Recent developments in **integral transform** approaches

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Integral Transforms as an alternative method for dealing with the continuum

Reactions to continuum

perturbative (electro-weak)

$\gamma^{(*)}$ + b ----> c + d +...

non-perturbative (hadronic)

a + b ----> c + d +...

a,b,c,d... are single nucleons or bound nuclear systems
These reactions involve the dynamics of A nucleons
A-BODY PROBLEM IN THE CONTINUUM!

Reactions to continuum

Framework:

- Energies in the non-relativistic regime

  → Non-Relativistic Quantum Mechanics

  (including ***Translation*** *,* ***Galileian*** *,* ***Rotational*** *… invariances)*

  $[H, \mathbf{P}_{cm}]=0$ $[H, \mathbf{R}_{cm}]=0$ $[H, \mathbf{J}]=0$ ...

- Degrees of freedom: all A nucleons ("microscopic" model)

- $H = T + V$ $\qquad V = \sum_{ij} v_{ij} + \sum_{ijk} v_{ijk} + \ldots$

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- **Linear** Response theory

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$$\sigma(\omega) \sim \quad |\langle n | \Theta | 0 \rangle|^2 \quad \delta(\omega - E_n + E_0)$$

$$H | n \rangle = E_n | n \rangle$$

Reactions to continuum

- **First order** perturbation theory *(Fermi-Golden Rule)*
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Energy transferred by the perturbative probe

perturbative (electro-weak)

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- **First order** perturbation theory *(Fermi-Golden Rule)*
- **Linear** Response theory

$$\sigma(\omega) \sim \quad |\langle n | \Theta | 0 \rangle|^2 \, \delta(\omega - E_n + E_0)$$

Ground state of the target
A-body bound state!

perturbative (electro-weak)

$\gamma^{(*)}$ + b ----> c + d +…

Reactions to continuum

- **First order** perturbation theory *(Fermi-Golden Rule)*
- **Linear** Response theory

$$\sigma(\omega) \sim \left| \langle n | \Theta | 0 \rangle \right|^2 \delta(\omega - E_n + E_0)$$

Fragmented target
A-body continuum state!

perturbative (electro-weak)

$\gamma^{(*)}$ + b ----> c + d +...

Reactions to continuum

- **First order** perturbation theory *(Fermi-Golden Rule)*
- **Linear** Response theory

$$\sigma(\omega) \sim \quad |\langle n | \Theta | 0 \rangle|^2 \; \delta(\omega - E_n + E_0)$$

Property of the target responsible of the interaction with the perturbative probe e.g. charge-current density

perturbative (electro-weak)

$\gamma^{(*)}$ + b ----> c + d +...

Reactions to continuum

- **First order** perturbation theory *(Fermi-Golden Rule)*
- **Linear** Response theory

$$\sigma(\omega) \sim \sum_n |\langle n | \Theta | 0 \rangle|^2 \delta(\omega - E_n + E_0)$$

perturbative (electro-weak) **INCLUSIVE**

$\gamma^{(*)} + b \dashrightarrow b^*$ ($b^* = c+d+..$ *or* $e+f+..$ *or* ...)

Reactions to continuum

- **First order** perturbation theory *(Fermi-Golden Rule)*
- **Linear** Response theory

$$\sigma(\omega) \sim \sum_n |\langle n | \Theta | 0 \rangle|^2 \delta(\omega - E_n + E_0)$$

$$\sum_n | n \rangle \langle n | = I$$

$$H | n \rangle = E_n | n \rangle$$

Reactions to continuum

PERTURBATIVE INCLUSIVE

$$S(\omega) = \sum_n |\langle n | \Theta | 0 \rangle|^2 \delta(\omega - E_n + E_0)$$

$S(\omega)$ represents the crucial quantity

Requires the solution of both

the bound and continuum A-body problem

We see next that in case of non perturbative reactions the crucial quantity for calculating the cross section has a very similar form

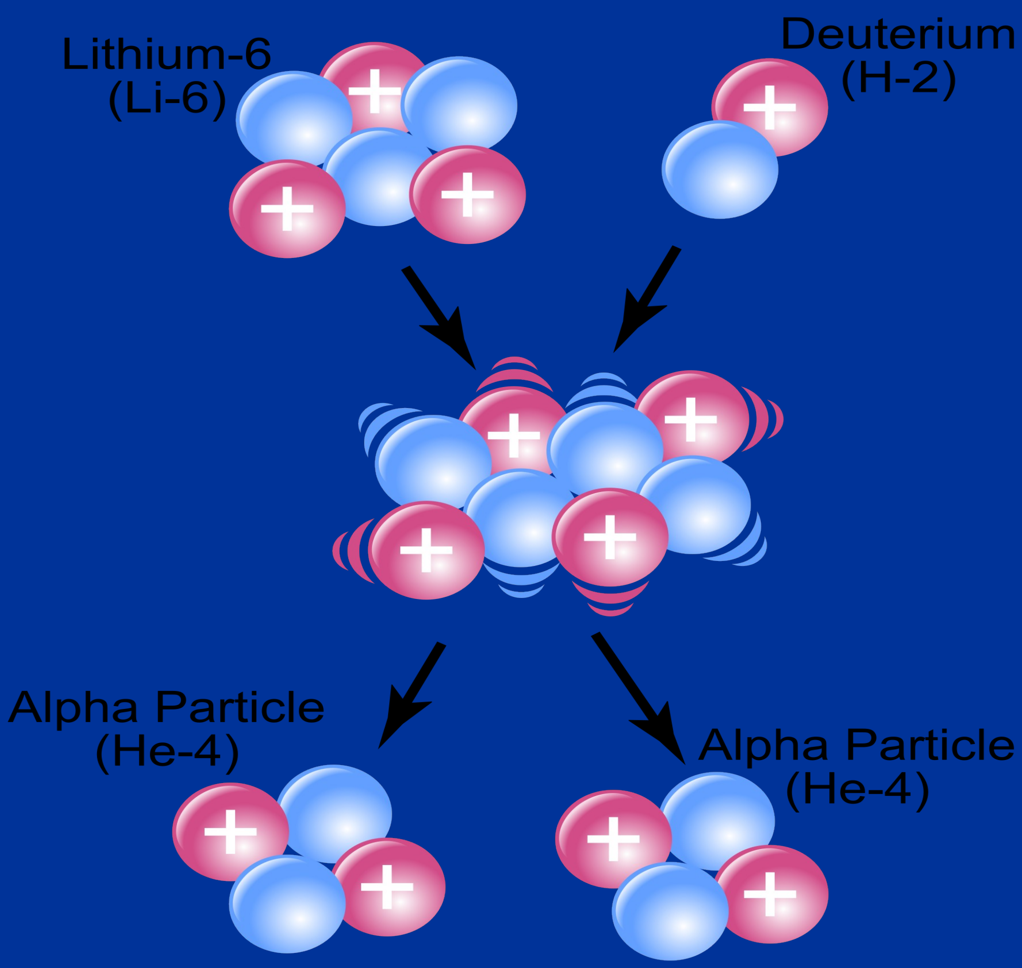
Reactions to continuum

non-perturbative (hadronic)

a + b ----> c + d +...

$$\sigma(\omega) \sim |T_{\beta\alpha}(E)|^2$$

H is the Hamiltonian of the 8-body system



Lithium-6 – Deuterium Reaction

Reactions to continuum

non-perturbative (hadronic)

a + b ----> c + d +...

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$$\sigma(\omega) \sim |T_{\beta\alpha}(E)|^2$$

General form of T-matrix

(notation as Goldberger-Watson Collision Theory)

$$T_{\beta\alpha}(E) = < \chi_\beta \mathcal{V}_\alpha \chi_\alpha > + < \chi_\beta \mathcal{V}_\beta (E - H + i\eta)^{-1} \mathcal{V}_\alpha \chi_\alpha >$$

Reactions to continuum

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A-body continuum energy

Reactions to continuum

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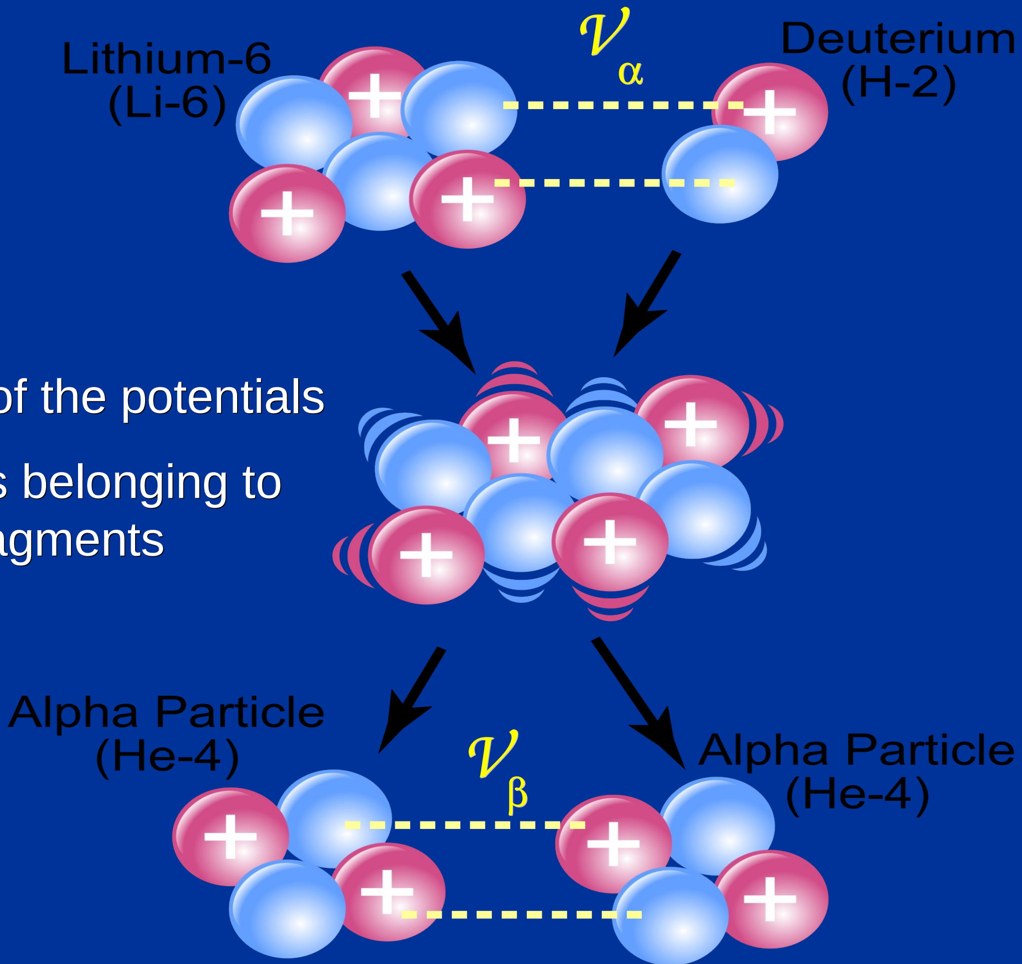
$\chi_{\beta,}$ and χ_α are the "channel functions" (with proper antisymmetrization), namely products of the **bound states** of **a** and b, times a relative Plane Wave

$$|\chi_\alpha> = \mathcal{A} |a>|b>|\mathcal{PW}>$$

H is the Hamiltonian of the 8-body system



$\mathcal{V}_{\alpha,\beta}$ is the sum of the potentials between particles belonging to different fragments

