_\bullet,x_\perp)$$
$$A_\mu(x)\stackrel{x_\bullet\to-\infty}{=}V_\mu(x_*,x_\perp),\quad \psi(x)\stackrel{x_\bullet\to-\infty}{=}\Sigma_b(x_*,x_\perp)$$

We start with $U_\mu+V_\mu$ and compute Feynman diagrams (with retarded propagators) with a source $J_\nu=D^\mu F^{\mu\nu}(U+V)~\sim~m_\perp^3$

Expansion at small momentum transfer

The solution of YM equations in general case (scattering of two "color glass condensates") is yet unsolved problem.
Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution as a series in $\frac{q_\perp}{Q}$.

Example:

$$A_\bullet = U_\bullet + \int dz (x|\frac{1}{p^2 + i\epsilon p_0} p_\bullet |z)[U_j, V^j](z) = U_\bullet + \frac{1}{2}\int dz (x|\frac{1}{\alpha - \frac{p_\perp^2}{\beta s} + i\epsilon}|z)[U_j, V^j](z)$$

The characteristic $\alpha \geq \alpha_q$ and $\beta \geq \beta_q$ so $\alpha \gg \frac{p_\perp^2}{\beta s}$

$$\Rightarrow (x|\frac{1}{\alpha - \frac{p_\perp^2}{\beta s} + i\epsilon}|z) = (x|\frac{1}{\alpha + i\epsilon}|z) + (x|\frac{1}{\alpha + i\epsilon}\frac{p_\perp^2}{\beta s}\frac{1}{\alpha + i\epsilon}|z) + \ldots$$

and in the leading order in $p_\perp/p_\parallel$ we get

$$\begin{aligned} A_\bullet(x) &= U_\bullet(x_\bullet, x_\perp) + \frac{1}{2}\int dz (x|\frac{1}{\alpha + i\epsilon}|z)[U_j, V^j](z) \\ &= U_\bullet(x_\bullet, x_\perp) - \frac{i}{2}\int_{-\infty}^{x_\bullet} dx'_\bullet \, [U_j(x'_\bullet, x_\perp), V^j(x_*, x_\perp)] \end{aligned}$$

Gluon fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

With the expansion

$$\frac{1}{p^2+i\epsilon p_0} = \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} = \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2+i\epsilon p_0}p_\perp^2\frac{1}{p_\parallel^2+i\epsilon p_0} + \ldots$$

the dynamics in transverse space is trivial.

$$\begin{aligned}
\text{Gluon fields :}\qquad F_{\bullet i}^{(-1)} &= V_{\bullet i}, \quad F_{*i}^{(-1)} = U_{*i},\\
F_{*\bullet}^{(-1)} &= U_{*\bullet} + V_{*\bullet} - \frac{is}{2}U_j^{ab}V^{bj}\\
F_{\bullet i}^{(0)a} &= U_{\bullet i}^a - iU_\bullet^{ab}V_i^b - \frac{i}{2(\alpha+i\epsilon)}\tilde{L}_i^{(0)} - \mathcal{D}_i^{ab}V_j^{bc}\frac{1}{2(\alpha+i\epsilon)}U^{cj},\\
F_{*i}^{(0)a} &= V_{*i}^a - iV_*^{ab}U_i^b - \frac{i}{2(\beta+i\epsilon)}\tilde{L}_i^{(0)} - \mathcal{D}_i^{ab}U_j^{bc}\frac{1}{2(\beta+i\epsilon)}V^{cj},\\
F_{ik}^{(0)} &= U_{ik} + V_{ik} - i[U_i,V_k] - i[V_i,U_k],
\end{aligned}$$

where

$$\begin{aligned}
L_i^{(0)a} &= -iU^{jab}V_{ji} - iV^{jab}U_{ji} - i\mathcal{D}_j^{ab}(U^{jbc}V_i^c + V^{jbc}U_i^c)\\
&\quad - \frac{2i}{s}\big(U_{*\bullet}^{ab}V_i^b - V_{*\bullet}^{ab}U_i\big) + \bar{\Sigma}_a t^a\gamma_i\Sigma_b + \bar{\Sigma}_b t^a\gamma_i\Sigma_a
\end{aligned}$$

Gluon fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

With the expansion

$$\frac{1}{p^2+i\epsilon p_0} = \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} = \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2+i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2+i\epsilon p_0} + \dots$$

the dynamics in transverse space is trivial.

$$\text{Gluon fields}: \quad F^{(-1)}_{\bullet i} = V_{\bullet i}, \quad F^{(-1)}_{*i} = U_{*i},$$

$$F^{(-1)}_{*\bullet} = U_{*\bullet} + V_{*\bullet} - \frac{is}{2} U^{ab}_j V^{bj}$$

$$F^{(0)a}_{\bullet i} = U^a_{\bullet i} - iU^{ab}_\bullet V^b_i - \frac{i}{2(\alpha+i\epsilon)} \tilde{L}^{(0)}_i - \mathcal{D}^{ab}_i V^{bc}_j \frac{1}{2(\alpha+i\epsilon)} U^{cj},$$

$$F^{(0)a}_{*i} = V^a_{*i} - iV^{ab}_* U^b_i - \frac{i}{2(\beta+i\epsilon)} \tilde{L}^{(0)}_i - \mathcal{D}^{ab}_i U^{bc}_j \frac{1}{2(\beta+i\epsilon)} V^{cj},$$

$$F^{(0)}_{ik} = U_{ik} + V_{ik} - i[U_i, V_k] - i[V_i, U_k],$$

We integrate over α without cutoff $\alpha > \sigma$ since the contour over α can be removed from the pole to the region of large α (if there is no pinch). Similarly, we integrate over all β's.

(Different from SCET where they keep the cutoffs $\alpha > \sigma_b$ and $\beta > \sigma_a$).

TMD factorization with power corrections (A. Tarasov and I.B.)

At the tree level

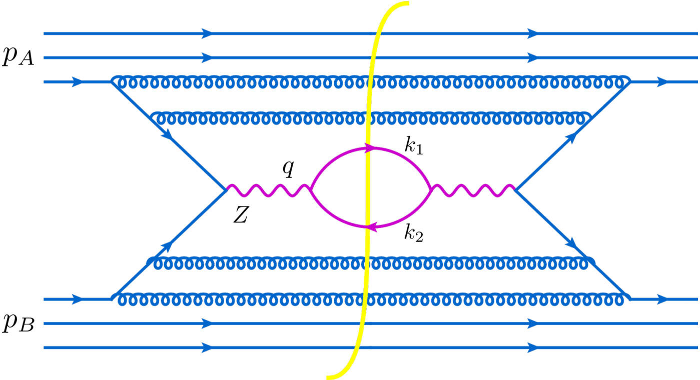
$$F^2(x) = \frac{8}{s}U_*^{ai}(x)V_{\bullet i}^a(x) + 2f^{mac}f^{mbd}\Delta^{ij,kl}U_i^a(x)U_j^b(x)V_k^c(x)V_l^d(x) + \ldots$$
$$\Delta^{ij,kl} \equiv g^{ij}g^{kl} - g^{ik}g^{jl} - g^{il}g^{jk}$$

$\Rightarrow$ in the region $s \gg Q^2 \gg Q_\perp^2$

$$W(p_A,p_B,q) = \frac{64/s^2}{N_c^2-1}\int d^2x_\perp\, e^{i(q,x)_\perp}\frac{2}{s}\int dx_\bullet dx_*\, e^{-i\alpha_q x_\bullet - i\beta_q x_*}$$
$$\times \Big\{\langle p_A|U_*^{mi}(x_\bullet,x_\perp)U_*^{mj}(0)|p_A\rangle\langle p_B|V_{\bullet i}^n(x_*,x_\perp)V_{\bullet j}^n(0)|p_B\rangle$$
$$- \frac{N_c^2}{N_c^2-4}\frac{\Delta^{ij,kl}}{Q^2}\int_{-\infty}^{x_\bullet} d\frac{2}{s}x'_\bullet\, d^{abc}\langle p_A|U_{*i}^a(x_\bullet,x_\perp)U_{*j}^b(x'_\bullet,x_\perp)U_{*r}^c(0)|p_A\rangle$$
$$\times \int_{-\infty}^{x_\bullet} d\frac{2}{s}x'_*\, d^{mnl}\langle p_B|V_{\bullet k}^m(x_*,x_\perp)V_{\bullet l}^n(x'_*,x_\perp)V_{\bullet r}^n(0)|p_B\rangle + x\leftrightarrow 0\Big\}$$

The correction is $\sim \frac{Q_\perp^2}{Q^2}$.

