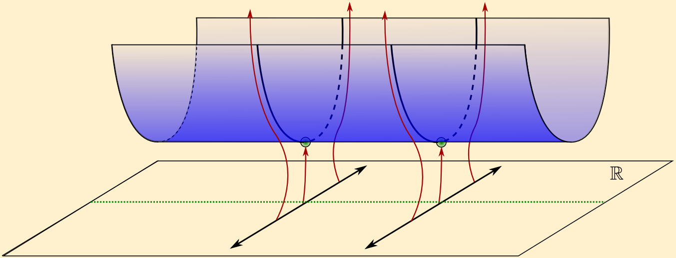
Complex Solutions to Sign Problems

Henry Lamm

w/ Andrei Alexandru, Gökçe Başar, Paulo Bedaque, Scott Lawrence, Neill Warrington

1709.01971, 1804.00697, 1807.02027, 180x.xxxxx



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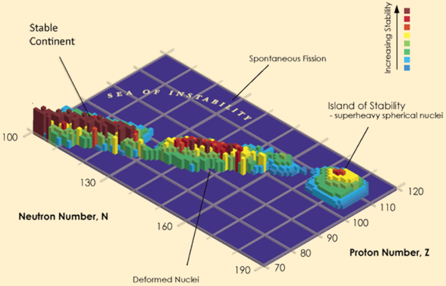
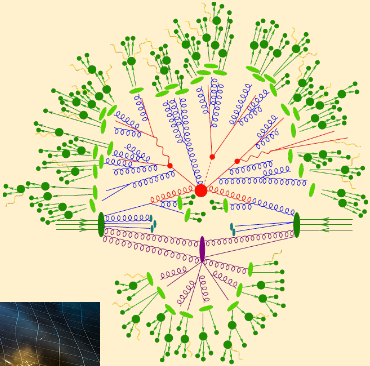
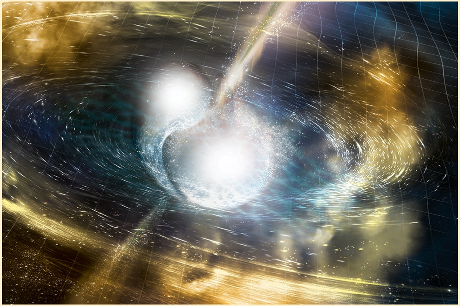
Nuclear Physics is plagued by sign problems







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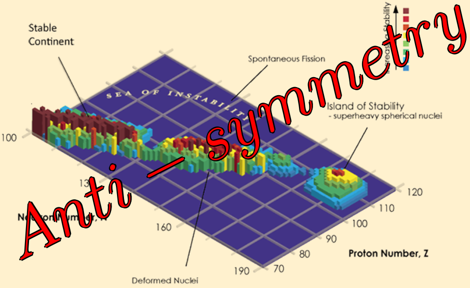

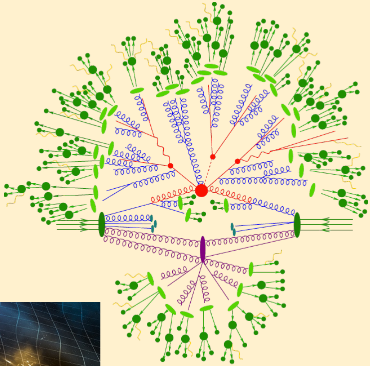

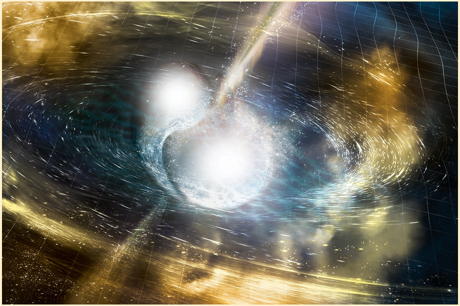

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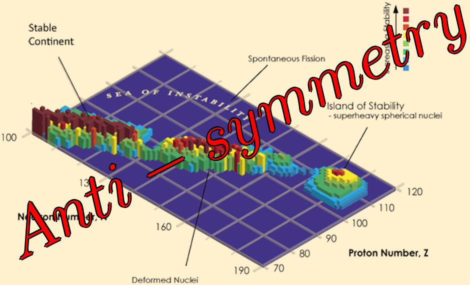

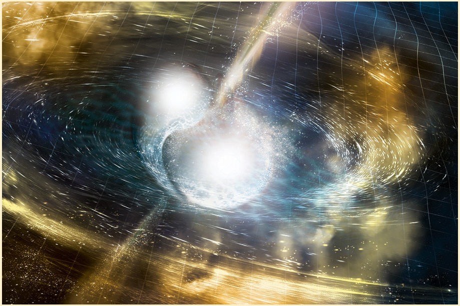

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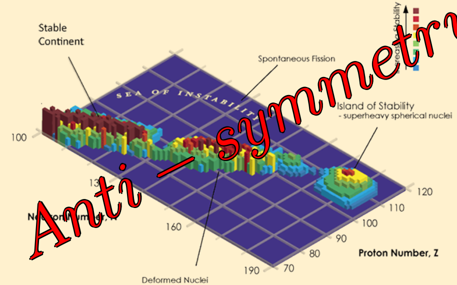

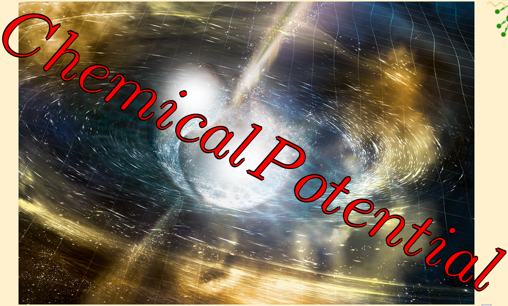

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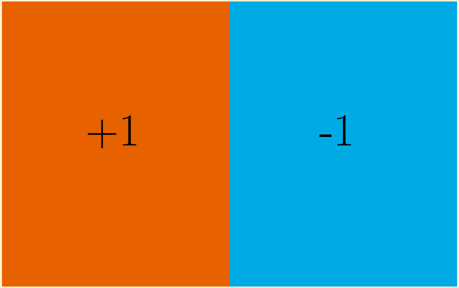

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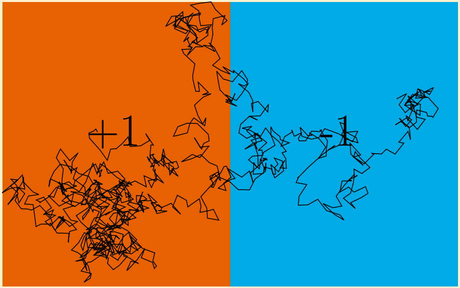

