Lattice MC calculations of unitary fermions: *odd-even staggering*

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Phys. Rev. A84 (2011) 043644, Endres, Kaplan, JWL, **Nicholson**

Phys. Rev. Lett. 107 (2011) 201601, Endres, Kaplan, JWL, Nicholson

Phys. Rev. A87 (2013) 023615, Endres, Kaplan, JWL, **Nicholson**

Working in progress (2018), JWL

August 6, 2018 @ Seattle, USA

Bosenova *• Cold atom experiments* m experiments

• Approximation to low-density nuclear matter

Neutron superfluid in crust

² (14)

Cⁱ (⌧) (18)

 $k_F a \sim -10$, $a \sim -7r_e$

Almost unitary fermi gas

N^c

P Fermions at unitarity

from Eq. ?? at large times, as was evident in Fig. ??? at large times, as was evident in Fig. ????????????????

Chronology of the Bertsch parameter finite temperature is the post of the property of \mathbf{r}

temperature. References $\mathcal{S}(\mathcal{S})$ and $\mathcal{S}(\mathcal{S})$ and $\mathcal{S}(\mathcal{S})$ and $\mathcal{S}(\mathcal{S})$ quoted the Bertsch

developed for studying large numbers of strongly interacting

methods used in the past in that it does not make use of

schemes in that one may use the ensemble generated to

reliably estimate all desired observables. Thus our approach

nonrelativistic spin- ¹

The Bertsch parameter is approaching 0.37 at a few percent level ! $\mathbf{F}_{\mathbf{1}}$ re Bertsch parameter is approaching 0.37 at a jew perce estimating a single observable for which it was designed. Our \emph{vel} !

Table VI; our value is indicated as the latest simulation data point. In the latest simulation data point. In

Pairing gap from cold atom experiments

NATURE PHYSICS DOI: 10.1038/NPHYS4187 LETTERS

dotted lines indicate the onset of the single-particle branch for ¯*h*!2 (*E*^F =12.1kHz). Inset: comparison of experimental and theoretical BA mode peaks.

nary BCS gap equation has a value close to the binding

energy of the paired fermions. This is new information

regarding the relation between thermodynamic quanti-

ties and the binding energy in the BCS region close to

the unitarity limit. Furthermore, such as experiments, α

ments involving the measurement of the Higgs mode of

the order parameter, will reveal the magnitude of the or-

der parameter as well as the relation between the order

ner, W. Ketterle, J. Thomas, T. E. Drake, and D. Jin for

providing their experimental data, R. Haussmann, W.

Zwerger, H. Hu, G. C. Strinati, A. Gezerlis, S. Gandolfi,

and J. Carlson for providing their theoretical calcula-

tions, and W. Zwerger for valuable discussions about the

critical behavior of contact density around the superfluid

transition point. MH would like to thank Y. Aratake for

assistance in conducting the experiments. However, \mathcal{A}

would like to thank P. van Wyk, R. Hanai, D. Kagami-

parameter and the binding energy.

BCS-MF calculation [44]. The green triangle and the green

Pairing gap from numerical simulations **Pairing gap from numerical simulations** = 0*.*504(30) (3)

Add one to fully-paired system

Any recent update from numerical simulations?

correlator.

process. The correlator has a nearly Log-Normal (LN) distribution and a long tail with

Outline

- 1) Model
- 2) Systematics
	- I. Unitarity limit and discretization/finite volume effects
	- II. Interpolating field overlap
	- II. Statistical overlap/noise
	- II. Thermodynamic limit
- 3) Numerical results
- 4) Summary and Conclusions