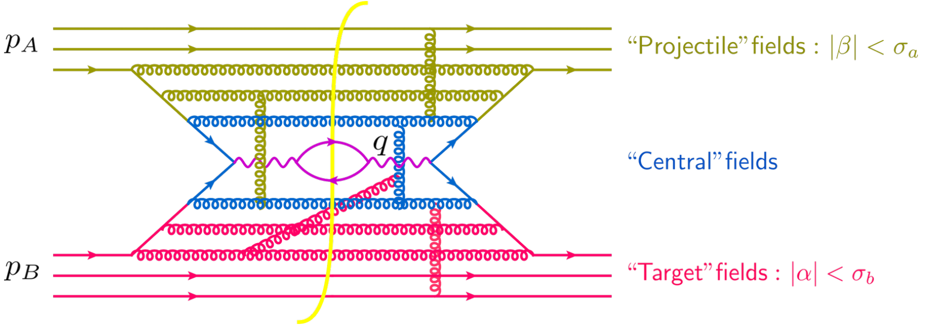
Z-boson production in pp scattering



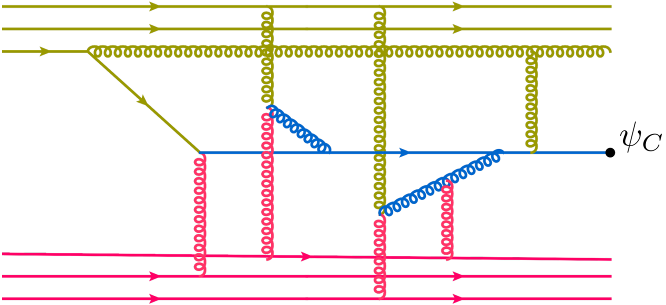
$$\frac{d\sigma}{dQ^2 dy d q_\perp^2} = \frac{e^2 Q^2}{192 s s_W^2 c_W^2} \frac{1 - 4s_W^2 + 8 s_W^4}{(m_Z^2 - Q^2)^2 + \Gamma_Z^2 m_Z^2} [-W_Z(p_A, p_B, q)],$$

$$W_Z(p_A, p_B, q) \equiv \frac{1}{(2\pi)^4} \sum_X \int d^4x \, e^{-iqx} \langle p_A, p_B | J_\mu(x) | X \rangle \langle X | J^\mu(0) | p_A, p_B \rangle$$

Same story: factorization + integration over central fields



Diagrams for classical solution



Classical solution = sum of perturbative diagrams with retarded propagators in the background of projectile and target fields.

Expansion of quark fields

Expanding it in powers of $p_\perp^2/p_\parallel^2$ as for gluons we get:

$$\Psi(x) \; = \; \Psi_A^{(0)} + \Psi_B^{(0)} + \Psi_A^{(1)} + \Psi_B^{(1)} + \ldots,$$

where

$$\begin{aligned}
\Psi_A^{(0)} &= \psi_A + \Xi_{2A}, \quad & \Xi_{2A} &= -\frac{g\not{p}_2}{s}\gamma^i B_i \frac{1}{\alpha + i\epsilon}\psi_A, \\
\bar\Psi_A^{(0)} &= \bar\psi_A + \bar\Xi_{2A}, \quad & \bar\Xi_{2A} &= -\big(\bar\psi_A \frac{1}{\alpha - i\epsilon}\big)\gamma^i B_i \frac{g\not{p}_2}{s}, \\
\Psi_B^{(0)} &= \psi_B + \Xi_{1B}, \quad & \Xi_{1B} &= -\frac{g\not{p}_1}{s}\gamma^i A_i \frac{1}{\beta + i\epsilon}\psi_B, \\
\bar\Psi_B^{(0)} &= \bar\psi_B + \bar\Xi_{1B}, \quad & \bar\Xi_{1B} &= -\big(\bar\psi_B \frac{1}{\beta - i\epsilon}\big)\gamma^i A_i \frac{g\not{p}_1}{s}.
\end{aligned}$$

Power corrections

$$
\begin{aligned}
&W(p_A, p_B, q)\\
&= -\frac{e^2}{8s_W^2 c_W^2 N_c}\int d^2k_\perp \Big[\Big\{(1+a_u^2)\Big[1-2\frac{(k,q-k)_\perp}{Q^2}\Big]f_{1u}(\alpha_z,k_\perp)\bar{f}_{1u}(\beta_z,q_\perp-k_\perp)\\
&\qquad\qquad + 2(a_u^2-1)\frac{k_\perp^2(q-k)_\perp^2}{m_N^2Q^2}h_{1u}^\perp(\alpha_z,k_\perp)\bar{h}_{1u}^\perp(\beta_z,q_\perp-k_\perp)\\
&+\frac{2k_\perp^2(q-k)_\perp^2}{(N_c^2-1)Q^2m_N^2}(a_u^2-1)[h_u^{\rm tw3}(\alpha_z,k_\perp)\bar{h}_u^{\rm tw3}(\beta_z,q_\perp-k_\perp)\\
&+\tilde{h}_u^{\rm tw3}(\alpha_z,k_\perp)\bar{\tilde{h}}_u^{\rm tw3}(\beta_z,q_\perp-k_\perp)]\\
&-\frac{N_c}{N_c^2-1}\frac{(k,q-k)_\perp}{Q^2}\\
&\times\Big(2(1+a_u^2)\big[j_{1u}^{\rm tw3}(\alpha_z,k_\perp)j_{2u}^{\rm tw3}(\beta_z,q_\perp-k_\perp)-\tilde{j}_{1u}^{\rm tw3}(\alpha_z,k_\perp)\tilde{j}_{2u}^{\rm tw3}(\beta_z,q_\perp-k_\perp)\big]\\
&+(1-a_u^2)\big[j_{1u}^{\rm tw3}(\alpha_z,k_\perp)j_{1u}^{\rm tw3}(\beta_z,q_\perp-k_\perp)+j_{2u}^{\rm tw3}(\alpha_z,k_\perp)j_{2u}^{\rm tw3}(\beta_z,q_\perp-k_\perp)\\
&+\tilde{j}_{1u}^{\rm tw3}(\alpha_z,k_\perp)\tilde{j}_{1u}^{\rm tw3}(\beta_z,q_\perp-k_\perp)+\tilde{j}_{2u}^{\rm tw3}(\alpha_z,k_\perp)\tilde{j}_{2u}^{\rm tw3}(\beta_z,q_\perp-k_\perp)\big]\Big)\\
&+(\alpha_z\leftrightarrow\beta_z)\Big\} + \Big\{u\leftrightarrow c\Big\}+\Big\{u\leftrightarrow d\Big\}+\Big\{u\leftrightarrow s\Big\}\Big]\Big(1+O\big(\frac{m_\perp^2}{s}\big)\Big)
\end{aligned}
$$

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times \frac{q_\perp^2}{Q^2} \times \left(1 + \frac{1}{N_c} + \frac{1}{N_c^2}\right)$.