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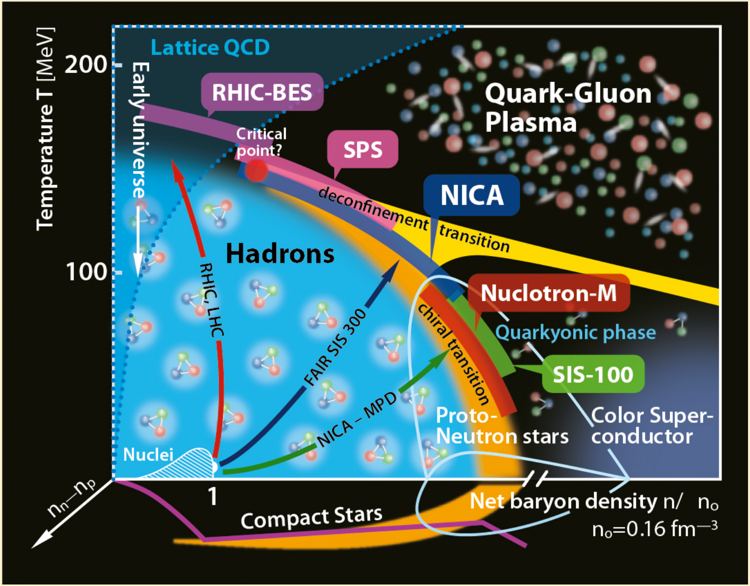

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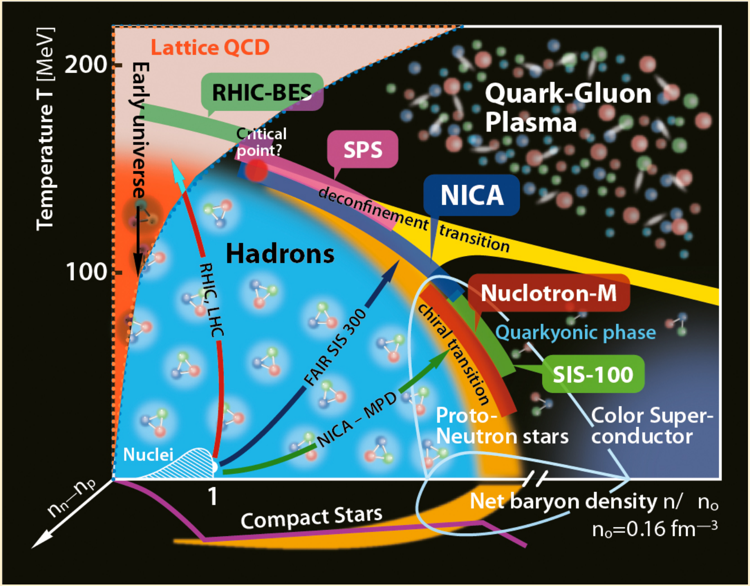

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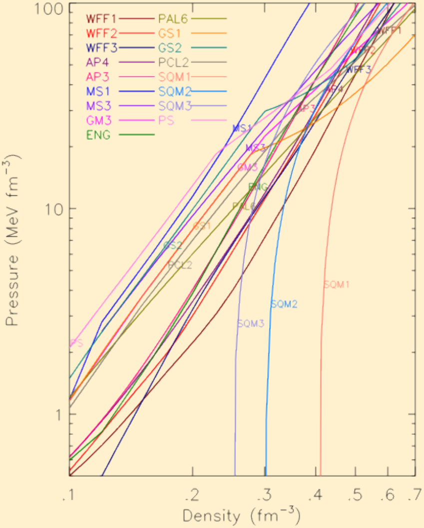

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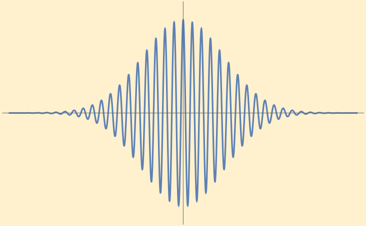

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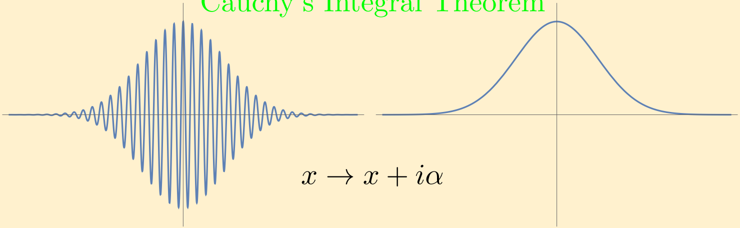

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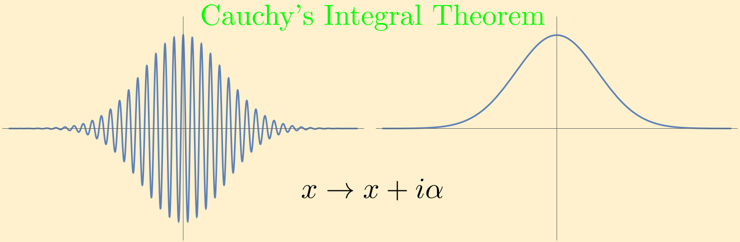

- CIT guarantees holomorphic $f(x)$ (*physics*) unchanged
- **Nonholomorphic** $f(x)$, like the average sign, $\langle\sigma\rangle$, can change!

$$\langle\sigma\rangle_{\mathcal{M}} = \frac{\int_{\mathbb{R}^N} \mathcal{D}\phi \; e^{-S_{\text{eff}}[\phi;\lambda]}}{\int_{\mathbb{R}^N} \mathcal{D}\phi \; e^{-\mathrm{Re}(S_{\text{eff}}[\phi;\lambda])}} = \frac{\text{holomorphic}}{\text{non-holomorphic}}$$

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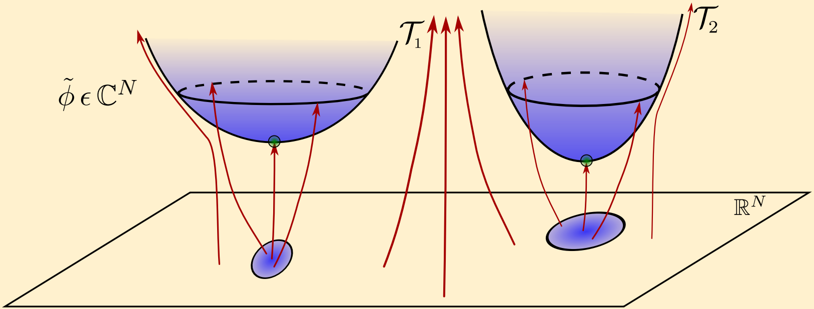

