Can one sum up all Feynman diagrams for the unitary Fermi gas?

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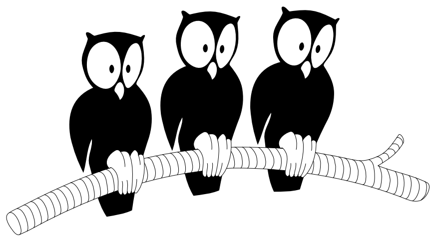
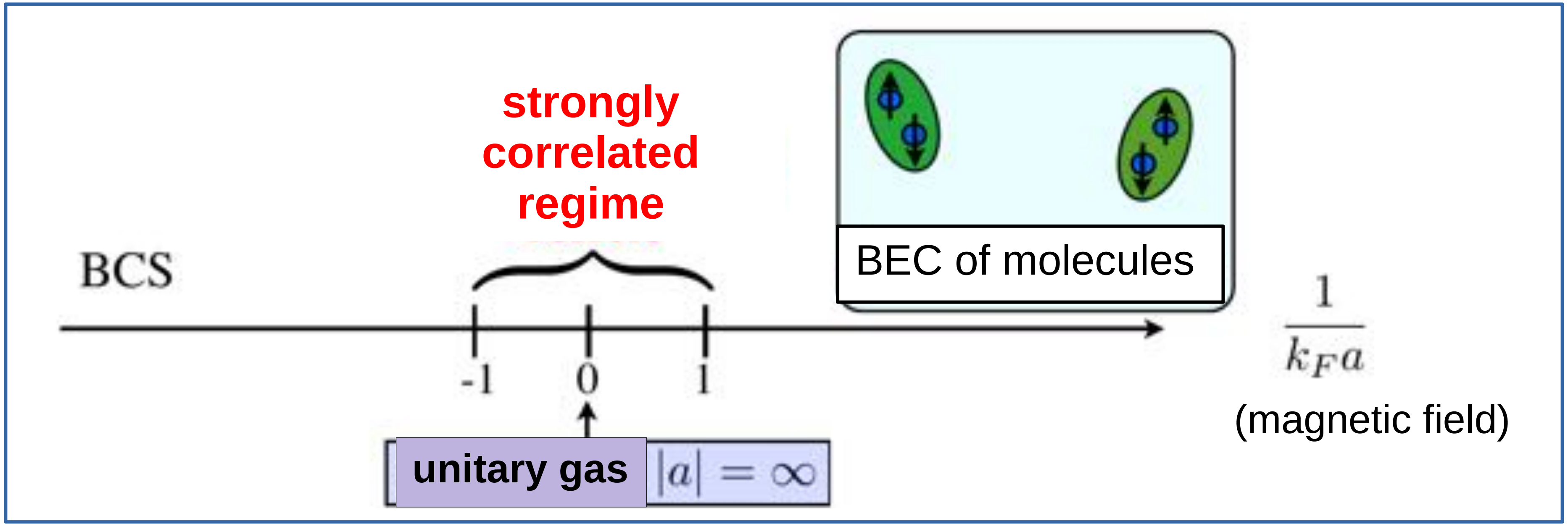


INT, Seattle, August 17, 2018

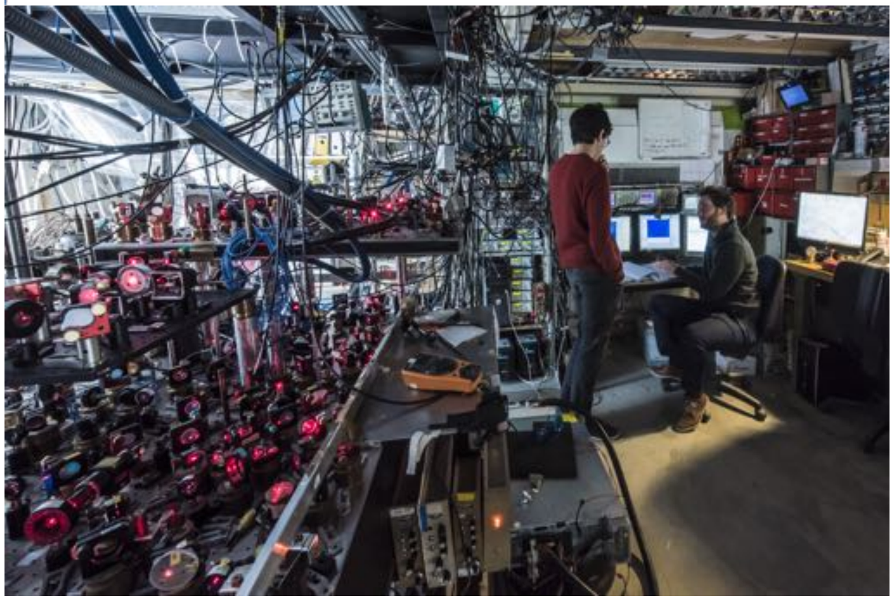
Collaborators: Félix Werner (LKB-ENS), Riccardo Rossi (CCQ NYC), Takahiro Ohgoe (Tokyo U.) Boris Svistunov (Umass Amherst), Nikolay Prokof’ev (Umass Amherst), Evgeny Kozik (King’s college)



cold atom experiments
fermionic atom, 2 internal states, Feshbach resonance

accurate comparison

zero-range theory



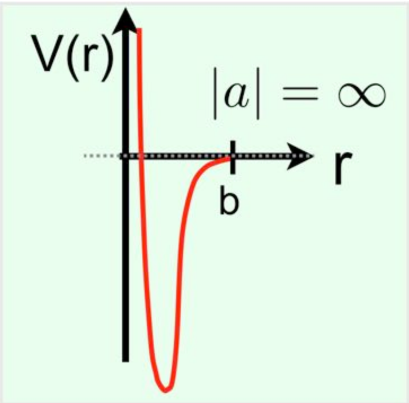
relevant for neutron matter

The unitary Fermi gas

Spin-1/2 fermions in 3D, interactions have

- infinite scattering length
- zero range

Universality hypothesis:



Zero-range limit: $\begin{cases} n^{-1/3} \gg b \\ \lambda \gg b \end{cases}$ $\quad \lambda \equiv \sqrt{\dfrac{2\pi\hbar^2}{mk_BT}}$

$\Longrightarrow$ Properties do not dependent on $V(r)$

$(N_\uparrow = N_\downarrow)$ $\boxed{n(T,\mu)\lambda^3 = \text{universal function of } \beta\mu}$

Construction from Hubbard model:

- $\frac{U}{t} = -7.913552\ldots$ (appearance of 2-body bound state)
- thermodynamic limit
- filling $\to 0$ with $\frac{T}{T_F}$ fixed (= continuum limit)

Feynman diagrams

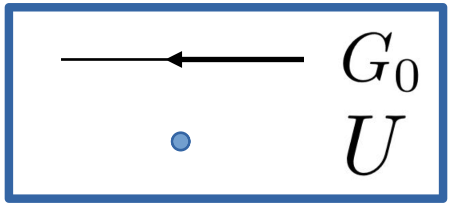
Q : intensive
e.g. pressure:

$$Q = \frac{T}{\mathcal{V}} \ln \mathrm{Tr}\, e^{-\beta(H-\mu N)}$$

$$Q_M = \sum_{N=0}^{M} a_N$$

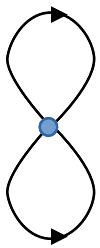
a_N → all order-N diagrams

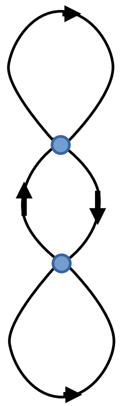
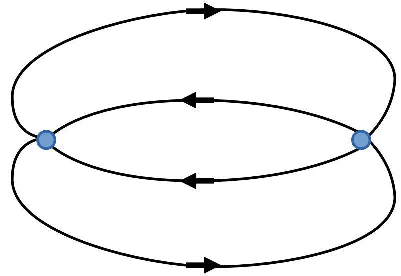
Bare scheme



Hubbard model:

$$H = -t \sum_{\langle \vec{r}, \vec{r}' \rangle} c^{\dagger}_{\vec{r},\sigma} c_{\vec{r}',\sigma} + U \sum_{\vec{r}} c^{\dagger}_{\vec{r},\uparrow} c^{\dagger}_{\vec{r},\downarrow} c_{\vec{r},\downarrow} c_{\vec{r},\uparrow}$$