(Pleasant) surprise: terms not suppressed by $\frac{1}{N_c}$ are determined by the leading-twist terms due to QCD equations of motion

Leading twist:

$$\frac{1}{8\pi^3 s}\int dx_\bullet d^2x_\perp\ e^{-i\alpha x_\bullet + i(k,x)_\perp}\ \langle A|\hat{\bar{\psi}}_f(x_\bullet, x_\perp)\ \not{p}_2 \hat{\psi}_f(0)|A\rangle\ =\ f_{1f}(\alpha, k_\perp^2)$$

Power correction:

$$\frac{g}{8\pi^3 s}\int dx_\bullet dx_\perp\ e^{-i\alpha_q x_\bullet + i(k,x)_\perp}$$
$$\times\ \langle A|\hat{\bar{\psi}}^f(x_\bullet, x_\perp)\ \not{p}_2[\hat{U}_i(x_\bullet, x_\perp) - i\gamma_5 \hat{\tilde{U}}_i(x_\bullet, x_\perp)]\hat{\psi}^f(0)|A\rangle$$

$$=\ -k_i f_{1f}(\alpha_q, k_\perp^2)\ +\ O(\alpha_q).$$

(Mulders & Tangerman, 1996)

Result:

$$\begin{aligned} W_Z(p_A,p_B,q) &= -\frac{e^2}{8s_W^2c_W^2N_c}\int d^2k_\perp \\ &\times \Big[\Big\{(1+a_u^2)\Big[1-2\frac{(k,q-k)_\perp}{Q^2}\Big] f_{1u}(\alpha_z,k_\perp)\bar{f}_{1u}(\beta_z,q_\perp-k_\perp) \\ &+ 2(a_u^2-1)\frac{k_\perp^2(q-k)_\perp^2}{m_N^2Q^2}h_{1u}^\perp(\alpha_z,k_\perp)\bar{h}_{1u}^\perp(\beta_z,q_\perp-k_\perp) + (\alpha_z\leftrightarrow\beta_z)\Big\} \\ &+ \Big\{u\leftrightarrow c\Big\} + \Big\{u\leftrightarrow d\Big\} + \Big\{u\leftrightarrow s\Big\}\Big]\Big(1 + O\big(\frac{1}{N_c}\big)\Big). \end{aligned}$$

$$a_{u,c} = (1-\tfrac{8}{3}s_W^2), \quad a_{d,s} = (1-\tfrac{4}{3}s_W^2)$$

Power correction is $\sim \frac{q_\perp^2}{Q^2}$.

Result:

$$W_Z(p_A,p_B,q) = -\frac{e^2}{8s_W^2c_W^2N_c}\int d^2k_\perp$$
$$\times\Big[\Big\{(1+a_u^2)\Big[1-2\frac{(k,q-k)_\perp}{Q^2}\Big]f_{1u}(\alpha_z,k_\perp)\bar{f}_{1u}(\beta_z,q_\perp-k_\perp)$$
$$+2(a_u^2-1)\frac{k_\perp^2(q-k)_\perp^2}{m_N^2Q^2}h_{1u}^\perp(\alpha_z,k_\perp)\bar{h}_{1u}^\perp(\beta_z,q_\perp-k_\perp) + (\alpha_z\leftrightarrow\beta_z)\Big\}$$
$$+\Big\{u\leftrightarrow c\Big\}+\Big\{u\leftrightarrow d\Big\}+\Big\{u\leftrightarrow s\Big\}\Big]\Big(1+O(\frac{1}{N_c})\Big).$$

$$a_{u,c} = (1-\tfrac{8}{3}s_W^2),\quad a_{d,s} = (1-\tfrac{4}{3}s_W^2)$$

Power correction is $\sim \frac{q_\perp^2}{Q^2}$.

($\frac{1}{N_c}$ and $\frac{1}{N_c^2}$ terms involve twist-3 quark-quark-gluon TMDs which do not reduce to leading-twist distributions).

Estimate of power corrections

If $Q^2 \gg k_\perp^2 \gg m_N^2$ we can approximate

$$f_1(\alpha_z, k_\perp^2) \simeq \frac{f(\alpha_z)}{k_\perp^2}, \quad h_1^\perp(\alpha_z, k_\perp^2) \simeq \frac{m_N^2 h(\alpha_z)}{k_\perp^4}$$

$$\Rightarrow W_Z(p_A, p_B, q) \simeq -\frac{e^2}{8s_W^2 c_W^2 N_c} \int d^2k_\perp \frac{1}{k_\perp^2 (q-k)_\perp^2} \Big[1 - 2\frac{(k, q-k)_\perp}{Q^2}\Big]$$
$$\times \sum_f (1 + a_f^2)[f_f(\alpha_z)\bar{f}_f(\beta_z) + \bar{f}_f(\alpha_z) f_f(\beta_z)]$$

Estimate of power corrections

If $Q^2 \gg k_\perp^2 \gg m_N^2$ we can approximate

$$f_1(\alpha_z,k_\perp^2) \simeq \frac{f(\alpha_z)}{k_\perp^2}, \quad h_1^\perp(\alpha_z,k_\perp^2) \simeq \frac{m_N^2 h(\alpha_z)}{k_\perp^4}$$

$$\Rightarrow W_Z(p_A,p_B,q) \simeq -\frac{e^2}{8s_W^2c_W^2N_c}\int d^2k_\perp \frac{1}{k_\perp^2(q-k)_\perp^2}\Big[1-2\frac{(k,q-k)_\perp}{Q^2}\Big]$$
$$\times \sum_f (1+a_f^2)\big[f_f(\alpha_z)\bar{f}_f(\beta_z)+\bar{f}_f(\alpha_z)f_f(\beta_z)\big]$$

With logarithmic accuracy

$$W_Z(p_A,p_B,q) = -\frac{\pi e^2}{4s_W^2c_W^2N_c}\Big[\frac{1}{q_\perp^2}\ln\frac{q_\perp^2}{m_N^2}+\frac{1}{Q^2}\ln\frac{Q^2}{q_\perp^2}\Big]$$
$$\times \sum_f (1+a_f^2)\big[f_f(\alpha_z)\bar{f}_u(\beta_z)+\bar{f}_f(\alpha_z)f_f(\beta_z)\big]$$

$\Rightarrow$ power correction reaches 10% level at $q_\perp \sim \frac{1}{4}Q \sim$ 20 GeV

1. Conclusions I
   - Higher-twist power correction to H and Z production at $s \gg q^2 \gg q_\perp^2$ are calculated. The estimate gives 10% corrections at $q_\perp \sim \frac{1}{4}Q$.
2. Outlook
   - Power corrections to $W_{\mu\nu}$ for Drell-Yan and SIDIS

Rapidity evolution: one loop

We study evolution of $\tilde{\mathcal{F}}_i^{a\eta}(x_\perp, x_B)\mathcal{F}_j^{a\eta}(y_\perp, x_B)$ with respect to rapidity cutoff η

$$\mathcal{F}_i^{a(\eta)}(z_\perp, x_B) \equiv \frac{2}{s}\int dz_*\, e^{ix_B z_*}[\infty, z_*]_z^{am} F^m_{\bullet i}(z_*, z_\perp)$$
$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

At first we study gluon TMDs with Wilson lines stretching to $+\infty$ (like in SIDIS).
Matrix element of $\tilde{\mathcal{F}}_i^a(k'_\perp, x'_B)\mathcal{F}^{ai}(k_\perp, x_B)$ at one-loop accuracy:
diagrams in the "external field" of gluons with rapidity $< \eta$.

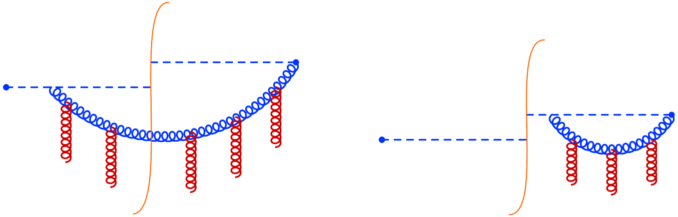
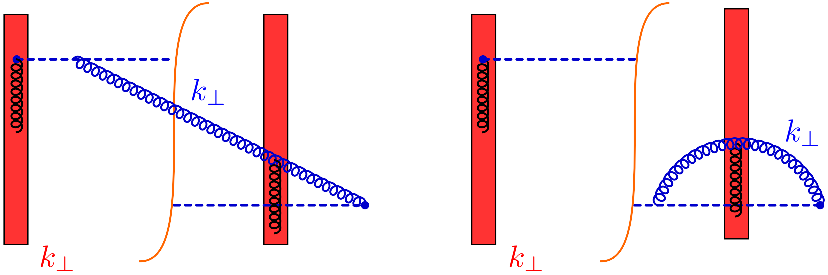


Figure : Typical diagrams for one-loop contributions to the evolution of gluon TMD.
(Fields $\tilde{\mathcal{A}}$ to the left of the cut and $\mathcal{A}$ to the right.)