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“easier” part

Very difficult part

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"easier" part

Very difficult part

If we denote $\mathcal{V}_{\alpha,\beta}\, \chi_{\alpha,\beta} = \phi_{\alpha,\beta}$

$\mathcal{V}_{\alpha,\beta}$ is the sum of the potentials between particles belonging to different fragments

General form of T-matrix

$$T_{\beta\alpha}(E) = < \chi_\beta\, \mathcal{V}_\alpha\, \chi_\alpha > \; + < \chi_\beta\, \mathcal{V}_\beta\, (E - H + i\,\eta)^{-1} \mathcal{V}_\alpha\, \chi_\alpha >$$

easier part

Very difficult part

$$< \phi_\beta |\; (E - H + i\,\eta)^{-1} | \phi_\alpha >$$

One can manipulate the non trivial part:

$$< \phi_\beta | \ (E - H + i\eta)^{-1} \ | \phi_\alpha > =$$

Step 1) Insert **completeness of eigenstates** |n> of H: $\Sigma_n |n><n| = I$

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$$= \int d\omega \, (E - \omega + i\eta)^{-1} \; F_{\alpha\beta}(\omega) =$$

the problem reduces to calculate the function $F_{\alpha\beta}(\omega)$

$$F_{\alpha\beta}(\omega) = \Sigma_n \delta(\omega - E_n) < \phi_\beta | n > < n | \phi_\alpha >$$

Similar expressions!

Non-Pert.

$$F_{\alpha\beta}(\omega) = \sum_n \langle \phi_\beta | n \rangle \langle n | \phi_\alpha \rangle \, \delta(\omega - E_n)$$

Pert.

$$S(\omega) = \sum_n \langle 0 | \Theta^+ | n \rangle \langle n | \Theta | 0 \rangle \, \delta(\omega - E_n + E_0)$$

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$|0\rangle$, $|\phi_\alpha\rangle$, $|\phi_\beta\rangle$ *Needs only to be able to calculate bound states!*

(Remember: $\phi_\alpha = \mathcal{V}_\alpha \, \chi_\alpha = \mathcal{V}_\alpha \, \mathcal{A} \, |a\rangle |b\rangle \, |\mathcal{PW}\rangle$)

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$| n \rangle$ *All eigenstates of the Hamiltonian,*
Bound and continuum

At present there are only few methods for reactions and they are limited to A=3,4. Why?

scattering many-body problem!

In configuration space (Schroedinger)

Difficulties in dealing with asymptotic conditions in the solution of the many coupled differential equations

scattering many-body problem

In momentum space (Faddeev-Yakubowski)

Difficulties in coping with complicated poles in the many coupled integral equations

The integral transform approach may help

Integral transforms:

$$\Phi(\sigma) = \int d\omega \; K(\omega,\sigma)\, F(\omega)$$

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$$\Phi(\sigma) = \int d\omega \; K(\omega,\sigma)\, F(\omega)$$

Sometimes problematic!

In fact:

$$\Phi(\sigma) = \int d\omega \;\; K(\omega,\sigma) \, F(\omega)$$

Suppose to have a different **F'(ω)**

↓

$$[F(\omega) + A \sin(\nu\omega)]$$

$$\Phi(\sigma) = \int d\omega \; K(\omega,\sigma) \, F(\omega)$$

Suppose to have a different **F'(ω)**

$$\Phi(\sigma) + A \, \Delta \Phi(\nu) = \int d\omega \; K(\omega,\sigma) \left[F(\omega) + A \sin(\nu\omega) \right]$$

$$\Phi(\sigma) = \int d\omega \; K(\omega,\sigma) \, F(\omega)$$

Error in the transform

Suppose an error

$$\Phi(\sigma) + A \, \Delta\Phi(\nu) = \int d\omega \; K(\omega,\sigma) \, [F(\omega) + A \sin(\nu\omega)]$$

for very large ν

0

independently on the amplitude A of the error!

More remarks later on this aspect

Remember the similar expressions of the crucial quantities for perturbative and Non-perturbative reactions:

Non-Pert.

$$F_{\alpha\beta}(\omega) = \sum_n <\phi_\beta | n><n|\phi_\alpha> \delta(E_n - \omega)$$

Pert.

$$S(\omega) = \sum_n <0|\Theta^+|n><n|\Theta|0> \delta(\omega - E_n + E_0)$$

$|n>$ *All eigenstates of the Hamiltonian, Bound and continuum*

Let us consider e.g. S(ω)

(similar for $F_{\alpha\beta}(\omega)$ with $\Theta\,|\,0>$ replaced by $|\,\phi_\alpha>$

and $<0\,|\,\Theta^{+}$ replaced by $<\phi_\beta\,|$)

$$S(\omega) = \sum_n\!\!\!\!\!\!\int |\langle n|\Theta|0\rangle|^2\,\delta(\omega - E_n + E_0)$$

Scattering states

Energies in the continum

$$\mathrm{S}(\omega) = \sum_n |\langle n|\Theta|0\rangle|^2 \, \delta(\omega - E_n + E_0)$$

$$\Phi(\sigma) = \int S(\omega) \, K(\omega,\sigma) \, d\omega =$$

1) exchange $\int$ and $\sum$

$$\Phi(\sigma) = \sum_n \int d\omega \, \delta(\omega - E_n + E_0) \, K(\omega,\sigma) \, |< 0 \,|\, \Theta \,|\, n >|^2$$

2) integrate the δ–function

$$\Phi(\sigma) = \sum_n K(E_n - E_0, \sigma) < 0 \,|\, \Theta^+ \,|\, n > < n \,|\, \Theta \,|\, 0 > =$$

$$= \sum_n < 0 \,|\, \Theta^+ \, K(H - E_0, \sigma) \,|\, n > < n \,|\, \Theta \,|\, 0 >$$

3) Use completeness of H eigenstates

$$\Sigma_n | n >< n | = I$$

$$\Phi(\sigma) = \sum\!\!\!\!\!\!\int_n < 0 | \Theta^+ K(H-E_0,\sigma) | n >< n | \Theta | 0 >$$

$$= \langle 0 | \Theta^+ K(H-E_0,\sigma)\, \Theta | 0 \rangle$$

$$\Phi(\sigma) = \int S(\omega)\, K(\omega,\sigma)\, d\omega =$$

$$\langle 0 |\, \Theta^{+}\, K(H-E_0,\sigma)\, \Theta\, | 0 \rangle$$

The calculation of **ANY** transform seems to require, **in principle**, only the knowledge of the ground state!

However,

$K(H-E_0,\sigma)$ can be quite a complicate operator.

$$\Phi(\sigma) = \boxed{\langle 0|\, \Theta^{+}\, K(H-E_0,\sigma)\, \Theta |0\rangle}$$

The calculation of **ANY** transform seems to require, **in principle**, only the knowledge of the ground state!

However,

$K(H-E_0,\sigma)$ can be quite a complicate operator.

So, which kernel is suitable for calculation of this?

$$\Phi(\sigma) = \langle 0 | \Theta^{+} K(H-E_0,\sigma)\, \Theta | 0 \rangle$$

One familiar example: sum rules!

Sum rules are transforms in terms of ***"Moments"***

$K(\omega,\sigma) = \omega^{\sigma}$ with σ integer

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One familiar example: sum rules!

Sum rules are transforms in terms of ***"Moments"***

$K(\omega,\sigma) = \omega^{\sigma}$ with σ integer

To obtain $S(\omega)$ the inversion of the transform

is equivalent to the reconstruction of $S(\omega)$

in terms of its moments (theory of moments)

One familiar example: sum rules!

Sum rules are transforms in terms of ***"Moments"***

$K(\omega,\sigma) = \omega^{\sigma}$ with σ integer

however,

$$\Phi(\sigma) = \int d\omega\, \omega^{\sigma} S(\omega)$$

may be **infinite** for some $\sigma > \bar{\sigma}$!!!

Another common example:

The Laplace Kernel:

$$\Phi(\sigma) = \int e^{-\omega\sigma} S(\omega)\, d\omega = \langle 0|\, \Theta^{+}\, e^{i(H-E_0)\sigma}\, \Theta|0\rangle$$

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$$\Phi(\sigma) = \int e^{-\omega\sigma} S(\omega)\, d\omega = \langle 0|\, \Theta^{+}\, e^{i(H-E_0)(i\sigma)}\, \Theta|0\rangle$$

In Condensed Matter Physics:

In QCD

In Nuclear Physics:

The Laplace Kernel:

$$\Phi(\sigma) = \int e^{-\omega\sigma} S(\omega)\, d\omega = \langle 0 | \Theta^{+} e^{-(H-E_0)\tau} \Theta | 0 \rangle$$

In Condensed Matter Physics:

In QCD

In Nuclear Physics:

$\sigma = \tau =$ it imaginary time!

$\Phi(\tau)$ is calculated with **Monte Carlo Methods**

The Laplace Kernel:

$$\Phi(\sigma) = \int e^{-\omega\sigma} S(\omega)\, d\omega = \langle 0 | \Theta^{+} \mathbf{e}^{-(H-E_0)\tau} \Theta | 0 \rangle$$

In Condensed Matter Physics:

In QCD

In Nuclear Physics:

$\sigma = \tau =$ it imaginary time!