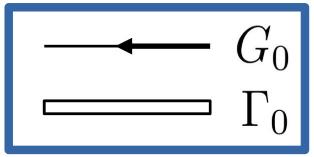
$a_1 =$ 

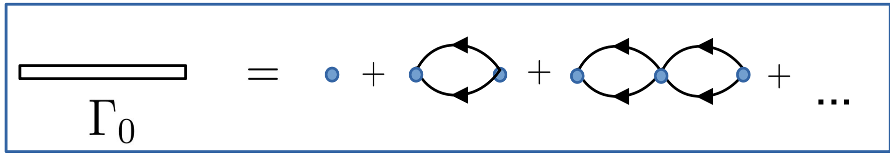
$a_2 =$  $+$ 

$$Q_M \underset{M\to\infty}{\longrightarrow} Q_\infty$$

Taylor series $(a_N \propto U^N)$
$\Rightarrow \; Q_\infty = Q_{\text{phys}}$

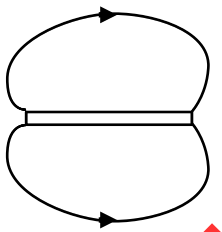
Ladder scheme



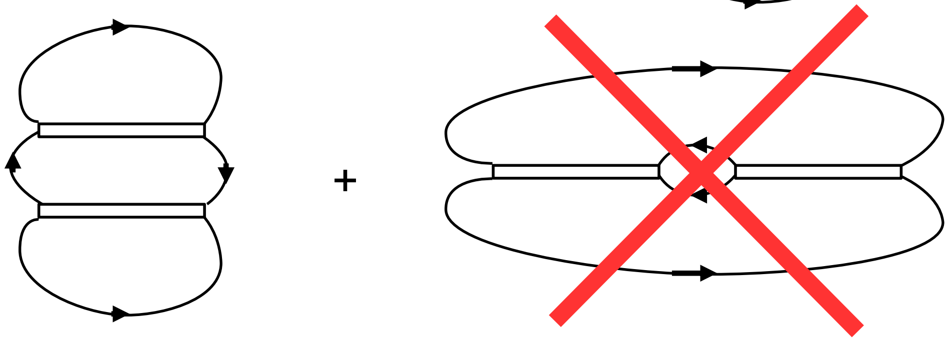


finite
in continuum limit

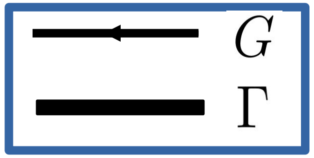
$a_1 =$



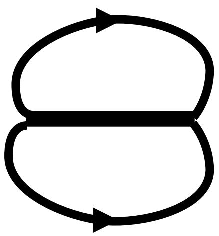
$a_2 =$



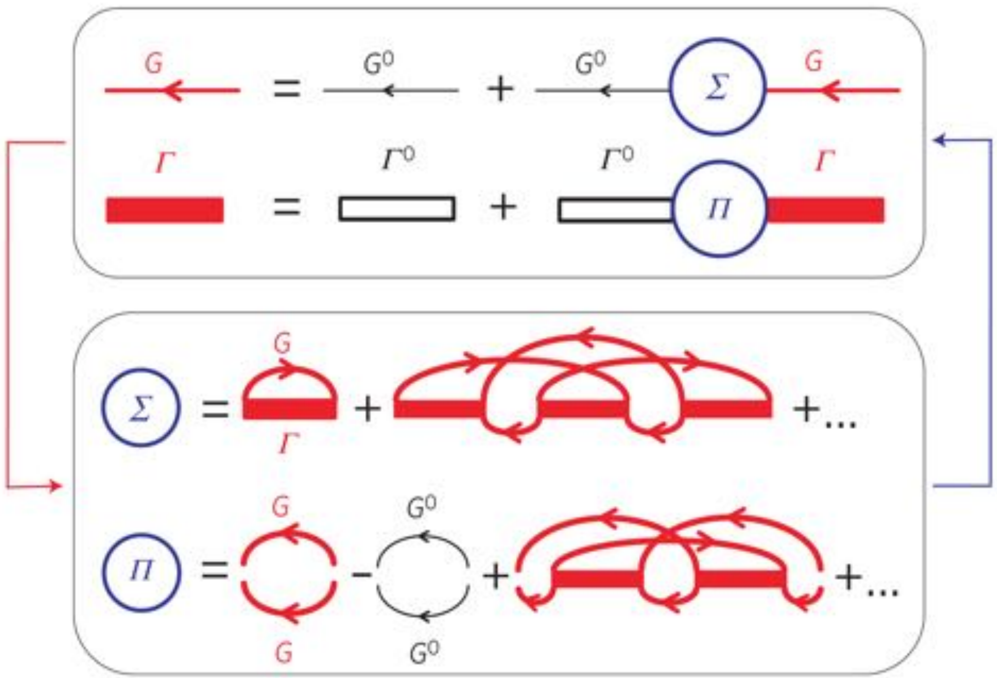
Bold scheme



$a_1 =$



$a_2 = 0$



$$a_N = \sum_{\text{topologies } \mathcal{T}} \int dX_1 \dots dX_N \mathcal{D}(\mathcal{T}; X_1 \dots X_N)$$

$$X = (\vec{r}, \tau)$$

$$\int dX = \sum_{\vec{r}} \int_0^\beta d\tau$$

Monte Carlo algorithms

- DiagMC [Van Houcke, Kozik, Prokof'ev, Svistunov, Phys. Proc. 2010]

configuration: $\mathcal{C} = (\mathcal{T}, X_1, \dots, X_N)$ probability: $P(\mathcal{C}) \propto |\mathcal{D}(\mathcal{T}; X_1 \dots X_N)|$

- CDet [Rossi, PRL 2017]

$\mathcal{C} = (X_1, \dots, X_N)$ $P(\mathcal{C}) \propto \left| \sum_{\mathcal{T}} \mathcal{D}(\mathcal{T}; X_1 \dots X_N) \right|$

Computational complexity

- traditional form of sign problem: error $\sim e^{\mathcal{V}}$

for connected Feynman diagrams: $\mathcal{V} = \infty$

- instead

$$\text{error} \sim \begin{cases} N! & \text{(DiagMC)} \\ e^{\# N} & \text{(CDet)} \end{cases}$$

- counteracted by convergence: $|a_N| \sim e^{-\# N}$

➔ $t(\epsilon) \sim \epsilon^{-\# \ln(\ln \epsilon^{-1})}$ (DiagMC),

$t(\epsilon) \sim \epsilon^{-\alpha}$ (CDet).

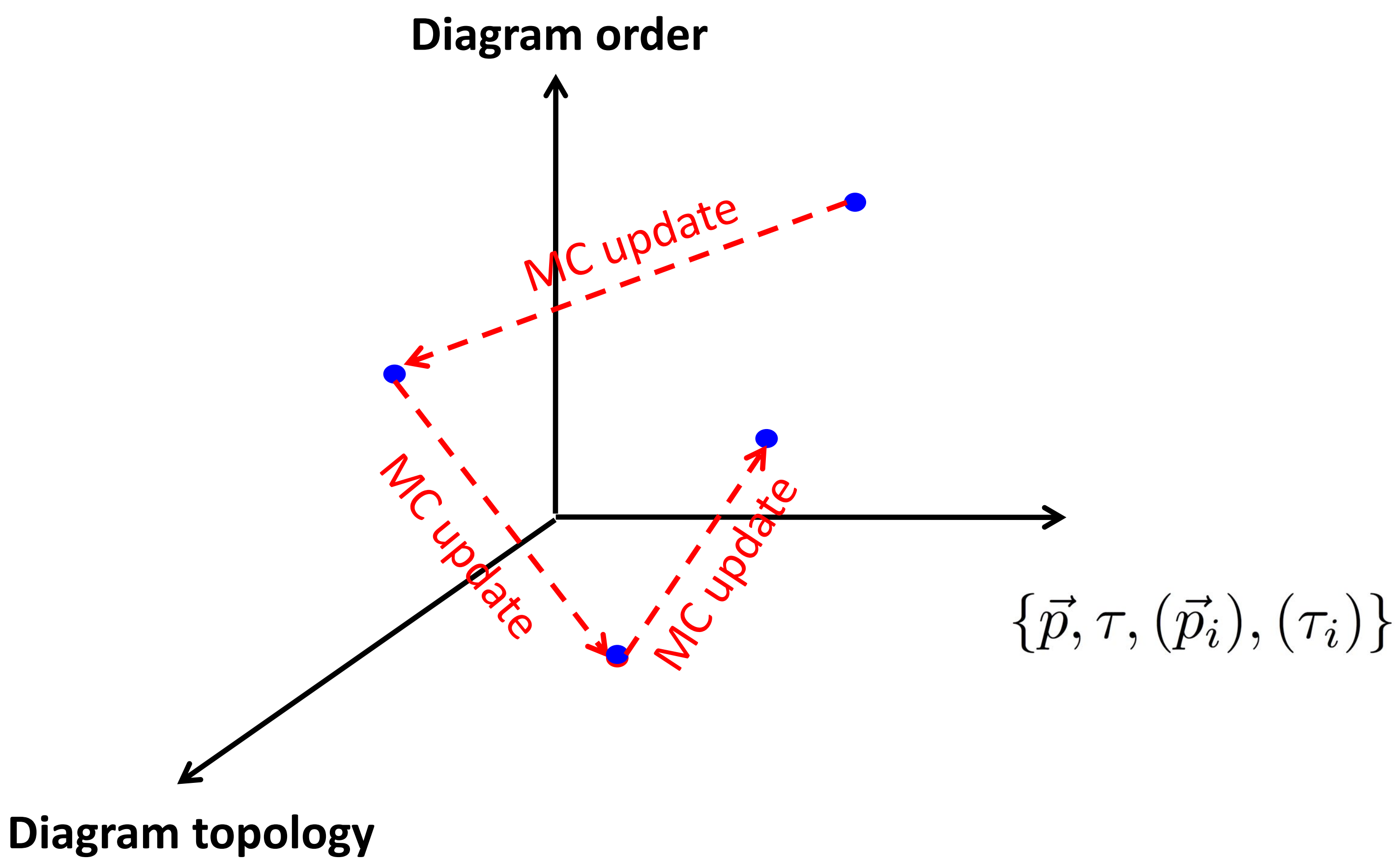
where ϵ = total error (statistical + truncation)

[Rossi, Prokof'ev, Svistunov, KVH, Werner, EPL 2017]

Diagrammatic Monte Carlo:

Importance sampling of the space of all possible diagram topologies and all values of internal and external variables.