Shock-wave formalism and transverse momenta

$\alpha \gg \alpha$ and $k_\perp \sim k_\perp \Rightarrow$ shock-wave external field



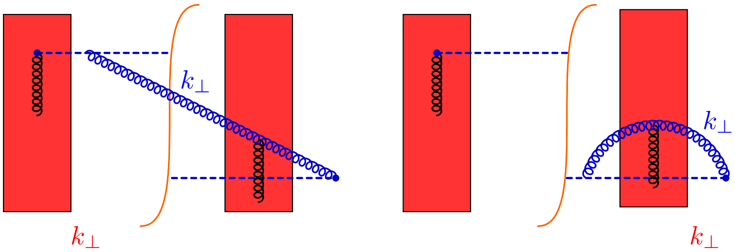
Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \leq k_\perp^2 \Rightarrow x_* \gg x_*$
$\Rightarrow$ Diagrams in the shock-wave background at $k_\perp \sim k_\perp$

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp \Rightarrow$ the external field may be wide



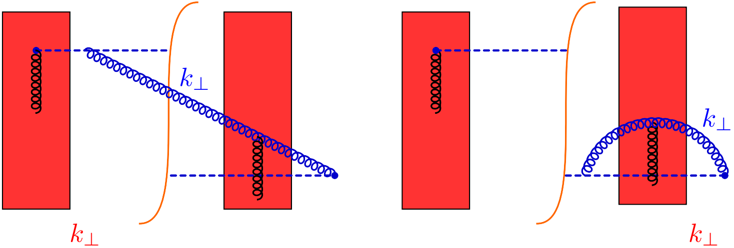
Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s} \Rightarrow x_* \sim \frac{\alpha s}{k_\perp^2}$

Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta s}$, $\beta \sim \frac{k_\perp^2}{\alpha s} \Rightarrow x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2 \Rightarrow x_* \sim x_* \Rightarrow$ shock-wave approximation is invalid.

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp \Rightarrow$ the external field may be wide



Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s} \Rightarrow \; x_* \sim \frac{\alpha s}{k_\perp^2}$

Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta s}$, $\beta \sim \frac{k_\perp^2}{\alpha s} \Rightarrow \; x_* \sim \frac{\alpha s}{k_\perp^2}$

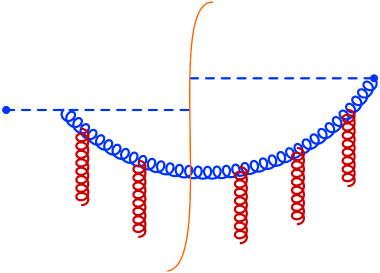
If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2 \Rightarrow x_* \sim x_* \Rightarrow$ shock-wave approximation is invalid.

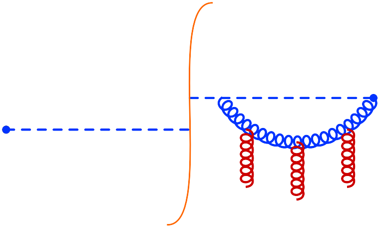
Fortunately, at $k_\perp^2 \gg k_\perp^2$ we can use another approximation

$\Rightarrow$ Light-cone expansion of propagators at $k_\perp \gg k_\perp$

Method of calculation

We calculate one-loop diagrams in the fast-field background





in following way:

if $k_\perp \sim k_\perp \Rightarrow$ propagators in the shock-wave background

if $k_\perp \gg k_\perp \Rightarrow$ light-cone expansion of propagators

We compute one-loop diagrams in these two cases and write down "interpolating" formulas correct both at $k_\perp \sim k_\perp$ and $k_\perp \gg k_\perp$

Reminder:

$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s}\int dz_* \, e^{-ix_B z_*} F^m_{\bullet i}(z_*, z_\perp)[z_*, \infty]_z^{ma}$$

At $x_B \sim 1$ $e^{-ix_B z_*}$ may be important even if shock wave is narrow.
Indeed, $x_* \sim \frac{\alpha s}{k_\perp^2} \ll x_* \sim \frac{\alpha s}{k_\perp^2} \Rightarrow$ shock-wave approximation is OK,
but $x_B \sigma_* \sim x_B \frac{\alpha s}{k_\perp^2} \sim \frac{\alpha s}{k_\perp^2} \geq 1 \Rightarrow$ we need to “look inside” the shock wave.

Reminder:

$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s}\int dz_* \, e^{-ix_B z_*} F^m_{\bullet i}(z_*, z_\perp)[z_*, \infty]_z^{ma}$$

At $x_B \sim 1$ $e^{-ix_B z_*}$ may be important even if shock wave is narrow.
Indeed, $x_* \sim \frac{\alpha s}{k_\perp^2} \ll x_* \sim \frac{\alpha s}{k_\perp^2} \Rightarrow$ shock-wave approximation is OK,
but $x_B \sigma_* \sim x_B \frac{\alpha s}{k_\perp^2} \sim \frac{\alpha s}{k_\perp^2} \geq 1 \Rightarrow$ we need to “look inside” the shock wave.

Technically, we consider small but finite shock wave:
take the external field with the support in the interval $[-\sigma_*, \sigma_*]$ (where $\sigma_* \sim \frac{\alpha s}{k_\perp^2}$), calculate diagrams with points in and out of the shock wave, and check that the σ_*-dependence cancels in the sum of “inside” and “outside” contributions.

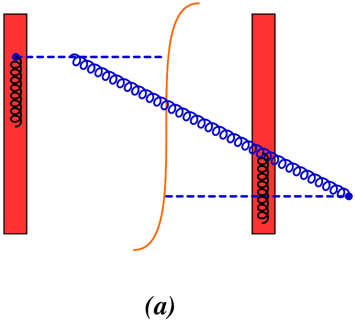


Figure : Typical diagrams for production (a) and virtual (b) contributions to the evolution kernel.

Real corrections: square of "Lipatov vertex"

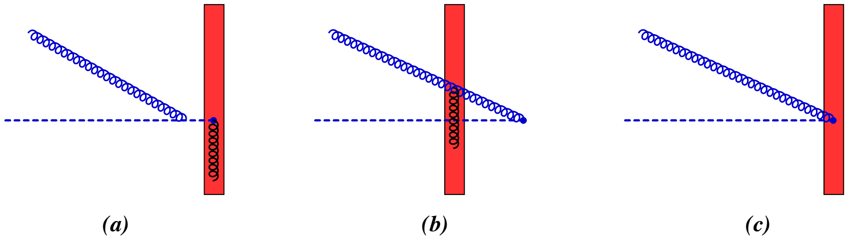


Figure : Lipatov vertex of gluon emission.

Definition

$$L^{ab}_{\mu i}(k, y_\perp, x_B) \;=\; i \lim_{k^2\to 0} k^2 \langle T\{A^a_\mu(k)\mathcal{F}^b_i(y_\perp, x_B)\}\rangle$$

Lipatov vertex in the shock-wave case

Result of calculation (in the background-Feynman gauge)

$$
\begin{aligned}
L^{ab}_{\mu i}(k, y_\perp, x_B) \; = \; & 2ge^{-i(k,y)_\perp}\Big(\frac{p_{2\mu}}{\alpha s} - \frac{\alpha p_{1\mu}}{k_\perp^2}\Big)\big[\mathcal{F}_i(x_B, y_\perp) - U_i(y_\perp)\big]^{ab} \\
& + g(k_\perp|g_{\mu i}\Big(\frac{\alpha x_B s}{\alpha x_B s + p_\perp^2} - U\frac{\alpha x_B s}{\alpha x_B s + p_\perp^2}U^\dagger\Big) \\
& \qquad + 2\alpha p_{1\mu}\Big(\frac{p_i}{\alpha x_B s + p_\perp^2} - U\frac{p_i}{\alpha x_B s + p_\perp^2}U^\dagger\Big) \\
& + \Big[2ix_B p_{2\mu}\partial_i U - 2i\partial^\perp_\mu U p_i + \frac{2p_{2\mu}}{\alpha s}\partial_\perp^2 U p_i\Big]\frac{1}{\alpha x_B s + p_\perp^2}U^\dagger - \frac{2\alpha p_{1\mu}}{p_\perp^2}U_i|y_\perp)^{ab}
\end{aligned}
$$

$U_i \equiv \mathcal{F}_i(0) = i(\partial_i U)U^\dagger$.