$\Phi(\tau)$ is calculated with **Monte Carlo Methods**
and then inverted with methods
based on Bayesian theorem (MEM)

Are there other kernels suitable for diagonalization methods on *finite norm basis functions*

???

$$\boxed{\Phi(\sigma)} = \int S(\omega)\, K(\omega,\sigma)\, d\omega =$$

$$\boxed{\langle 0 |\, \Theta^{+}\, K(H-E_0,\sigma)\, \Theta \,| 0 \rangle}$$

Notice !!!

$$S(\omega) = \sum\!\!\!\!\!\!\int_n |\langle n|\Theta|0\rangle|^2\, \delta(\omega - E_n + E_0)$$

S(ω) =

$$= -1/\pi \ \mathrm{Im} \left[\Sigma_n <0|\Theta^+|n><n|\,\Theta\,|0> \right] (\omega - E_n + E_0 + \iota\varepsilon\,)^{-1}]$$

$$= 1/\pi \ \mathrm{Im} \left[\Sigma_n <0|\,\Theta^+(H - \omega - E_0 + \iota\varepsilon\,)^{-1} |n><n| \Theta\,|0> \right]$$

$$= -1/\pi \ \mathrm{Im} \left[<0|\,\Theta^+(H - \omega - E_0 + \iota\varepsilon\,)^{-1}\, \Theta\,|0> \right]$$

If we had to deal with a **"confined"** system one could represent H on **bound states eigenfunctions |ν >**

$$\langle 0|\Theta^{+}\, K(H-E_0,\sigma)\, \Theta|0\rangle = \Phi(\sigma) =$$

$$\Sigma_{\mu\nu}\, \langle 0|\, \Theta^{+}\, |\mu\rangle\, \langle\mu|\, K(H_{\mu\nu}-E_0,\sigma)|\nu\rangle\langle\nu|\, \Theta|0\rangle$$

If we had to deal with a **“confined”** system one could represent H on **bound states eigenfunctions** $|\nu>$

$$\langle 0|\Theta^+ K(H-E_0,\sigma)\,\Theta|0\rangle = \Phi(\sigma) =$$

$$\Sigma_{\mu\nu} \langle 0|\,\Theta^+\,|\mu\rangle\langle\mu|\,K(H_{\mu\nu}-E_0,\sigma)|\nu\rangle\langle\nu|\,\Theta|0\rangle$$

After diagonalizing $H_{\mu\nu}$ the transform would be simply

$$\Sigma_\lambda\ K(\varepsilon_\lambda - E_0,\sigma)\ |\langle\lambda|\,\Theta|0\rangle|^2 = \Phi(\sigma)$$

However, a nucleus is NOT **“confined”!**

The nuclear **H** has positive energy eigenstates

and therefore, in general, CANNOT be represented

on **b.s. eigenfunctions** $|\nu>$

(Continuum discretization approximation)

THE GOOD NEWS:

The representation of H on **b.s. eigenfunctions $|\nu>$**
and therefore the calculation of the transform via

$$\Phi(\sigma) = \boxed{\Sigma_\lambda \; K(\varepsilon_\lambda - E_0, \sigma) \; |\langle\lambda|\,\Theta|0\rangle|^2}$$

is **allowed** for **specific kernels $K(\omega,\sigma)$!**

No approximation!

Conditions required:

1) $\Phi(\sigma) = \int S(\omega)\, K(\omega,\sigma)\, d\omega < \infty$

2) $\int S(\omega)\, d\omega < \infty$

3) $K(\omega,\sigma) \neq \delta(\omega - \sigma)$

Conditions required N.1:

1) $\Phi(\sigma) = \int S(\omega)\, K(\omega,\sigma)\, d\omega < \infty$

This is required in order to change $\sum \quad \int$

remember ...

$$S(\omega) = \sum\!\!\!\!\!\!\int_n |\langle n|\Theta|0\rangle|^2 \, \delta(\omega - E_n + E_0)$$

$$\Phi(\sigma) = \int S(\omega)\, K(\omega,\sigma)\, d\omega =$$

1) exchange $\int$ and $\sum\!\!\!\!\!\!\int$

$$\Phi(\sigma) = \sum\!\!\!\!\!\!\int_n \int d\omega\, \delta(\omega - E_n + E_0)\, K(\omega,\sigma)\, |< 0 | \Theta | n >|^2$$

2) integrate the δ–function

$$\Phi(\sigma) = \sum\!\!\!\!\!\!\int_n K(E_n - E_0, \sigma) < 0 | \Theta^+ | n > < n | \Theta | 0 > =$$

$$= \sum\!\!\!\!\!\!\int_n < 0 | \Theta^+ K(H - E_0, \sigma) | n > < n | \Theta | 0 >$$

Condition required N.2:

2) $\int S(\omega)\, d\omega < \infty$

since $\int S(\omega)\, d\omega = \langle 0|\Theta^+ \Theta|0\rangle$

It means that $\Theta|0\rangle$ has finite norm

and therefore has bound state asymptotic behaviour

Therefore it is allowed to insert
a complete **bound basis**

$$\langle 0|\Theta^+ K(H-E_0,\sigma)\,\Theta|0\rangle = \Phi(\sigma) =$$

$$\Sigma_{\mu\nu} \langle 0|\,\Theta^+\,|\mu\rangle\,\langle\mu|\,K(H_{\mu\nu}-E_0,\sigma)|\nu\rangle\langle\nu|\,\Theta|0\rangle$$

After diagonalizing $H_{\mu\nu}$ the transform would be simply

$$\Sigma_\lambda\; K(\varepsilon_\lambda - E_0,\sigma)\; |\langle\lambda|\,\Theta|0\rangle|^2 = \Phi(\sigma)$$

Condition required N.3:

3) $K(\omega,\sigma)$ not $\delta(\omega - \sigma)$!!!

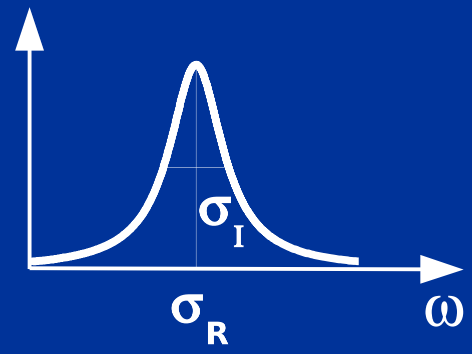
$$\Phi(\sigma) = \int S(\omega)\, \delta(\omega - \sigma)\, d\omega = S(\sigma)$$

$$\langle 0 |\, \Theta^{+}\, \delta(H - E_0 - \sigma)\, \Theta\, | 0 \rangle =$$

$$-1/\pi \, \mathrm{Im} \left[<0|\, \Theta^{+} (H - \sigma - E_0 + \iota\varepsilon)^{-1}\, \Theta\, |0> \right]$$

In this case the insertion of a complete bound basis would correspond to ***Continuum discretization approximation*** *!!!*

The LIT Kernel satisfies the condition N.1



$$\Phi(\sigma_R,\sigma_I) = \sigma_I/\pi \int [(\omega-\sigma_R)^2+\sigma_I^{\,2}]^{-1} S(\omega)\, d\omega < \infty$$

The LIT Kernel satisfies the conditions....

Moreover it is not a δ-function, but
one of **its representations.**
Therefore makes the inversion easier!!

ε initesimal!

$$S(\omega) = -1/\pi \, \mathrm{Im} \left[<0| \Theta^{+} (H - E_0 - \omega + \iota\varepsilon)^{-1} \Theta |0> \right]$$

$$\Phi(\sigma) = -\Gamma/\pi \, \mathrm{Im} \left[<0| \Theta^{+} (H - E_0 - \sigma_R + \iota\sigma_I)^{-1} \Theta |0> \right]$$

σ_I finite!

Illustration of the problem:

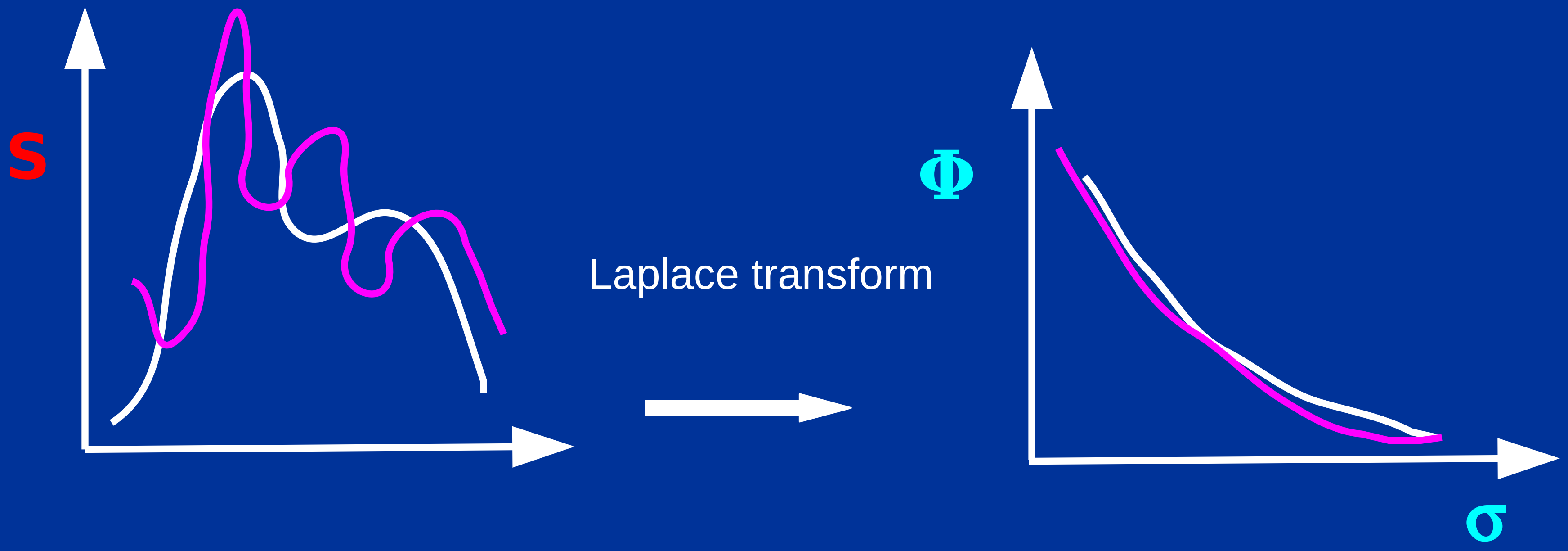


Illustration of the problem:

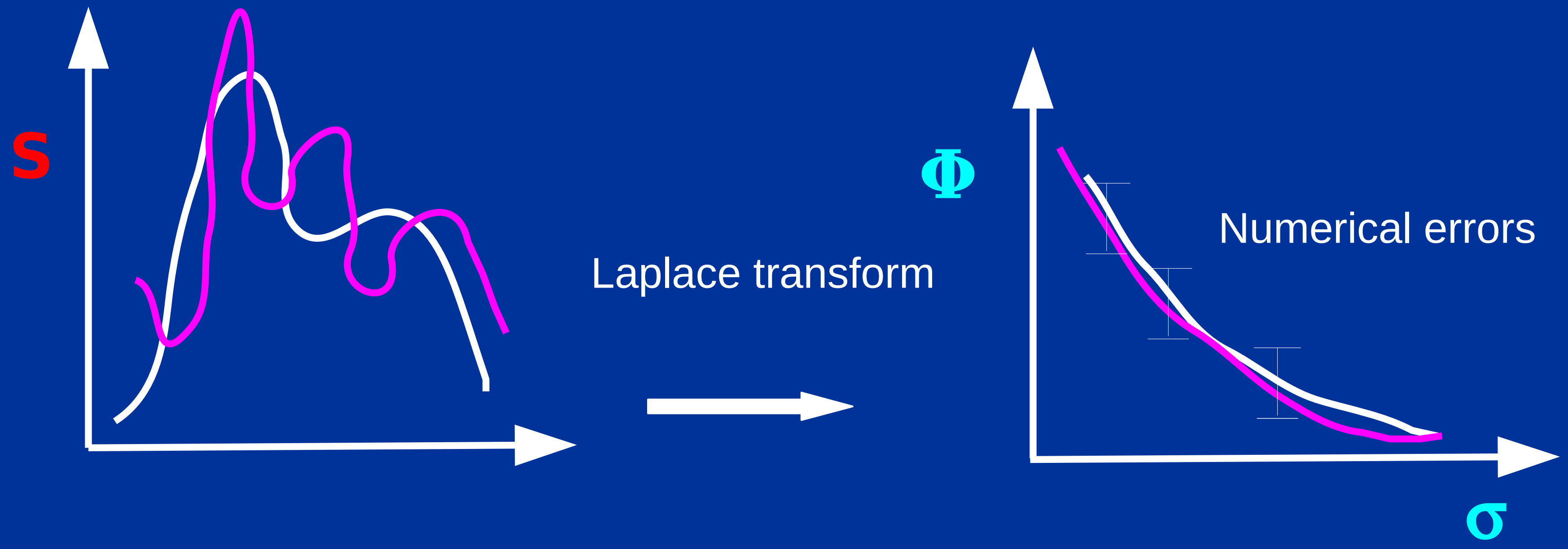
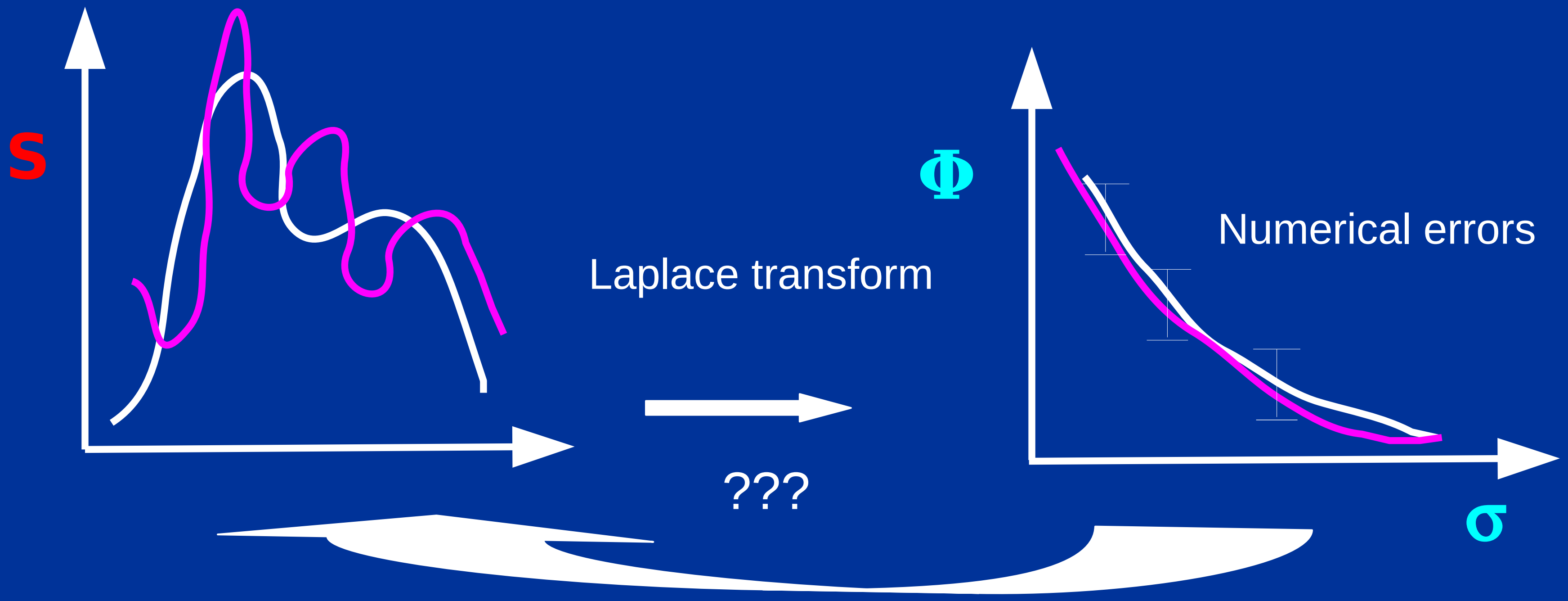
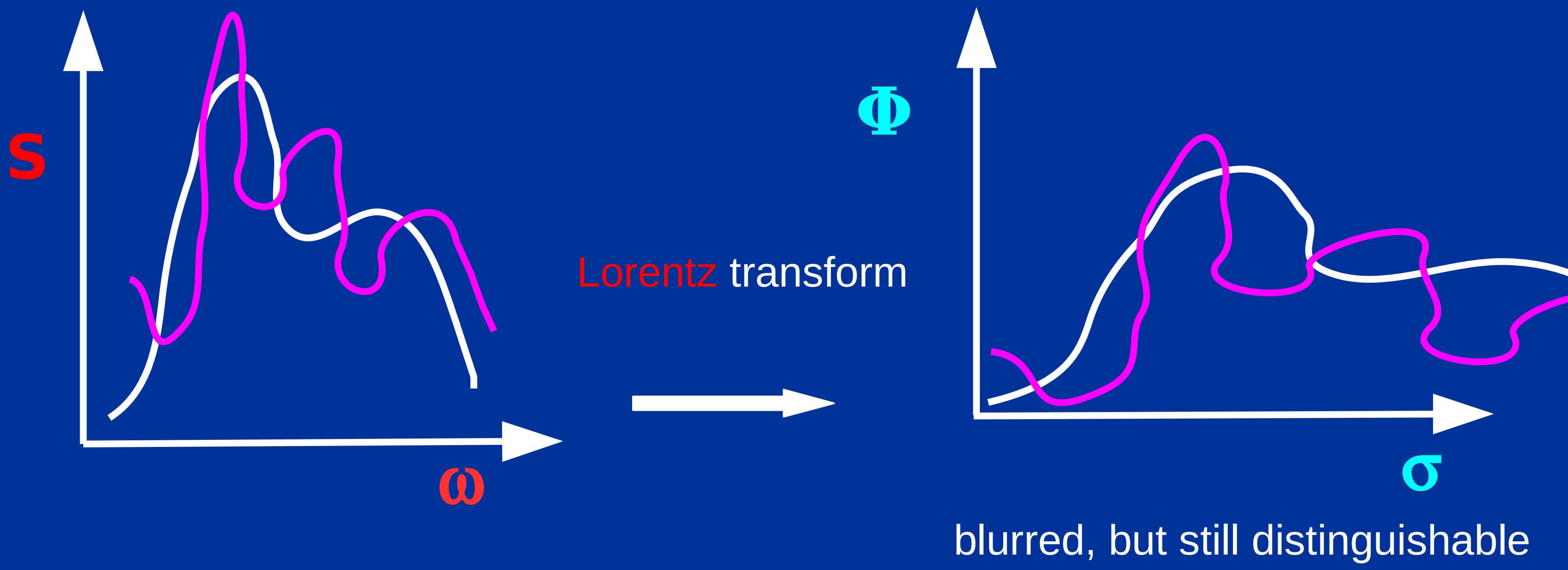


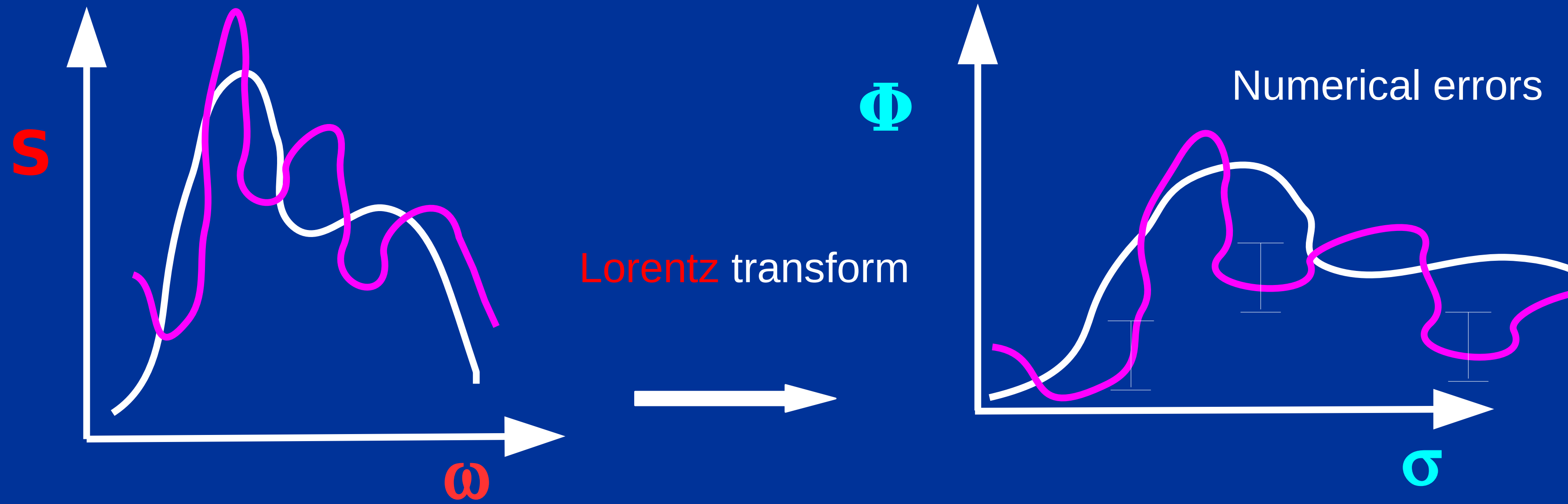
Illustration of the problem:



Advantages of a delta function representation:

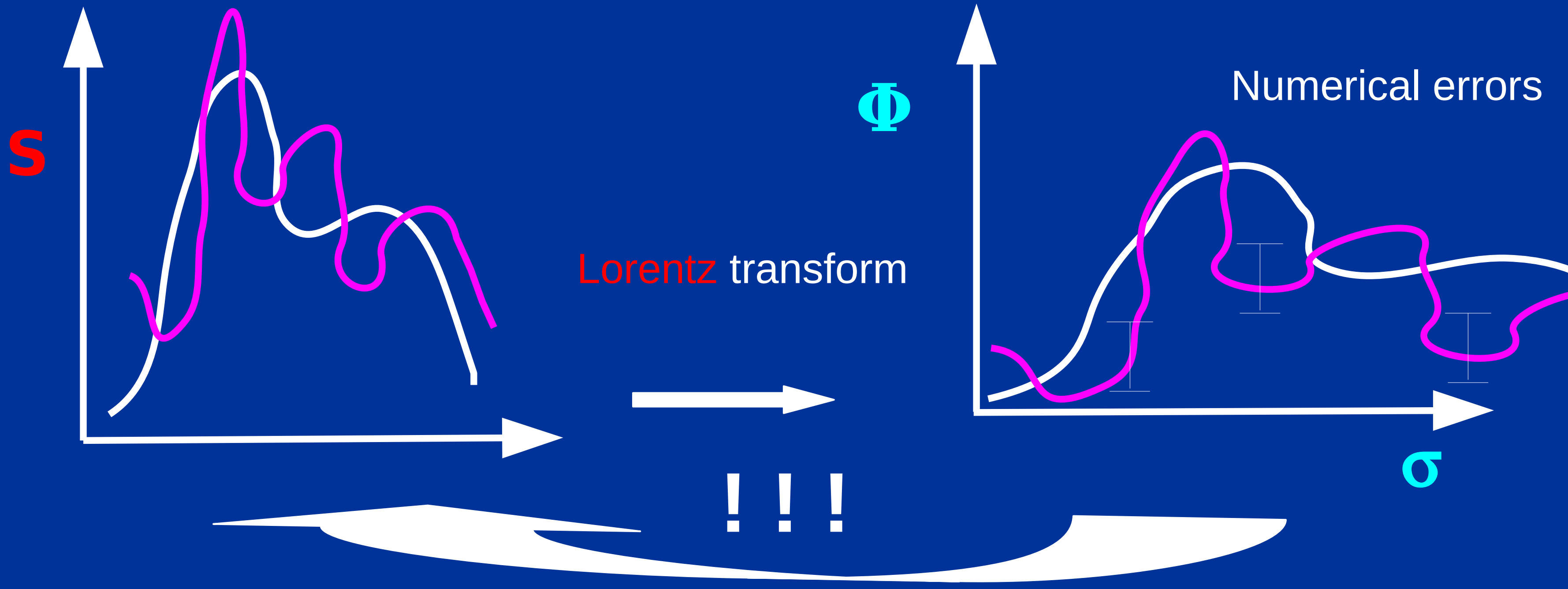


Advantages of a delta function representation:



blurred, but still distinguishable
also with errors!