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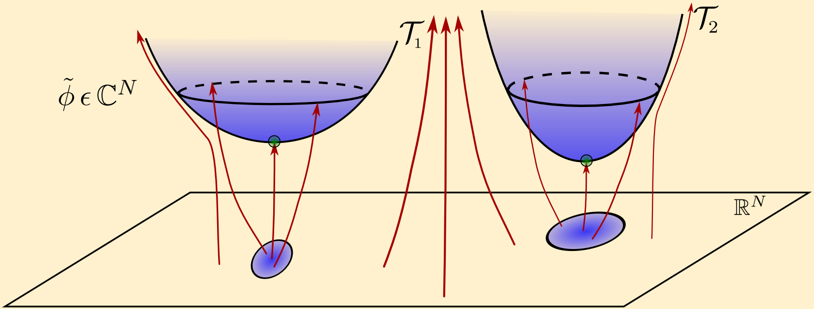

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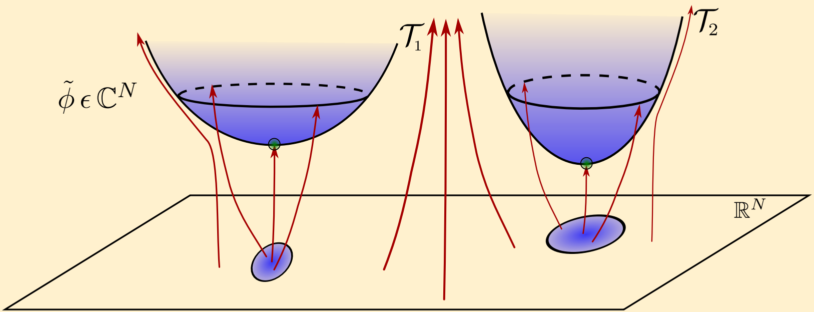

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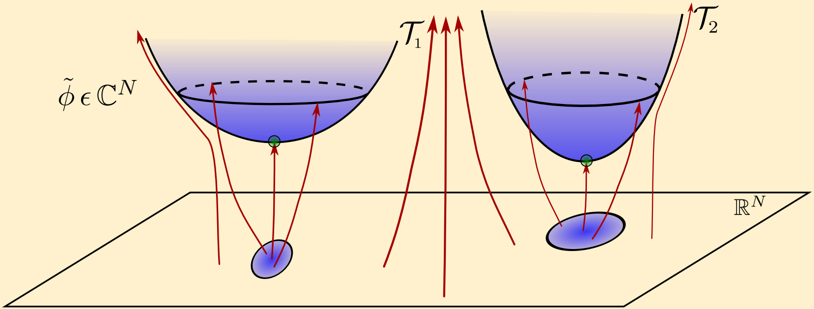

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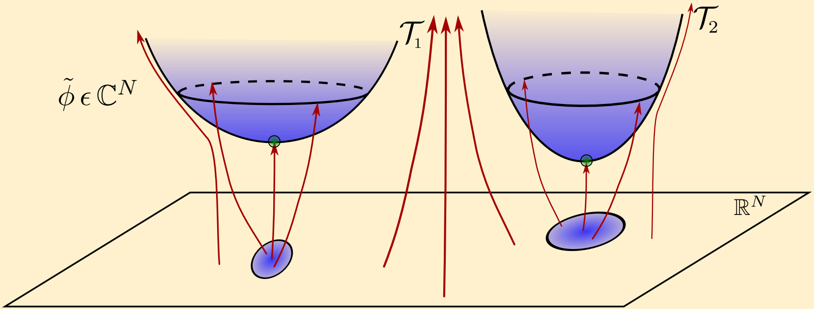

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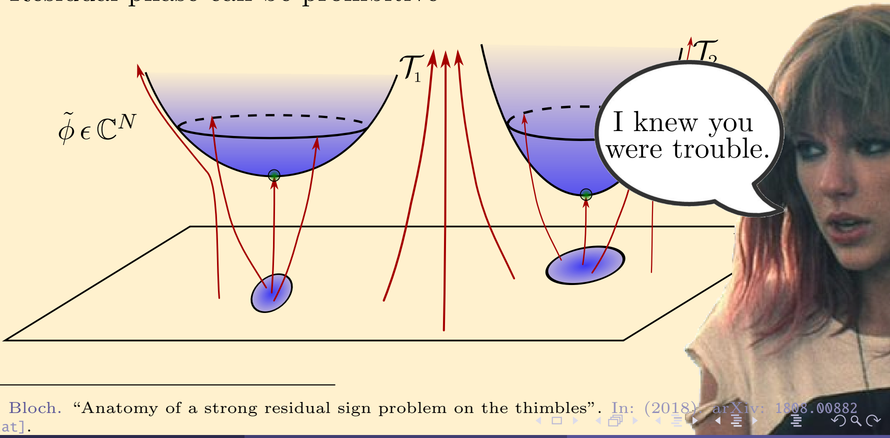

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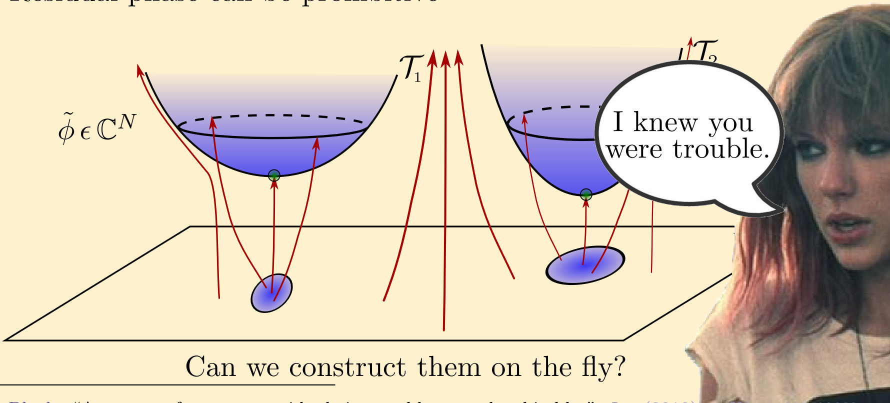

Can we construct them on the fly?