Each configuration is visited with a probability proportional to the absolute value of its contribution to $\Sigma_\sigma(\mathbf{p},\tau)$ as given by the Feynman rules.



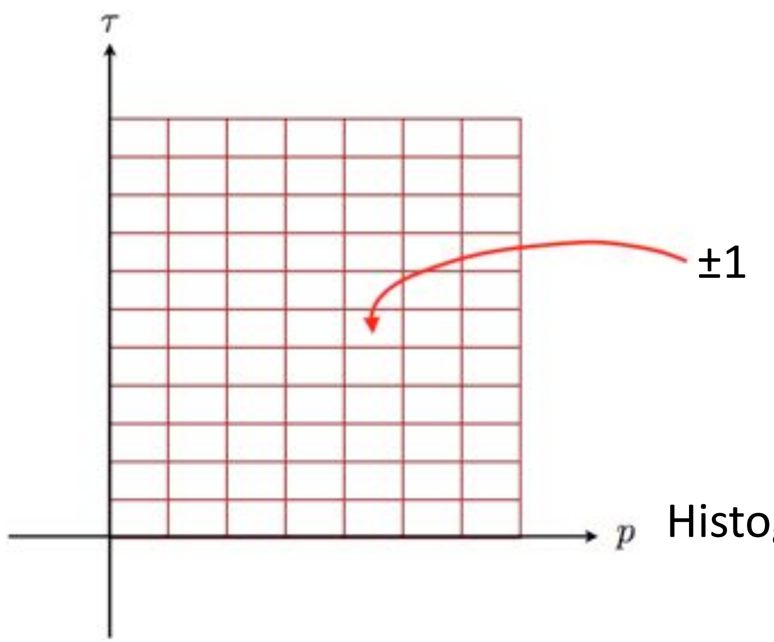
This is NOT: write diagram after diagram, compute its value, sum

Diagrammatic Monte Carlo:

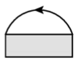
Importance sampling of the space of all possible diagram topologies and all values of internal and external variables.

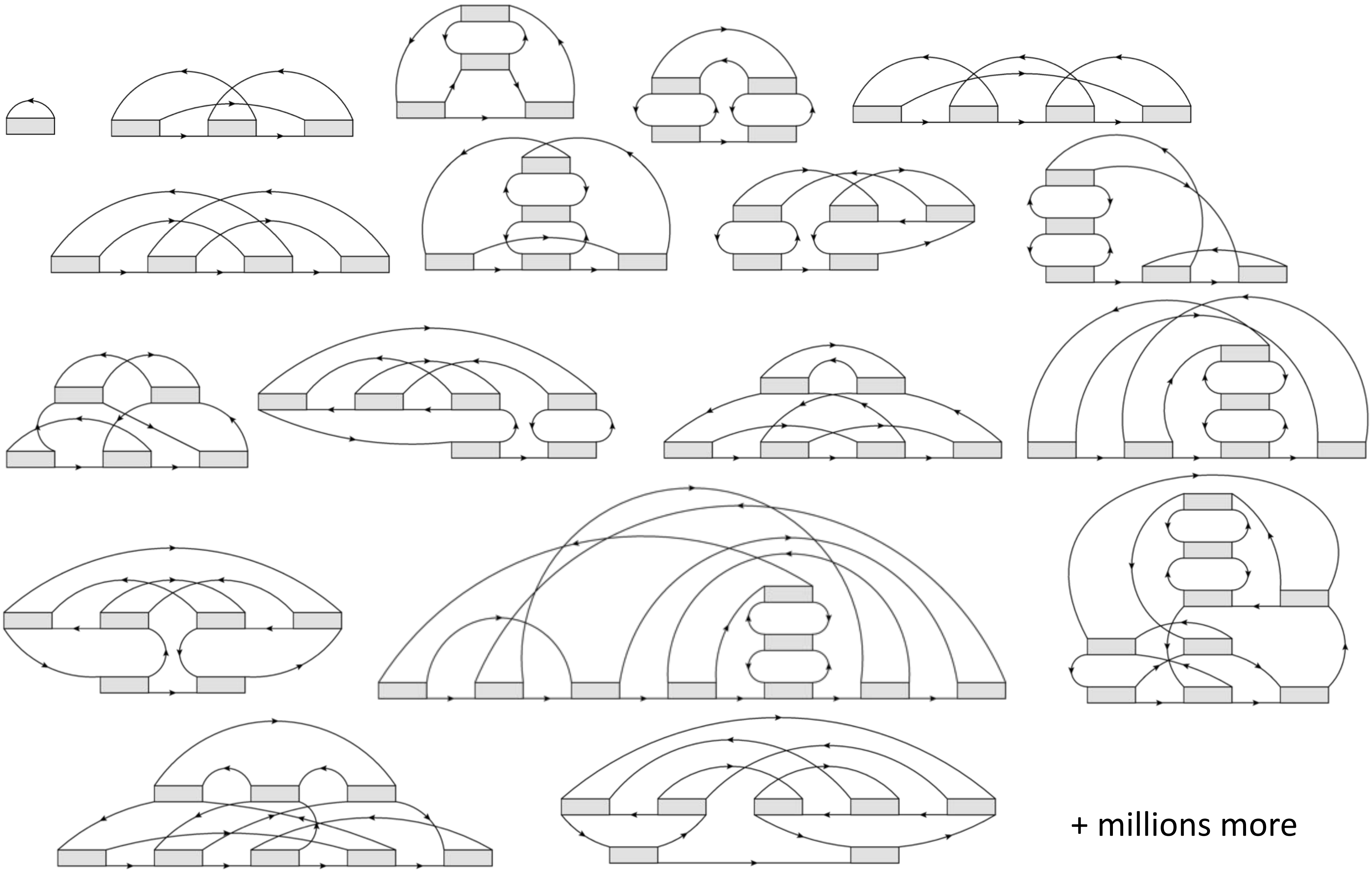
Each configuration is visited with a probability proportional to the absolute value of its contribution to $\Sigma_\sigma(\mathbf{p},\tau)$ as given by the Feynman rules.

After each MC update:



Histogram for $\Sigma_\sigma(\mathbf{p},\tau)$

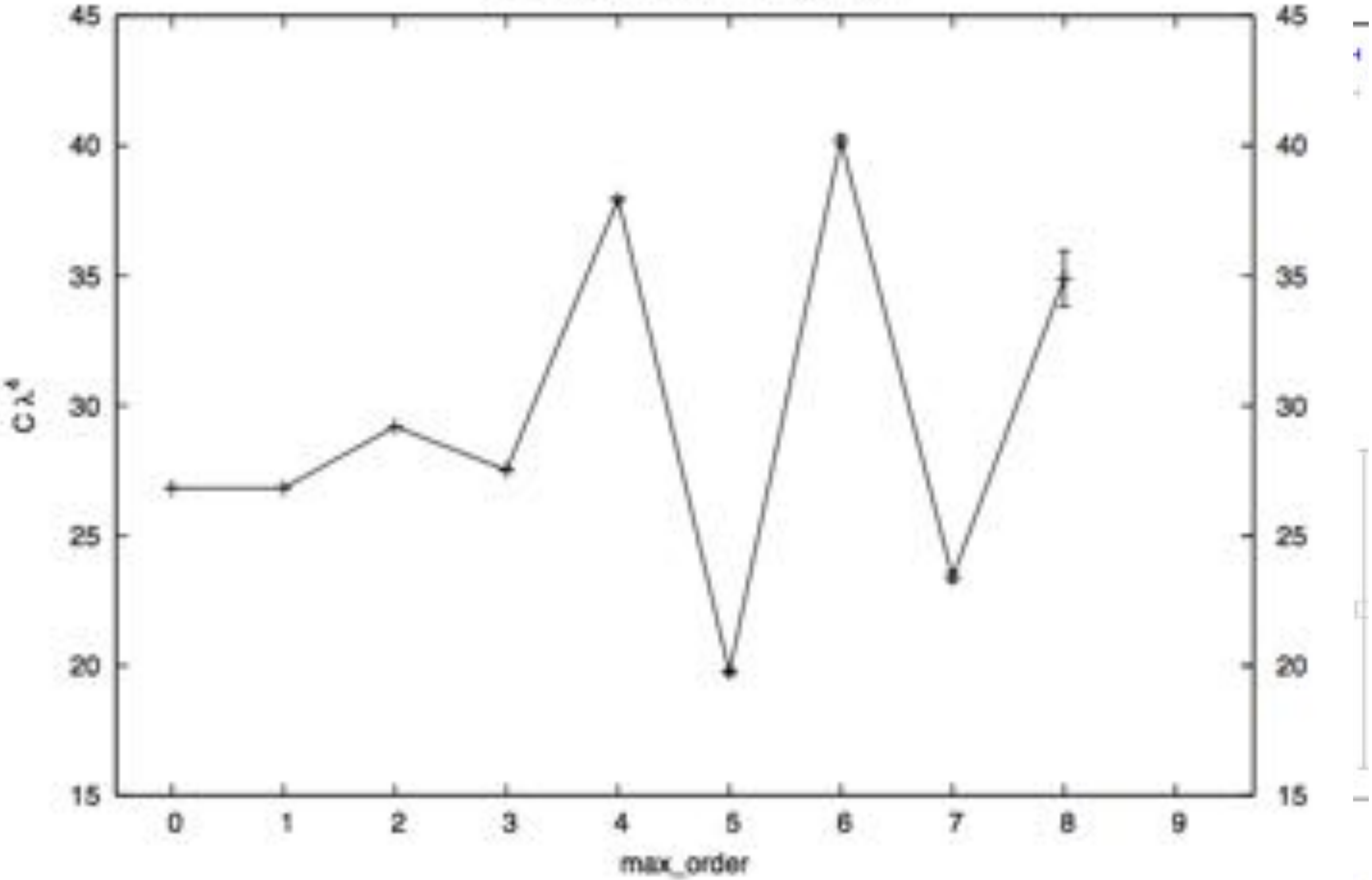



+ millions more

Problem: series divergence

Contact $\mathcal{C} := \lim_{k\to\infty} n(k)k^4 = -\Gamma(\mathbf{r}=\mathbf{0}, \tau = 0^-)$

