# DEFORMING PATH INTEGRALS: APPLICATION TO THE (2+1) THIRRING MODEL

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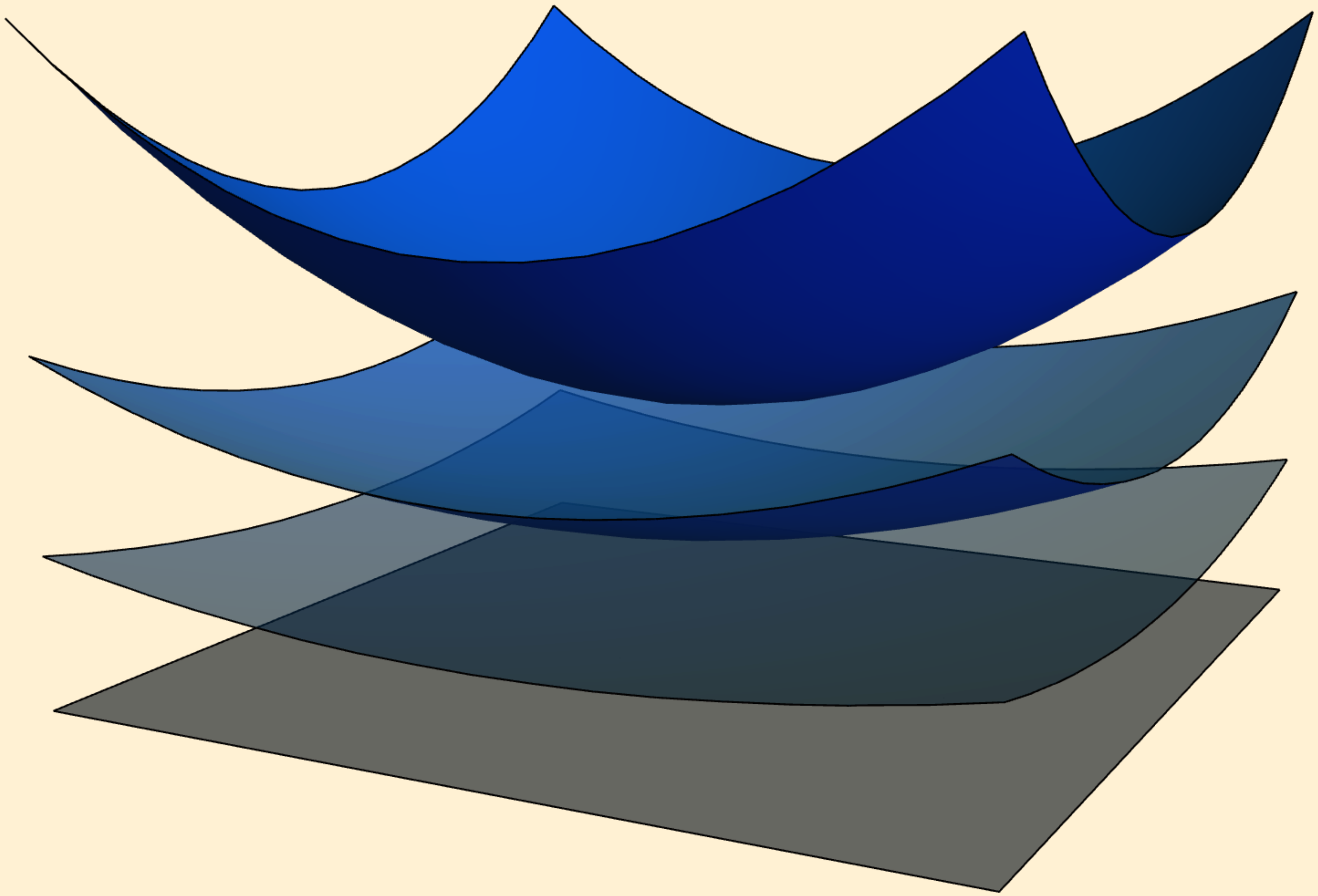
# Our flavor of Monte Carlo:

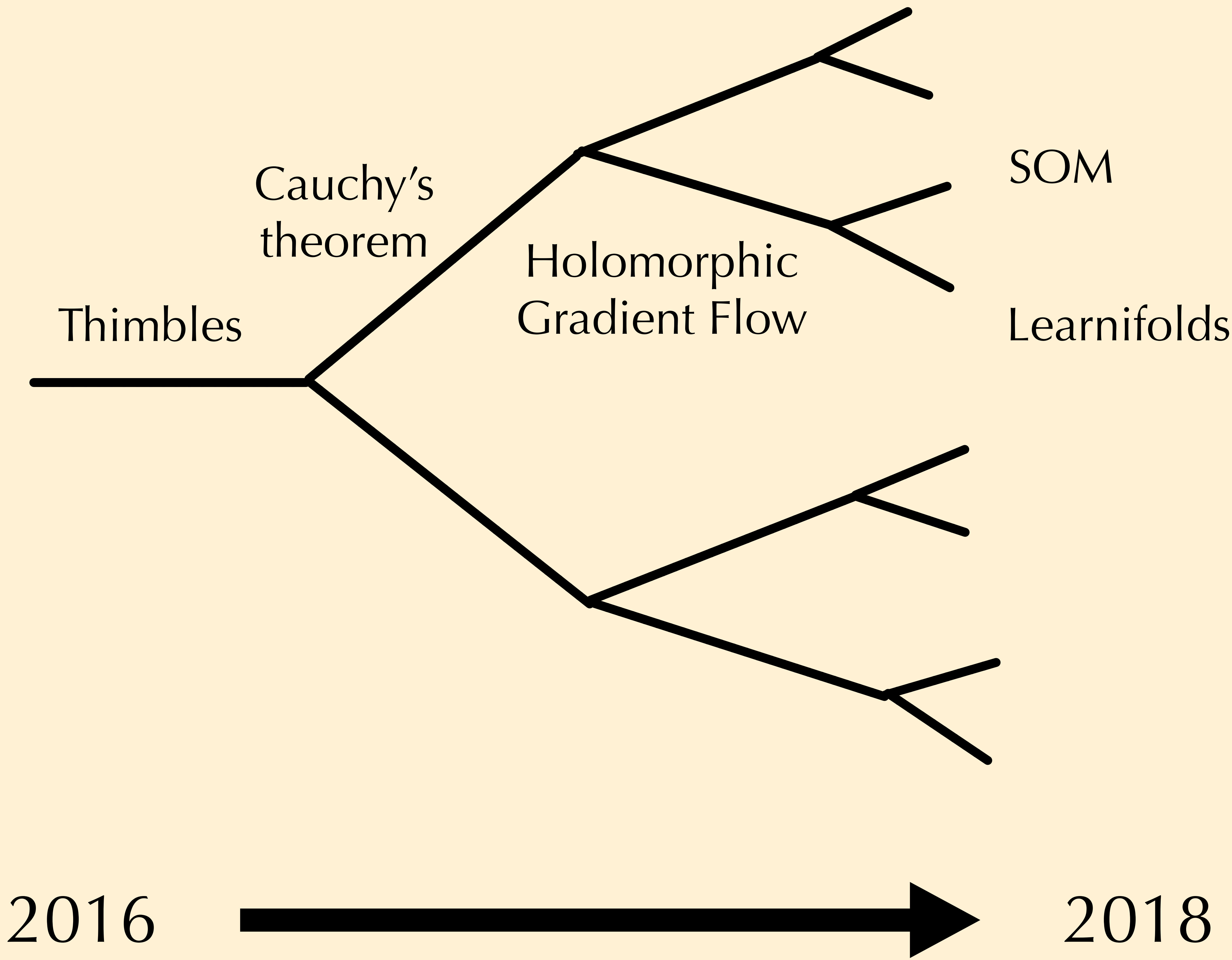
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \mathrm{tr}(\mathrm{e}^{-\beta(H-\mu N)} \mathcal{O}) = \frac{1}{Z} \int D\phi \; \mathrm{e}^{-S(\phi)} \mathcal{O}(\phi)$$