Advantages of a delta function representation:



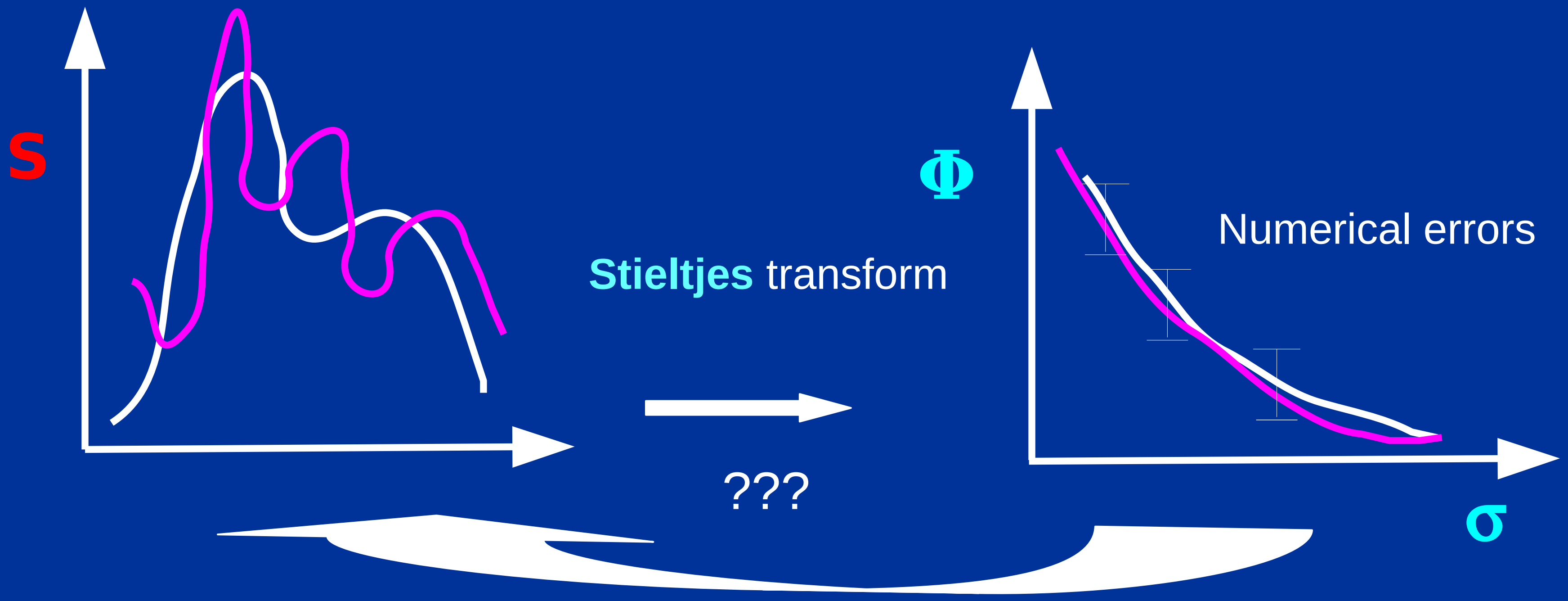
Moreover,
if the width of the Lorentzian were
the same as the experimental resolution
one would not need to invert the transform!

Other kernels?

The Stieltjes Kernel:

$$K(\omega, \sigma) = (\omega + \sigma)^{-1}$$

Illustration of the problem:
Same as Laplace!



However, it may be useful for another purpose:

In fact:

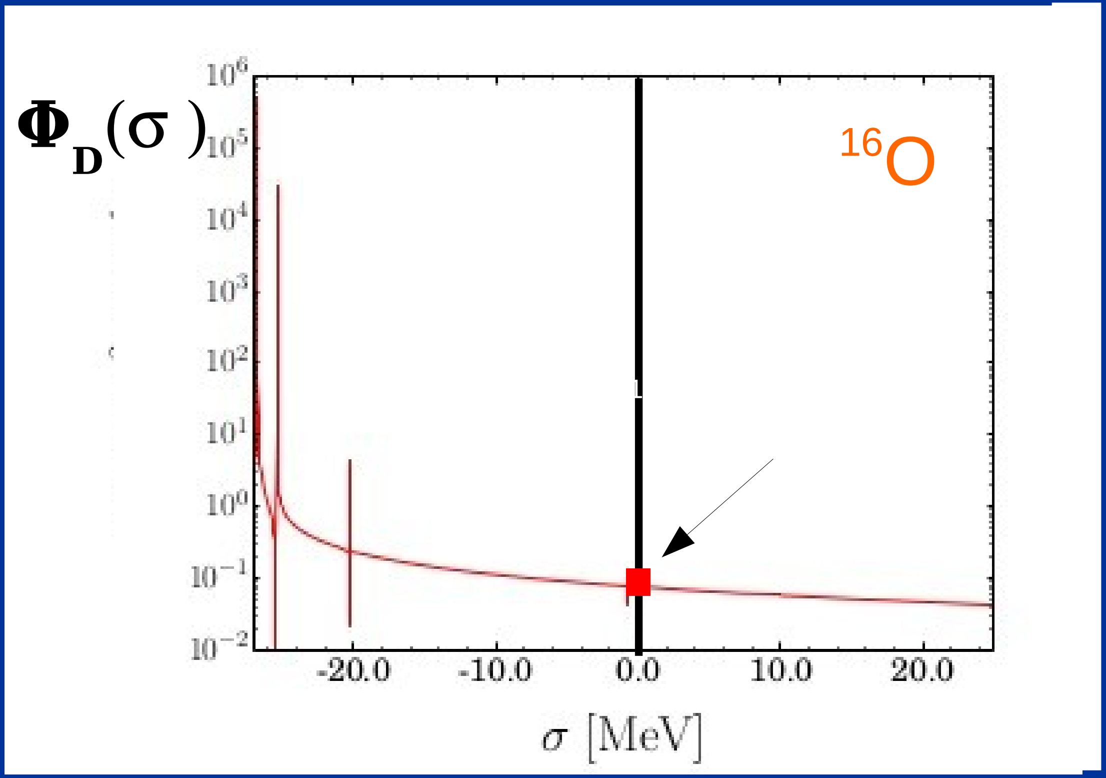
$$\lim_{\sigma \to 0} \Phi(\sigma) = \int S(\omega)\, \omega^{-1}\, d\omega = \alpha_{\Theta}$$

"generalized polarizability"

e.g. electric polarizability, magnetic susceptibility, compressibility etc... depending on Θ

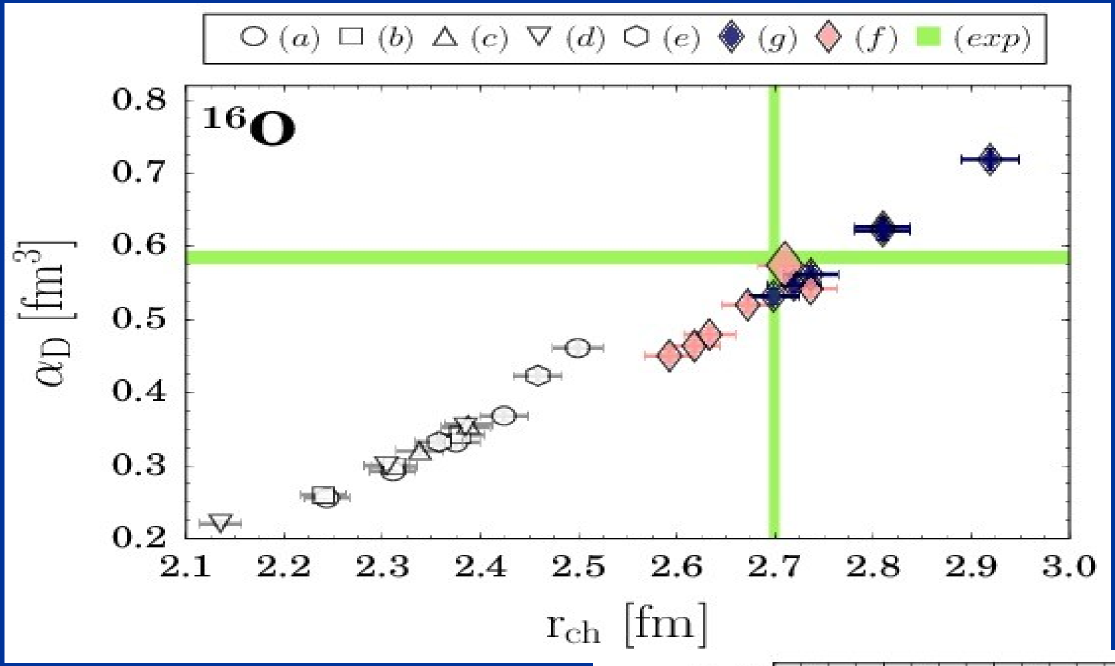
Recent results on α_Θ with $\Theta = D$ (El. Dipole Polarizability)