$\beta\mu = 0$ ($T/T_F = 0.6$), Ladder scheme



$$Q \hat{=} \sum_{N=0}^{\infty} a_N \quad \textbf{diverges}$$

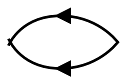
ladder scheme
or
bold scheme

$$\text{construct } Q(z) \ / \ \begin{cases} Q(z) \underset{z\to 0^+}{=} \sum_{N=0}^{M} a_N\, z^N + O(z^{M+1}) \\ Q(z=1) = Q_{\text{phys}} \end{cases}$$

$$Q(z) = \frac{T}{\mathcal{V}} \ln\left\{ \# \int \mathcal{D}\varphi\, \mathcal{D}\eta\ e^{-S^{(z)}[\varphi,\eta]} \right\}$$

ladder scheme:

$$S^{(z)}[\varphi,\eta] = -\int dX \sum_{\sigma=\uparrow,\downarrow} \bar{\varphi}_\sigma(X)(G_0^{-1}\varphi_\sigma)(X) - \int dX \bar{\eta}(X)(\Gamma_0^{-1}\eta)(X)$$
$$+ \sqrt{z}\int dX \Big[(\bar{\eta}\varphi_\downarrow\varphi_\uparrow)(X) + (\bar{\varphi}_\uparrow\bar{\varphi}_\downarrow\eta)(X)\Big] - \underbrace{z\int dX \bar{\eta}(X)(\Pi_0\eta)(X)}_{\text{counterterm}}$$

$\Pi_0 =$ 

$$Q \hat{=} \sum_{N=0}^{\infty} a_N \quad \textbf{diverges}$$

ladder scheme
or
bold scheme

$$\text{construct } Q(z) \ / \ \begin{cases} Q(z) \underset{z\to 0^+}{=} \sum_{N=0}^{M} a_N \, z^N + O(z^{M+1}) \\ Q(z=1) = Q_{\text{phys}} \end{cases}$$

$$Q \hat{=} \sum_{N=0}^{\infty} a_N \quad \textbf{diverges}$$

ladder scheme
or
bold scheme

construct $Q(z)$ / $\begin{cases} Q(z) \underset{z\to 0^+}{=} \sum_{N=0}^{M} a_N z^N + O(z^{M+1}) \\ Q(z=1) = Q_{\text{phys}} \end{cases}$

$$\{a_N\} \xrightarrow{?} Q(z=1) = Q_{\text{phys}}$$

- *naive approach* [Van Houcke *et al.*, Nature Phys. 2012]: assuming $|a_N| < C^N$

$$Q(z=1) = \lim_{\epsilon \to 0^+} \sum_{N=0}^{\infty} a_N \, e^{-\epsilon N \ln N} \qquad \text{(Lindelöf)}$$

- *serious approach* [Rossi, Ohgoe, KVH, Werner, PRL 2018.]
  - $a_N \sim (N!)^{1/5}$
  - series resummable by conformal-Borel

compute the ***sum of all diagrams of order n***
in the ***large n*** *limit*

using the ***functional integral*** *representation*
and the ***saddle-point method***

Bosonic field theories (ϕ^4):
Lipatov
Brézin, Le Guillou, Zinn-Justin

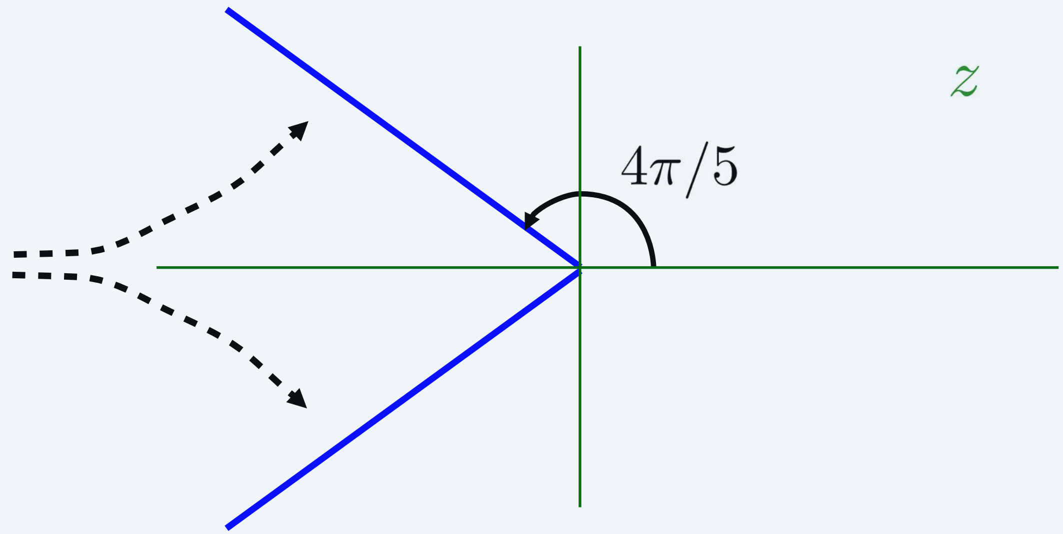
Fermionic theories (QED):
Parisi, Itzykson, Zuber, Balian

$$Q(z) = \int \mathcal{D}\eta \underbrace{\int \mathcal{D}\varphi \; e^{-S^{(z)}[\eta,\varphi]}}_{e^{-S_B^{(z)}[\eta]}}$$

quasi-local approximation for $|\eta| \to \infty, \;\; z \to 0$

$$\frac{\delta S_B^{(z)}[\eta]}{\delta \eta} = 0$$ instanton

$$\text{Disc } Q(z) \underset{|z|\to 0}{\sim} \exp\left[-\left(\frac{A}{|z|}\right)^5\right]$$



$$a_N \underset{N\to\infty}{\sim} (N/5)! \;\; A^{-N} \;\; \cos\left(\frac{4\pi}{5}N\right)$$

Conformal Borel transformation

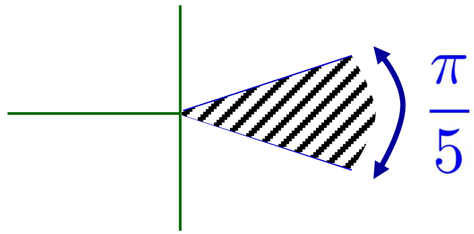
$$a_N \underset{N\to\infty}{\sim} (N/5)!\; A^{-N} \cos\left(\frac{4\pi}{5}N\right)$$

$$\text{Borel transform}: \quad B(z) := \sum_{N=0}^{\infty} \frac{a_N}{(N/5)!}\, z^N \qquad |z| < A$$

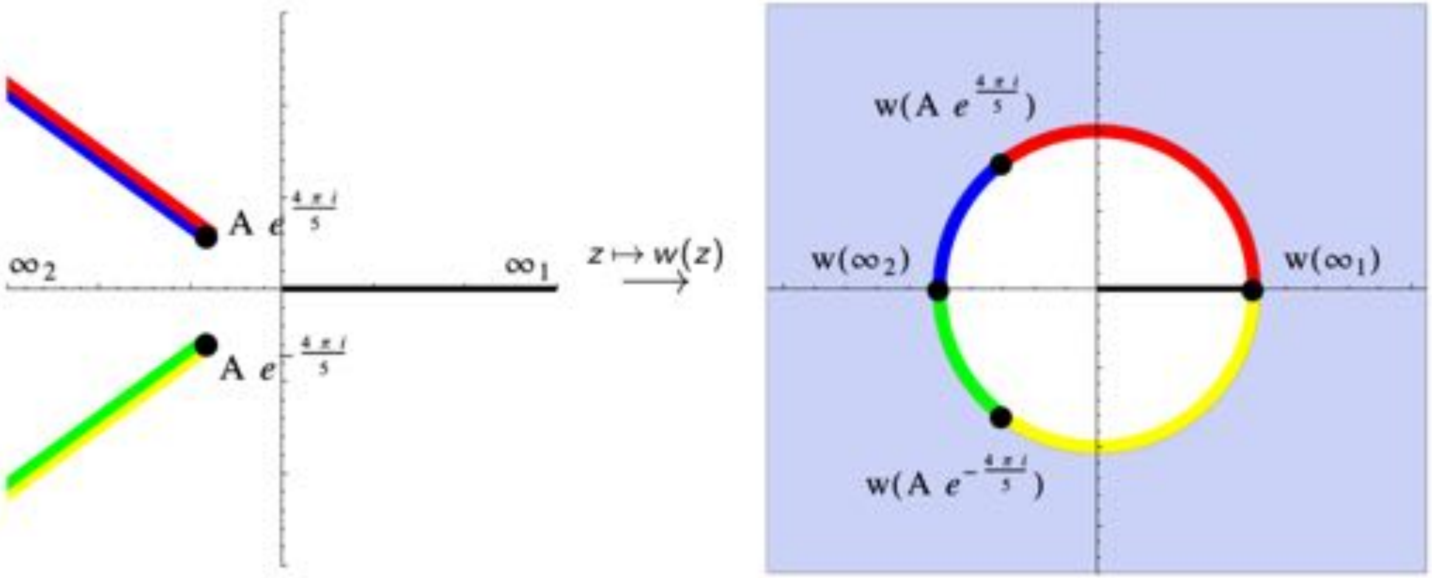
$$\text{inverse Borel transform}: \quad Q(1) \stackrel{?}{=} \int_0^\infty dz\, z^4\, e^{-z^5}\, B(z)$$