1. Sample according to $\mathrm{Pr}(\phi) = \dfrac{\mathrm{e}^{-\mathrm{Re} S(\phi)}}{Z}$

2. Reweigh the phase $\langle \mathcal{O} \rangle = \dfrac{\left\langle \mathcal{O} \; \mathrm{e}^{-i \mathrm{Im}(S)} \right\rangle_R}{\left\langle \mathrm{e}^{-i \mathrm{Im}(S)} \right\rangle_R}$

$$\int_{\mathbb{R}^N} D\phi \, \mathrm{e}^{-S(\phi)} = \int_{\mathcal{M}} D\tilde{\phi} \, \mathrm{e}^{-S(\tilde{\phi})}$$




Cauchy's
theorem
Holomorphic
Gradient Flow
SOM
Thimbles
Learnifolds
2016
2018

# Complex Manifolds so far

(0+1) Fermions ✓

(1+1) Fermions ✓

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How about (2+1)?

# (2+1) Thirring Model

$$S = \int dt d^2x \left[ \overline{\psi}(\partial\!\!\!/ + m + \mu\gamma^0)\psi + \frac{g^2}{4}(\bar{\psi}\gamma^\mu\psi)^2 \right]$$

## What is known?

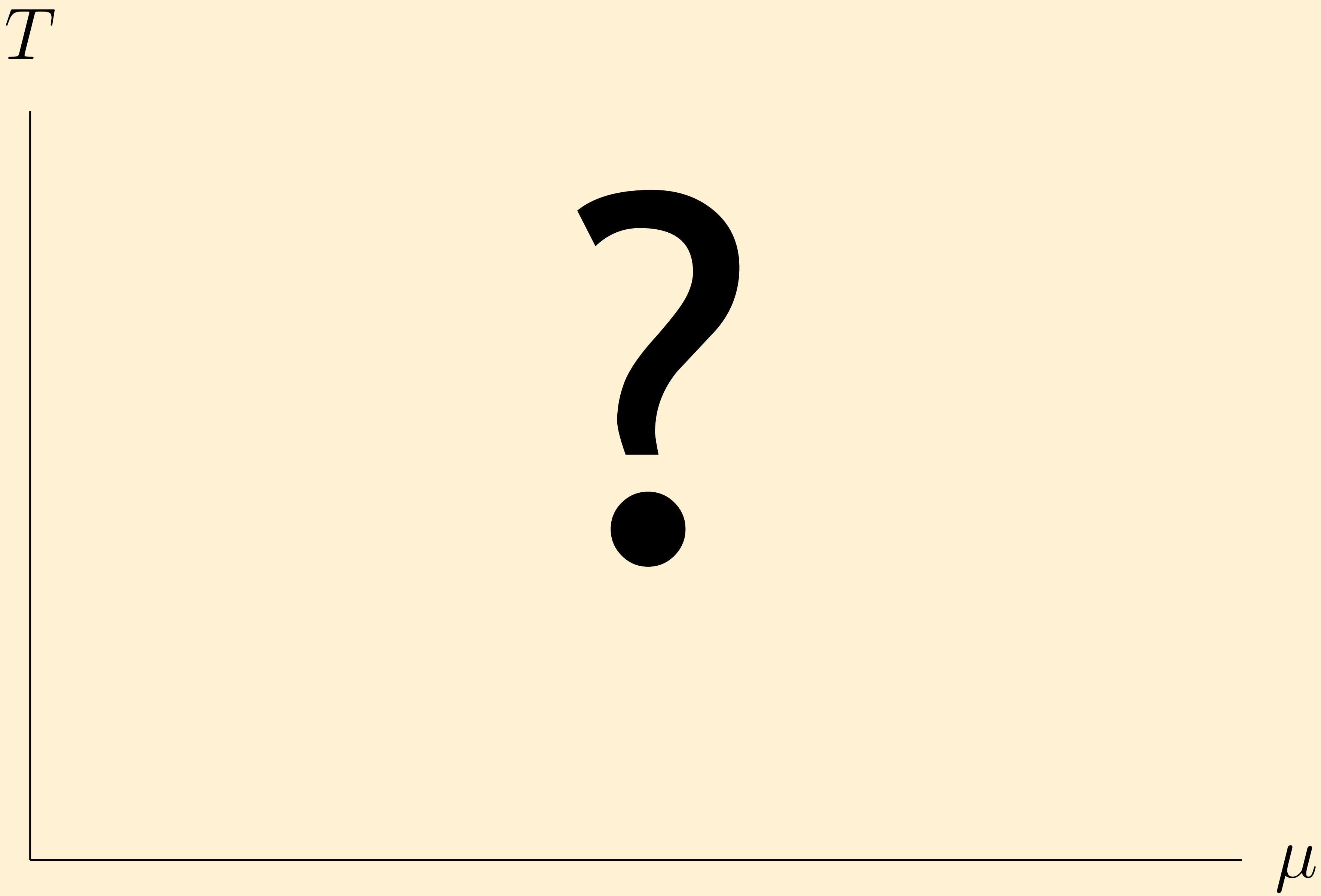
$\mu = 0$

- $\langle\bar{\psi}\psi\rangle \neq 0$ spontaneously at $m = 0$ Hands et. al. [9512013]
- Chiral restoration at large $N_f$ Hands et. al. [0701016]