Schwinger's notations $(x_\perp|\mathcal{O}(\hat{p}_\perp, \hat{X}_\perp)|y_\perp) \equiv \int d^2p\,\mathcal{O}(p_\perp, x_\perp)e^{-i(p,x-y)_\perp}$

Result of calculation (in the background-Feynman gauge)

$$L^{ab}_{\mu i}(k,y_\perp,x_B)\rangle \;=\; \frac{2ge^{-i(k,y)_\perp}}{\alpha x_B s+k_\perp^2}\mathcal{F}^{ab}_l(x_B+\frac{k_\perp^2}{\alpha s},y_\perp)$$
$$\times\Big[\frac{\alpha x_B s}{k_\perp^2}\big(\frac{k_\perp^2}{\alpha s}p_{2\mu}-\alpha p_{1\mu}\big)\delta^l_i-\delta^l_\mu k_i+\frac{\alpha x_B s g_{\mu i}k^l}{k_\perp^2+\alpha x_B s}+\frac{2\alpha k_i k^l}{k_\perp^2+\alpha x_B s}p_{1\mu}\Big]$$

$$\text{NB}: \quad k^\mu L^{ab}_{\mu i}(k,y_\perp,x_B) \;=\; 0$$

for both shock-wave and light-cone Lipatov vertices.

It is convenient to write Lipatov vertex in the light-like gauge $p_2^\mu A_\mu=0$ by replacement $\alpha p_1^\mu\rightarrow\alpha p_1^\mu-k^\mu \;=\; -k_\perp^\mu-\frac{k_\perp^2}{\alpha s}$

$$L^{ab}_{\mu i}(k,y_\perp,x_B)^{\text{light-like}} \;=\; 2ge^{-i(k,y)_\perp}\Big[\frac{k^\perp_\mu\delta^l_i}{k_\perp^2}$$
$$-\frac{\delta^l_\mu k_i+\delta^l_i k^\perp_\mu-g_{\mu i}k^l}{\alpha x_B s+k_\perp^2}-\frac{k_\perp^2 g_{\mu i}k^l+2k^\perp_\mu k_i k^l}{(\alpha x_B s+k_\perp^2)^2}\Big]\mathcal{F}^{ab}_l(x_B+\frac{k_\perp^2}{\alpha s},y_\perp)\;+\;O(p_{2\mu})$$

"Interpolating formula" between the shock-wave and light-cone Lipatov vertices

$$L^{ab}_{\mu i}(k, y_\perp, x_B)^{\text{light}-\text{like}}$$
$$= g(k_\perp|\mathcal{F}^j\Big(x_B + \frac{k_\perp^2}{\alpha s}\Big)\Big\{\frac{\alpha x_B s g_{\mu i} - 2k^\perp_\mu k_i}{\alpha x_B s + k_\perp^2}(k_j U + U p_j)\frac{1}{\alpha x_B s + p_\perp^2}U^\dagger$$
$$- 2k^\perp_\mu U\frac{g_{ij}}{\alpha x_B s + p_\perp^2}U^\dagger - 2g_{\mu j}U\frac{p_i}{\alpha x_B s + p_\perp^2}U^\dagger + \frac{2k^\perp_\mu}{k_\perp^2}g_{ij}\Big\}|y_\perp)^{ab} + O(p_{2\mu})$$

This formula is actually correct (within our accuracy $\alpha_{\text{fast}} \ll \alpha_{\text{slow}}$) in the whole range of x_B and transverse momenta

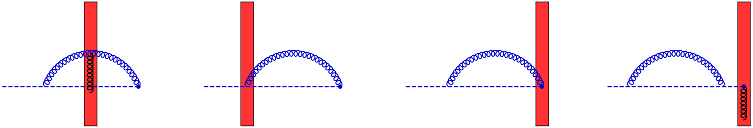


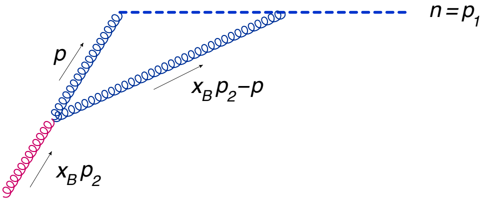
Figure : Virtual gluon corrections.

Result of the calculation (in light-like and background-Feynman gauges)

$$\langle \mathcal{F}_i^n(y_\perp, x_B)\rangle^{\text{Fig. 4}} = -ig^2 f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\bar{}\,\alpha}{\alpha}(y_\perp| - \frac{p^j}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l + U_l)$$
$$\times\ (2\delta_j^k\delta_i^l - g_{ij}g^{kl})U\frac{1}{\alpha x_B s + p_\perp^2}U^\dagger + \mathcal{F}_i(x_B)\frac{\alpha x_B s}{p_\perp^2(\alpha x_B s + p_\perp^2)}|y_\perp)$$

NB: with $\alpha < \sigma$ cutoff there is no UV divergence.

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I = \int \frac{d^4p}{\pi^2 i} \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)} \frac{x_B p_2 \cdot n}{(x_B p_2 - p)^2 - m^2 + i\epsilon}$$

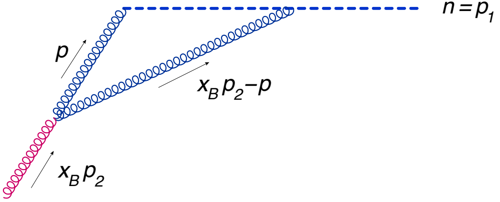
Regularization # 1 (ours): $n = p_1$, $|\alpha| < \sigma$

$$I_1 = -i\frac{s}{2\pi^2}\int_{-\sigma}^{\sigma} d\alpha \int \frac{d\beta}{\beta - i\epsilon}\int d^2p_\perp \frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon} \frac{x_B}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$= \frac{1}{\pi}\int_0^{\sigma} d\alpha \int d^2p_\perp \frac{1}{m^2 + p_\perp^2}\frac{1}{\alpha + \frac{m^2 + p_\perp^2}{sx_B}} = \int_0^{\sigma}\frac{d\alpha}{\pi\alpha}\ln\Big(1 + \frac{\alpha s x_B}{m^2}\Big) = \frac{1}{2}\ln^2\frac{\sigma s x_B}{m^2} + \frac{\pi^2}{6}$$

Double log of σ, no UV.

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

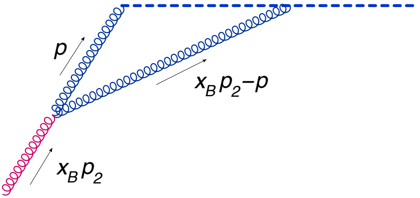
$$I \;=\; \int \frac{d^4p}{\pi^2 i}\, \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)}\, \frac{x_Bp_2\cdot n}{(x_Bp_2 - p)^2 - m^2 + i\epsilon}$$

Regularization # 2 (by slope of Wilson line): $n = p_1 + \gamma p_2$, $\gamma \ll 1$

$$I_2 \;=\; -i\frac{s}{2\pi^2}\int d\alpha d\beta \int d^2p_\perp \frac{1}{\beta + \gamma\alpha - i\epsilon}\, \frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon}\, \frac{1}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$\Rightarrow\; I_2 \;\stackrel{(p_2\cdot n)^2 \gg m^2n^2}{\rightarrow}\; \frac{1}{2}\ln^2\frac{x_Bs^2}{m^2n^2} + \frac{\pi^2}{6} \;=\; \frac{1}{2}\ln^2\frac{x_Bs}{m^2\gamma} + \frac{\pi^2}{6}$$

Double log of σ, no UV.