Yes, because [Ramis theorem]:

- $Q(z)$ analytic in
- $\frac{1}{N!}\left|\frac{d^N Q(z)}{dz^N}\right| \lesssim (N/5)!$ in



conformal mapping
$\int_0^\infty dz = \int_0^1 dw$

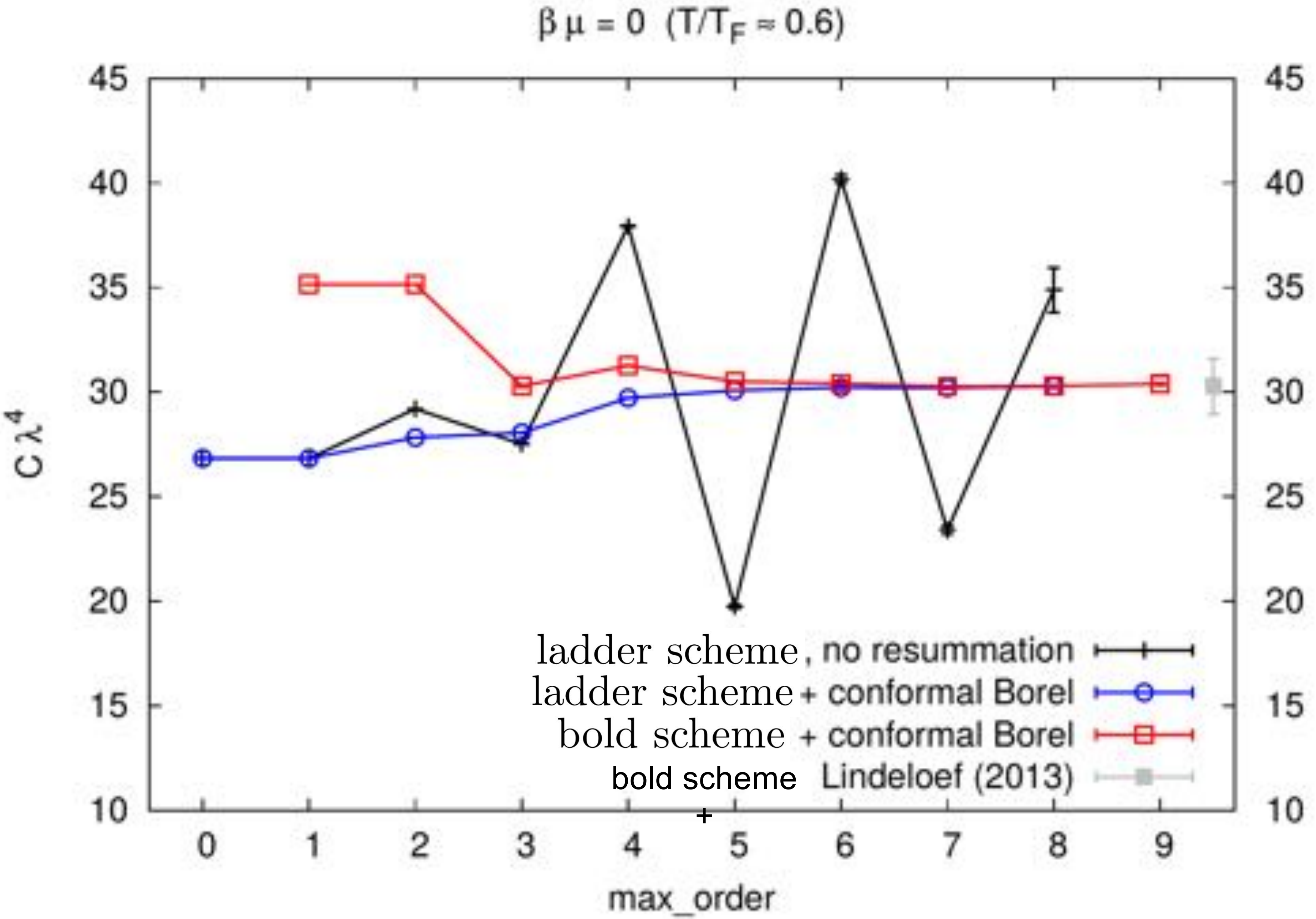


applicable not only for $Q = P$

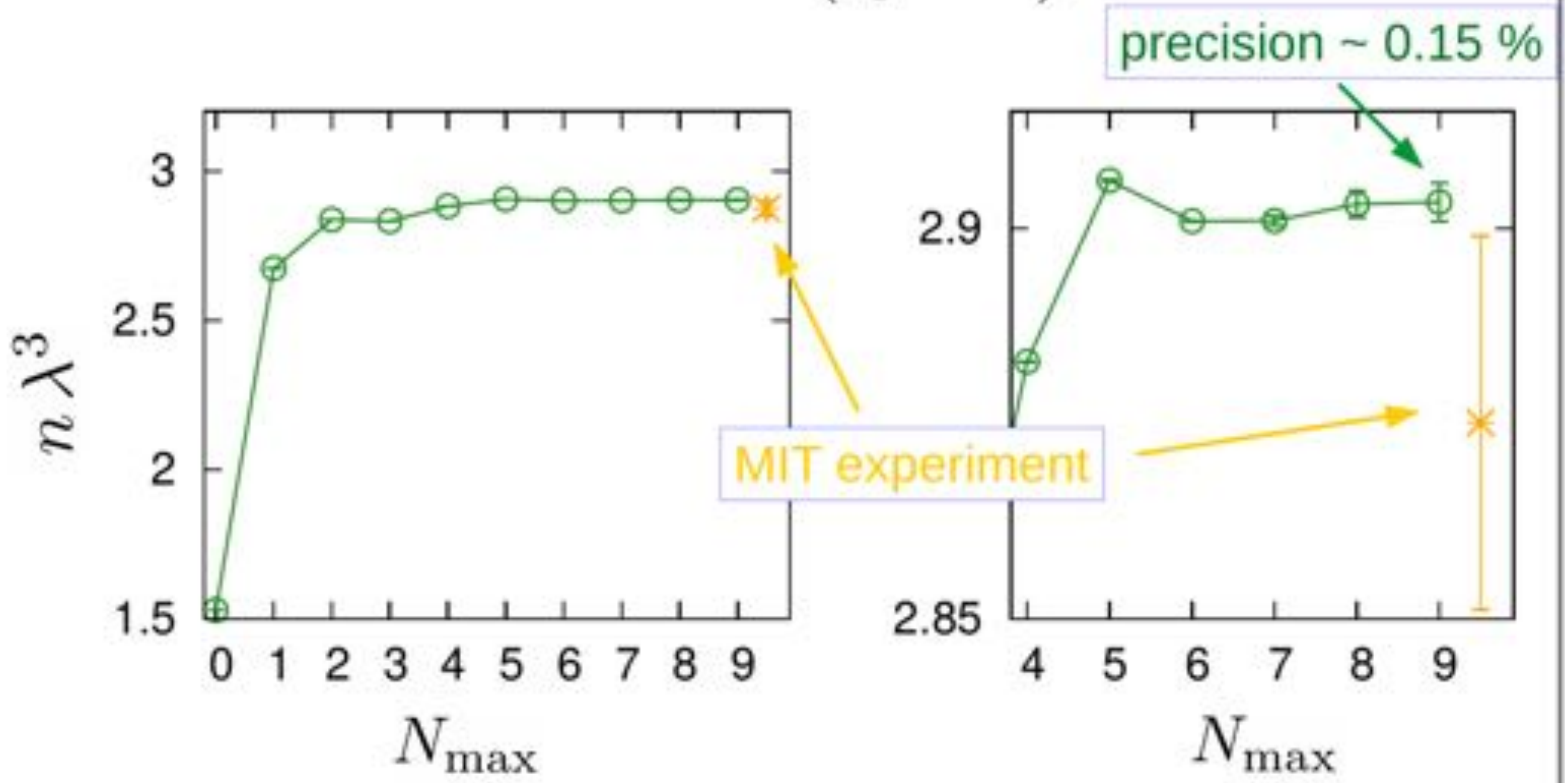
$$\text{but also}\quad Q = \begin{cases} \mathrm{G} \text{ or } \Gamma & \text{(ladder scheme)} \\ \Sigma \text{ or } \Pi & \end{cases}$$

Contact $\quad \mathcal{C} := \lim_{k\to\infty} n(k)k^4 = -\Gamma(\mathbf{r}=\mathbf{0}, \tau = 0^-)$