$\mu \neq 0$

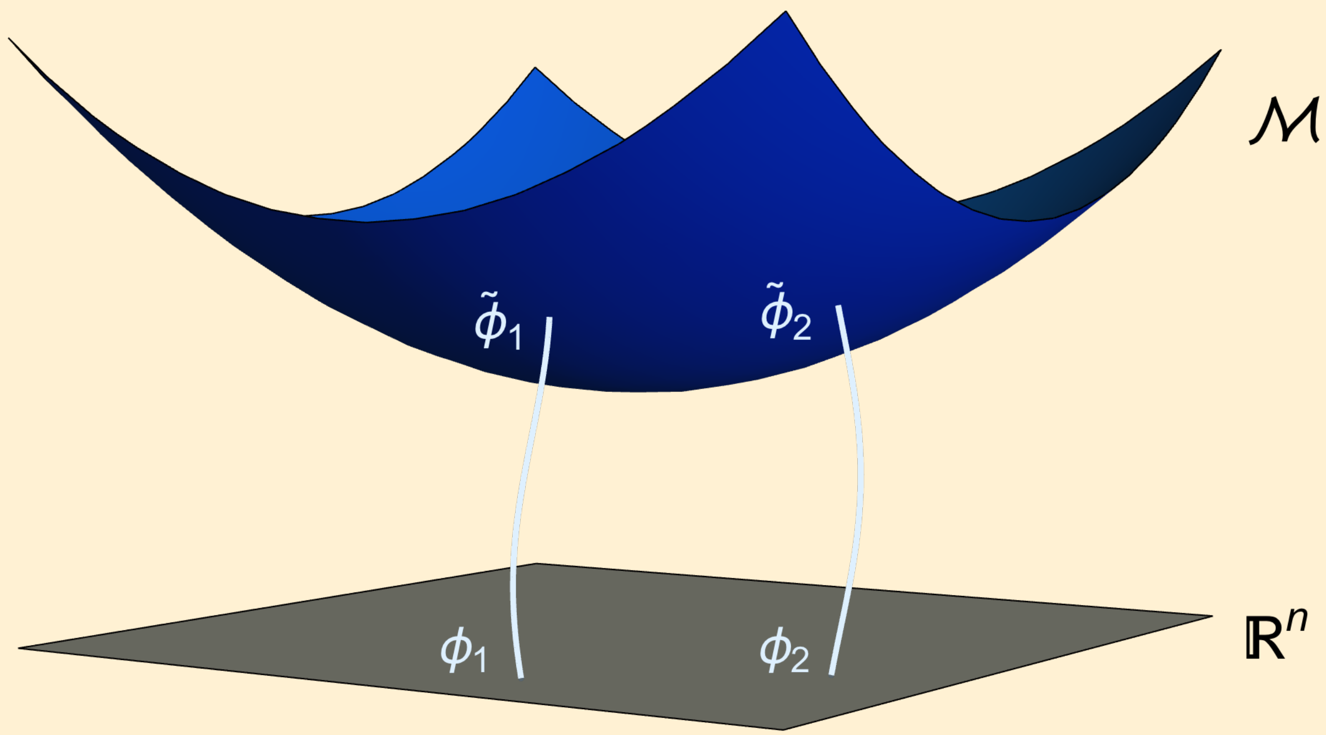
- Complex Langevin + Heavy Dense Pawlowski et. al. [1302.2249]
- Complex Langevin + Fermion bags Li [1608.03141]

# What is not known



## Holomorphic Gradient Flow

## **S**ign **O**ptimized **M**anifolds




Tangent Plane
+
Heavy Dense

$$\frac{d\phi}{ds} = \frac{\partial S}{\partial \phi}^*$$

# Sign Optimized Manifolds

**1. Propose family of manifolds**

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$$\tilde{\phi}_x(\phi) = \phi_x + i\Big(\lambda_1 + \lambda_2 \cos(\phi_x) + \lambda_3 \cos(2\phi_x)\Big)$$

three parameter family

# Sign Optimized Manifolds

## 1. Propose family of manifolds

$$\tilde{\phi}_x(\phi) = \phi_x + i\Big(\lambda_1 + \lambda_2 \cos(\phi_x) + \lambda_3 \cos(2\phi_x)\Big)$$

three parameter family

Feature: Manifold is factorizable

$$J_{xy} = \frac{\partial \tilde{\phi}_x}{\partial \phi_y} = \delta_{xy} * g(\phi_x)$$

# Sign Optimized Manifolds

**2. Maximize average sign over family**

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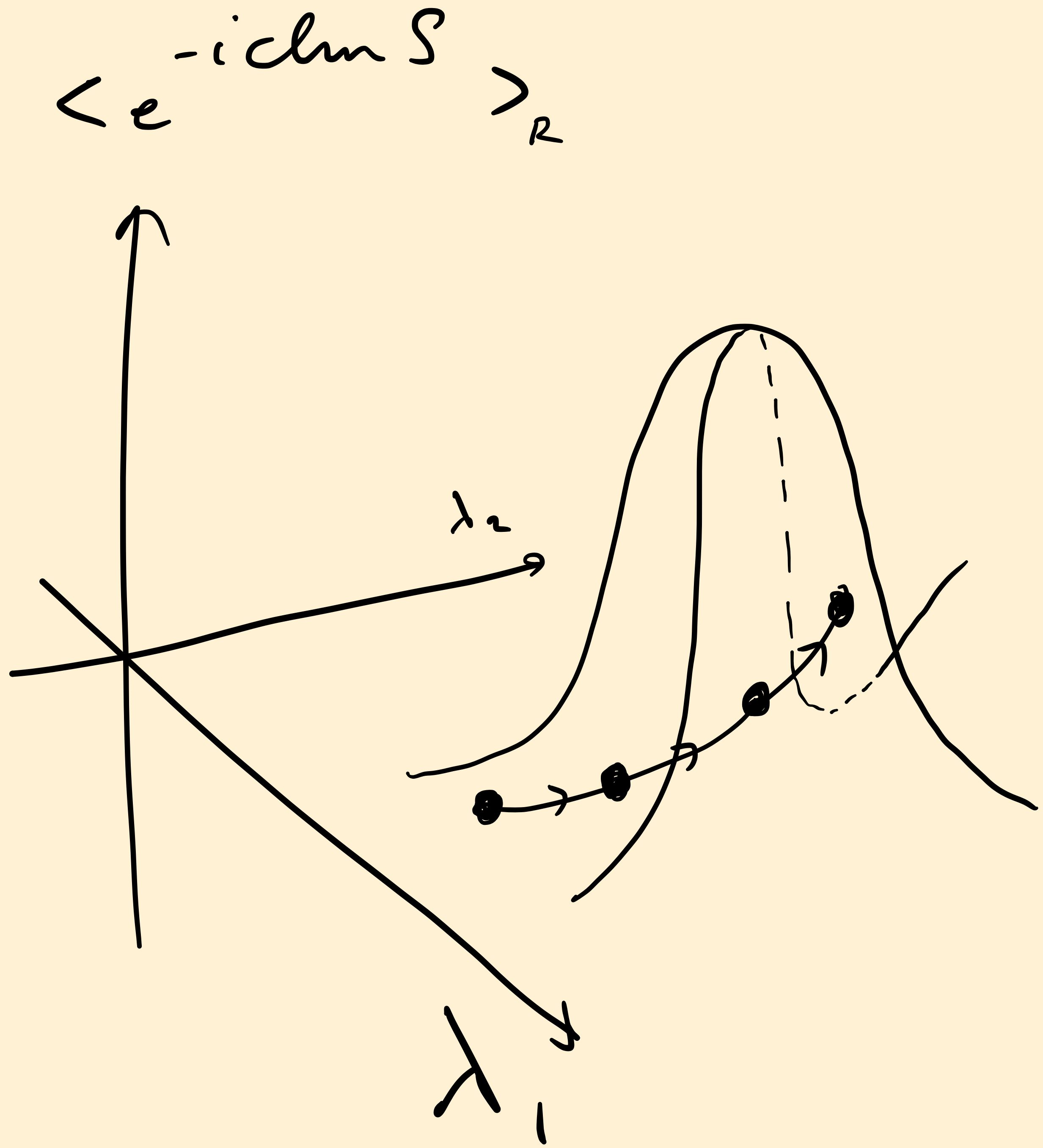