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I = \int \frac{d^4p}{\pi^2 i} \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)} \frac{x_B p_2 \cdot n}{(x_B p_2 - p)^2 - m^2 + i\epsilon}$$

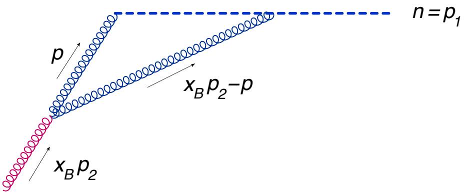
Regularization # 3: $n = p_1$, $\beta > b$

$$I_3 = -i\frac{s}{2\pi^2}\int d\alpha d\beta \int d^2p_\perp \frac{1}{\beta - i\epsilon} \frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon} \frac{x_B}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$\Rightarrow I_3 = \frac{1}{\pi}\int_b^{x_B} \frac{d\beta}{\beta} \int \frac{d^2p_\perp}{m^2 + p_\perp^2} = \ln\frac{x_B}{b} \ln\frac{\mu_{\rm UV}^2}{m^2}$$

UV $\times$ single log of the cutoff

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I = \int \frac{d^4p}{\pi^2 i} \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)} \frac{x_B p_2 \cdot n}{(x_B p_2 - p)^2 - m^2 + i\epsilon}$$

Regularization # 1 $\Rightarrow$ Regularization # 3:

$$\text{change of variables} \quad \beta = \frac{x_B(m^2 + p_\perp^2)}{\alpha s x_B + m^2 + p_\perp^2}$$

$$\int_0^\sigma d\alpha \int d^2p_\perp \frac{1}{m^2 + p_\perp^2} \frac{1}{\alpha + \frac{m^2 + p_\perp^2}{s x_B}} = \int_b^{x_B} \frac{d\beta}{\beta} \int \frac{d^2p_\perp}{m^2 + p_\perp^2}, \quad b = \frac{x_B(m^2 + p_\perp^2)}{\sigma s x_B + m^2 + p_\perp^2}$$

Evoltuion equation for the gluon TMD operator

A. Tarasov and I.B.

$$\frac{d}{d\ln\sigma}\left(\tilde{\mathcal{F}}^a_i(x_\perp,x_B)\mathcal{F}^a_j(y_\perp,x_B)\right)^{\ln\sigma}$$

$$= \ -\alpha_s\int d\!\!{}^-{}^2k_\perp\ \mathrm{Tr}\{\tilde{L}_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}$$

$$-\ \alpha_s\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}$$

$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp)$$

$$+\ (x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2}$$

$$+\ \tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}\ +\ O(\alpha_s^2)$$

This expression is UV and IR convergent.

It describes the rapidity evolution of gluon TMD operator in for any x_B and transverse momenta!

Evoltuion equation for the gluon TMD

$$\frac{d}{d\ln\sigma}\langle p|\big(\tilde{\mathcal{F}}_i^a(x_\perp,x_B)\mathcal{F}_j^a(y_\perp,x_B)\big)^{\ln\sigma}|p\rangle$$

$$= -\alpha_s\int đ^2k_\perp\ \langle p|\mathrm{Tr}\{\tilde{L}_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}\theta\Big(1-x_B-\frac{k_\perp^2}{\alpha s}\Big)L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}$$

$$-\ \alpha_s\langle p|\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}$$

$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp)$$

$$+\ (x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta_i^k\delta_m^l-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2}$$

$$+\ \tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}|p\rangle\ +\ O(\alpha_s^2)$$

The factor $\theta\big(1-x_B-\frac{k_\perp^2}{\alpha s}\big)$ reflects kinematical restriction that the fraction of initial proton's momentum carried by produced gluon should be smaller than $1-x_B$

Light-cone limit

$$\langle p|\tilde{\mathcal{F}}^n_i(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma} = \frac{\alpha_s}{\pi}N_c\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\int_0^\infty d\beta\,\Big\{\theta(1-x_B-\beta)$$
$$\times\Big[\frac{1}{\beta}-\frac{2x_B}{(x_B+\beta)^2}+\frac{x_B^2}{(x_B+\beta)^3}-\frac{x_B^3}{(x_B+\beta)^4}\Big]\langle p|\tilde{\mathcal{F}}^n_i(x_B+\beta,x_\perp)$$
$$\times\,\mathcal{F}^{ni}(x_B+\beta,x_\perp)|p\rangle^{\ln\sigma'}-\frac{x_B}{\beta(x_B+\beta)}\langle p|\tilde{\mathcal{F}}^n_i(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma'}\Big\}$$

In the LLA the cutoff in $\sigma \Leftrightarrow$ cutoff in transverse momenta:

$$\langle p|\tilde{\mathcal{F}}^n_i(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{k_\perp^2<\mu^2} = \frac{\alpha_s}{\pi}N_c\int_0^\infty d\beta\int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}}\frac{d\alpha}{\alpha}\,\{\text{same}\}$$

$\Rightarrow$ DGLAP equation $\Rightarrow$ $(z'\equiv\frac{x_B}{x_B+\beta})$

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(x_B,0_\perp,\eta)$$

DGLAP kernel

$$=\frac{\alpha_s}{\pi}N_c\int_{x_B}^1\frac{dz'}{z'}\,\Big[\Big(\frac{1}{1-z'}\Big)_+ +\frac{1}{z'}-2+z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{x_B}{z'},0_\perp,\eta\Big)$$

Low-x regime: $x_B = 0$ + characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$

$\Rightarrow$ in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1$

$\Rightarrow$ the kinematical constraint $\theta\big(1 - \frac{k_\perp^2}{\alpha s}\big)$ can be omitted

$\Rightarrow$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2)$$

$$= -\frac{g^2}{8\pi^3}\mathrm{Tr}\Big\{(-i\partial_i^{z_1} + \tilde{U}_i^{z_1})\Big[\int d^2z_3(\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger - 1)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}(U_{z_3}U_{z_2}^\dagger - 1)\Big](i\overleftarrow{\partial}_j^{z_2} + U_j^{z_2})\Big\}$$

where $\eta \equiv \ln\sigma$ and $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$ is the dipole kernel

Low-x case: BK evolution of the WW distribution

Low-x regime: $x_B = 0$ + characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$

$\Rightarrow$ in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1$

$\Rightarrow$ the kinematical constraint $\theta\big(1 - \frac{k_\perp^2}{\alpha s}\big)$ can be omitted

$\Rightarrow$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2)$$

$$= -\frac{g^2}{8\pi^3}\mathrm{Tr}\Big\{(-i\partial_i^{z_1} + \tilde{U}_i^{z_1})\Big[\int d^2z_3(\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger - 1)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}(U_{z_3}U_{z_2}^\dagger - 1)\Big](i\overleftarrow{\partial}_j^{z_2} + U_j^{z_2})\Big\}$$

where $\eta \equiv \ln\sigma$ and $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$ is the dipole kernel

This eqn holds true also at small x_B up to $x_B \sim \frac{(x-y)_\perp^{-2}}{s}$ since in the whole range of evolution $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ one can neglect $\sigma x_B s$ in comparison to $p_\perp^2$ in the denominators $(p_\perp^2 + \sigma x_B s)$ $\Leftrightarrow$ effectively $x_B = 0$.

Sudakov double logs

Sudakov limit: $x_B \equiv x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV.

One can show that the non-linear terms are power suppressed $\Rightarrow$

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle$$
$$= 4\alpha_s N_c \int \frac{d\!\!{}^-{}^2 p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(x_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(x_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle$$
$$- \frac{\sigma x_B s}{\sigma x_B s + p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow \frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

Sudakov double logs

Sudakov limit: $x_B \equiv x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV.

One can show that the non-linear terms are power suppressed $\Rightarrow$

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle$$
$$= 4\alpha_s N_c\int\frac{{d\!\!{}^-}^2p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(x_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(x_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle$$
$$-\frac{\sigma x_B s}{\sigma x_B s+p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow\ \frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

$\Rightarrow$ Sudakov double logs

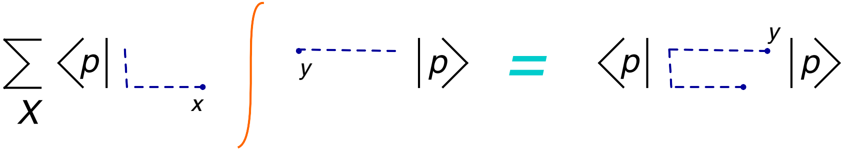
$$\mathcal{D}(x_B,k_\perp,\ln\sigma) \sim \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\Big\}\mathcal{D}\Big(x_B,k_\perp,\ln\frac{k_\perp^2}{s}\Big)$$

Gluon TMD with gauge links to $-\infty$

$$\alpha_s\mathcal{D}(x_B,z_\perp) \;=\; -\frac{\alpha_s}{2\pi(p\cdot n)x_B}\int du\; e^{-ix_Bu(pn)}\langle p|\tilde{\mathcal{F}}^a_\xi(z_\perp+un)[z_\perp,0]_{-\infty}\mathcal{F}^{a\xi}(0)|p\rangle$$

$$\mathcal{F}^a_\xi(z_\perp+un) \;\equiv\; [-\infty n+z_\perp, un+z_\perp]^{am} n^\mu F^m_{\mu\xi}(un+z_\perp)$$
$$\tilde{\mathcal{F}}^a_\xi(z_\perp+un) \;\equiv\; n^\mu F^m_{\mu\xi}(un+z_\perp)[un+z_\perp, -\infty n+z_\perp]^{ma}$$

To calculate, use double functional integral:



One-loop diagrams are the same as before.