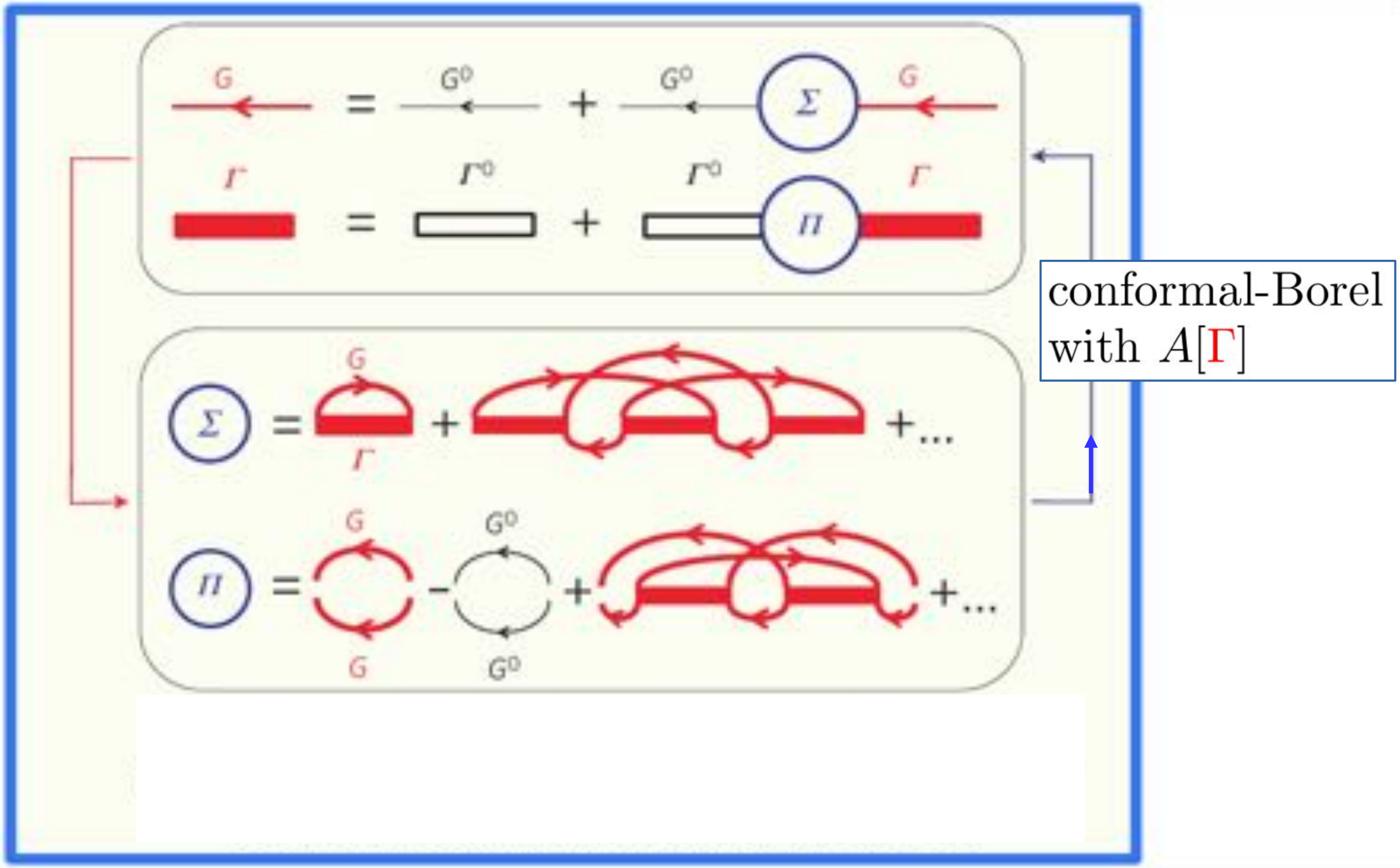
$$\langle\, \hat{n}_\uparrow(\mathbf{r})\, \hat{n}_\downarrow(\mathbf{0})\,\rangle \underset{r\to 0}{\sim} \frac{\mathcal{C}}{(4\pi\, r)^2}$$