Electric Dipole Polarizability as limit of the Stieltjes transform for σ ---> 0



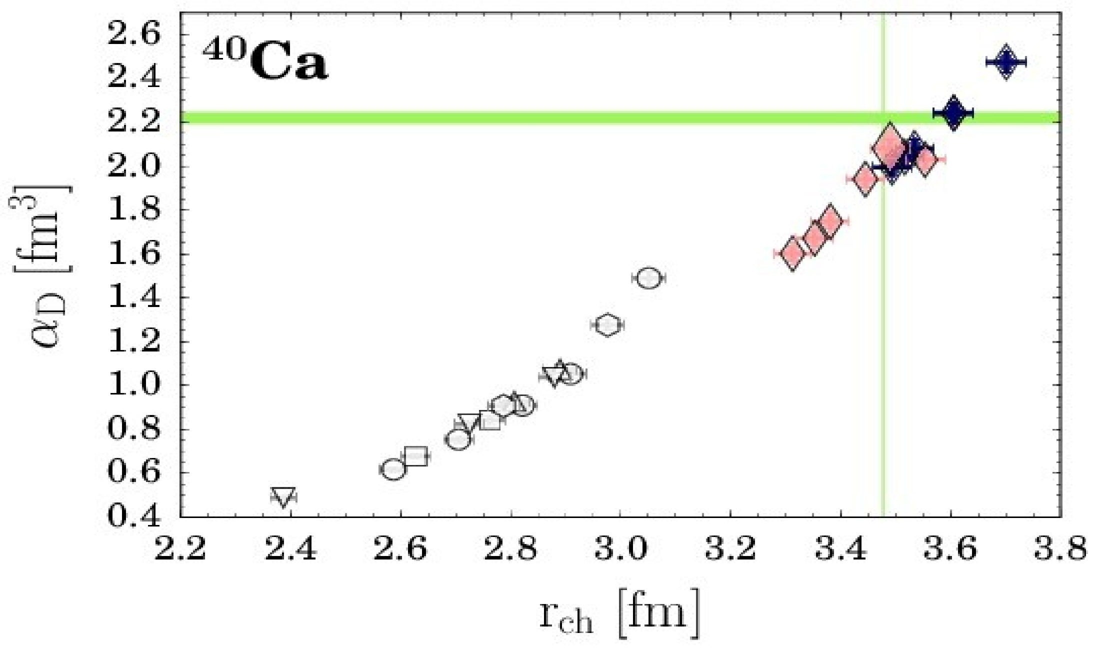
M.Miorelli et al. Phys. Rev. C 94, 034317 (2016)

b.s. expansion: Coupled Cluster



Interesting correlation with the proton charge radius

Role of 3b-force



G. Hagen et al.
Nature Phys. 2016

A Transform with a kernel suitable for Monte Carlo methods:

[A.Roggero, F. Pederiva, G.O. Phys. Rev. B 88, 115138 (2013)]

combination of Sumudu kernels:

$$K(\omega , \sigma, P) = N \sigma \left(\frac{e^{-\mu\, \omega/\sigma}}{\sigma} - \frac{e^{-\nu\, \omega/\sigma}}{\sigma} \right)^{P}$$

$\nu/\mu = b/a$ $\qquad \nu - \mu = \dfrac{ln\,[b] - ln\,[a]}{b - a}$ $\qquad b > a > 0$ integer

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$$\nu/\mu = b/a \qquad \nu-\mu = \frac{\mathit{ln}\,[b] - \mathit{ln}\,[a]}{b-a} \qquad b > a > 0 \text{ integer}$$

$$K(\omega\,,\sigma,\mathbf{P}) \xrightarrow[\mathbf{P}\,\longrightarrow\,\infty]{} \delta\,(\omega-\sigma)$$

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$$= N \sum_k^P (-1)^k \binom{k}{P} e^{-\tau(P,k,\sigma)\, \omega}$$

Finite sum of Laplace Kernels!

A Transform with a kernel suitable for Monte Carlo methods:

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$$= N \sum_{k}^{P} (-1)^{k} \binom{k}{P} e^{-\tau(P,k,\sigma)\,\omega}$$

$$\tau(P,k,\sigma) = \log\,(b/a)\,[P\,a/(b-a) + k\,]\,/\,\sigma$$

Small width ---> large P ---> **large** imaginary time

Applied only to bosonic system: Liquid Helium

The transform is calculated with AFDMC and then inverted with MEM

See A.Roggero, F. Pederiva, G.O. Phys. Rev. B 88, 115138 (2013)]

Let's remember:

$$\Phi(\sigma) = \int d\omega \; K(\omega,\sigma)\, S(\omega)$$

In order to obtain $S(\omega)$ one needs to invert the transform

Problem:

Sometimes the “inversion” of $\Phi(\sigma)$ may be problematic

New Kernels?

What about “wavelets”?

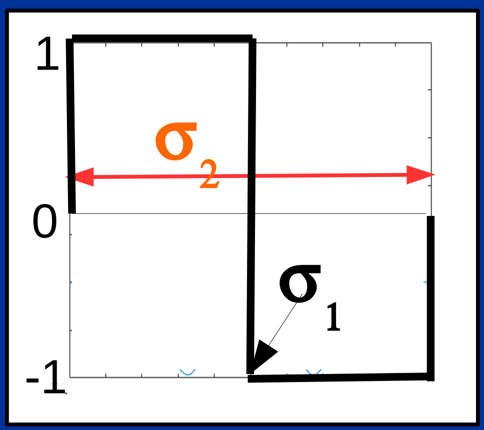
A **wavelet** Kernel is an oscillating function but with a **"window"**.

It has 2 parameters:

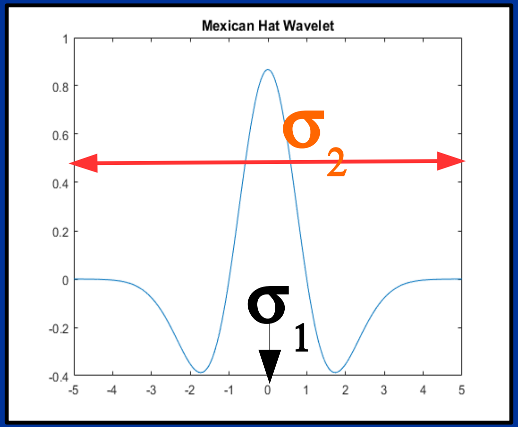
σ_2 drives the frequency of the oscillation

σ_1 drives the position of the window over the ω range

discrete



continuous



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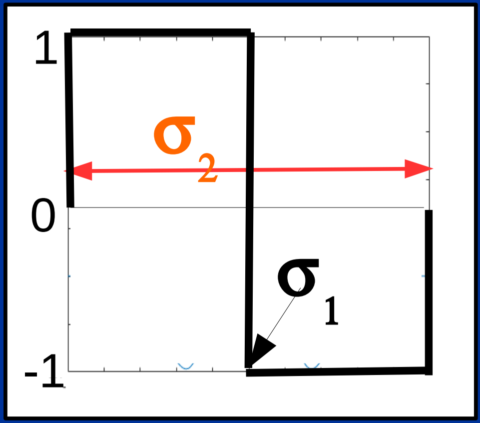
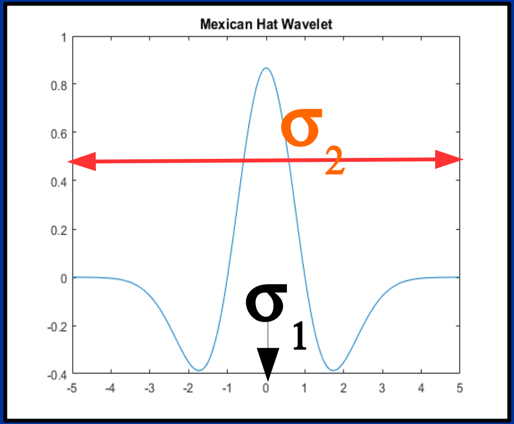


continuous



They combine the power of the Fourier Kernel

(in detecting frequencies of oscillations)

and the Lorentz Kernel

(in picking the information around specific ω ranges)

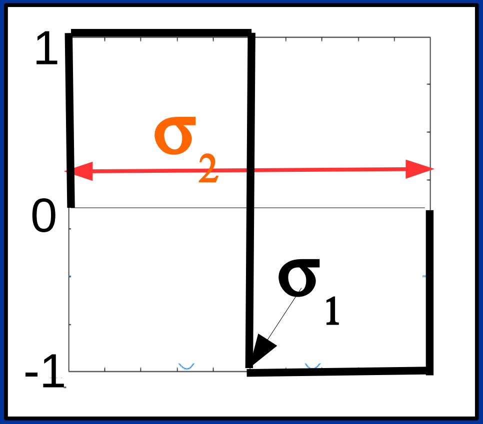
A **wavelet** Kernel is an oscillating function but with a **"window"**.

It has 2 parameters:

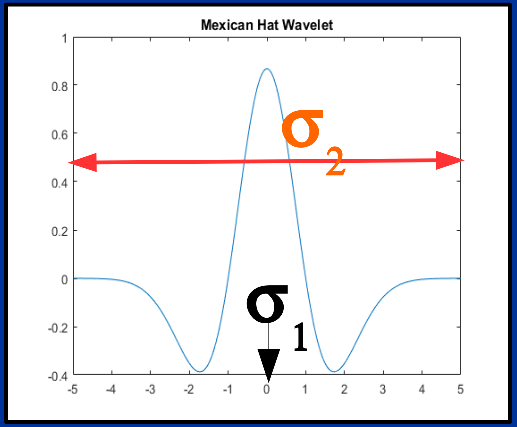
σ_1 drives the frequency of the oscillation

σ_2 drives the position of the window over the ω range

discrete



continuous



Since wavelets are orthonormal functions in principle their inversion is straightforward !

[linear combination of $\Phi(\sigma_1, \sigma_2)$]

Integral transform

$$\Phi(\sigma) = \int d\omega \; K(\omega,\sigma)\, S(\omega)$$

If $K(\omega,\sigma) \equiv K_\sigma(\omega)$ are the elements of an orthogonal basis

$\Phi(\sigma) = \Phi_\sigma$ are the coefficients of the expansion of $S(\omega)$ on that basis

then $$S(\omega) = \sum_\sigma \Phi_\sigma K_\sigma(\omega)$$

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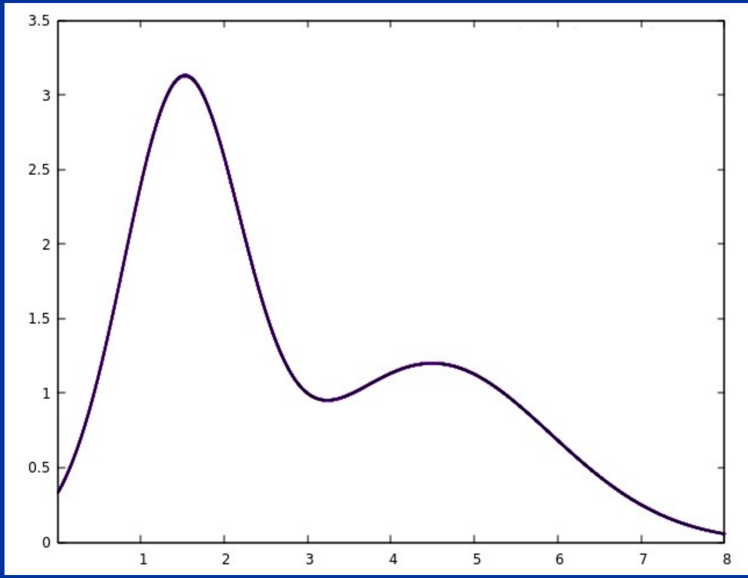
$\Phi(\sigma) = \Phi_\sigma$ are the coefficients of the expansion of $S(\omega)$ on that basis

then $$S(\omega) = \sum_\sigma \Phi_\sigma K_\sigma(\omega)$$

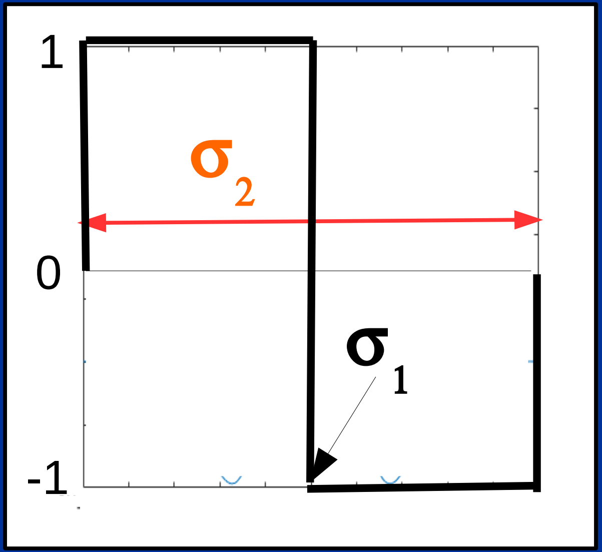
Namely the inversion is straightforward!