Lipatov vertex for gauge links going to $-\infty$

$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{\rm light-like}_{-\infty} ~=~ g(k_\perp|U\mathcal{F}^j\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)$$

$$\times\Big[\frac{\alpha\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha\beta_B s+k_\perp^2}\frac{(p+k)_j}{\alpha\beta_B s+p_\perp^2}-\frac{2k^\perp_\mu g_{ij}+2g_{\mu j}p_i}{\alpha\beta_B s+p_\perp^2}\Big]+2U\frac{p^\perp_\mu}{p_\perp^2}\mathcal{F}_i\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)|y_\perp)$$

Lipatov vertex for gauge links going to $-\infty$

$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{\rm light-like}_{-\infty} ~=~ g(k_\perp|U\mathcal{F}^j\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)$$
$$\times\Big[\frac{\alpha\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha\beta_B s+k_\perp^2}\frac{(p+k)_j}{\alpha\beta_B s+p_\perp^2}-~\frac{2k^\perp_\mu g_{ij}+2g_{\mu j}p_i}{\alpha\beta_B s+p_\perp^2}\Big]+2U\frac{p^\perp_\mu}{p_\perp^2}\mathcal{F}_i\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)|y_\perp)$$

Compare to L. vertex for gauge links going to $+\infty$

$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{\rm light-like}_{+\infty} ~=~ g(k_\perp|\Big\{\frac{\alpha\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha\beta_B s+k_\perp^2}\mathcal{F}^j\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)U\frac{(p+k)_j}{\alpha\beta_B s+p_\perp^2}U^\dagger$$
$$-~2\mathcal{F}^j\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)U\frac{g_{ij}k^\perp_\mu+g_{\mu j}p_i}{\alpha\beta_B s+p_\perp^2}U^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}\mathcal{F}_i\big(\beta_B+\frac{k_\perp^2}{\alpha s}\big)\Big\}|y_\perp)^{ab}$$

Replace

$\infty n \to -\infty n$ everywhere

and

$x_B \to -x_B$ in the virtual correction:

$$\frac{d}{d\ln\sigma}\langle p|\big(\mathcal{F}_i^a(x_\perp,x_B)\mathcal{F}_j^a(y_\perp,x_B)\big)^{\ln\sigma}|p\rangle$$

$$= -\alpha_s\int d\!\!{}^-{}^2k_\perp\ \langle p|\mathrm{Tr}\{L_i{}^\mu(k,x_\perp,x_B)^{\mathrm{light-like}}\theta\Big(1-x_B-\frac{k_\perp^2}{\alpha s}\Big)L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-lik}}$$

$$-\ \alpha_s\langle p|\mathrm{Tr}\Big\{\mathcal{F}_i(x_\perp,x_B)(y_\perp|U^\dagger\frac{1}{\sigma x_B s-p_\perp^2+i\epsilon}U(2\delta_m^k\delta_j^l-g_{jm}g^{kl})(i\partial_l+U_l)\mathcal{F}_k(x$$

$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_B s}{p_\perp^2(\sigma x_B s-p_\perp^2+i\epsilon)}|y_\perp)$$

$$+\ (x_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_l(x_B)(i\overleftarrow{\partial}_k+U_k)(2\delta_i^k\delta_m^l-g_{im}g^{kl})U^\dagger\frac{1}{\sigma x_B s-p_\perp^2-i\epsilon}U$$

$$+\ \mathcal{F}_i(x_B)\frac{\sigma x_B s}{p_\perp^2(\sigma x_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}|p\rangle\ +\ O(\alpha_s^2)$$

1. Conclusions
   - The evolution equation for gluon TMD at any x_B and transverse momenta.
   - Interpolates between linear DGLAP and Sudakov limits and the non-linear low-x BK regime
2. Outlook
   - TMD factorization at the one-loop level.

TMD factorization in the coordinate space (ignoring indices of $\mathcal{F}_i$)

$$
\begin{aligned}
&\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\mathcal{F}^2(x)\mathcal{F}^2(y)\rangle \\
&\quad = \langle \mathcal{O}(z_{1-}, z_{1_\perp})\mathcal{O}(z_{2-}, z_{2_\perp})\mathcal{F}(x_+, x_\perp)\mathcal{F}(y_+, y_\perp)\rangle \\
&\qquad \times \langle \mathcal{O}(z_{3+}, z_{3_\perp})\mathcal{O}(z_{4+}, z_{4_\perp})\mathcal{F}(x_-, x_\perp)\mathcal{F}(y_-, y_\perp)\rangle
\end{aligned}
$$

Regge limit: $z_{1-}, z_{3+} \to \infty$, $z_{2-}, z_{4+} \to -\infty$, everything else fixed.

In the Regge limit

$$
\langle \mathcal{O}(z_{1-}, z_{1_\perp})\mathcal{O}(z_{2-}, z_{2_\perp})\mathcal{F}(x_+, x_\perp)\mathcal{F}(y_+, y_\perp)\rangle = \int d\nu\, F(\nu, \alpha_s)\Phi(r_1, \nu) R_1^{\frac{\omega(\nu, \alpha_s)}{2}}
$$

where $\omega(\nu, \alpha_s)$ = pomeron intercept,

$$
\begin{aligned}
R_1 &= \frac{(z_1 - x)(z_2 - y)}{z_{12}^2 (x-y)^2} \stackrel{\text{Regge limit}}{\rightarrow} \frac{z_{1-} z_{2-} x_+ y_+}{z_{12_\perp}^2 (x-y)_\perp^2} && \to \infty \\
r_1 &= \frac{(x_+ - y_+)^2}{x_+ |y_+| (x-y)_\perp^2} \frac{[z_{1-}(x-z_2)_\perp^2 - z_{2-}(x-z_1)_\perp^2]^2}{z_{1-}|z_{2-}| z_{12_\perp}^2} && \sim 1 \ \ \text{(fixed)}
\end{aligned}
$$

and $\Phi(r_1, \nu)$ is some function (hypergeometric).

Conformal properties of TMD factorization

The conformal ratios R_1 and r_1 are invariant under the inversion

$$z_{1-} \rightarrow \frac{z_{1-}}{z_{1_\perp}^2}, \quad z_{2-} \rightarrow \frac{z_{2-}}{z_{2_\perp}^2}, \quad x_+ \rightarrow \frac{x_+}{x_\perp^2}, \quad y_+ \rightarrow \frac{y_+}{y_\perp^2} \qquad (*)$$

Similarly, for the bottom correlator

$$\langle \mathcal{O}(z_{3+}, z_{3_\perp})\mathcal{O}(z_{4+}, z_{4_\perp})\mathcal{F}(x_-, x_\perp)\mathcal{F}(y_-, y_\perp)\rangle = \int d\nu\, F(\nu, \alpha_s)\Phi(r_2, \nu)R_2^{\frac{\omega(\nu,\alpha_s)}{2}}$$

where

$$R_2 = \frac{(z_3-x)(z_4-y)}{z_{34}^2(x-y)^2} \overset{\text{Regge limit}}{\rightarrow} \frac{z_{3+}z_{4+}x_-y_-}{z_{34_\perp}^2(x-y)_\perp^2} \qquad \Rightarrow \infty$$

$$r_2 = \frac{(x_- - y_-)^2}{x_-|y_-|(x-y)_\perp^2}\frac{[z_{3+}(x-z_4)_\perp^2 - z_{4+}(x-z_3)_\perp^2]^2}{z_{3+}|z_{4+}|z_{34_\perp}^2} \qquad \sim 1 \;\; \text{(fixed)}$$