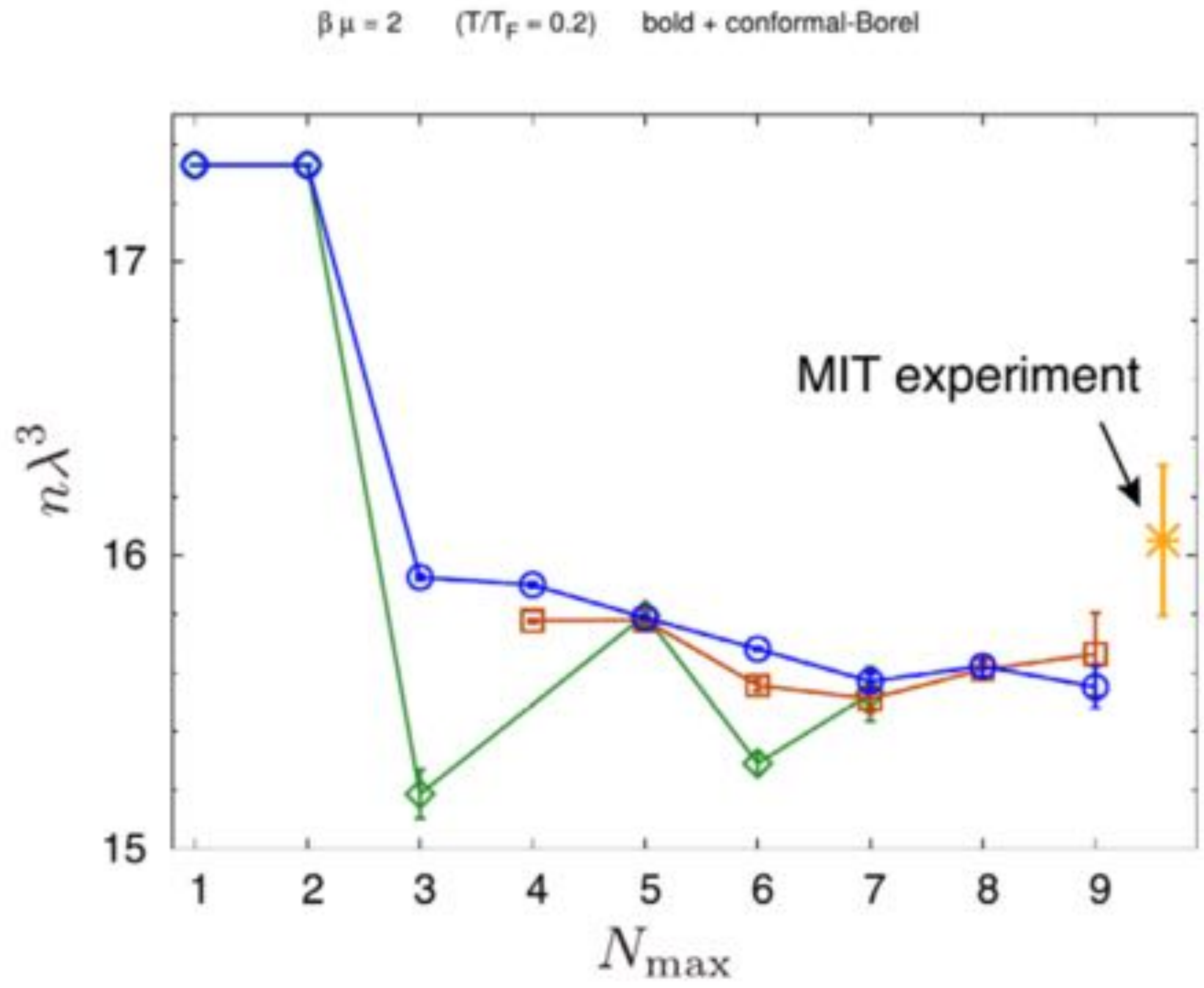


$\mu = 0 \qquad \left(\frac{T}{T_F} \approx 0.6\right)$

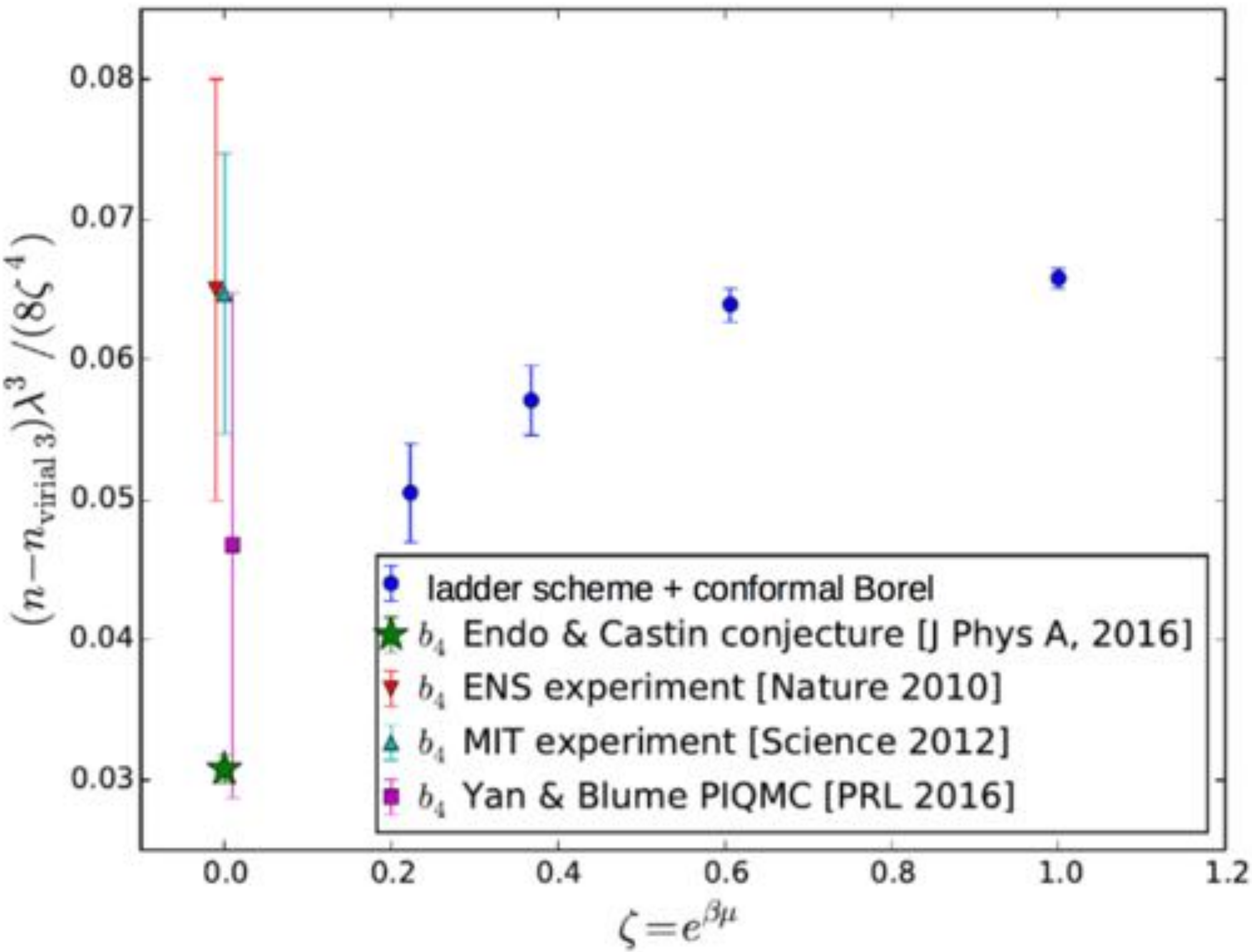


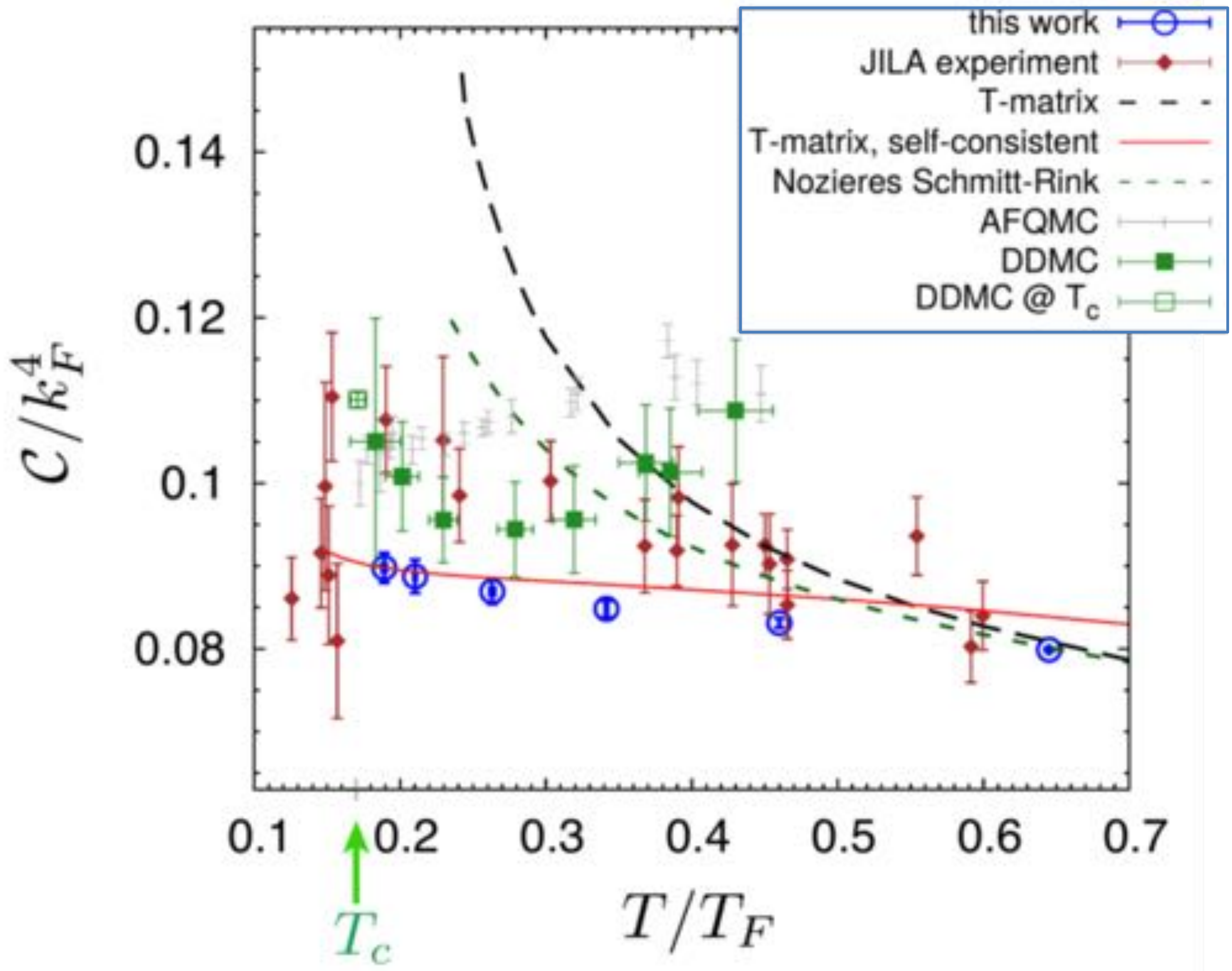
bold scheme



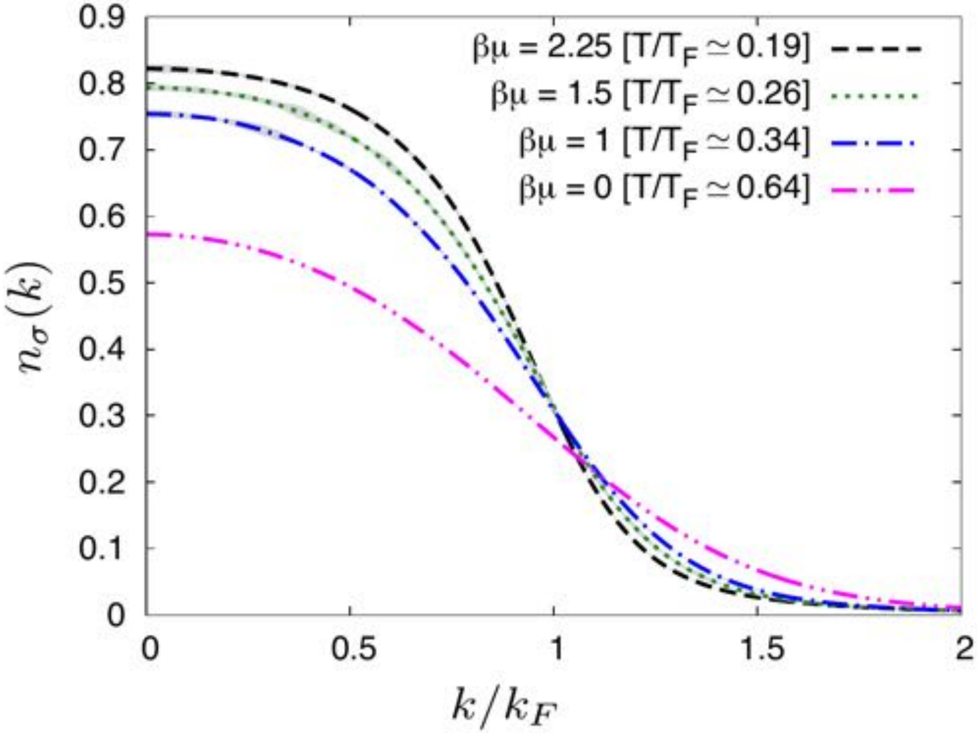

β μ = 2 (T/T_F = 0.2) bold + conformal-Borel
MIT experiment
17
16
15
1
2
3
4
5
6
7
8
9
nλ^3
N_max