A model study
(discrete wavelets)

Our model S(ω)



A wavelet kernel
(Haar wavelets)

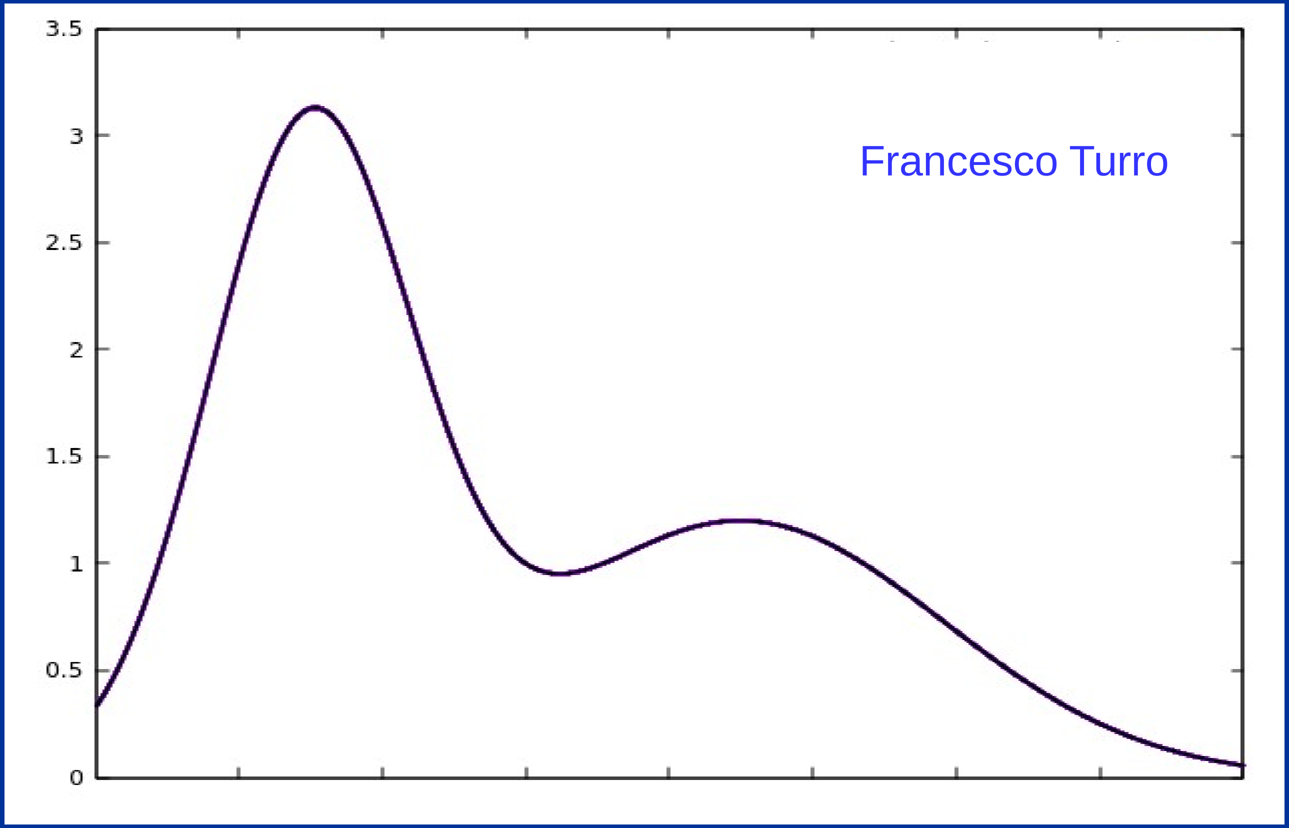


$K(\omega, \sigma_1, \sigma_2)$

F. Turro, G.O. Few-Body Sys.58 (2017) 76

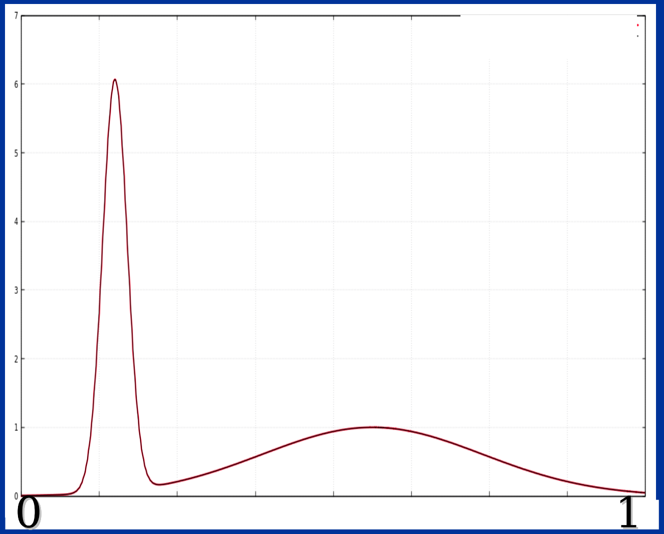
Model S(ω) and reconstructed from wavelet transform:

identical!

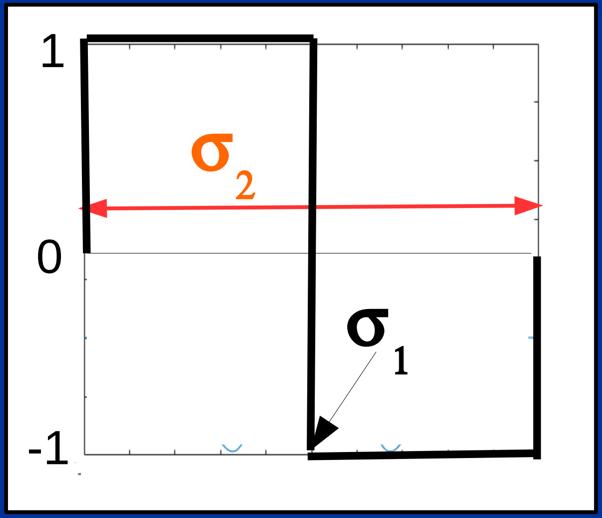


Another model study
(narrow resonance, discrete wavelets)

Our model S(ω)



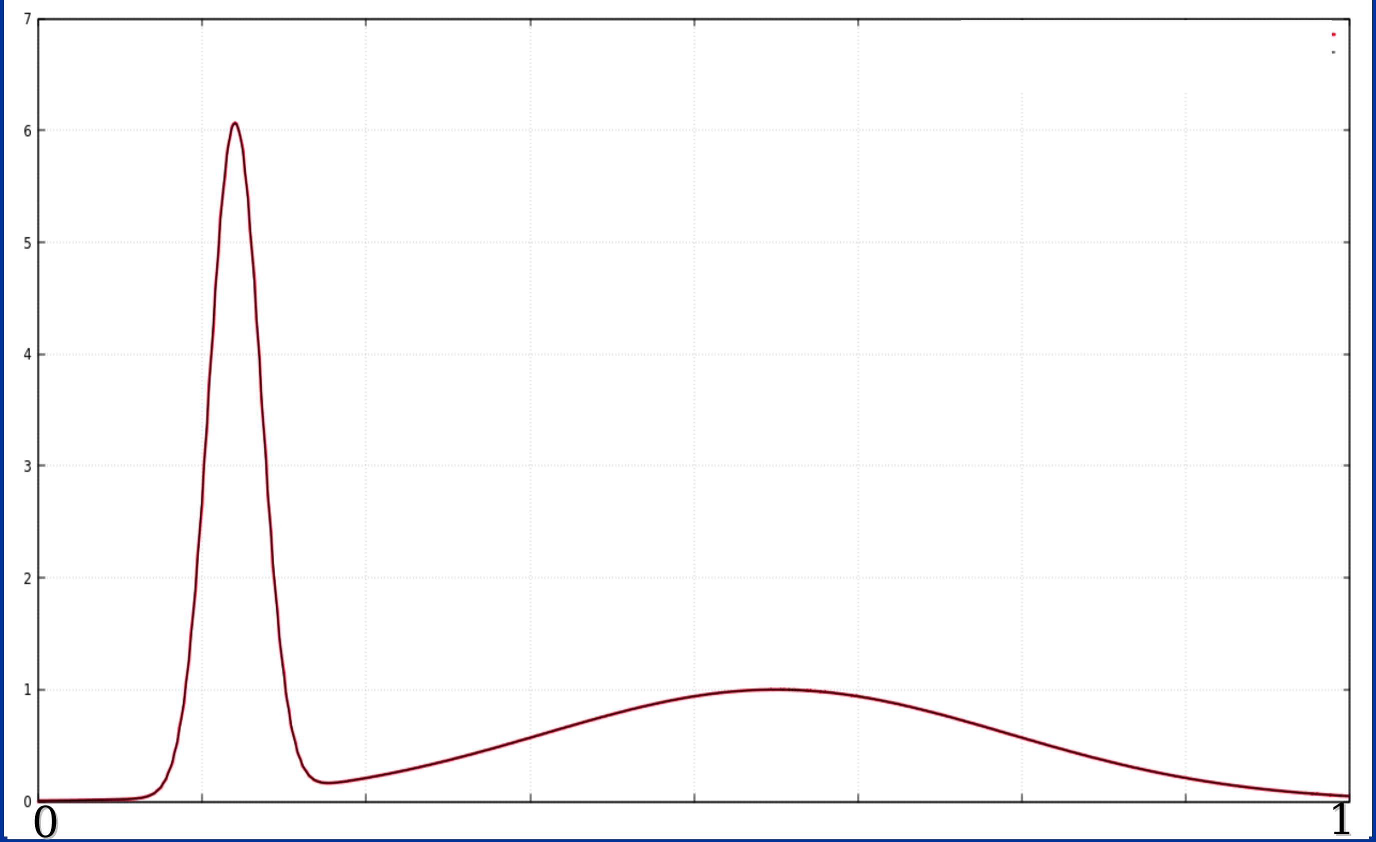
A wavelet kernel



$K(\omega, \sigma_1, \sigma_2)$

Model S(ω) and reconstructed from wavelet transform:

again identical!