$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O}\ \mathrm{e}^{-i\mathrm{Im}(S)} \rangle_R}{\langle \mathrm{e}^{-i\mathrm{Im}(S)} \rangle_R}$$

need precision
if small

# Sign Optimized Manifolds

## **2. Maximize average sign over family**

$$\langle \sigma \rangle \equiv \langle \mathrm{e}^{-i\mathrm{Im}S} \rangle_R$$

$$\nabla_\lambda \langle \sigma \rangle = |\langle \sigma \rangle| \frac{\int D\phi \; \mathrm{e}^{-\mathrm{Re}(S)} \left[ \nabla_\lambda \mathrm{Re}(S) - \mathrm{tr}(J^{-1} \nabla_\lambda J) \right]}{\int D\phi \; \mathrm{e}^{-\mathrm{Re}(S)}}$$

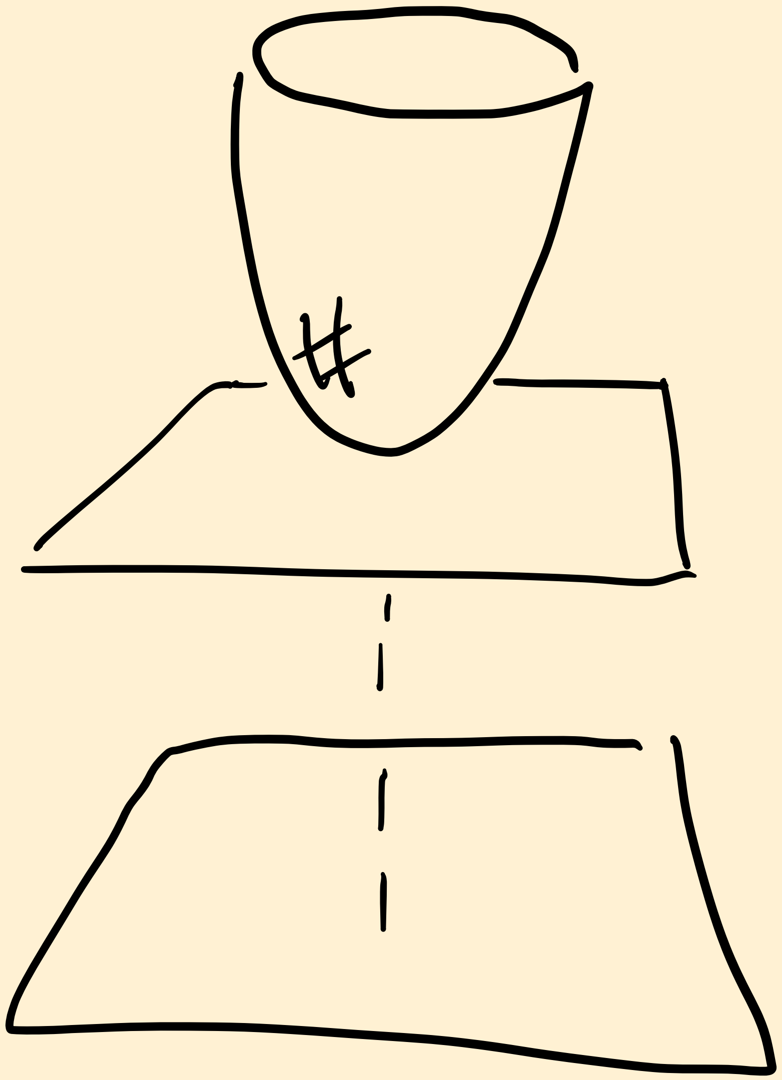
↑

sign problem free = cheap

# Sign Optimized Manifolds

**3. Integrate over optimal manifold**

# Tangent Shift Heavy Dense




$$Z = \left( \int d\phi \; \mathrm{e}^{-S(\phi)} \right)^{\beta V}$$

$$\tilde{\phi}_x(\phi) = \phi_x + i\Big(\lambda_1 + \lambda_2 \cos(\phi_x) + \lambda_3 \cos(2\phi_x)\Big)$$

# Strategy

- Use sign opt. manifold
- HMC on manifold

# HMC on Manifolds

$$Z = \int D\phi \, \mathrm{e}^{-S} = \int_{\mathcal{M}} D\tilde{\phi} \, \mathrm{e}^{-S} = \int_{\mathbb{R}^N} D\phi \, \det J(\phi) \mathrm{e}^{-S}$$

↑ Cauchy's Theorem

↑ Choose coordinates

Want to sample as $\Pr(\phi) \propto |\det J(\phi)| \, \mathrm{e}^{-S_R}$

# HMC on Manifolds

1. $H(\pi,\phi) = \frac{1}{2}\sum \pi_x [J(\phi)J^\dagger(\phi)]^{-1}_{xy}\pi_y + S(\phi)$