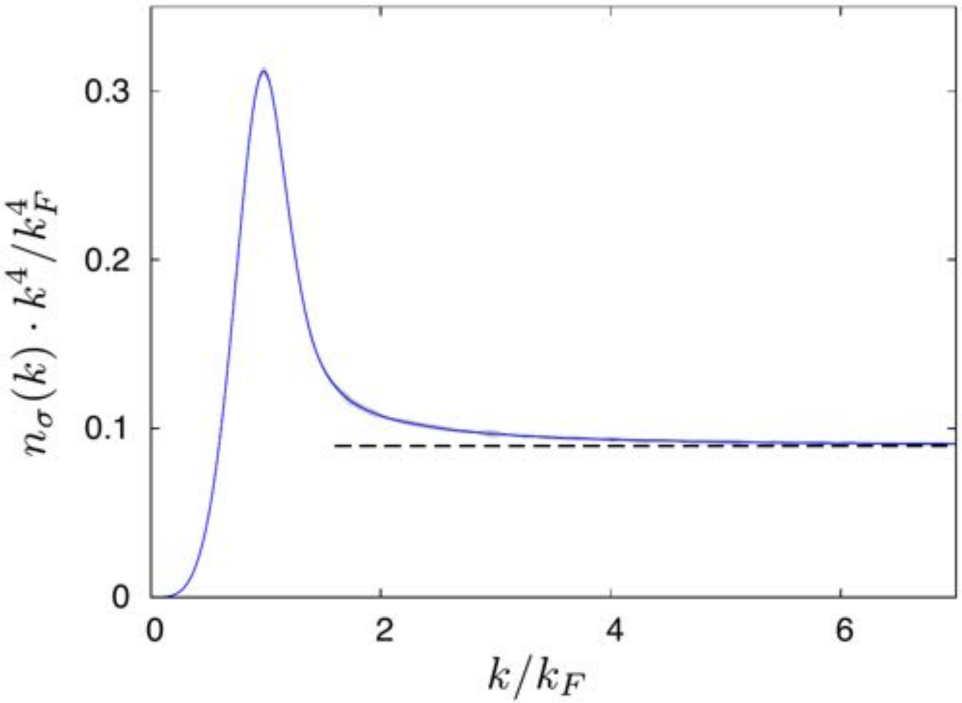
Equation of state and 4th virial coefficient

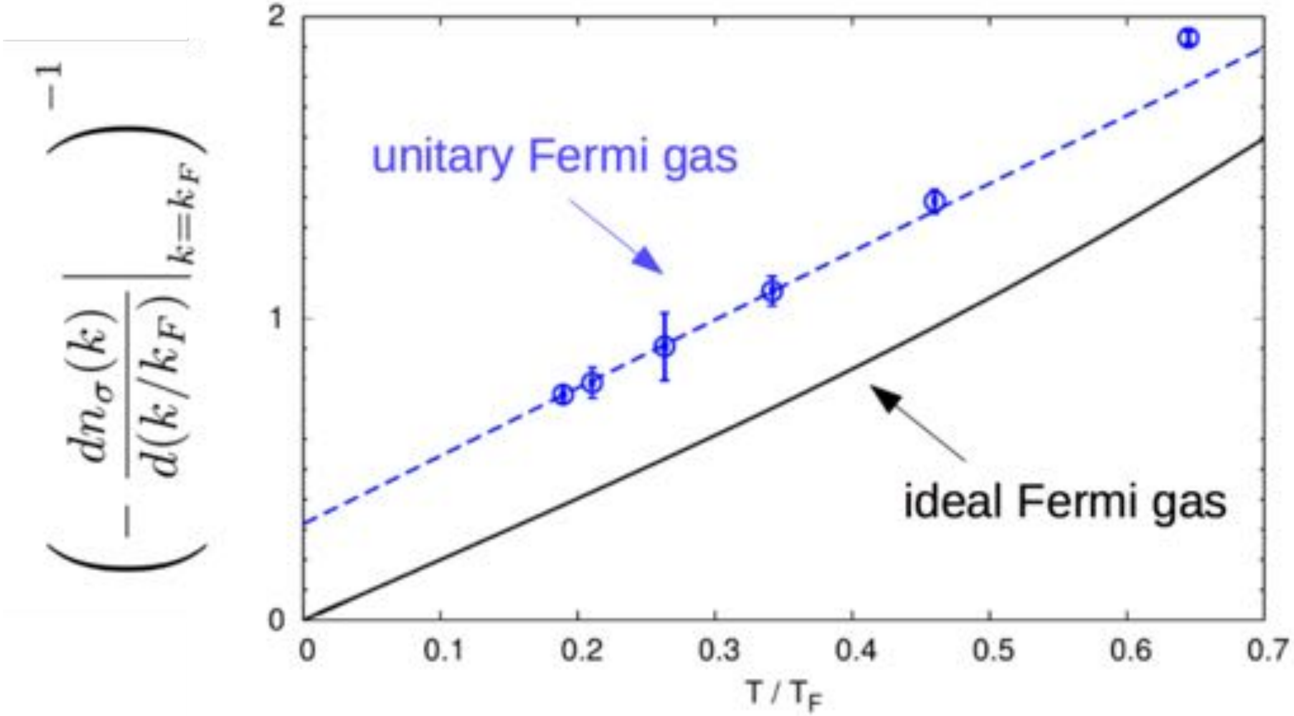




this work
JILA experiment
T-matrix
T-matrix, self-consistent
Nozieres Schmitt-Rink
AFQMC
DDMC
DDMC @ T_c
C/k_F^4
0.14
0.12
0.1
0.08
0.1
0.2
0.3
0.4
0.5
0.6
0.7
T/T_F
T_c



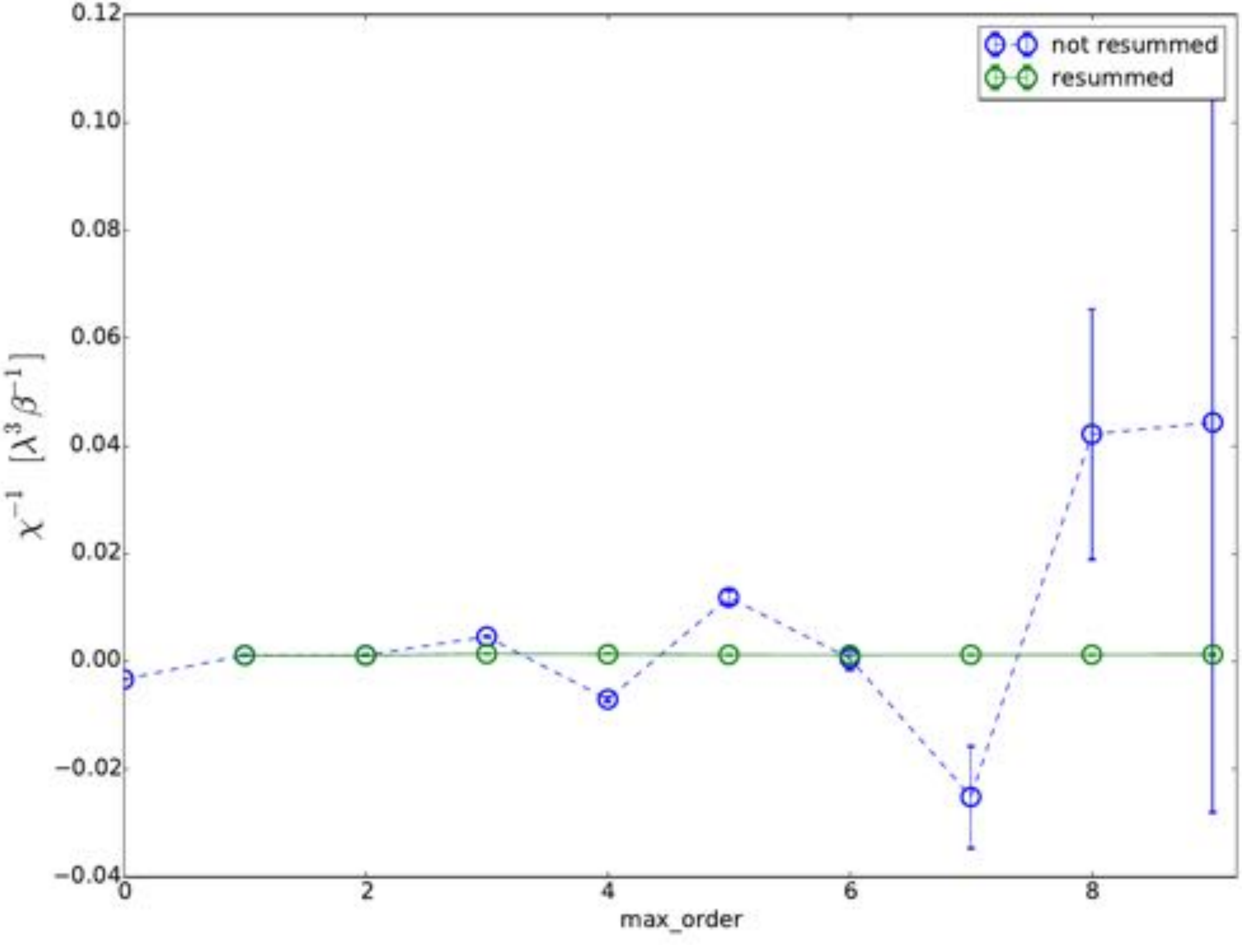




non
Fermi liquid
behavior

Order Parameter Susceptibility $\quad \chi = -\Gamma(\mathbf{p} = \mathbf{0}, \Omega_n = 0)$

$\beta\mu = +2 \quad (T/T_F \approx 0.2)$, bold series



conclusion:
controlled summation of diagrammatic series
with zero convergence radius
for a strongly correlated fermionic field theory

Outlook:

Precise and controlled theoretical study
of unitary Fermi gas and BEC-BCS crossover
Spectral function ⇒ Fermi liquid / pseudogap regimes
T_c vs. polarisation and scattering length
2D
4-leg diagrams ⇒ p-wave superfluid transition, transport properties (shear viscosity)
superfluid phases (anomalous propagators)

other fermionic models in continuous-space (electron gas) ?

new MC algorithms → higher orders?

$$a_n = \int dX \sum_{\mathcal{T} \in \mathcal{S}_n} \mathcal{D}(\mathcal{T}, X)$$

R. Rossi, PRL 2017.