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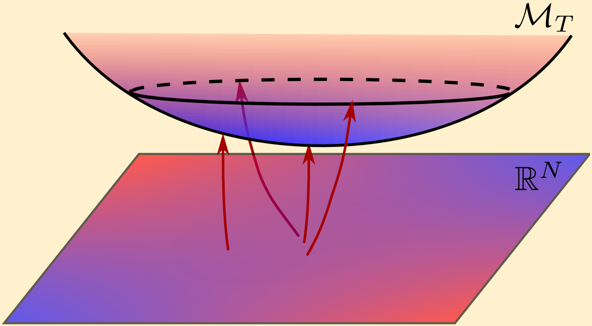

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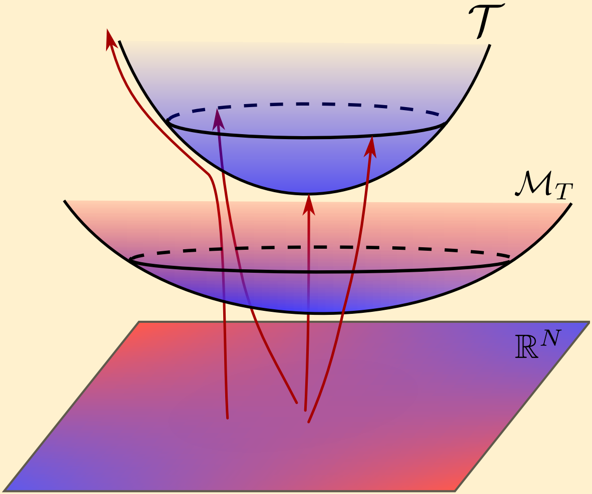

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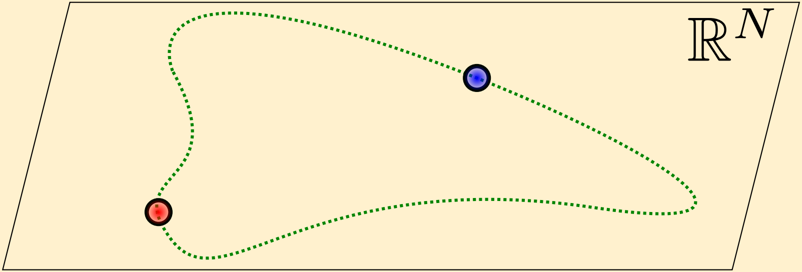

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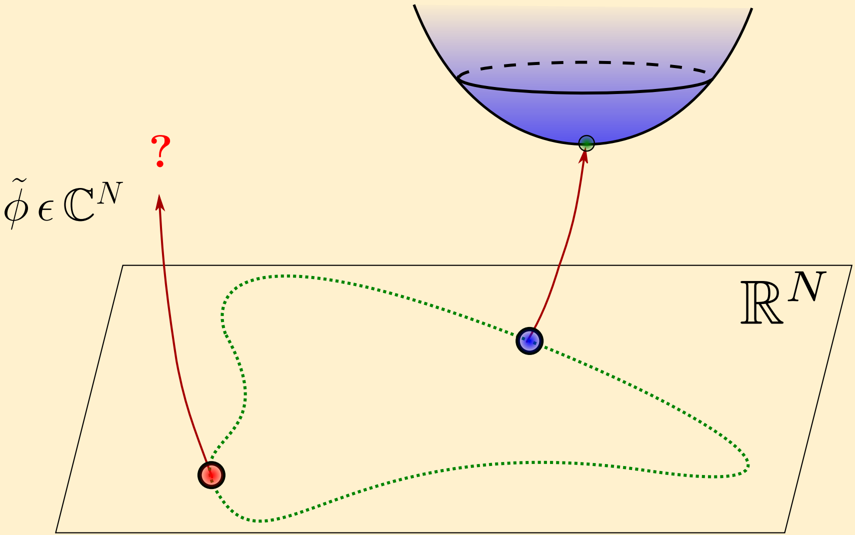

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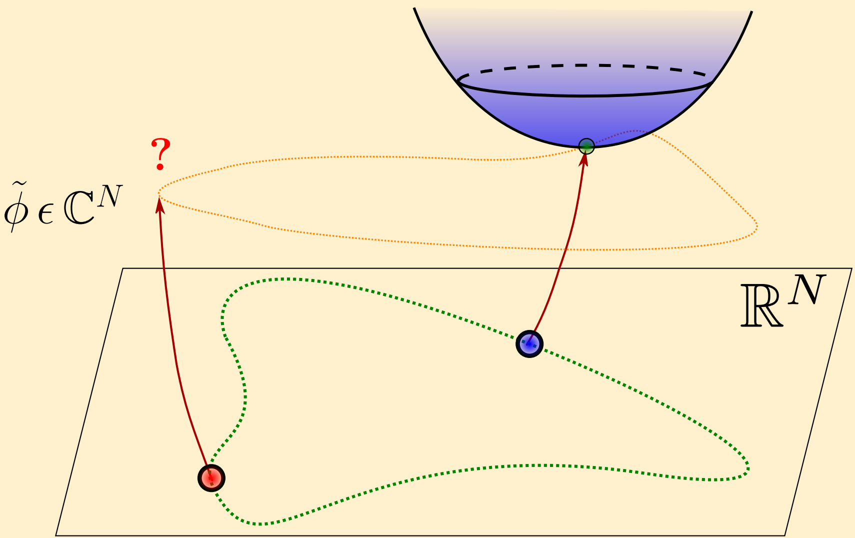

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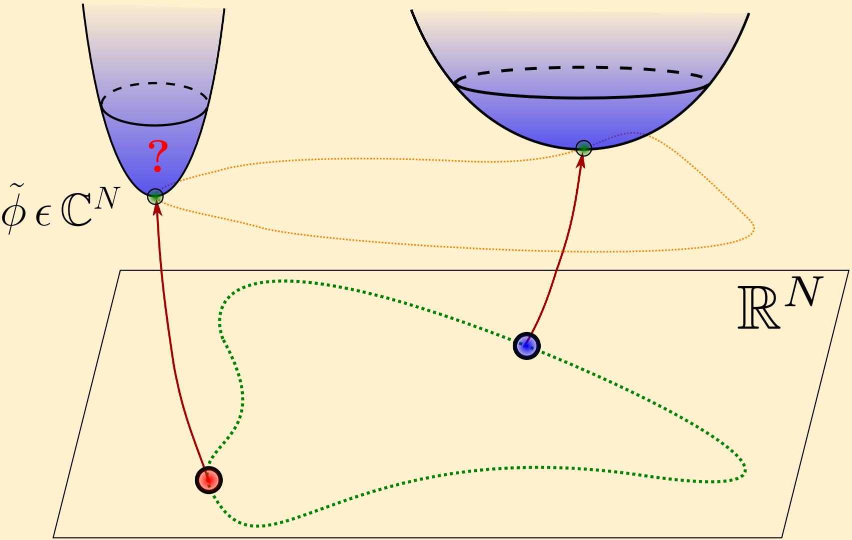