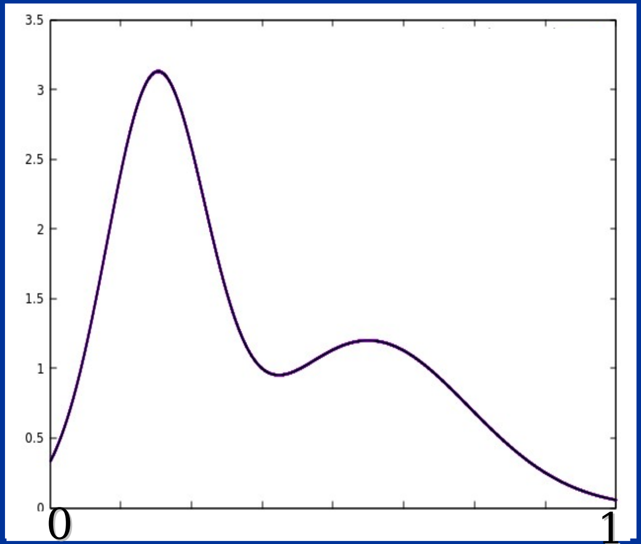


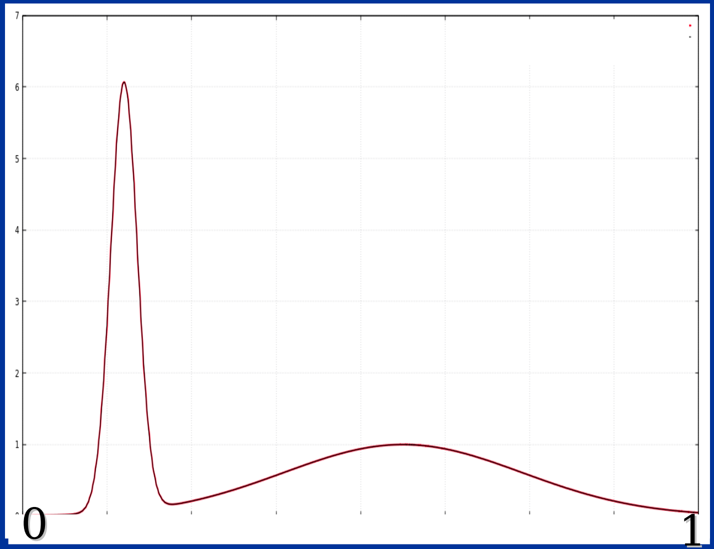
Which information has been used to reconstruct S(ω) ???

Which information has been used to reconstruct $S(\omega)$???

values of $K(\omega, \sigma_1, \sigma_2)$ with different widths

$\sigma_2 = 1/2^J$, $J=1\text{-}5$





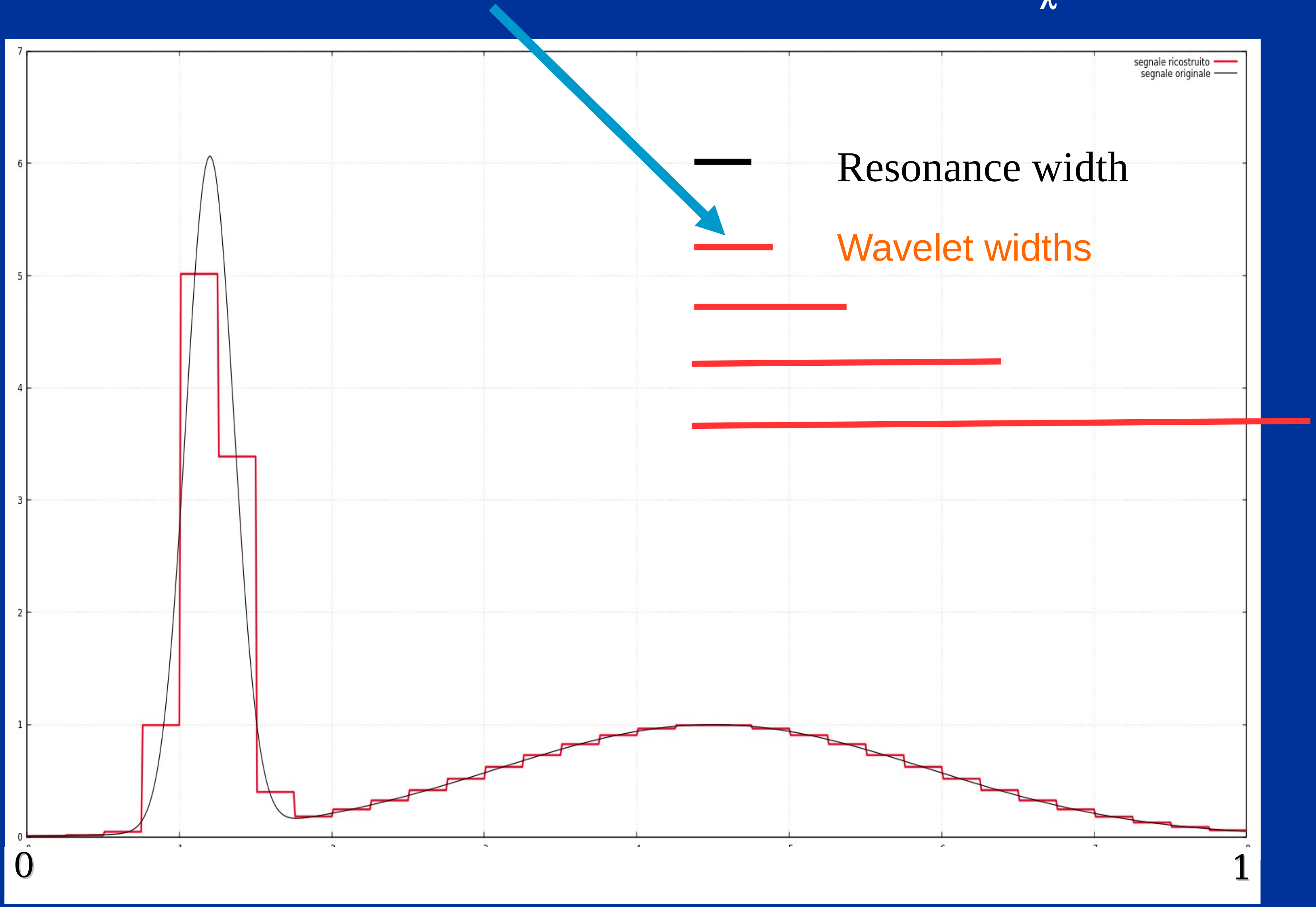
namely a lot of different resolutions up to $\sigma_2 = 0.03$!!!

This may not be possible with diagonalization in realistic cases!

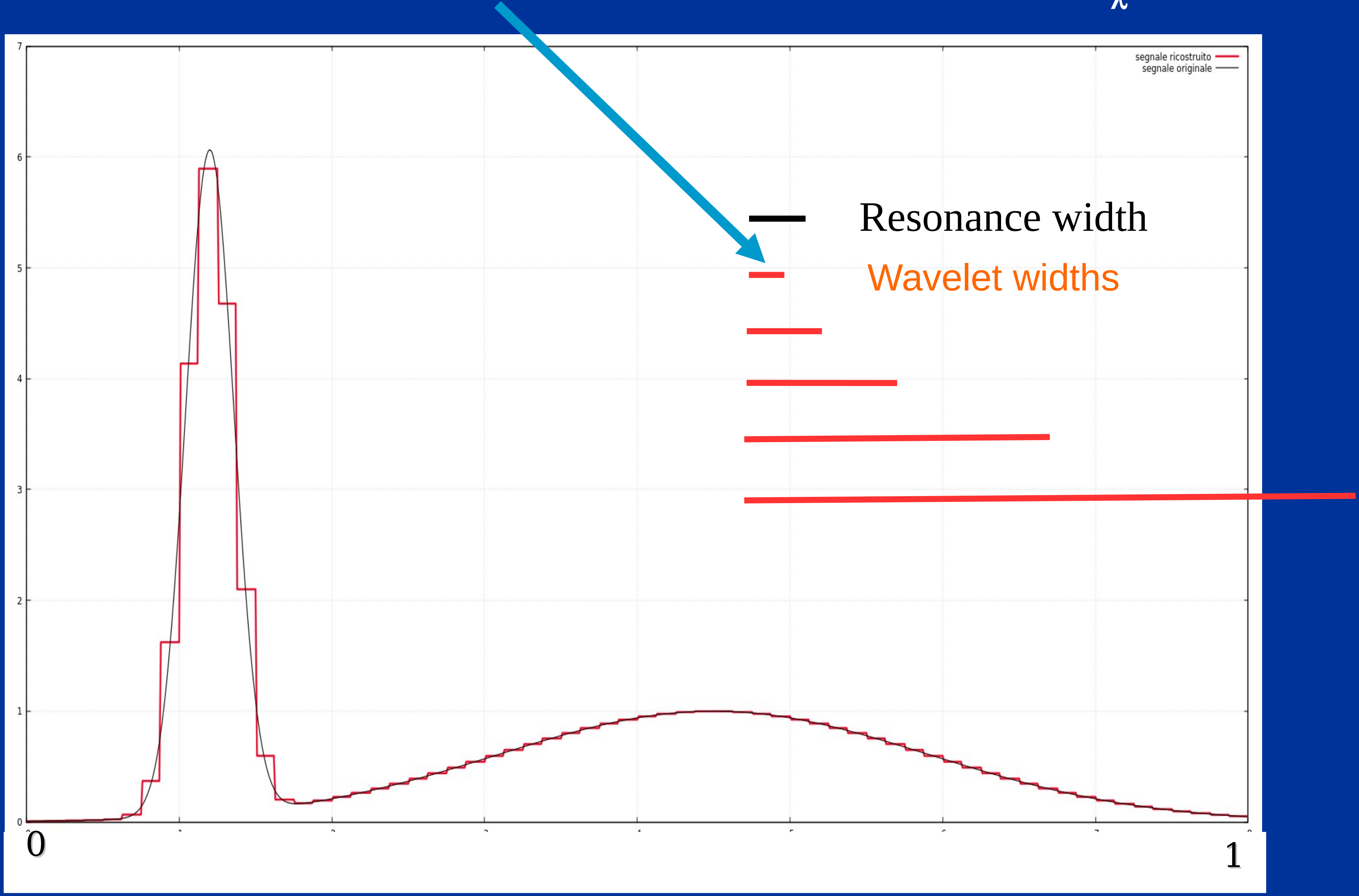
Remember Winfried's talk:

<<one must have enough Hamiltonian eigenvalues In the energy range of interest!>>

Hp. on smallest "resolution" (low density of ε_λ):



Hp. on smallest "resolution" (higher density of ε_λ):



Summary:

- IT methods are alternative approaches to overcome the many-body scattering problem
- They are suitable for perturbative (inclusive and exclusive) as well as non-perturbative reactions
- New kernels have to be explored
- Big potentialities not yet exploited

Acknowledgements

to all people who have taken part in the
IT adventure over 20 years

- **Victor Efros**
- Winfried Leidemann
- Nir Barnea
- *Sonia Bacca*
- *Sofia Quaglioni*
- Ed Tomusiak
- The CC people (Gaute Hagen, Thomas Papenbrock, *Mirko Miorelli*...)
- The MC people (Francesco Pederiva, *Alessandro Roggero)*
- *...*

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