2. Sample as $P(\pi,\phi) \propto e^{-H(\pi,\phi)}$

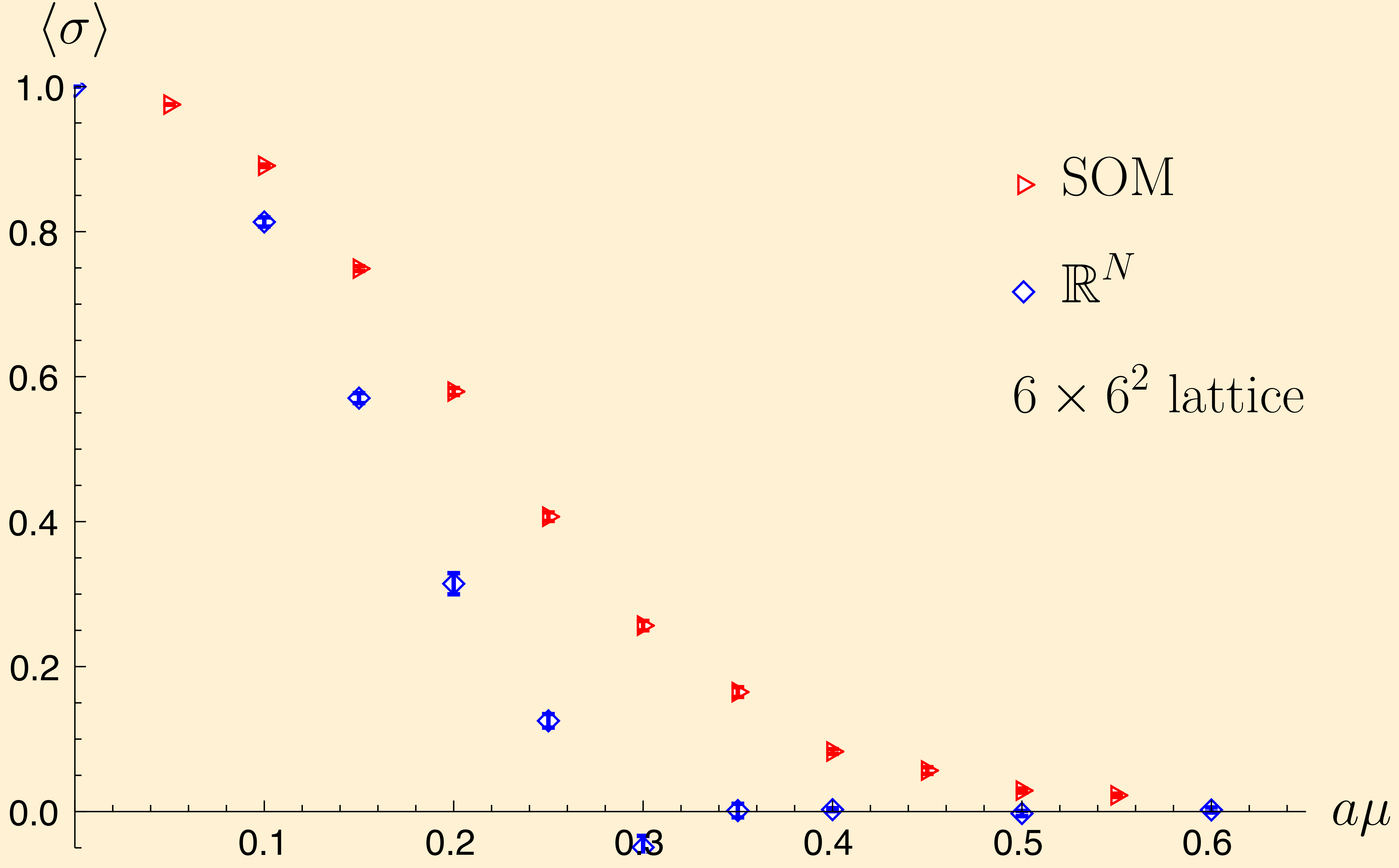
3. Marginalize: $\sum_\pi P(\pi,\phi) \propto |\mathrm{det}J(\phi)|\mathrm{e}^{-S(\phi)}$

$$H(\pi, \phi) = \frac{1}{2} \sum \pi_x [J(\phi) J^\dagger(\phi)]^{-1}_{xy} \pi_y + S(\phi)$$