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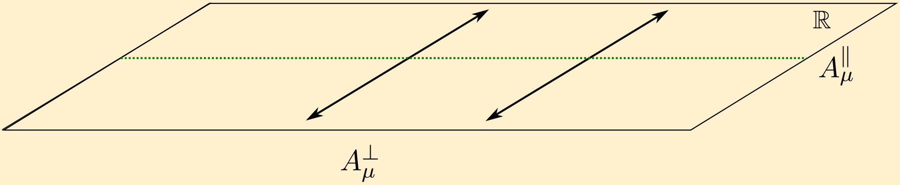

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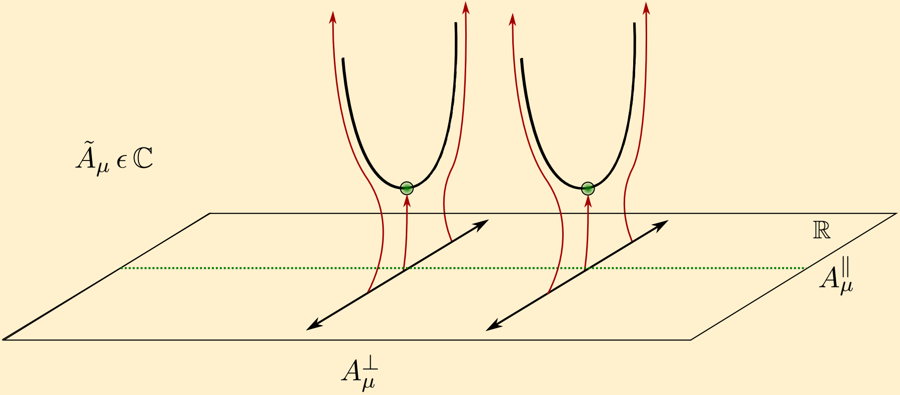

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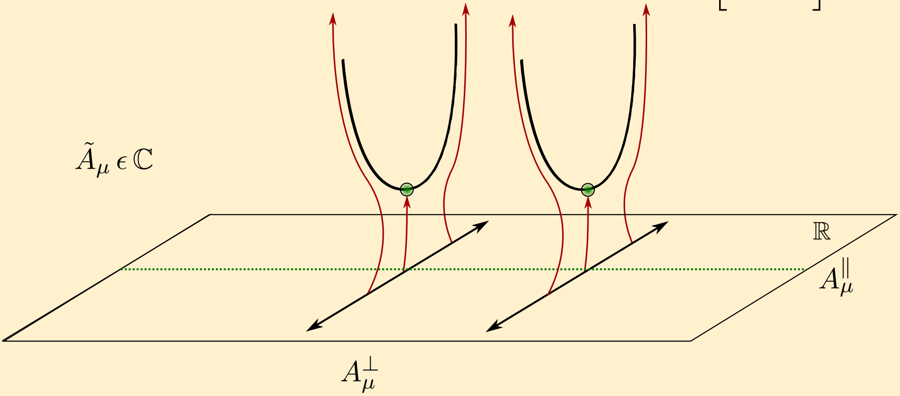
$$\left[\frac{\partial S}{\partial A^{\perp}}, \mathcal{G}\right] = 0$$



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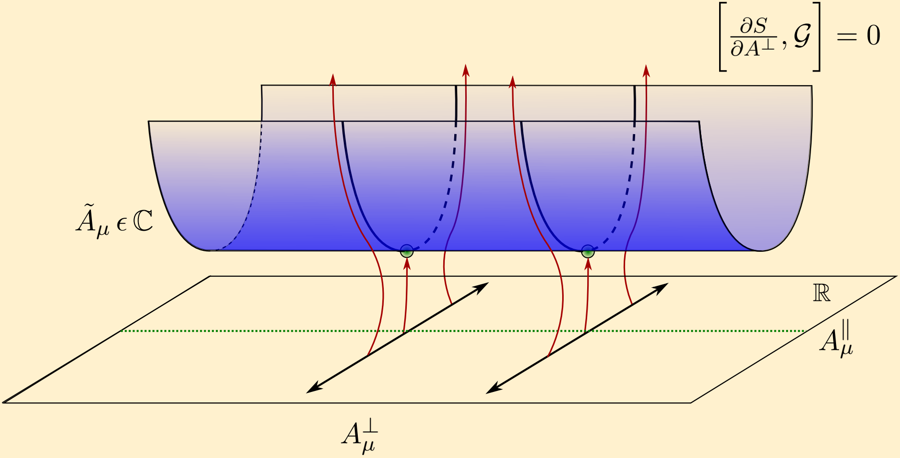
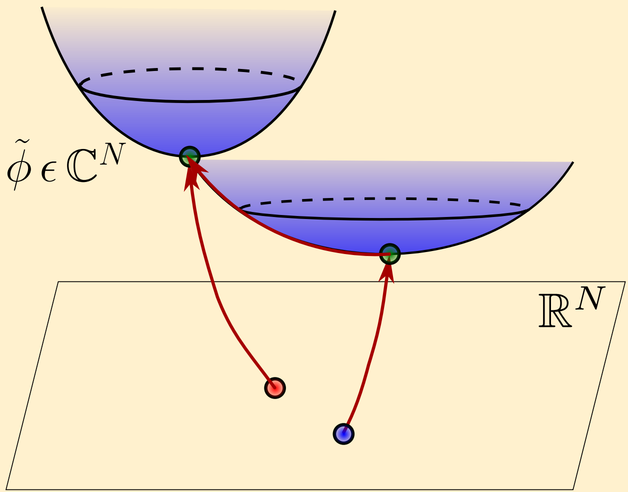


Manifold under flow is just $\mathcal{M}_g \oplus \mathcal{G}$!