Outline

1) Model

2) Systematics

I. Unitarity limit and discretization/finite volume effects

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4) Summary and Conclusions

 $\frac{1}{2}$ the assionless to the $C_0(u) \equiv \mathcal{L} \mathcal{L} C_0(u)$ and take usual definition of the *^A*¹ ⁼ *C*⁰ \overrightarrow{CD} of \overrightarrow{R} \overrightarrow{H} \overrightarrow{H} \overrightarrow{C} <u>triviaĉ, 43, fixedĉ, point, and</u> **Cy Cation + because of the Contract Control** et *µ dpt@ken*
and theory $\hat{\beta}$. (2015.10) $\partial \phi$ $\sin \phi$; $\frac{4\pi}{\sqrt{2\pi}}$ $\frac{4\pi}{\sqrt{2\pi}}$ $\frac{4\pi}{\sqrt{2\pi}}$ $\frac{4\pi}{\sqrt{2\pi}}$ $\frac{4\pi}{\sqrt{2\pi}}$ **Fined by** $\frac{4\pi}{\sqrt{2\pi}}$ **00 PLAN** add
8. H *dµ .* (1.28) ⁰ ⇤ *µ* ⁰ = *C*2 ⁰ = 0 *.* (1.29) ⁰ = *C*2 ⁰ = 0 *.* (1.29) *E/*2 + *q*⁰ q2*/*2*M* + *i*⇥ *E/*2 *q*⁰ q2*/*2*M* + *i*⇥ *Mµ C*² *Mµ C*² *^Iⁿ* ⁼ *i*(*µ/*2)4*^D* $\frac{1}{4}$ **M** $\frac{1}{4}$ **CO** $\frac{1}{4}$ **C** $\frac{1}{4}$ This -function become zero when *C*⁰ = 0, a trivial IR fixed point, and *a* ⌅ ⇧, a **Ansee Rome** Unit $\frac{1}{2}$ + $\frac{1}{2}$ + $\frac{1}{2}$ + $\frac{1}{2}$ + $\frac{1}{2}$ + $\frac{1}{2}$ $\theta_{\mu} = \theta_{\mu}$ ($\text{Tr} \text{Pvi}$ a) $\hat{\theta}_{0}$ fixed $\hat{\theta}_{0}$ point, and $a = \text{Hom}$ and $\hat{\theta}_{0} = -\mu C_{0}$ *M I* definitional
definition contractive theory on lattice. The main features at $\frac{\partial \mathcal{L}_0}{\partial t}$ t**i**ce **dute 1 Fhe main features of our** This -function become zero when *C*⁰ = 0, a trivial IR fixed point, and *a* ⌅ ⇧, a Function of $\frac{4\pi\omega_1 m}{4\pi}$ when $4\pi\omega_1 M$ - Chis S-function become zero when **Externait** $\frac{1}{5}$ $\frac{1}{2}$ \mathcal{L} *d* (*p*) \mathcal{L} (*p*) \mathcal{L} (*p*) \mathcal{L} = $-\mu C_0$ ting non-relativistic fermions, leads for us to develop a new fattice r
Fi*ttle* sewidatuge and properties definition of the p-funging **potential currical currical pursus on tatule.** I he mant learnies \mathcal{Y}_ν is $\frac{1}{2}$ Θ *n*=0 *P*₁, *C***₂***n***₁, ***C*₂*n*₁, *C₂<i>n*₁, *C*₂*n*₁, *C*₂*n*₁, *C*₂*n*₁, *C* μ E_duch de loop integral diagram in the bubble diagram in Fig. 1.1 and for μ in Fig. $\widetilde{\mathbf{G}}$ $\widetilde{\mathbf{B}}$ $\widetilde{\mathbf{B}}$ (2⌅)*^D* ^q2*ⁿ* SIRMSH@H&LWISTJC.TGKIMIOAS.H&JQIS. D \overline{D} \overline{D} (ЛЮТИОН)
D a new **131-1311** P *i* ideen ot ነ⊧
₩≼ ³ *^D* ⇥ (*µ/*2)4*^D* 4π **Low** 4π **and** $M - \epsilon$ **the Remix function focus** And a systematic approach to studyease temps make the state of the value of the control of \mathbb{C}^3 $\frac{HCH}{VCH}$ to $\frac{HCH}{VCH}$ $\begin{bmatrix} \text{Detter} & \text{unust} \end{bmatrix}$ $\frac{1}{2}$ *N*(μ *)* and take µµ
|入 $\mathrm{d}\widehat{\text{e}}$ n $\widehat{\text{e}}$ $\frac{1}{2}$ *C*⇤⁰ 1 ⇥ *^C*⁰ ⇥ *Mµ* t er $\begin{bmatrix} \mathbf{a} & \text{nonnegative} \ \mathbf{a} & T\beta\mathbf{x} \neq \mathcal{U} \mathbf{B} \mathbf{u} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \ \mathbf{a} & \mathbf{c} & \mathbf{c} \end{bmatrix}$ a $\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{c} \ \mathbf{a} & \mathbf{c} & \mathbf{c} & \mathbf{c} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{c}$ \overline{U} in \overline{U} is a small \overline{U} tan'i Germandie 4π β_0 $\overline{4\pi a/M}$ $\lim_{\Omega} \operatorname{zero}$ when $C_0 = \mathbb{R}_{\mathbb{R}} = \mathbb{R}$ frivial $\lim_{\Omega} \operatorname{fixed}$ fried $\lim_{\Omega} \lim_{\Omega}$, and a_{Ω} . $\max_{\{1,1\}}\max_{\{1,2\}}\max_{\{1,3\}}\min_{\{1,4\}}\min_{\{1,5\}}\min_{\{1,6\}}\min_{\{1,7\}}\min_{\{1,8\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}\min_{\{1,9\}}$ the <u>lo</u>w-relativistic regnions,leads for us to develop a new fattice i
The main features of the main features of the p-functio www.urg.
teracting non-*fe*lativistic fermion, Meads for inster detengoment that is gisted. Fermi contact interaction we also facilities and the praise field field to the actures of our $\overline{0}$. $\overline{1}$ ˆ ⌅*C* \bigcup $\frac{47}{1}$ <u>ivA</u> \overline{CD} $/M - 1$ $\dot{\mathbb{H}}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\sqrt{\frac{1}{2}}$ $\frac{1}{2}$ $\mathcal{C}_0(\mu)$ and take us he dengt der and a systematic fermions to study system $\frac{4\pi}{4}$ the latting numeration is the lattice of the main and socion formi**n features of the main features of our lattice** Francisco can by studence a temperature and a can be a auxiliary field of the inde $\frac{1}{2}$ ₀ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ ⌅*µ* $\frac{4}{7}$ *M* **1.4 A New Lattice Approach to starting the Starting Approach** $\mu_{\text{F}} = \mu_{\text{C}} \mu_{\text{F}}$ and a series to star definition of the μ_{F} study μ_{F} The low energy nuclear theory of the contract of our lattice. are followed by We prepare a $\tilde{T} \times L^3$ Euclidente and for a trip terminative and
srapppoach rost udkesvstings motels and state atexting as the formulation PLUIT COMPARE INTERNATION CREATES RELEASE TREPPERED & CONTRATT RELEASE $\frac{1}{\text{ess}}$ *C*₀(μ) = $\mathcal{L}G(\mu)$ and take usus definition of the $T_{\text{max}} = \frac{4\pi}{4}$ = $T_{\text{max}} = 0,$ and $T_{\text{max}} = 0,$ and $T_{\text{max}} = 0,$ and $T_{\text{max}} = 0,$ ne zero when $C_0 = \mathbb{R}$, $=$ if \mathbb{R} is \mathbb{R} for \mathbb{R} in \mathbb{R} is \mathbb{R} and a better illustration of \mathbb{R} flow, dimensional sine a new $C_0(u) = -u$ $\mathcal{C}_0(\mu)$ and take usual definition of the β -function. This \mathbf{C} indy definition of \mathbf{q}_1 and \mathbf{q}_2 and \mathbf{q}_3 and \mathbf{q}_4 and \mathbf{q}_5 and \mathbf{q}_7 and \mathbf{q}_8 and \mathbf{q}_7 and \mathbf{q}_8 and |
| g
ji Ω_{ξ} $\frac{4\pi}{4\pi}$ \int ist ∕
St positing non-relativistic fermion *Meads* for inster aleting non-weat the isolated $\frac{4\pi}{100}$ $\frac{80}{4\pi} \frac{4\pi a}{M}$ $\frac{1}{200}$ $\frac{1}{4\pi a}$ $\frac{1}{M}$ $\frac{1}{2\pi}$ $\frac{1$ P iv **A**e **P P R O axed** point rive d \ ˆ ⁰(*µ*) ⇤ *µC*0(*µ*) and take usual definition of the -function. This non-trivial UV fixed point. For a better illustration of RG flow, we may define a new better illustration of RG flow, we may not earning $C_0(\mu) \equiv -\mu C_0(\mu)$ \cdot \cdot \overrightarrow{G} \det_{Ω} 、
CW . ⌅*µ* $\bigoplus\limits_{i=1}^{n}$ ˆ ice
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ungtio $\frac{1}{\sqrt{2}}$ *M* 4⇥*a/M C*⁰ α^{11} 6g (Anti-Lail CILCUIVE picol y on laughc. $\partial \overline{\Omega}$ ˆ ergy puelear effective theory on lattice. The \mathbf{A} ting non-relativistic fermions, leads for **t**
d *i* PA = Vew *Catul*e and program is ual de $|t^{\prime}_{\rm e}|$ β $\overline{\beta}$ $\frac{1}{6} \oint \frac{3}{\mu}$ ueraction can purey $\overline{0}$ Q_{μ} Q_{μ} Q_{μ} Q_{μ} Q_{μ} Q_{μ} Q_{μ} Q_{μ} Q_{μ} Q_{μ} \vec{H} 11 Cent_a $\hat{\mathbf{\theta}}$ $a \overrightarrow{T} \overrightarrow{\partial} \overrightarrow{a} \overrightarrow{a} \overrightarrow{b} \overrightarrow{b} \overrightarrow{c} \overrightarrow{d} \overrightarrow{d} \overrightarrow{b} \overrightarrow{d} \overrightarrow{b} \overrightarrow{c} \overrightarrow{d} \overrightarrow{d} \overrightarrow{d} \overrightarrow{b} \overrightarrow{d} \overrightarrow{d} \overrightarrow{b} \overrightarrow{d} \overrightarrow{$ Teads for us to develop a new lattide mother the theory, which said the UH fixed \int_{0}^{1} **MP** io $\left(\frac{29}{2} \right)^{40}$ \overline{on} \mathbf{m} e zero w *.* (1.29) **EPPROFICE pointrivial UV** fixed apoint. For a be FPETTIMPATIVE and 3 SVETEMMLG approach to SURIY
sual definition of test 3-function. This ^C T ー
・ ⌅*µ* = *C* ˆ n teatur
for to stu g energy puclear effective theory on lattice. The main features ϕ_{μ} str $\mathbf{\hat{u}}$ $\frac{1}{\tau}$ $\frac{1}{2}$ Γ **1974 A Proprime Boy 10 Conce** anteraction interactivity and completed to develop the excession of the excession of the excession of the excession of the station o the low called the low equipment of the materials of our main when $\frac{c}{c}$ $\Delta \mu$ \equiv \neq μ *C*₀(μ) and take usuga definition of the μ $\begin{bmatrix} 0 & 1 \end{bmatrix}$ $\sqrt{M - E}$ **W** 4⇥ 4⇥*a/M C*⁰ **F**-WC
11S achato starke The active - a computer when the computation become $\beta_0 = 0, \alpha_1$ $\frac{1}{2}$ $\frac{4\pi}{M}$ **C**₀ $\frac{4\pi}{M}$ **C**₀ $\frac{4\pi}{M}$ and $\frac{4\pi}{M}$ and the $\frac{4\pi}{M}$ and the $\frac{4\pi}{M}$ $\frac{1}{2}$ $\frac{1}{2}$ **ADALIER ACUICE SAT DICOGAI** ⌅*µ* eegs wee \mathbf{IX}_{\cdot} **Francisco** *1.4 A New Lattice Approxy A New Yorks Application Application Application Application Application Application* The demand of the demand of the demand and a strongly systematic and a systematic approach to strongly systems of strongly systems of strongly systematic approximation of strongly systematic approximation of strongly syste μ) $\equiv -\mu$ C₀ \vert
a to study dimensionless coupling *C* i 10a
GMT *M* This Conduction Can be completed by intrivial process and $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{$ $\frac{1}{4}$ PODOOACIV TAULICOUGWIOHCDE PERTURBAND FOR CHIC ADOOPY OPPHATHE OU ecome zero when α *Tero* when $C_0 = M = 3$ *Trivia* C_0 *in* $R = 3$ (C_0 point, and a *unions, leads for develop a new fattice A* uC111
11Ct1 $\frac{4\pi}{4}$ Man Altice and assign erdettbackty be denfer $\tilde{\mathbf{z}}$ **DEUTHRY ITHET OCTUSHY IS UC ATENTIFIES HEACLS ACAITA iVALIDAT** α beint, and α -**The South Beautiful Service Company of Service Point Company IR for an and a** trial and and a now energy muclear effective theory on attice, The main features of our contact interaction can be addeved by introducing a auxiliary field to the man of the second with a second second with a second second with a s ل
14 d *1 p* = ut the coupling coupling of the will be a 1 post party of the set of the set of the set of the set of the method of the set of the method of the coupling of the set of the method of the method of the method of th $\overline{\mathbf{f}}$ μ = Δ ² μ ¹ (μ) $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ **们** AenVew Lattice in profecta pointriana $W \rightarrow$ fixed apoint. For a be T_{SUSC} increased of T_{SUSC} is relevant to unitary C_{U} *dC*⁰ ting non-relativistic fermions leads for us to develop a new lattice A \overline{a} $\rm ic$ armi $\lim_{\epsilon \to 0} \frac{M}{2}$ eads fo \int $\frac{1049044}{100}$ $\partial \hat{W}$ *µ*
⊥1. → $\frac{1}{2}$ ³
W la ¹ *^M* -44 *C*ˆ0 宴 **. (1.30) Band and Alger Band And Algeria and Algeria Algeria Algeria and Algeria and Algeria Algeria and Algeri 1**
0.000 ⌅*µ* 4⇥ mand for a non-perturbative and a systematic approach to study. ∂g wing: We **Reeterinea** *Tox 1* a Flundidean cattlee and assign fermionic fermion dimensionless coupling *C* \sum $\mathcal{O}(\mu) \equiv \mathcal{G} \mathcal{H}^C(\mu)$ and take usual definition of the \mathcal{G} gives **AP** $\hat{\beta}$ $\bigcup_{\mathcal{P}} \equiv \partial_{\mathcal{U}}$ ⌅*C* ˆ $\overline{\mathbf{p}}$ **Mp** H_0C $\hat{\bigcirc}$ $\overline{0}$ ¹ *^M* 4跏 **C** $\hat{\bigcirc}$ $\overline{\mathrm{Q}}$ $\sqrt{4}$ $\lim_{a\to a} \frac{\text{fixed}}{\text{argmax}}$ (i.g. and a -**A LATTE ALTER AND ACHILICATE** he demând for a non-perturbative and a systematic approach to study systematic teracting non-relativistic fermions leads for insteraction properties is to develop a new data time is to deve the low energy ner active theory of control of the main features of our a following? (1) We prepare a $T\times L^3$ Euclidean lattice tor a thore perturbative and
approach to stu**d resustence refusing privi**stic farmions feads tor he to $\lim_{\alpha \to 0} \frac{1}{\alpha} \sum_{i=1}^n \frac{1}{\alpha} \$ $I_{\rm s}4$ $A_{\rm o}$ $N_{\rm F}$ $N_{\rm F}$ $N_{\rm F}$ $N_{\rm F}$ $I_{\rm s}$ $I_{\$ T he demand for a non-perturbative and $T_{\rm eff}$ the low theory teatures of our distinction for the main \mathcal{L} **THRINOTS FEARS TORTUP ITAL**
THRINOLOGY DIGIORIC AND ITAL FILCOTYC UICOPY-OIP TAU DICC. $\overline{{\cal B}_{0}^{=-\pm}}$ *Mby* $\frac{1}{4}\pi$ $\overline{C_{0}^2}$ C_0 <u>ර්</u> $4\pi q/M - C_0$ *.*
8-function become zero when nand-ter a non-pertuiteative and a systematic approach to study.
Cau Land fake usual definition of ter 3-function. "Bhis" *gus Alger the MCAttige And take usual definition of the* β *-function* $\bigotimes\limits_{\Delta \mathrm{V}}$ $\hat{\mathcal{B}}_{0}$ **Ree eig** ∂C relativistic fermions Meads for Θ $\partial \mathcal{H}$ **EXTERNAL** ˆ <u>।</u>
। rmions
1**1 didaen**
1967 - Hagr 细声 **C** $\hat{\mathfrak{g}}$ ttlce and assign fermionf**enicle**s
In lattice : The main features of our *1.4 A New Lattice Approach* \Box This R-function become zero when dimensionless coupling *C* .★
) ($\overline{\text{Detter}}$ illustration of RG flow, $\overline{\text{dip}}$ angion $\overline{\text{dip}}$ completing $\overline{C}_0(\mu) \equiv -\mu \overline{C}_0(\mu)$ $\mathcal{C}_0(\mu)$ and take usual definition of test β -function. This β $\widetilde{\beta}$ θ ¹ $\overline{\text{Tr}}$ es \bigoplus $\overline{0}$ $\partial \psi$ 4π **1944 Oli Adultica, Filicandal de Negotia (d. 1956)** ↑ Level Aπ
△ L3 Luclidean attice and assign fermionic neighed leads for us to develop a new lattice mothod by adopting the latter action *Mµ* $\frac{1}{\pi}$ $\mathcal{C}^\mathbf{A}\mathbf{S}^\mathbf{I} \mathbf{O} \mathbf{I} \ \mathbf{S}^\mathbf{A} \overline{\mathbf{S}^\mathbf{A}}$ $\frac{1}{2}$ $\frac{4\pi a}{4\pi}$ $\frac{4\pi}{6}$ $\mathcal{L}_0(\mu) = \mathcal{L}_0(\mathcal{L}_0(\mu))$ and vane (1.29) **C** ˆ 0 ena $\hat{\partial}$ $\boldsymbol{\theta}$ $\begin{minipage}{0.4\textwidth} \begin{picture}(1,0) \label{fig:2} \end{picture} \begin{picture}(1,0) \label{fig:2} \end{picture} \begin{picture}(1,0) \label{fig:2} \end{picture}$ 4π \mathfrak{C} $\hat{\widehat{\mathbb{D}}}$ $\boldsymbol{0}$ ⇥ erturbative and a systematic appl EV(μ) \mathbb{E} and \mathbb{E} field \mathbb{E} the cory on lattice. The main definition of the usual definition of the usual development of the usual definition of the unit of the usual definition of the last of the second the main features of our contact interaction can be eachleved by introven ₹₹ **195 4 G**
1 *A* Can dena
2 Can de 12 **ALS** 1d tor a thon-perturbative and
Photossigh fermionic never lea In Fig. 1.2, we draw this beta function with RG flows. The IR fixed point at *C*ˆ⁰ = 0 corresponds to a trivial non-interacting theory, while the UV fixed point at *C*ˆ⁰ = 4⇤*/M* Fine-ting non-felativistic fermion Fine-ds for inster aleting manitatives **Free** $\underline{\mathcal{H}}$ intensionless \mathcal{H} \mathcal{H} $\underline{\mathcal{H}}$ $C_0(\mu) \equiv \mathcal{H}$ $C_0(\mu)$ and take usual definition $\zeta_{\text{res}}^{\text{max}}$ \sim $\frac{1}{4\pi}$ $\frac{1}{2\pi}$ $\frac{1}{$ 100+pertumba **10W, COUNTERVIERENT CHAPTING CON** t ting non-relativistic fermic *M ^µ* + 1*/a*⇥ neory on lat<u>tice</u>. <u>I ne main</u> i <u>₽ ₩₩ ₽</u>
U **dd**
Der t interaction can be eath \mathfrak{X}_q **A**
MILG **Garce CORTAIL CALLES AND ASSISTED TELLE** T a I β κ \pm γ Ethelear and becomes T λ I 3 4π , and I and a for a trivial perimeter $\operatorname{graph}(G)$ Gall_G is a better in Gyl_G in Gyl_G of Gyl_G a Gyl_G in Gyl_G BLOT CALL OG atentewege, M. Prinsege