It looks like the factorazed correlator $\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\mathcal{F}^2(x)\mathcal{F}^2(y)\rangle$ is a function of four conformal ratios R_1, R_2, r_1, r_2 instead of 9
(In general, n-point correlator is a function of $4n-15$ conformal ratios)

However, there are more "conformal ratios" invariant under inversion ($*$)
Example: correlator of two currents and conformal dipole with rapidity cutoff
$a \sim \alpha_{\max} \sim e^{\eta_{\max}}$

$$\langle \mathcal{O}(z_{1-}, z_{1_\perp}) \mathcal{O}(z_{2-}, z_{2_\perp}) \mathcal{U}^a_{\rm conf}(x_\perp, y_\perp) \rangle = \int d\nu \, F(\nu, \alpha_s) \Phi(r, \nu) \Big(\frac{z_{1-} z_{2-}}{z^2_{12_\perp}} \Big)^{\frac{\omega(\nu,\alpha_s)}{2}} a^{\frac{\omega(\nu)}{2}}$$

$$r = \frac{z^2_{12_\perp} (x-y)^2_\perp}{z_{1-} z_{2-} \Big[\frac{(x-z_1)^2_\perp}{z_{1-}} - \frac{(x-z_2)^2_\perp}{z_{2-}} \Big] \Big[\frac{(y-z_1)^2_\perp}{z_{1-}} - \frac{(y-z_2)^2_\perp}{z_{2-}} \Big]} \quad \sim 1 \text{ in the Regge limit}$$

Q: How many such "conformal ratios" for TMD correlators?

However, there are more "conformal ratios" invariant under inversion ($*$)
Example: correlator of two currents and conformal dipole with rapidity cutoff $a \sim \alpha_{\max} \sim e^{\eta_{\max}}$

$$\langle\mathcal{O}(z_{1-}, z_{1_\perp})\mathcal{O}(z_{2-}, z_{2_\perp})\mathcal{U}^a_{\rm conf}(x_\perp, y_\perp)\rangle = \int d\nu\, F(\nu,\alpha_s)\Phi(r,\nu)\Big(\frac{z_{1-}z_{2-}}{z^2_{12_\perp}}\Big)^{\frac{\omega(\nu,\alpha_s)}{2}} a^{\frac{\omega(\nu)}{2}}$$

$$r = \frac{z^2_{12_\perp}(x-y)^2_\perp}{z_{1-}z_{2-}\Big[\frac{(x-z_1)^2_\perp}{z_{1-}} - \frac{(x-z_2)^2_\perp}{z_{2-}}\Big]\Big[\frac{(y-z_1)^2_\perp}{z_{1-}} - \frac{(y-z_2)^2_\perp}{z_{2-}}\Big]} \sim 1 \ \text{ in the Regge limit}$$

Q: How many such "conformal ratios" for TMD correlators?

Formally, TMDs are invariant under the inversion ($*$):

$$\begin{aligned}\mathcal{F}^m_i(z_\perp, z_+) &= [\infty_+ + z_\perp, z_+ + z_\perp]^{mn}_z F^n_{\bullet i}(z_+, z_\perp)\\ \to\ \Big[\infty_+ + \frac{z_\perp}{z^2_\perp}, \frac{z_+}{z^2_\perp} + \frac{z_\perp}{z^2_\perp}\Big]^{mn} F^n_{\bullet i}\Big(\frac{z_+}{z^2_\perp}, \frac{z_\perp}{z^2_\perp}\Big) &= \mathcal{F}^m_i(z_\perp, z_+)\end{aligned}$$

but the rapidity cutoff spoils this invariance.

Sudakov regime in the coordinate space

In Sudakov regime $\alpha_{\max}\beta_B s \gg (x-y)^{-2}$ and $\beta_B \sim \frac{1}{x_*} \sim \frac{1}{y_*}$

$$\Phi(x_*,y_*;\lambda) \equiv F^m_{\bullet i}(x_*,x_\perp)[x,\infty]^{ml}_x[\infty,y_*]^{ln}_y F^n_{\bullet j}(y_*,y_\perp), \qquad \lambda \equiv \frac{(x-y)^2_\perp \alpha_{\max}s}{4}$$

In Sudakov regime $\lambda \gg x_*, y_*$

Evolution equation

$$\begin{aligned}
&\lambda\frac{d}{d\lambda}\Phi(x_*,y_*;\lambda)\\
&= \Big[\int_{x_*}^{\infty} dx'_*\frac{1}{x'_*-y_*}e^{i\frac{\lambda}{x'_*-y_*}}\Phi(x_*,y_*;\lambda) - \int_{y_*}^{\infty} dy'_*\frac{\Phi(x_*,y_*;\lambda)-\Phi(x_*,y'_*;\lambda)}{y'_*-y_*}\\
&+ \int_{y_*}^{\infty} dy'_*\frac{1}{y'_*-x_*}e^{i\frac{\lambda}{y'_*-x_*}}\Phi(x_*,y_*;\lambda) - \int_{x_*}^{\infty} dx'_*\frac{\Phi(x_*,y_*;\lambda)-\Phi(x'_*,y_*;\lambda)}{x'_*-x_*}\Big]
\end{aligned}$$

Sudakov regime in the coordinate space

Solution ($\bar\alpha_s \equiv \frac{\alpha_s N_c}{4\pi}$)

$$
\begin{aligned}
\Phi(x_*, y_*; \lambda) \; &= \\
= \; &-e^{-\frac{\bar\alpha_s}{4}[\ln^2 \frac{\lambda^2}{x_* y_*} - \ln^2 \frac{\lambda_0^2}{x_* y_*}] + 4\bar\alpha_s \psi(1) \ln \frac{\lambda}{\lambda_0}} \int dx'_* dy'_* \; \Phi(x'_*, y'_*; \lambda_0) (x_* y_*)^{-\bar\alpha_s \ln \frac{\lambda}{\lambda_0}} \\
\times & \left[\frac{\Gamma\big(1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}\big)}{(x_* - x'_* + i\epsilon)^{1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}}} + \frac{\Gamma\big(1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}\big)}{(x'_* - x_* + i\epsilon)^{1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}}} \right] \\
\times & \left[\frac{\Gamma\big(1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}\big)}{(y_* - y'_* + i\epsilon)^{1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}}} + \frac{\Gamma\big(1 - \bar\alpha_s \ln \frac{\lambda}{\lambda_0}\big)}{(y'_* - y_* + i\epsilon)^{1 - \bar\alpha_s \ln \frac{a}{\lambda_0}}} \right]
\end{aligned}
$$

Does not look conformal

Sudakov regime in the coordinate space

If we use rapidity cutoff at $\alpha_{\max} = \frac{4\sigma}{|x-y|_\perp\sqrt{s}} \quad \Rightarrow \lambda = \sigma|x-y|\sqrt{s}$

$$\begin{aligned}
&\Phi(x_*, y_*; \sigma) \\
&= -e^{-\frac{\bar{\alpha}_s}{4}\left(\ln^2 \frac{(x-y)_\perp^2 s\sigma^2}{x_* y_*} - \ln^2 \frac{(x-y)_\perp^2 s\sigma_0^2}{x_* y_*}\right)} e^{4\bar{\alpha}_s\psi(1)\ln\frac{\sigma}{\sigma_0}} \int dx'_* dy'_* \; \Phi(x'_*, y'_*; \sigma_0) \\
&\times (x_* y_*)^{-\bar{\alpha}_s \ln\frac{\sigma}{\sigma_0}} \left[\frac{\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\sigma}{\sigma_0}\right)}{(x_* - x'_* + i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}} + \frac{\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\sigma}{\sigma_0}\right)}{(x'_* - x_* + i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}}\right] \\
&\times \left[\frac{\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\sigma}{\sigma_0}\right)}{(y_* - y'_* + i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}} + \frac{\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\sigma}{\sigma_0}\right)}{(y'_* - y_* + i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}}\right]
\end{aligned}$$

is conformally invariant (under the inversion $x_* \to \frac{x_*}{x_\perp^2}$, $y_* \to \frac{y_*}{y_\perp^2}$).