Stokes' phenomenon prevent thimble decomposition



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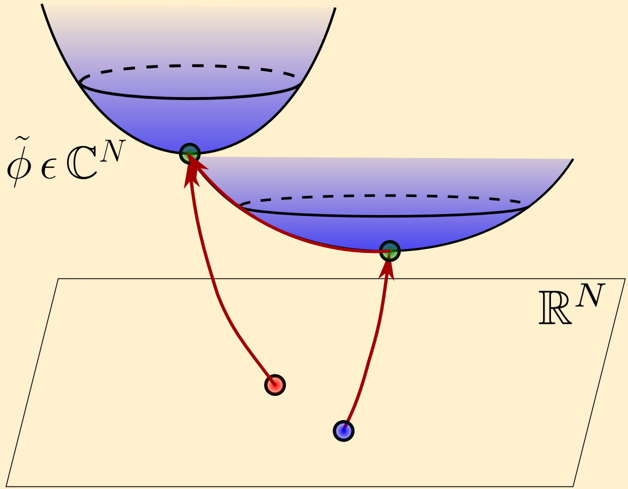
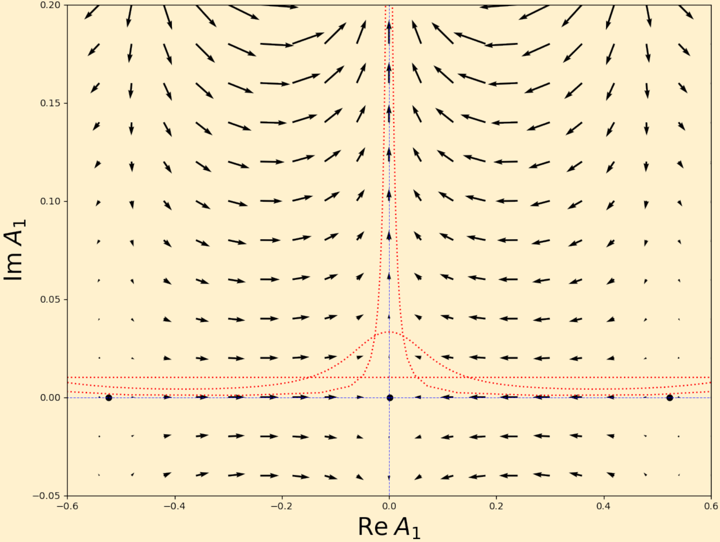


Effect of Stokes' phenomenon on flow is $\langle \sigma \rangle < 1$ due to "bumps"

Ain't no thang but a flow thang.



QED_{1+1} with $N_f = 3$ staggered fermions

In the continuum:

$$S = \int \mathrm{d}^2x \; \left[F_{\mu\nu}F^{\mu\nu} + \bar{\psi}^a(\partial\!\!\!/ + \mu_Q\gamma_0 + m - gQ_a A\!\!\!/)\psi^a\right]$$

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which we discretize and integrate out the fermions to obtain:

$$S = \frac{1}{g^2}\sum_r (1 - \cos P_r) - \sum_a \ln\det D^{(a)}_{xy}$$

$$P_r \equiv A_1(r) + A_0(r+\hat{x}) - A_1(r+\hat{t}) - A_0(r)\,.$$

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- $g = 0.50$ and $m_a = 0.05$, $Q_1 = Q_+ = +2$ and $Q_{2,3} = Q_- = -1$

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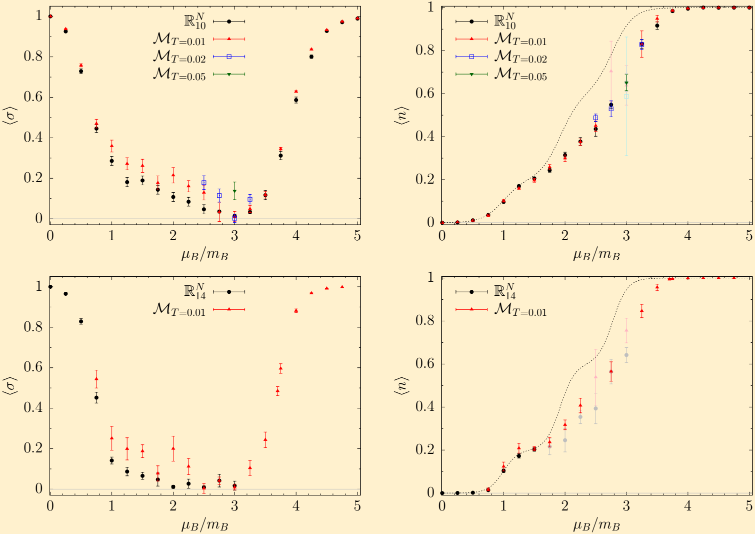
- $g = 0.50$ and $m_a = 0.05$, $Q_1 = Q_+ = +2$ and $Q_{2,3} = Q_- = -1$
- *Baryon* with $am_B \approx 0.6$

QED_{1+1}



$\langle\sigma\rangle$ and $\langle n\rangle$ as a function of μ for $10, 14 \times 10$ for QED_{1+1} with $N_f = 3$

The Jacobian must be flowed as well: $\frac{dJ_{ij}}{dt} = \overline{\frac{\partial^2 S}{\partial\zeta_i \partial\zeta_k}} \bar{J}_{kj}$

[4] A. Alexandru et al. "Fast estimator of Jacobians in the Monte Carlo integration on Lefschetz thimbles". In: *Phys. Rev.* D93.9 (2016), p. 094514. arXiv: 1604.00956 [hep-lat].

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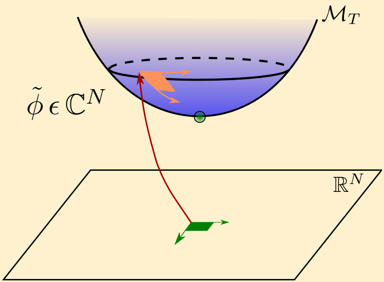

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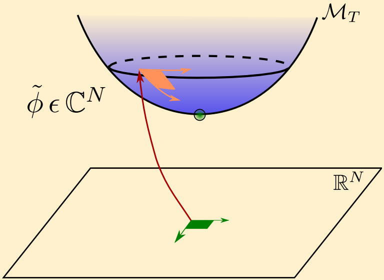

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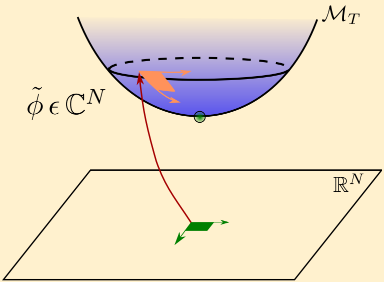