interacting non-relativistic fermions leads for us to develop a new lattice method by adopting

ⁿ(*ME ⁱ*⇥)

the low energy nuclear energy nuclear energy nuclear energy on Ω

ˆ

 \mathbf{r} are following:

(*D*3)*/*2

Calilean interaction for non-relativistic fermions The starting point for our construction is a highly improved variant of the non-experimental $\mathcal{L}_\mathcal{D}$ The derivative operator appearing in Eq. 1 represents a backward dierence operator in

• Interaction & Lattice action fermions in another. Thus, boosted pairs of particles would see an interaction which did not

iton
J. -W. Chen, D. B. Kaplan (2004)

. (1)

, ⌅⇤) with equal mass *M*

$$
S = b_{\tau} b_s^3 \sum_{\tau, \mathbf{x}} \left[\bar{\psi}_{\mathbf{x}, \tau} (\partial_{\tau} \psi)_{\mathbf{x}, \tau} - \frac{1}{2M} \bar{\psi}_{\mathbf{x}, \tau} (\nabla^2 \psi)_{\mathbf{x}, \tau} + \underbrace{(\sqrt{C} \phi)_{\mathbf{x}, \tau} \bar{\psi}_{\mathbf{x}, \tau} \psi_{\mathbf{x}, \tau-1}}_{\text{Interacting out}}
$$

poses no challenge in a numerical simulation of Eq. 1.

choice of boundary conditions forbids the introduction of a chemical potential and as a

 $T \times L^3$ Euclidean Lattice and spatial discretization errors. The spatial discretization errors. The spatial discretization \sim boundary conditions in the time direction with time labeled by integers \overline{a} 0 *D X*(*T* 2) 0 *...* 0 L³

, (4)

and periodic boundary conditions in the spatial directions with positions \mathcal{L}_max

0 0 *D X*(*T* 3) *...* 0

between the ingoing and outgoing fermion momenta. This is important, since tuning a

Integrating out

choice of boundary conditions forbids the introduction of a chemical potential and as a

1, however in some sections in some space of the lattice space of the latti

$$
\langle \phi_{{\bf x},\tau} \rangle = 0 \;, \qquad \langle \phi_{{\bf x},\tau} \phi_{{\bf x}',\tau'} \rangle = \delta_{{\bf x},{\bf x}'} \delta_{\tau,\tau'} \qquad \qquad
$$

 \Box (2) Open b.C. in this and periodic b.C. \Box source (2) Open B.C. in time and periodic B.C. in space \mathcal{L} distribution in space \mathcal{L} and \mathcal{L} and \mathcal{L} are \mathcal{L} and \mathcal{L}

5

x and *a* R ², *R*₂*, in*², *R*₂*, the <i>j*₂*, temporature* conditions, *a n*², 3. As a *n*², 3. As a *n*² Restricted to zero temperature

where the matrix *K* is given by *^C*⇥)x*,* (⇤⇤¯)x*,* ⇤ (⇤⇤¯)x*,* (*C*⇤⇤¯)x*, ,* (3) matrix *K* are given in block-matrix form by: $\overline{}$ *D X*(*T* 1) 0 0 *...* 0 **Lattice construction for non-relativistic fermions**

• *Fermion matrix*
\n
$$
S = \bar{\psi} K \psi \qquad K = \begin{pmatrix}\nD & -X(T-1) & 0 & 0 & \dots & 0 \\
0 & D & -X(T-2) & 0 & \dots & 0 \\
0 & 0 & D & -X(T-3) & \dots & 0 \\
0 & 0 & 0 & D & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & -X(0) \\
0 & 0 & 0 & 0 & \dots & D\n\end{pmatrix}
$$
\nOpen B.C. in time
\n
$$
D = 1 - \frac{\nabla^2}{2M}, \qquad X(\tau) = 1 - \sqrt{C}\Phi(\tau)
$$
\ndet K is independent of the auxiliary field

auxiliary field . auxiliary field

²*^M , X*()=1 ⌅

. .