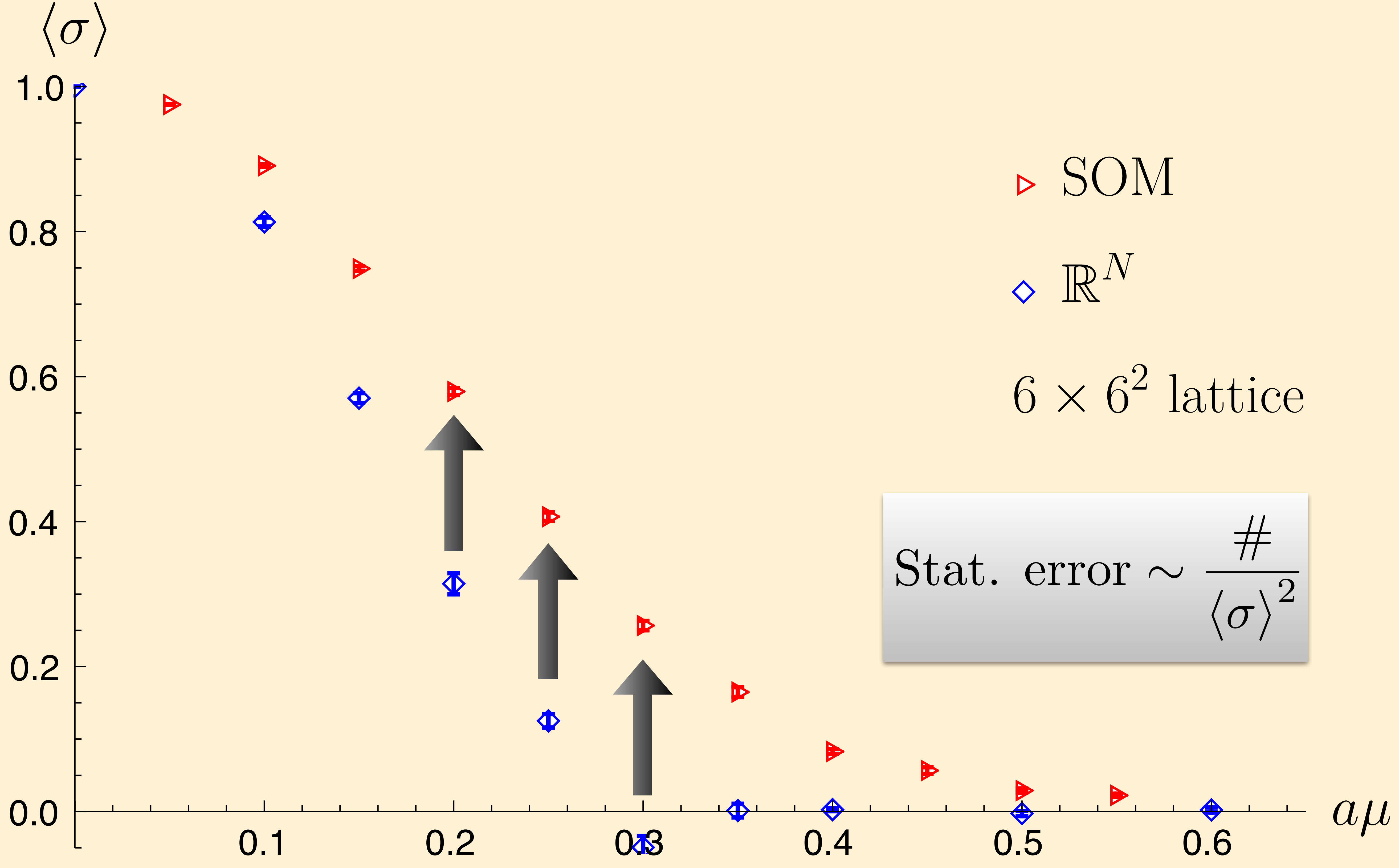
↑

Separable manifolds are cheap

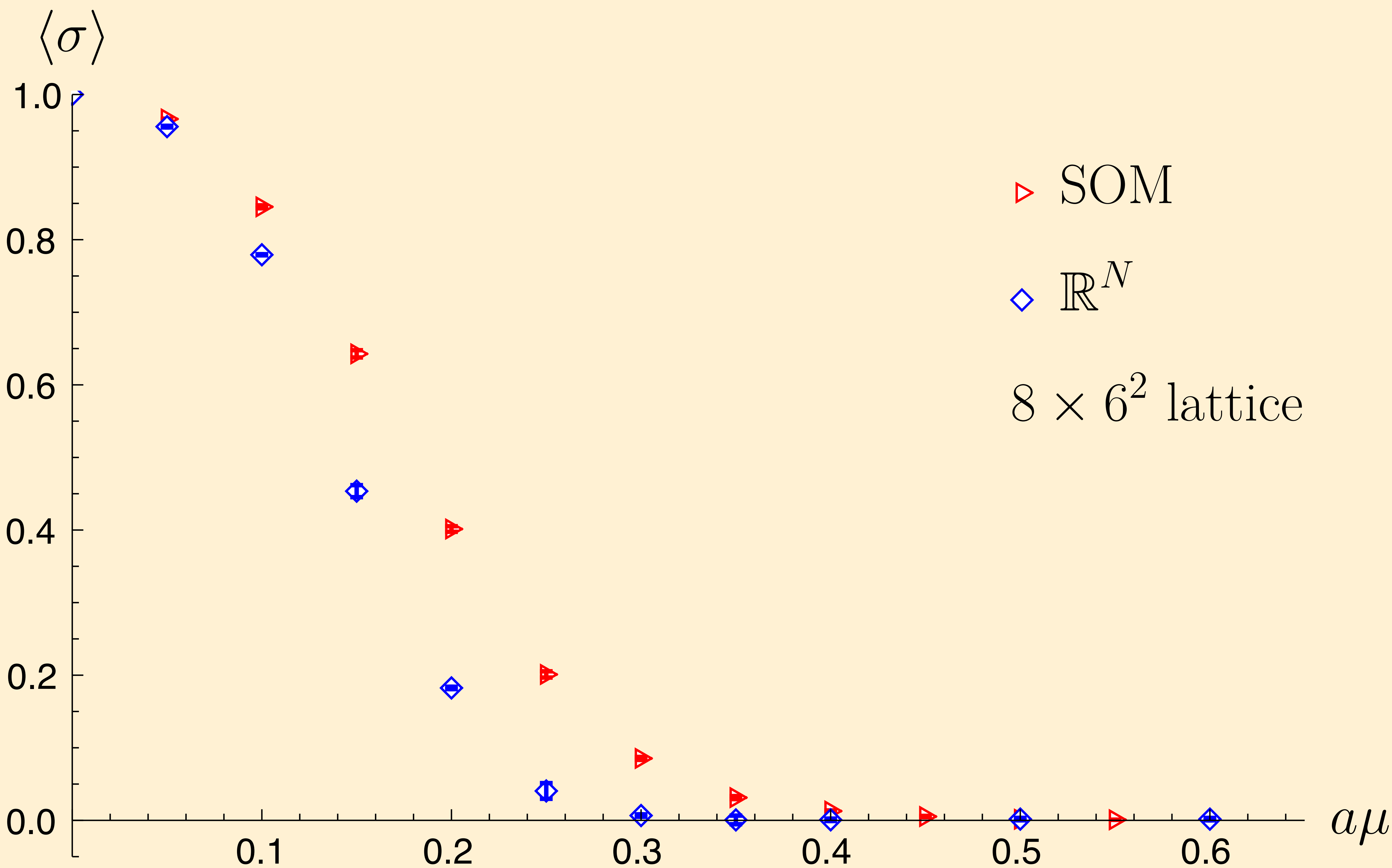
# Sign Increases



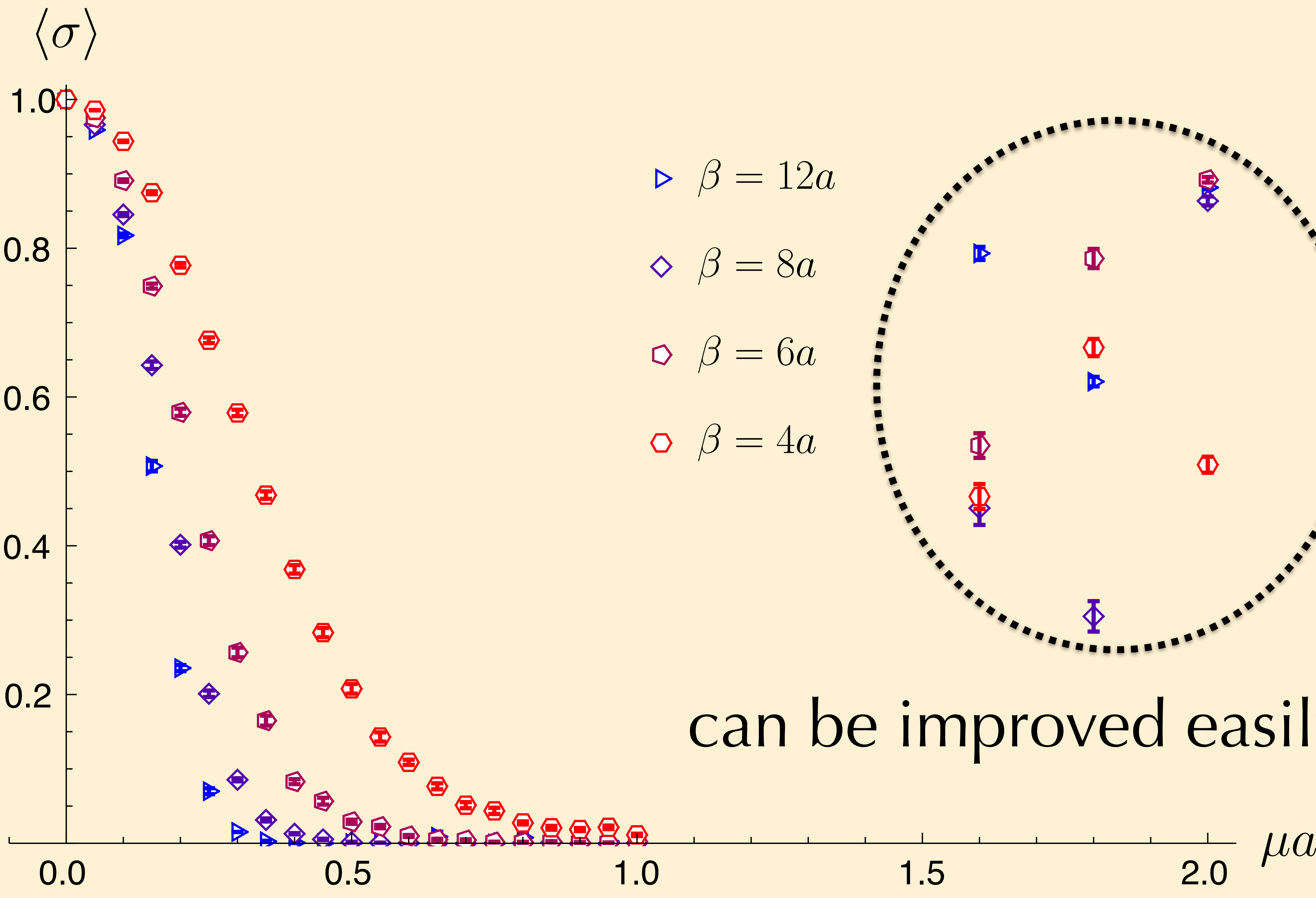
# Sign Increases



# Sign Increases

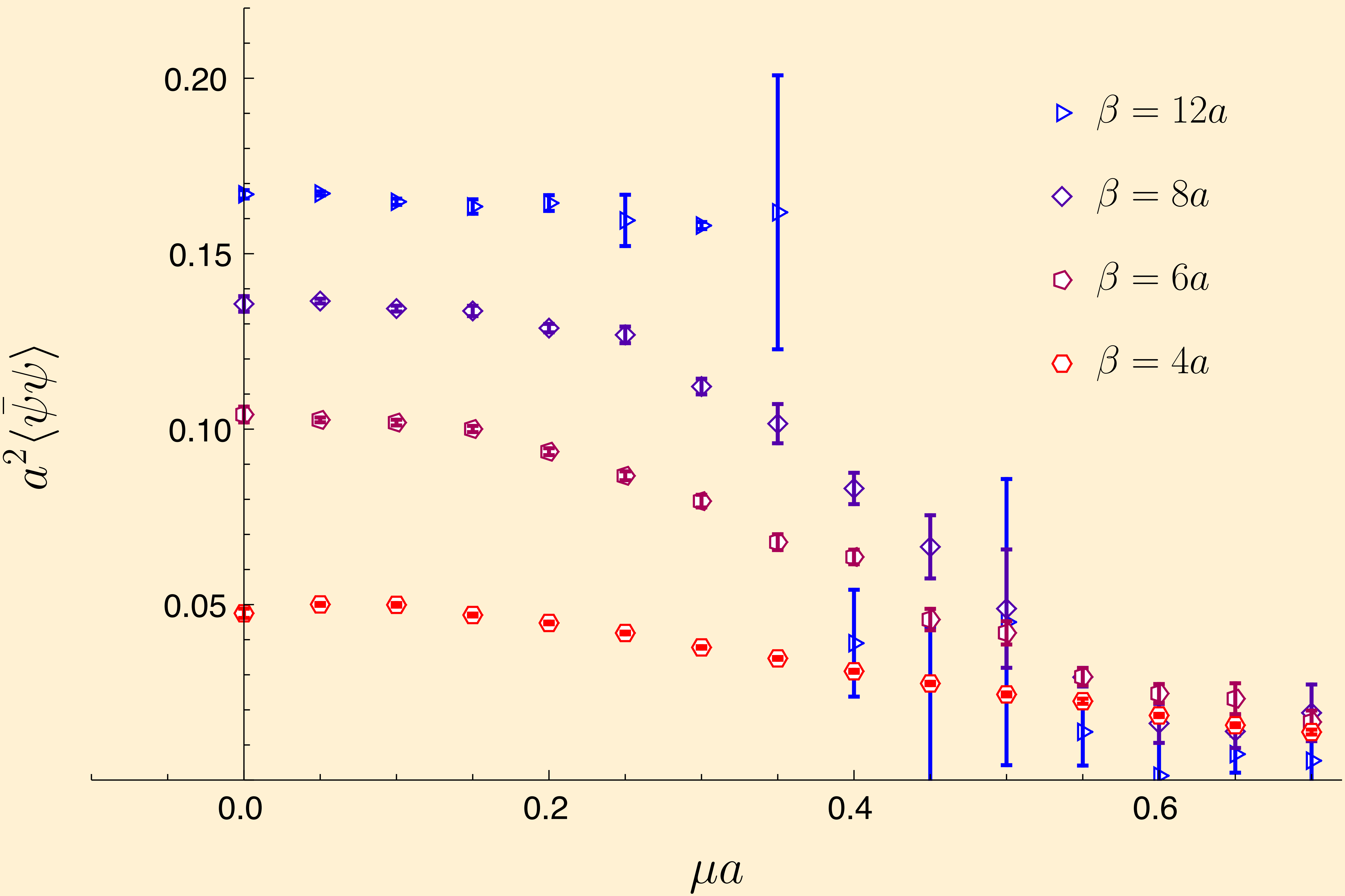