Prohibitive reweighting from $W - J$ from **exceptional** configurations when T is large

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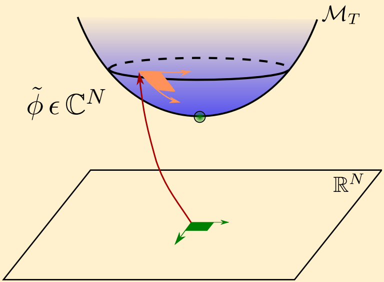

Prohibitive reweighting from $W - J$ from **exceptional** configurations when T is large

Gets **worse** in: $d > 1$, $g \to \infty$, gauge theories, and at transition[5]

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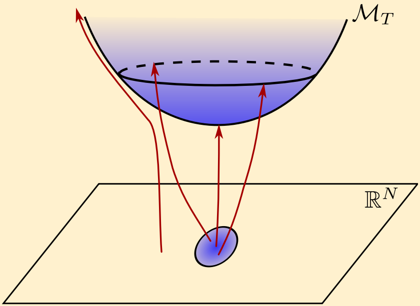

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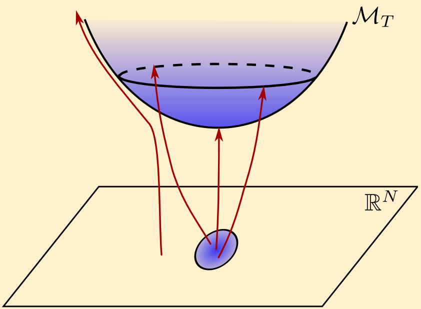

$$\tilde{\phi}_i(\phi) = \phi_i + if(T_i\phi) \tag{1}$$

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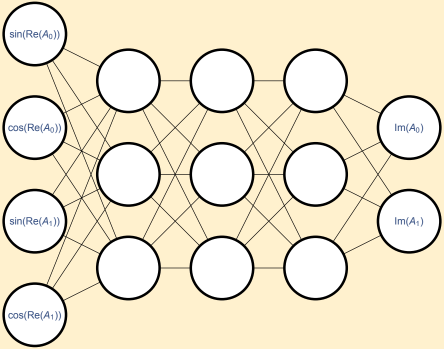

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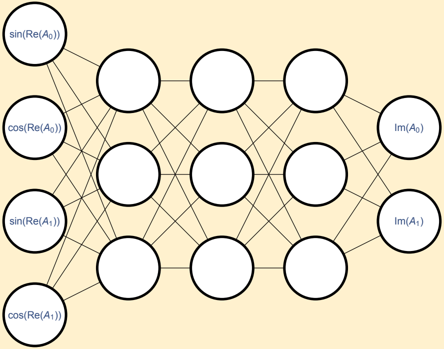

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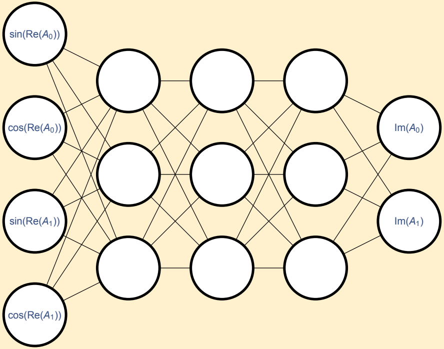

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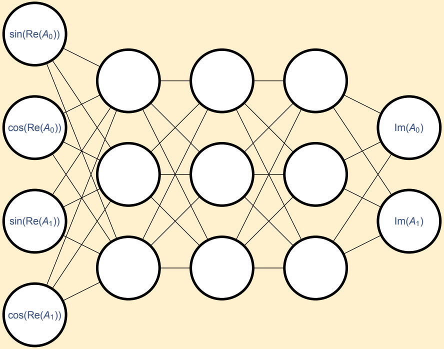

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Perform a backward propagate with gradient descent to adjust b_j, w_{ij}

1+1 Thirring model with Wilson fermions

$$S = \sum_{x,\nu} \frac{N_F}{g^2}(1 - \cos A_\nu(x)) + \sum_{x,y} \bar{\psi}^\alpha(x) D^W_{xy}(A) \psi^\alpha(y)\,, \qquad (2)$$

and with the hopping parameter $\kappa = 1/(2m+4)$,

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The integration over the fermion fields leads to

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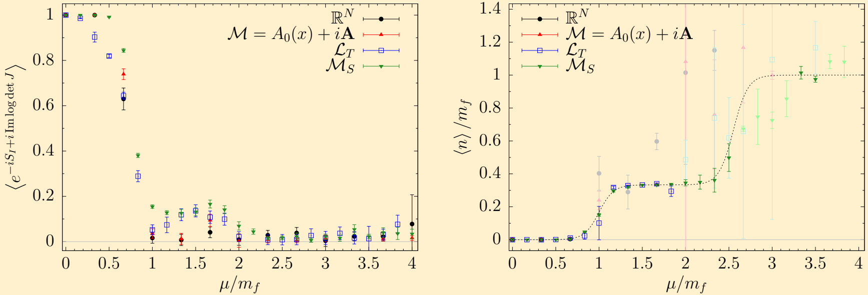
- $am_f = 0.30(1)$ and $am_b = 0.44(1) \implies m_b/m_f = 1.5(2)$

1+1 Thirring 40×10



$\langle e^{-iS_I + i\,\mathrm{Im}\log\det J}\rangle$ and $\langle n\rangle/m_f$ as a function of μ/m_f for 40×10 with $am_f = 0.30(1)$. The dashed curve represents the free fermion gas with the same mass.

What about avoiding flow all together?