^D = 1 ⇧²

diagonal blocks, det *K* = (det *D*)*^T*which is independent of the auxiliary field. Therefore the

A consequence of the open boundary condition is that the matrix *K* is an upper triandiagonal matrix of the Quenched simulation & sign free *K*2 *Department of the probability measure*
Duanahad aimulation 8 aign free

We may express Eq. 1 succinctly as *S = ∴*
In time components of the time components of the fermion of the fermion of the fermion of the fermion of the f

⇤¯*K,*+1⇤+1

• Propagator (K^{-1}) & Transfer matrix (T) *-1 • Propagator (K) & Transfer matrix (T)*

sampling is necessary.

vanish automatically.

for a $\frac{1}{2}$ independent measure substantially simplified. The full numerical cal simulation of E \sim 2.1, which is equivalent to \sim 1, \sim 1 $p = \frac{1}{2} \left(\frac{1}{2} \right)^{N-1/2}$ \mathcal{L} block matrix, its determinant is given by the product of determinants of determinants of determinants of \mathcal{L} $K^{-1}(\tau; 0) = D^{-1/2} T^{\tau} D^{-1/2} \qquad T = D^{-1/2} X(\tau) D^{-1/2}$

C() *.* (5)

More practical and powerful way of extractions the ground state energy is to define a genergy is to define a g $$ $\overline{}$ **nergy** *C*() **Measurement of the ground state energy** result is Nconf ≫ e3 400 minutes and 400 minutes
The Conf ≫ e3 400 minutes and 400 minutes an Measurement of the ground state energy

of configurations required for a given value of α and α and α and α and α and α and α . The α

$$
C_{N_{\psi},N_{\uparrow}}(\tau) = Z_0 e^{-E_0 \tau} + Z_1 e^{-E_1 \tau} + Z_2 e^{-E_2 \tau} + ... \qquad \tau
$$
 is the Euclidean time.

 $m_{eff}(\tau) = \frac{1}{\Delta}$ $\Delta \tau$ $\log \left[\frac{C(\tau')}{C(\tau)} \right]$ $C(\tau)$ $m_{eff}(\tau) = \frac{1}{\tau} \log \left[\frac{C(\tau')}{\gamma} \right]$ im $m_{eff}(\tau) = E_0$ \overline{P} \sim 32 in Fig. 32 in Fig. 32 in Fig. 19 obtained the \overline{P} $p_{\text{ref}}(t) = \Delta \tau^{\text{log}} \left[C(\tau) \right]$ $\lim_{\tau \to \infty} m_{\text{eff}}(\tau) = E_0$

alized equal to the control of the contro

alized e⇥ective mass by

$$
\lim_{\tau \to \infty} m_{eff}(\tau) = E_0
$$

. The *mef f* () satisfies *energy of ground state* The traditional technique for avoiding difficulties associ-

Dashed line: Y. Castin et. al. (2007) sum in Eq. (C3) may be expressed as

, (3.3)

cumulant expansion

 \mathbf{r} = \mathbf{r}

normal-like distribution may be devised by considering the

 $\mathcal{L}(\mathcal{L})$ is the n-th cumulant of the distribution for the distribution for the distribution for $\mathcal{L}(\mathcal{L})$

ln C α

In this expansion, systematic uncertainties associated with the

 \mathbf{t}

Remarks

- 1) Canonical approaches on an Euclidean space-time lattice
- 2) Zero temperature (open b. c.)
- 3) No trapping potentials

4) Ground state energies of N_{up} = N_{down} and N_{up} +1= N_{down} unitary fermions

5) Numerical results for $N_{up}+1=N_{down}$ unitary fermions are very preliminary (single volume $V=16^3$, selected values of N).

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vemer advantage of multiple correlations of \mathcal{A} **Improvement: single particle sector**

advantage of multiple correlations α multiple correlations α of α and α of α

simultaneous fit

advantage of multiple correlations $\mathcal{O}(\mathcal{A})$

advantage of multiple correlations $\mathcal{O}(\mathcal{A})$

simultaneous fit

simultaneous fit

$$
D = 1 - \frac{\nabla^2}{2M}
$$

\n
$$
T = D^{-1} \text{ and } E = \ln D
$$

\n**Standard:**
$$
1 + \frac{2\sin^2(\mathbf{p}/2)}{M}
$$

\n
$$
E = \frac{\mathbf{p}^2}{2M}(1 + O(b_s^2))
$$

 $F_{\rm eff}$ is simultaneous fit of three correlation function function function function function function μ

^s) (C.5)

^M (C.6)

We choose *D* to have the form in the form in the form in model with $\frac{1}{2}$ Separable interaction and FFT algorithm *T* = *D*¹ (C.4) *• Perfect:*

where *p^j* = 2⇤*mj/L* for integers *m^j* ⌃ [*L/*2*, L/*2 1] and *j* = 1*,* 2*,* 3. The parameter

= ⇤ ⇥ (1 10⁵) is a hard momentum cuto⇥ imposed on the fermions; a small shift away

from 下from " has been introduced introduced in the cuto_n introduced inclusion of momenta lying on momenta lying
★ has been introduced in order to avoid inclusion of momenta lying on the cuton of momenta lying on the cut

$$
\begin{cases} e^{\mathbf{p}^2/(2M)} & |\mathbf{p}| < \Lambda \\ \infty & |\mathbf{p}| \ge \Lambda \end{cases} \qquad E = \frac{\mathbf{p}^2}{2M}
$$

C always comes in pairs of two. This property is generally true for any *N*-60 **Tuning: four-fermi interaction**

 $\sqrt{a^2 + b^2}$ Galilean-invariant interaction

with unknown coeºcients C_2 to be determined from scattering data. Our choice of basis C_1

*,*p⇥ ^q

*^O*2*n*(p) = *^Mⁿ*

 $\mathcal{F}_{\mathcal{A}}$, and the three-dimensional $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$ -functional $\mathcal{F}_{\mathcal{A}}$ -functional $\mathcal{F}_{\mathcal{A}}$

Critical to a numerical simulation is the ability to tune the

 $\mathcal{F}_{\mathcal{A}}$, and the three-dimensional $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$ -functional $\mathcal{F}_{\mathcal{A}}$ -functional $\mathcal{F}_{\mathcal{A}}$

Critical to a numerical simulation is the ability to tune the

P Realization of unitarity
(Fig.) Two-body interactions are induced by the periodic field by the periodic field by the periodic field by the periodic field η

a three-momentum pj $\mathcal{L}_\mathcal{A}$ is 2π $L_\mathcal{A}$, where $\mathcal{L}_\mathcal{A}$ is $L_\mathcal{A}$ is $L_\mathcal{A}$ is $L_\mathcal{A}$ if $L_\mathcal{A}$ is $L_\mathcal{A}$ if $L_\mathcal{A}$ is $L_\mathcal{A}$ if $L_\mathcal{A}$ is $L_\mathcal{A}$ if $L_\mathcal{A}$ is $L_\mathcal{A}$ i

for a periodic spatial lattice (assuming even L).

due to short-range interactions, then α is analytic interactions, then α is analytic in p 2 at an alytic in p 2

FIG. 2: Left: implied *p* cot ⁰ obtained from exact lattice eigenvalues of (using *L* = 32 and *M* = 5) - For a few low-lying energy eigenvalues systematically tune the operators $\mathcal{O}_{2n}(\mathbf{p})$ to reproduce $\begin{array}{ccc} \hline \ \hline \ \hline \ \hline \ \end{array}$

 $p\cot\delta_0=0$ for sufficiently small p

found that we are close to unitarity even heyond - Found that we are close to unitarity even beyond the exactly tuned states. - Found that we are close to unitarity even beyond the exactly tune found that we are elose to unitarity even beyond the exactly tank - Found that we are close to unitarity ϵ

- For non-zero net CoM is the tuning affected by the hard momentum cut-Gilbreth & Jensen in private conversation off for the single particle?