# $\mu \longrightarrow \infty$ limit solved

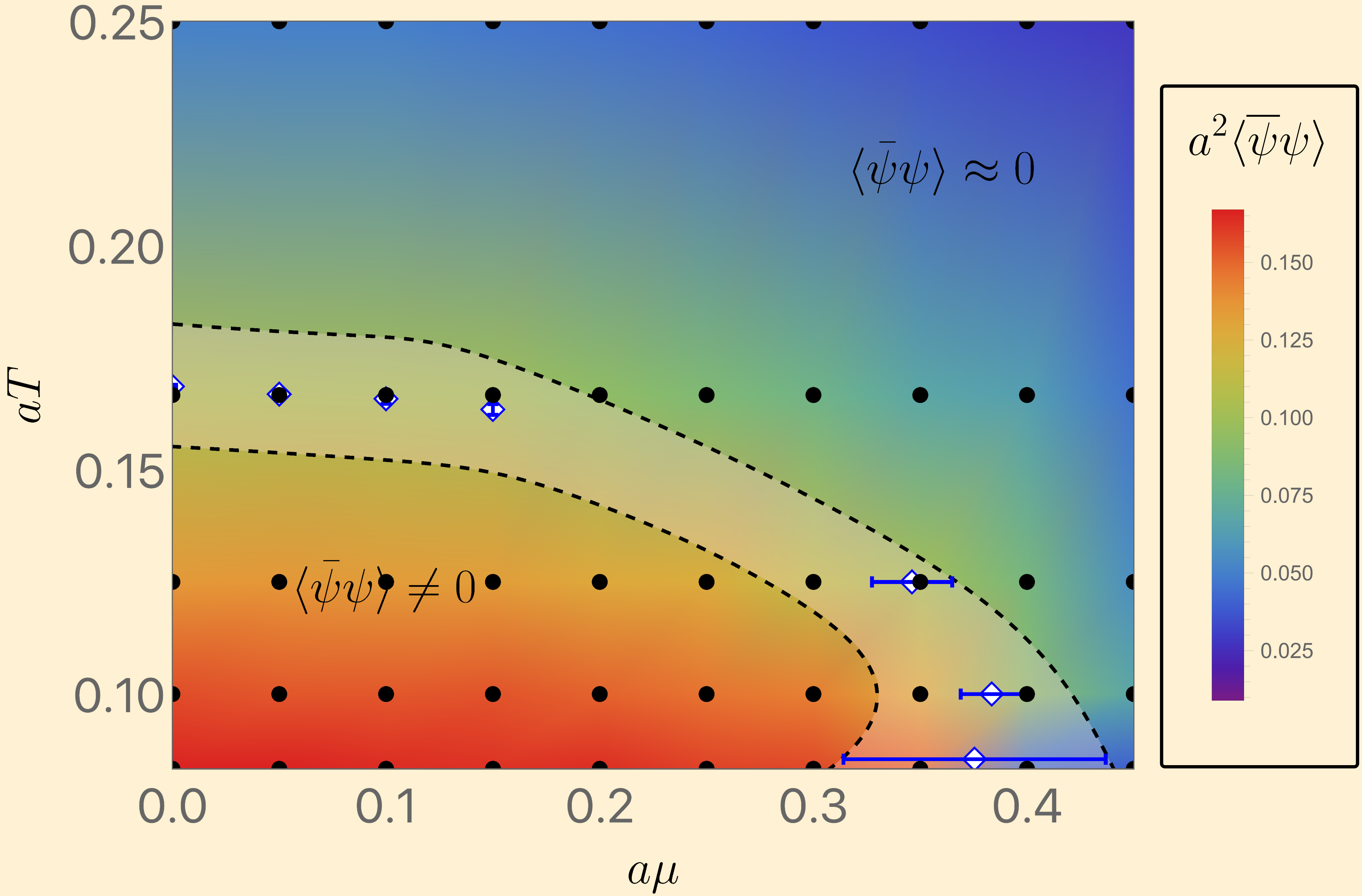



can be improved easily

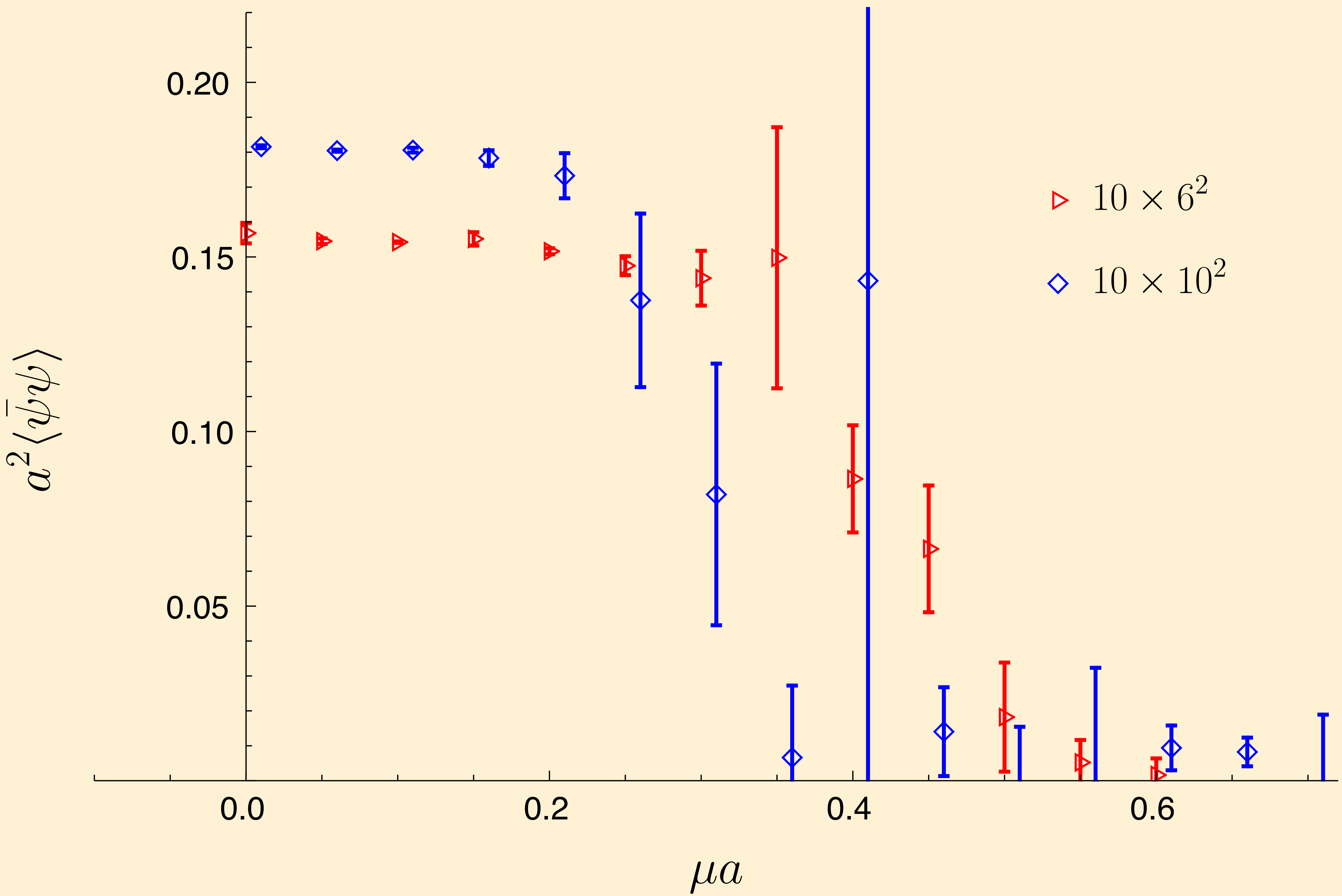
# Chiral Condensate



# $(T,\mu)$ Phase Diagram



# Current Limit



# Take Home

- Complex manifolds in (2+1) ✓
- SOM is fastest approach so far
- Up to $\sim 10 \times 10 \times 10$ possible
- $(T, \mu)$ phase diagram computed ab initio

# More Applications

## Thermodynamics:

Phys. Rev. D. 94, 045017
Phys. Rev. D. 95, 014502

## Real time dynamics:

Phys. Rev. Lett. 117, 081602
Phys. Rev. D 95, 114501

## Gauge Theories:

arXiv:1807.02027

Thanks!