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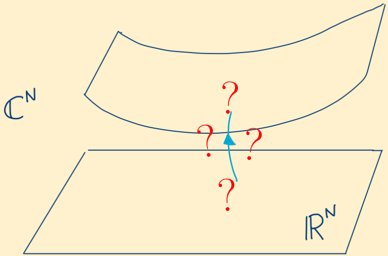

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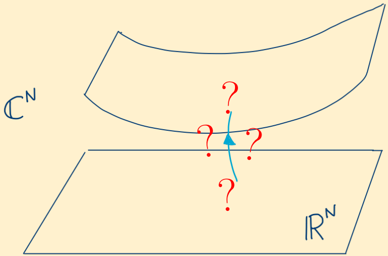

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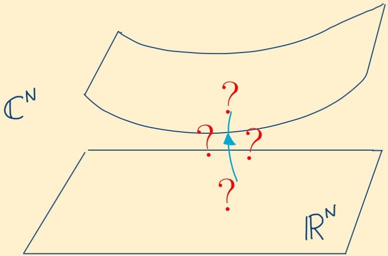

- What if thimbles aren't optimal? Perhaps we instead seek the exact surface of maximal $\langle\sigma\rangle$[7,8]

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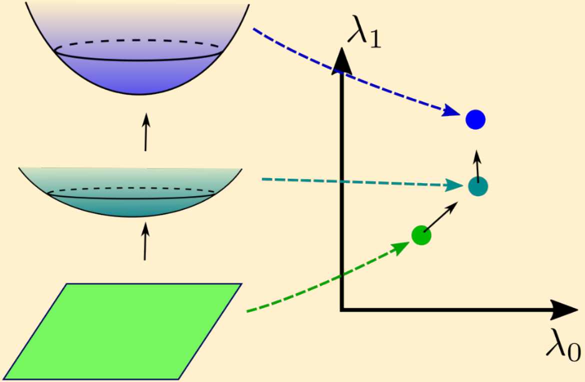
Sign-optimized manifold methods: Find $\mathcal{M}_S \subset \mathcal{M}_\lambda$[9]



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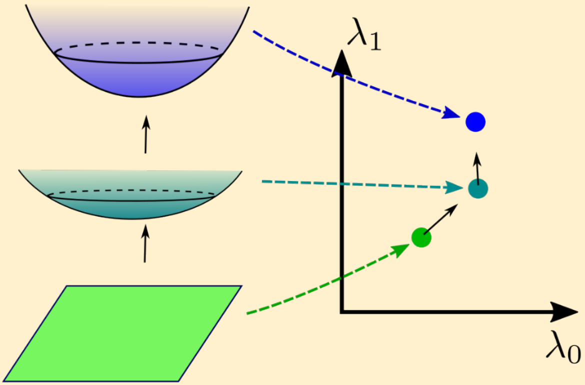

$$\nabla_\lambda |\langle\sigma\rangle| = |\langle\sigma\rangle| \frac{\int_{\mathbb{R}^N} \mathcal{D}\phi\, e^{-\mathrm{Re}(S_{\mathrm{eff}}[\phi;\lambda])} \left[\nabla_\lambda S_R - \mathrm{Re}\mathrm{Tr} J^{-1} \nabla_\lambda J\right]}{\int_{\mathbb{R}^N} \mathcal{D}\phi\, e^{-\mathrm{Re}(S_{\mathrm{eff}}[\phi;\lambda])}}$$

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- For fermionic theories, we find a useful $\mathcal{M}_\lambda$ to be

$$\begin{aligned}\tilde{A}_0(A_0, A_i 1) &= A_0 + i\left(\lambda_0 + \lambda_1 \cos A_0 + \lambda_2 \cos 2A_0\right) \\ \tilde{A}_i(A_0, A_i) &= A_i\end{aligned}$$

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- For bosonic theories, other groups have shown that nearest-neighbor correlations are important[10]

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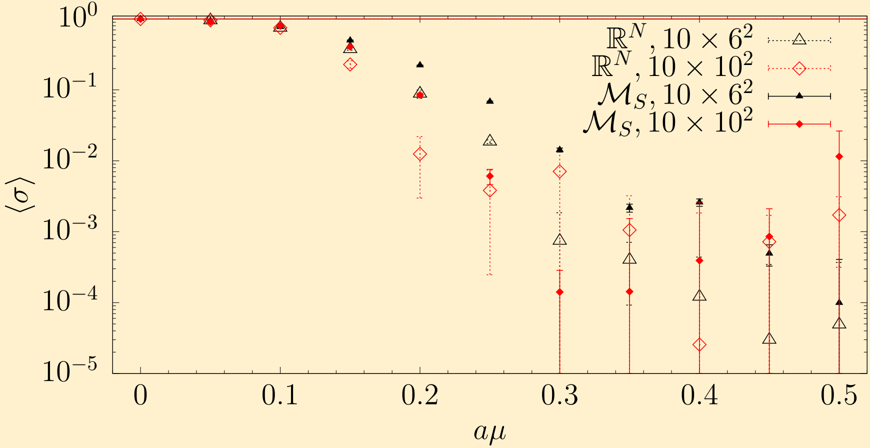
Within our $\mathcal{M}_\lambda$, we can analytically compute $[J(A)J^\dagger(A)]^{-1}_{xy}$
$\Rightarrow$ so $\mathcal{M}_S$ as fast as $\mathbb{R}^N$


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\langle\sigma\rangle$
0
0.1
0.2
0.3
0.4
0.5
$a\mu$
$\mathbb{R}^N, 10\times 6^2$
$\mathbb{R}^N, 10\times 10^2$
$\mathcal{M}_S, 10\times 6^2$
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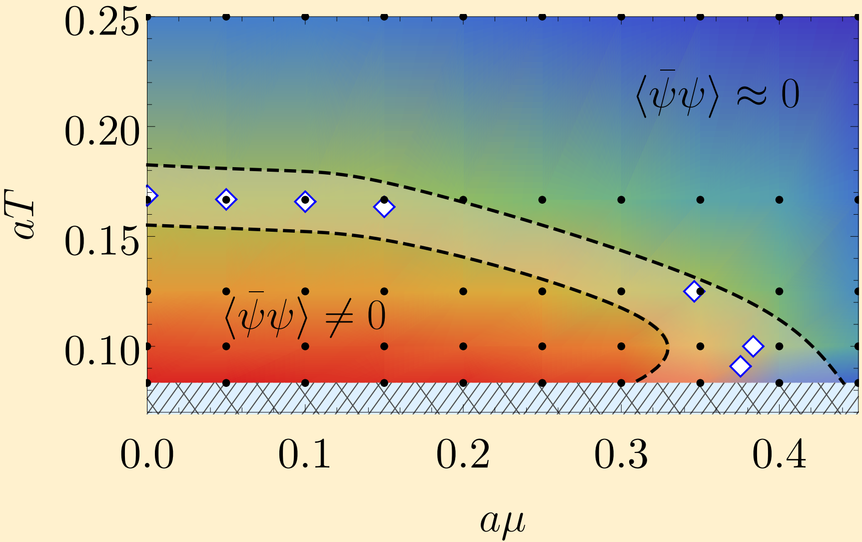
Prelim. 2+1 Thirring with Staggered Fermions $N_S = 6^2$



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