Gilbreth & Jensen in private conversation Γ for pop gare not CoM is the tuning ϵ every **P** and **r** α and α are the α

analyticity, at least up to the α 30th shell, at which point momenta are very close to the α

 $\overline{\mathcal{O}}$ the lattice, the energy eigenvalues are defined from $\overline{\mathcal{O}}$ the energy eigenvalues are defined from $\overline{\mathcal{O}}$

Beyond two-body sector both statistical and systematic errors. Black dashed lines indicate the error band obtained from *E*⌧ ! \overline{e} "
" \overline{O} **Beyond two-body sector**

fermions are not restricted to a specific angular momentum state. The subleading volume

represent fit results to *N^O* = 1 and *N^O* = 5 data as discussed in the text, with error bands reflecting

• Improvement in 2-body sector (S-wave): $1/L^9$ with 5 operators tuned σ -hody sector (S-waye): $1/I^9$ with r operators tuped • Improvement in 2-body sector $(S\text{-wave})$: $1/L^2$ with 5 operators tuned *M^N M*⇡*t* = 8 (29) *M^N M*⇡*t* = 8 (29)

 $f_{\text{N=4}}$ are not restricted to a specific angular momentum state. The subset of \sim

wave and d-wave two-body operators, respectively. By considering the leading *L*-dependence

- No improvement in 2-body sector (P-wave): $(1/L^3)$ $1/L^3$) of 1*/L*3. Blue data points and associated error bars were obtained from numerical simulation, $\left(\begin{array}{ccc} 1 & 1 & 1 \end{array} \right)$ is a contracted to L/L include statistical and fitting systematic errors combined in quadrature. The blue and yellow bands represent fit results to *N^O* = 1 and *N^O* = 5 data as discussed in the text, with error bands reflecting $\overline{\text{rc}}$ **Let** *No* improvement in 2-body sector (P-wave): $(1/I^3)$ *M^N* 2*M^N* 3*M^N* 4*M^N* (30) *M^N* + 3*M*⇡ 3*M*⇡ 6*M*⇡ (31)
- $N=4$ $N=3$ three fermion transfer matrix. Red error band in \mathcal{N} in \mathcal{N} in \mathcal{N} in \mathcal{N}

II. EXACT RESULTS FOR FEW ARTICLES FOR FEW

volume result of Pricoupenko and Castin reported in [57].

• Contribution of 3-body operators :
$$
\underbrace{\left(L^{-3.55}\right)}_{\text{S. Tan (2004)}}\left(\ell=1\right) \quad L^{-4.33}\left(\ell=0\right)
$$

dependence is therefore expected to scale as *L*3*.*⁵⁵. Additional subleading terms scale as

 $N-A$

N=4 N=3

a many-body correlator which is generated from stochastic process. As shown in process. As shown in previous α

state energy at *L* = 1, we therefore used *c*⁰ + *c*1*/L* as our fit function for the extrapolation.

for the ground state energy of four unitary fermions are consistent with the benchmark

at infinite volume we obtain *E/EF ree* = 0*.*2130(26). Both our *N^O* = 1 and *N^O* = 5 results

case for three fermions, however, the lowest dimension three-fermion operator is expected

ⁱ 2 [1*,* 1] (23)

both statistical and systematic errors. Black dashed lines indicate the error band obtained from

represent fit results to *N^O* = 1 and *N^O* = 5 data as discussed in the text, with error bands reflecting

both statistical and systematic errors. Black dashed lines indicate the error band obtained from

fermions are not restricted to a specific angular momentum state. The subleading volume

 $d = 3$

of these operators, we use the fit function: *c*⁰ + *c*1*/L*³ + *c*2*/L*³*.*⁵⁵ to extrapolate the energy

and infinite volume extrapolation of exact benchmark calculation of exact benchmark calculations reported in $[61.6, 1]$

ⁱ 2 [1*,* 1] (23)

|
|
|

M^N 2*M^N* 3*M^N* 4*M^N* (30)

M^N + 3*M*⇡ 3*M*⇡ 6*M*⇡ (31)

section, the correlator has a nearly Log-Normal (LN) distribution and a long tail with

II. EXACT RESULTS FOR FEW PARTIES FOR FEW PARTIES.

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I. Unitarity limit and discretization/finite volume effects

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Slater-determinants. Thus correlation functions of *N* = *N*⇥ + *N*⇤ fermions may be expressed

ponentially like that of free fermions with a Z-factor near unity. A better approach is to

⁸ Higher order time discretizations errors may be achieved with with the use of higher order decompositions

of operator exponentials. For interacting fermions, this would require additional auxiliary fields at each

the species. In order to satisfy Fermi-Dirac statistics, fermions of the same species must

to impose the proper anti-symmetrization requirements on \mathcal{C} is to use on \mathcal{C} is to use of u

have dierent quantum numbers. As is well-known from quantum mechanics, a simple way

at early times, where few interactions have occurred, the correlations have occurred, the correlation function

ponentially like that of free fermions with a Z-factor near unity. A better approach is to

of operator exponentials. For interacting fermions, this would require additional auxiliary fields at each

⁸ Higher order time discretizations errors may be achieved with with the use of higher order decompositions

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time-like link, however, such constructions are beyond the scope of this work.

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• Slater-Determinant to take account of Fermi-Dirac Statistics \overline{C} \overline{C}

 \overline{N}

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• Slater-Determinant to take account of Fermi-Dirac Statistics \overline{C} \overline{C}

 \overline{N}

Solution Setup Constrainers and a set of the body correlators of the Improvement: N-body correlators Slater-determinants. Thus correlation functions of *N* = *N*⇥ + *N*⇤ fermions may be expressed Slater-determinants. Thus correlation functions of *N* = *N*⇥ + *N*⇤ fermions may be expressed For *N*⇤ = *N*⌅ = *N/*2, these considerations lead us to study correlation functions of the k \sim *k* \sim

*C*_{*N*} (*C*) α *Slater-Determinant to take account of Fermi-Dirac Statistics* \overline{C} \overline{C} \overline{D} e *• Slater-Determinant to take account of Fermi-Dirac Statistics*

S⌅⇤

$$
C_{N_{\downarrow},N_{\uparrow}}(\tau) = \langle \det S^{\downarrow}(\tau) \det S^{\uparrow}(\tau) \rangle
$$

$$
S_{i,j}^{\sigma}(\tau) = \langle \alpha_i^{\sigma} | K^{-1}(\tau, 0) | \alpha_j^{\sigma} \rangle
$$

$$
S_{i,j}^{\downarrow \uparrow}(\tau) =
$$

the species. In order to satisfy Fermi-Dirac statistics, fermions of the same species must

to impose the proper anti-symmetrization requirements on \mathcal{C} is to use on \mathcal{C} is to use of u

have dierent quantum numbers. As is well-known from quantum mechanics, a simple way

Slater-determinants. Thus correlation functions of *N* = *N*⇥ + *N*⇤ fermions may be expressed

lap with the unitary Γ

⁸ Higher order time discretizations errors may be achieved with with the use of higher order decompositions

of operator exponentials. For interacting fermions, this would require additional auxiliary fields at each

$$
C_{N_{\downarrow},N_{\uparrow}}(\tau) = \langle \det S^{\downarrow\uparrow}(\tau) \rangle
$$

$$
S_{i,j}^{\downarrow\uparrow}(\tau) = \langle \Psi | K^{-1}(\tau,0) \otimes K^{-1}(\tau,0) | \alpha_i^{\downarrow} \alpha_j^{\uparrow} \rangle
$$

^j ⌃ (35)

◆

^j ⌃ *.* (36)

, (37)

^j ⌃ *.* (36)

i*,* (18)

(⇤*,* 0)*|*⇤

For
$$
N_{\uparrow} = N_{\downarrow} - 1
$$
, replace j-th row by
\n $\langle \alpha_j^{\downarrow} | K^{-1}(\tau, 0) | \alpha_1^{\downarrow} \rangle$, $\langle \alpha_j^{\downarrow} | K^{-1}(\tau, 0) | \alpha_2^{\downarrow} \rangle$, ..., $\langle \alpha_j^{\downarrow} | K^{-1}(\tau, 0) | \alpha_{N_{\downarrow}}^{\downarrow} \rangle$

Unitary fermions have small	In momentum space
wave function overlap with	\n $\Psi_{untrapped}(\mathbf{p}) = \begin{cases} \frac{e^{-bp}}{p^2}, & p \neq 0 \\ \psi_0, & p = 0 \end{cases}$ \n

 $\frac{1}{2}$ = 2 $\frac{1}{2}$ = 2 $\frac{1}{2}$

Consider a correlator *C* with positive definite real number and a new variable *Z* = ln *C*

Numerical evidence suggests that the best choice for (r*rel*) is a lattice approximation to

which possess a 1*/|*r*rel|* singularity. We therefore consider a momentum space wave-function

d

*|*p*[|]*

the two-particle s-wave solution to the continuum Schrodinger equation for unitary fermions, μ

where p is the momentum of each fermion in the center of momentum frame, *p* = *|*p*|*, and

b and ⇥⁰ are tunable parameters. The free parameter *b* and ⇥⁰ may be tuned to maximize

the overlap with the true ground state α is the only regime is the only regime is α in α

which possess a 1*/|*r*rel|* singularity. We therefore consider a momentum space wave-function

 \sim \sim $\frac{V_{unitary}(r)}{T_{unitary}(r)}$ of the sources points \sim 1/r (\cdot) in the Cartesian basis of the sources used in our simulations is provided in our simulations in our simulations is provided in our simulations in our simulations in our simulations in our simulations in \mathcal{L} Typically multi-particle sources constructed from single particle states possess poor over- $\mathcal{L}_{unitary}(r) \sim 1/r$

time-like link, however, such constructions are beyond the scope of this work.

Unitary fermions have small and single poor over-Wave function overlap with $\Psi_{untramped}(p) = \emptyset$ $\psi_{untramped}(\mathbf{p}) = \mathbf{p}$ $\frac{1}{2}$ where $\frac{1}{2}$ pon-int **non-interacting fermions.**

ponentially like that of free fermions with a Z-factor near unity. A better approach is to

of operator exponentials. For interacting fermions, this would require additional auxiliary fields at each

⁸ Higher order time discretizations errors may be achieved with with the use of higher order decompositions

of operator exponentials. For interacting fermions, this would require additional auxiliary fields at each

Improvement: N-body correlators

20

b and ⇥⁰ are tunable parameters. The free parameter *b* and ⇥⁰ may be tuned to maximize the overlap with the states on large state near unitary results. *Large overlap of the wave function significantly*

density, physically one expects μ 1*/n*¹/₃. In practice, we consider the discrete three discret

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Statistical noise & overlap problems

• Noise & drifted upward at large Euclidean time

• Worse for larger number of fermions

Conventional method for small N

Good agreement for ~4M configurations, but systematically deviated for smaller number of configurations

FIG. 6: Logarithm of the N=4 fermion correlations function for untrapped unitary fermions of *The error bar doesn't represent the uncertainty of the estimator correctly.*

mass *M* = 5 on an *L* = 10 lattice as a function of sample size. Dashed line indicates the result

Correlator distribution

Most configurations are far off the true ground state.

Insufficient sampling for the long tail leads to the shift in the true ground state energy. $\frac{1}{2}$ fig. 18: $\frac{1}{2}$ for the long-tail leads to the shift in the true for unitary fermions of mass *ground state energy.* The log-correlator of \mathcal{L} is the log-correlator of \mathcal{L}

Distribution overlap problem

Log-correlator distribution

for unitary fermions of mass *M* = 5 on an *L* = 10 lattice. Solid curves in the log-correlator

distribution plot correspond to Gaussian fits to the distribution.

FIG. 18: *N* = 4 fermion correlator and log-correlator distributions at various time separations Conventional sample average and the estimate of errors by standard deviation work well.

mass *M* = 5 on an *L* = 10 lattice as a function of sample size. Dashed line indicates the result obtained using *Nconf* = 2*B* configurations. **Log-normal distribution?**

plot correspond to Gaussian fits to the distribution. *not exactly*

formations at several values of the formation of a long the formation of the late time limit. *cf*: Product of random numbers has a log-normal distribution.

 A lso shown is a correlation for the distribution for the distribution for the logarithm of the correlation for the corr

which has some probability distribution *P*(*Z*). Then it is generally true that *General relation between* $ln < C$ *and* $\lt ln C$ Consider a correlator *C* with positive definite real number and a new variable *Z* = ln *C* which are some probability of the some that is generally the source of the source of the source of the source that is generally the source of the source

$$
\ln \langle C \rangle = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!}.
$$

small probability. This has motivated us to consider a new statistical measurement for the

section, the correlator has a nearly Log-Normal (LN) distribution and a long tail with

where
$$
\kappa_1 = \langle \ln C \rangle
$$
, $\kappa_2 = \langle (\ln C)^2 \rangle - \langle \ln C \rangle^2$, etc.

• Lowest few cumulants don't suffer from *<u>Z</u> Z*(*t*) *Dr Dr obtent* & *Converge quickly.* $1 \tI \tI \tC \tC \t1$ *• Lowest few cumulants don't suffer from the distribution overlap problem & converge quickly.*

 \mathbb{R}^2

P(*Z*)*dZ* +

Define ⇥ = *it*, then we have *<u>Decumulant exhansion at which the argument of* \overline{a} *and* \overline{b} *argument of* \overline{b} **</u>** *truncate the cumulant expansion at which the statistical error and truncation error are comparable.*

^P(*Z*)*Z*2*dZ* ⇥ ²

2! ⁺ *···*

P(*Z*)*ZdZ*⇥ +

Example: *N=46* **unitary fermions**

Convergence of cumulant expansion

Purple band represents the energy calculated using conventional method

Ground state energies: (Nup+1,Ndown) unitary fermions **33.** (IN_{UP}+L,INdown) UTIITATY TELTIITOTIS ✏thermo unitary!

Not all of correlators are positive.

Nc But, the long-tail develops in the distribution for positive correlators.

A (14)

$$
C_{N_{\uparrow},N_{\downarrow}-1}(\tau) = \frac{1}{N_{\rm c}} \sum_{i}^{N_{\rm c}} C_{\phi_{i}}(\tau)
$$

=
$$
\frac{1}{N_{\rm c}} \sum_{i}^{N_{\rm c}^{-}} C_{\phi_{i}}(\tau) + \frac{N_{\rm c}^{+}}{N_{\rm c}} \left(\frac{1}{N_{\rm c}^{+}} \sum_{i}^{N_{\rm c}^{+}} C_{\phi_{i}}^{+}(\tau) \right)
$$

=
$$
\frac{1}{N_{\rm c}} \sum_{i}^{N_{\rm c}^{-}} C_{\phi_{i}}(\tau) + \frac{N_{\rm c}^{+}}{N_{\rm c}} \exp \left[\sum_{n}^{\infty} \frac{\kappa_{n}^{+}(\tau)}{n!} \right]
$$

 $\kappa_n^+(\tau)$ is the *n*-th cumulant of the distribution for ln $C^+_\phi(\tau)$.

For fermions near unitarity, the Bertsch parameter has been determined from exper-

iments and Monte Carlo simulations with great precision. However, It is challenging or

impossible to take ensemble average of some data which don't have a Gaussian distribution.

One of our interest is measuring a many-body correlator which is generated from stochastic

Ground state energies: (N_{up}+1,N_{down}) unitary fermions

For positive configurations cumulant expansion method and the conventional approach agree to each other at large Euclidean time, so do for all configurations.

Ground state energies: (N_{up}+1,N_{down}) unitary fermions

A contribution from negative correlators is getting smaller as the number of fermions increases.

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Shell structure at finite N: Bertsch parameter

1 Introduction

¹ *^E*finite

unitary!

Shell structure at finite N: Pairing gap

$$
\varepsilon^{\text{unitary}}(n) = \Delta \varepsilon_F^{\text{free}}(n)
$$

 E _{Eu}nitary(*n*) = [n, *Eu*nitary(*n*) (1) = [n, *Eu*nitary(*n*) (1) = [n, *Eu*nitary(*n*) = [n, *Eu*

 $\epsilon^{\text{unitary}}(2N_{\uparrow}+1) = E(N_{\uparrow}, N_{\uparrow}+1) - \frac{E(N_{\uparrow}, N_{\uparrow}) + E(N_{\uparrow}+1, N_{\uparrow}+1)}{2}$ small probability. This have motivated us to consider a new statistical measurement for the ω $\frac{(1+|1|+2)(1+|1|+2)}{2}$

correlator.

correlator.

Shell structure at finite N: Pairing gap

Blue:
$$
\left(1 - \frac{\varepsilon_F^{\text{finite}}}{\varepsilon_F^{\text{thermo}}}\right) \times 100
$$
 Red: $\left(1 - \frac{\varepsilon_{\text{unitary}}^{\text{finite}}}{\varepsilon_{\text{unitary}}^{\text{thermo}}}\right) \times 100$

a
List

 $\mathbf{1}_{\mathbf{1}_{\mathbf{1}}}$, $\mathbf{1}_{\mathbf{1}_{\mathbf{1}}}$

unitary!

Shell structure at finite N: Pairing gap unitary **n** \bullet N: Pa uiring gaptatus and a series of the series

and thermodynamic mint **Finite N correction and thermodynamic limit**

$$
\delta \Delta_{k=k_F}(N) = \Delta^{\mathrm{Fit}}_{k=k_F}(N) - \Delta^{\mathrm{data}}_{k=k_F}(N)
$$

1

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Odd-even Staggering at unitarity and a finite box of the box \blacksquare

that "*˜*

QUANTUM MONTE CARLO STUDIES OF SUPERFLUID… PHYSICAL REVIEW A **70**, 043602 (2004)

*/"*r*= 0 at *r*= 0; its value is 6 in the limit *L*→-.

that we have to allow simultaneous variation of all the pa-

rameters, methods based on unguided variation become dif-

ficult, if not infeasible. Again, we rely on the GFMC proce-

are the compart of the compart of the secondary contractor of Odd-even staggering in the ground state energies of unitary $f_{\rm{e}}$ fermions has been found both in QMC and Lattice MC calculations. $T_{\rm eff}$ solid line represents the fitting range for the ex-

tracted value of ∆.

discussed in Ref. [4], the ground state energy for odd *N* depends on the momentum of the unpaired

spin up fermion. For *N* ≤ 37, the minimum energies are obtained by placing the unpaired fermion

anticle dispersion relation at unitarity Single particle dispersion relation at unitarity

kF

a

 $\overline{}$

1

(4)

Add one to fully-paired system

"thermo

F

Preliminary result from the ensemble with largest N and V shows that the pair-breaking energies are lower than QMC results for all k . $\frac{1}{2}$

Chronology of Pairing gap at unitarity

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Carlean 8 Baddy Bhys Bay Lett. 100, 150403 (2009) - Hoinka et al, Nature 13, 943-946 (2017)
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Summary and conclusions

- Highly improved lattice technique for strongly interacting nonrelativistic fermions at unitarity
	- *eliminate finite lattice spacings/finite volume effects systematically*
	- *improve the overlap of the interpolating field*
	- *use the cumulant expansion method to calculate the energies of ground states with high precision by improving the statistical overlap*
- Results of the pairing gap are still preliminary.
	- *agree with the recent cold-atom experiments*
	- *take account for the finite V and N effects carefully*

Thanks!

Backup

Effective mass plot of single particle excitation

Red and blue colors denote two different sinks.

No source dependence at large Euclidean time.

Signal-to-Noise (S/N) problems in numerical simulations of many-body quantum systems **Example: Consider to Noise (S/N) problems in numerical** simulations of many-body quantum syst *i*=1 **E 1** Jar antum *ⁿ*! (25)

X

 $C_{\rm{max}}$

• Lattice QCD calculation for many baryons (a canonical approach with zero chemical potential) *n*=1

nucleus of A_{-hari} *A* **Estimate of A-baryon correlation function:** ϵ and approach and the set of ϵ <u>Compute correlation of New York with 2012</u> $\overline{\mathbf{C}}$ **Estimate of A-baryon correlation function:** <u>relation f</u> $\frac{1}{2}$ (27) $\frac{1}{2}$ (2

E

gauge field *^A* has magnitude ⇥ *^e*3*A/*2*m*⇥

 $\overline{}$

Actual QCD data *Actual QCD data Actual QCD data*

znal to noise *t* $\frac{1}{t}$ $\frac{u}{\sqrt{t}}$ $\frac{1}{\bar{x}}$ $\frac{1}{0} \sim A(n)$ ι_{N} $\overline{}$ to noise: $-\frac{1}{t}$ *t* $\ln \frac{\sigma(t)}{t}$ $\bar{x}(t)$ \sim *A* $\left(m_N-\frac{3}{2}\right)$ 2 m_π \setminus **Signal to noise:**

Parisi/Lepage Parisi/Lepage *qualitatively* **agrees with** Lepage argument

D. B. Kaplan ~ INT Gauge Field Dynamics ~ 3/16/12

Why do log-normal like distribution arise? ⌧ *N* $\overline{}$ *<u>c</u> dist*

Typically, multiplicative stochastic process <u>2</u> $rac{1}{\sqrt{2}}$ <u>tochast</u>

E $\mathbf{\dot{c}}$ *n*=1 *Remember that correlators are products of t and N n*(a) = cumulants of the local porton cumulants of the cumulants of α *matrices of the form*

 $e^{-K}(1 - \sqrt{2})$ $\overline{C}\Phi$) e^{-K} *Random diagonal matrix constant matrix*

ⁱ 2 [1*,* 1] (23)

Little has been known about products of random **matrices**

representations (irreps) of the octahedral group *Oh*. Using these results, we have performed

exact digonalizations of the *N* = 1 + 1, *N* = 2 + 1 and *N* = 2 + 2 unitary fermion transfer

Toy model - one particle, one spatial site as the projection operators onto the center of mass (CM) frame and various irreducible

where *ⁿ* is the *nth* cumulant of *Y* . This relation can

ⁿ is ln *^Y* (*t*) where *^Y* (*t*) = ^h*eY t*ⁱ ⁼ ^h*X^t*

⁰ ¹⁰ ²⁰ ³⁰ ⁴⁰ ⁵⁰ ⁶⁰ 0.8

$$
C(\tau) = \prod_{i}^{\tau} (1 + g\phi_i), \quad 0 \le g \le 1 \qquad (1 + g\phi_{\tau})
$$

 U is the set of the set of the set of the seems to be known about products to be known about products μ

diagerer the Second Term Define "Energy": $\mathcal{E}_{\tau} = -\frac{1}{\tau} \ln \langle C_{\tau} \rangle$ add is $\begin{array}{c} \hline 7 \ (1 + g\phi_2) \end{array}$ $\mathcal{G}\varphi_i$) are given by $\mathcal{G}\varphi_i$ Exact tivation for investigating eq. (7) is that if the distribution *P* $C(T)$ **a i n** ϕ_i Fine "Energy": $\mathcal{E}_{\tau} = -\frac{1}{\tau}\ln \langle C_{\tau}\rangle$ t_{1} + *g*^{*i*}) (1 + *g*^{*p*}₂) (1 $\prod_{i=1}^{n} \frac{(1+g\varphi_i)}{1-q}$. The exact value for the exact value of the exact value for the exact val $\begin{array}{llll} \left[\begin{matrix} \texttt{L} & 1 \end{matrix} \right], & \textbf{then} & \mathcal{E}_{\tau} = 0 & \textbf{(1} + g \phi_1) \end{array}$ average of the correlation of the correlation of the correlation of the correlation of the cumulants of the cumulants of the cumulants of the correlation of the correlation of the correlation of the cumulants of the cumula $C(T) = \prod_{i=1}^{n} (1 + g\phi_i)$ *T* $i = 1$ $\phi_i \in [-1, 1]$ \cdots \cdots Define ' $\sqrt{2}$ ϵ **Energy" :** $\mathcal{E}_{\tau} = -\frac{1}{\tau}$ τ $\ln \langle C_\tau \rangle$ **b if** $\oint_{\mathcal{C}} \varphi_i \in \mathcal{C}^{\frac{1}{4}-1}$, 1, **then** $\mathcal{E}_{\tau} = 0$ $\begin{array}{ccc} \varphi_i \in [-1,1] & \ \varphi_i \in \mathsf{Inif}\ \mathsf{a} \mathsf{r} \mathsf{m} \mathsf{D} \mathsf{i} \mathsf{c} \mathsf{r} & \ \mathsf{F} \mathsf{v} \mathsf{a} \mathsf{c} \mathsf{t} \end{array}$ tribution on the interval interval intervalue for the intervalue for the interval intervalue for the exact value f
Intervalue for the intervalue f (1 + *g*2) (21) $\frac{T}{\prod_{i=1}^{n} (1 + q\phi_2)}$ $\phi_i \in \left[-1, 1\right]$, then $\mathcal{E}_{\tau} = 0$ $(1 + q\phi_1)$ *Uniform Dist. Define "Energy" :* (1 + $q\phi_2$) $[-1, 1]$, then $\mathcal{E}_{\tau} = 0$ (1 ab) *Exact* • If $\phi_i \in [-1, 1]$, then

*n*max for *g* = 1*/*2, and shows that even though the distri-

systematic error in eq. (7) when truncated at *n* = *nmax*,

bution is not log-normal (*n>*² 6= 0) the convergence is

⌘ (2 tanh1(*g*))

lnh*C*⌧ i (8)

(1 + *g*1) (20)

(1 + *g*⌧) (22)

 $40/9\pi$ *n*! **ulants of (In C)** $\overline{\text{}}$ of $\overline{\text{}}$ $\overline{\text{}}$ **•** Analytical result of cumulants of (In C) Can compute cumulants of (ln C) analytically: energy is the statistical energy is obviously **Extending to the statistical energy is the statistical energy in**
140/9π **e** Analytical result of cumulants of (In C)

$$
\begin{array}{lll}\n\cdots & \cdots & \cdots \\
\hline\n\frac{\partial}{\partial \pi} & \cdots & \kappa_1 & = & \tau \left[\frac{1}{2} \log \left(1 - g^2 \right) + \frac{\tanh^{-1}(g)}{g} - 1 \right], \\
\frac{\kappa_n}{50} & \frac{\kappa_n}{60} & n! & = & \tau \left(\frac{(-1)^n}{n} - \text{Li}_{1-n} \left(\frac{1+g}{1-g} \right) \frac{\left(2 \tanh^{-1}(g) \right)^n}{n!} \right) & n > 1\n\end{array}
$$

⌧

fore we analyze instead a toy model where we define a

a

2

n=1

Simulation with finite sample size N detion with finite sample size N

• Measurement of the energy

i=1

as the projection operators onto the center of mass (CM) frame and various irreducible

representations (irreps) of the octahedral group *Oh*. Using these results, we have performed

III. EXACT RESULTS FOR FEW-BODY STATES

 α conventional method α

$$
\mathcal{E}_{\tau} \longrightarrow -\frac{1}{\tau} \left[\frac{1}{N} \sum_{i=1}^{N} C(\tau, \phi_i) \right]
$$

i

(1 + *g*2) (21)

i

(1 + *g*⌧) (22)

as the projection operators on the projection operators on \mathbb{R}^n frame and various irreducible \mathbb{R}^n

representations (irreps) of the octahedral group *Oh*. Using these results, we have performed

(24)

(1 + *g*1) (20)

(1 + *g*2) (21)

(1 + *g*⌧) (22)

ⁿ! (25)

ⁱ 2 [1*,* 1] (23)

t exbans O $\overline{}$ *n*=1 *Fruncated cumulant expansion method* 1 <u>tea</u> $\frac{1}{1}$ $\overline{}$ *C*(⌧*, i*) *truncated cumulant expansion method* **E**

 $\mathcal{E}_{\tau} \longrightarrow -\frac{1}{\tau} \sum \frac{n(n+1)}{n!}$ $\kappa_n(\tau) =$ 1 τ \sum ∞ *n*=1 $\kappa_n(\tau)$ $\frac{i}{n!}$ $\kappa_n(\tau)$ = cumulants of 1 $\kappa_n(\tau)$ = cumulants of $\ln C(\tau, \phi)$

1

n(⌧)

estimate the lowest few cumulants from sample II. EXACT RESULTS FOR FEW PARTIES FOR FEW ARRANGEMENTS FOR FEW ARRANGEMENTS FOR FEW ARRANGEMENTS. **estimate the lowest few cumulants from sample**

as the projection operators on the projection operators on \mathbb{R}^n frame and various irreducible \mathbb{R}^n

representations (irreps) of the octahedral group *Oh*. Using these results, we have performed exact digonalizations of the *N* = 1 + 1, *N* = 2 + 1 and *N* = 2 + 2 unitary fermion transfer **Then, compare results with exact answers !**

matrices on small to moderate lattice volumes. ³ Armed with exact numerical results for the

eigenstates and energies, we may investigate the systematic errors associated with partial

8 Effective mass plot for toy model w_{max} random matrices. The weakly random matrices w_{max}

N V ⇥*^V* matrices of the form *^eK/*²(1+*g*')*eK/*², where

K is a constant matrix (the spatial kinetic operator), ' is

of random matrices, beyond the study in $[9]$ which deals the study in $[9]$ which deals the study in $[9]$

⌘ (2 tanh1(*g*))

Are distributions approaching log-normal appearing in QCD? **Appearance of LN distribution in lattice QCD**

Yes, at early time in a Lambda-Lambda correlator

 $ln(C_{\Lambda\Lambda}(t))$

Appearance of LN distribution in lattice QCD *N* **E OI LIN GISLINGULION IN RELLICE QUD** e of LN distribution in lattice OCD (1 + *g*2) (21)

 $C_{\rm{max}}$ \sim $C_{\rm{max}}$

(24)

(29)

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Odd moments die out faster...expect almost symmetric

At very late time, however, the distribution of correlator for A baryons presumably goes like a Gaussian with large **variance and small mean.** The cumulant state of last α ا:
ا $\frac{1}{2}$ *n*, however, the distribution of correlator nce and small mean. Juniuwiy sucu inte
|| maan rale linie, huwever, life uislinu

Consider the real part of correlator for A baryons μ part of e C ²) *compared to real determination* and *correspondent and areas of the correlation* $x \equiv Re[C_A(T)]$ $\hat{x} \equiv \text{Re}[\text{C}$ *Re* $(X \times 3q)^T$ for A baryons *^x Re*[*C^A*3*q*(*T*)] Real part of Euclidian correlator for a yuns 1 ⌧ *N* ida r the r or \overline{P} *E*⌧ ! *ⁿ*! (25)

E

E⌧ !

III. EXACT RESULTS FOR FEW-BODY STATES

III. EXACT RESULTS FOR FEW-BODY STATES even moments: $\langle x^{2k} \rangle$ \sim $e^{-A3kM_{\pi}T}$ *x*² *e f f c e f F E epage* odd moments: $\langle x^{2k+1} \rangle$ \sim $e^{-AM_N T} e^{-A 3 k m_{\pi} T}$ $\left(r^{2k}\right)$ *e*^{*A*3*kM_πT*} P. Lepage & M. Savage **even moments:** $\langle x \rangle \sim e$ **ients:** $\langle x^{-n+1} \rangle$ \sim e^{rrative}

expect symmetric distribution at late time. \blacksquare distribution at late time
distribution at late time d moments die out fas Since $M_N \gg M_\pi$, odd moments die out faster and we

as the projection operators onto the center of mass (CM) frame and various irreducible

representations (irreps) of the octahedral group *Oh*. Using these results, we have performed

Appearance of LN distribution in lattice QCD 1 X *E*⌧ ! **Appea earance of LIN distributed** \mathcal{L} is next take effective masses and, under a jackknife, compute \mathcal{L} ● Appearance of LN distribution in lattice QUD **i** 2 *i* and in lattice OCD *I* IMULIVII III **E** OT LT |
∎ $\overline{}$ <u>19</u> *N* \mathbf{z} ibution in lattice QCD <u>ais</u> **n**
*I*strip *ⁿ*! (25) ⌧ *n*=1 *ⁿ*! (25) *E*⌧ ! (1 + *g*⌧) (22) *ⁱ* 2 [1*,* 1] (23) ⌧ *N ities* Ω CD 1 $\overline{}$ **ELICE** *ⁱ* 2 [1*,* 1] (23)

n(⌧)

C(⌧*, i*)

of k2 for log C(t) which is linear in t. Fig. 7 shows that in this behavior α for hadron quite nicely for hadron α

Propy 1 x and SU(N) baryon correlator a fit to See Eq. 9) \overline{P} ⌧ *N* **SU(N)** Daryon ary *n* barvon co *n* a SU(N) baryon correlator
T. DeGrand (21 **C**) Daryon correlator
T DeCre 1 *i*=1

X 1

the asymptotic form may appear only at very late time.

1

E

Recall parameters of Fig. 6, showing the evolution of $\overline{\text{F}}$. DeGrand (2012) **r correlator**
T. DeGrand (2012) ⌧ *N i*=1 *n*. DeGrand (2012) $\overline{}$ aGrand *n*d (2012) DeGrand ⌧ *N i*=1 T. DeGrand (2012)

 (24)

"

N

C(⌧*, i*)

(1 + *g*2) (21)

(1 + *g*⌧) (22)

 $\overline{}$

n=1

 \mathcal{L} 1

n=1

C(⌧*, i*)

as the projection operators on the projection operators on \mathbb{R}^n frame and various irreducible \mathbb{R}^n

representations (irreps) of the octahedral group *Oh*. Using these results, we have performed

exact digonalizations of the *N* = 1 + 1, *N* = 2 + 1 and *N* = 2 + 2 unitary fermion transfer

matrices on small to moderate lattice volumes. ³ Armed with exact numerical results for the

both ki and k2 roughly been consistent with Leparent consistent scale ~t arg pseudoscalar or the rho, the lightest state is the lightest state is the two-pseudoscalar state and C matrices on small to moderate volumes. 3 Argu exact digonalizations of the *N* = 1 + 1, *N* = 2 + 1 and *N* = 2 + 2 unitary fermion transfer

matrices on small to moderate lattice volumes. ³ Armed with exact numerical results for the

FIG. 1: Histogram of values of log C(t) for the propagator of a J = 7/2 baryon in SU(7). Panels

Г

N

eigenstates and energies, we may investigate the systematic errors associated with partial

i=1

be roughly a constant for the pseudoscalar, roughly increasing exponentially as exp((m^ρ − $\mathbf{f}(\mathbf{f})$ for the reduced states in a box includes and interactions and interact

II. EXACT RESULTS FOR FEW-BODY STATES FOR FEW PLANTS FOR FEW PLANTS FOR FEW PLANTS FOR FEW PLANTS FOR FEW PLAN
In the states for th as the projection operators onto the center of mass (CM) frame and various irreducible I. Mass spectrum of moments

ⁱ 2 [1*,* 1] (23)

both KI and K2 roughly consistent with Lepage In Appendix A we derive exact the Scale \sim the M-particle transfer matrix, argument or just noise? \blacksquare representation on the center of mass (CM) frame and various irreducible center of the original group \mathbf{r} and \mathbf{r} as the projection operators onto the center of mass (CM) frame and various irreducible $\mathbf{I} = \mathbf{I} \times \mathbf{A}$ we derive exact expressions for the N-particle transfer matrix, as well as we \mathbf{p} the projection operators on \mathbf{p} frame and various irreducible \mathbf{p} representations (irreps) of the octahedral group *Oh*. Using these results, we have performed **roughly** consistent with Lepage argument or just noise: as the projection operators on the projection operators on \mathbb{R}^n frame and various irreducible \mathbb{R}^n **argument or just noise?**

matrices on small to moderate lattice volumes. ³ Armed with exact numerical results for the

exact digonalizations of the *N* = 1 + 1, *N* = 2 + 1 and *N* = 2 + 2 unitary fermion transfer

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representations (irreps) of the octahedral group *Oh*. Using these results, we have performed