Hydrodynamics and QCD Critical Point in Magnetic Field

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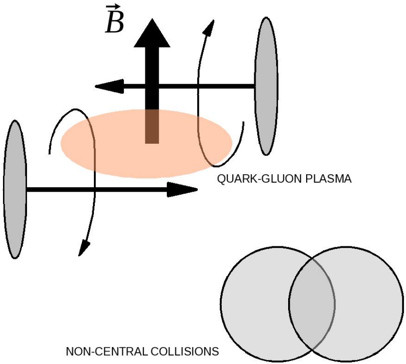
May 25, 2018

INT Workshop
Multi Scale Problems Using Effective Field Theories

Reference: Phys.Rev. D97 (2018) no.5, 056024 (with Shiyong Li)
and **Phys.Rev. D97 (2018) no.1, 016003**
and **arXiv:1711.08450** (with K. Hattori, Y. Hirono, Y. Yin)

The magnetic field in heavy-ion collisions



In heavy-ion collisions, two magnetic fields from the projectiles overlap along the same direction out of reaction plane

$$eB \sim (300\,\mathrm{MeV})^2 \sim T^2,$$

but the life time may be short $\tau \lesssim 1$ fm/c

Variables in hydrodynamics

Variables of hydrodynamics are local equilibrium parameters related to conserved quantities

Eg: $T(x)$, $u^\mu(x)$, $\mu(x)$, **etc, such that when they are constant, the system doesn't evolve**

The system changes arbitrarily slowly when these parameters vary arbitrary small in gradient expansion

Hydrodynamics with *External, Non-dynamic* Magnetic Field?

We will see that the transverse components of fluid velocity perpendicular to the magnetic field is not a hydrodynamic variable in a new emerging hydrodynamics at sufficiently low energy regime

Remark: This is not the case when we have dynamical electromagnetic fields in magneto-hydrodynamics

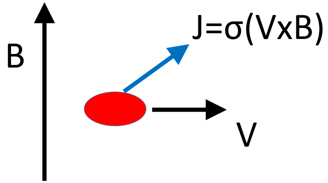
Let us start from the conventional hydrodynamics with a background magnetic field $\boldsymbol{B}$

$$j^\mu = \sigma E^\mu \,, \quad E^\mu = F^{\mu\nu} u_\nu$$

In non-relativistic limit $u^\mu = \gamma(1, \boldsymbol{v}) \approx (1, \boldsymbol{v})$ we have

$$\boldsymbol{j} = \sigma \boldsymbol{v}_\perp \times \boldsymbol{B}$$

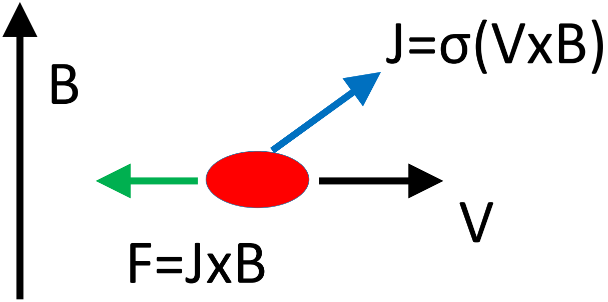
This can be understood from the fact that in the rest frame of the fluid with velocity $\boldsymbol{v}_\perp$, there is an electric field $\boldsymbol{E} = \boldsymbol{v}_\perp \times \boldsymbol{B}$, so there is an Ohmic current $\boldsymbol{j}_\perp = \sigma \boldsymbol{E}$. Or simply, the microscopic origin is the Lorentz force.



From the spatial component of the energy-momentum conservation $\partial_\mu T^{\mu\nu} = F^{\nu\alpha} j_\alpha$ with the constitutive relation $T^{0\perp} = (\epsilon + p)\boldsymbol{v}_\perp$, we get

$$\partial_t \boldsymbol{v}_\perp = \frac{1}{(\epsilon+p)} \boldsymbol{j} \times \boldsymbol{B} = -\frac{\sigma |B|^2}{(\epsilon+p)} \boldsymbol{v}_\perp \equiv -\frac{1}{\tau_R} \boldsymbol{v}_\perp$$

This is also the Lorentz force



We see that the transverse velocity $v_\perp$ relaxes to zero with a relaxation time $\tau_R = \frac{(\epsilon+p)}{\sigma|B|^2}$

They are no longer the hydrodynamic variables below this time scale, rather they are quasi-normal modes

The basic reason is the loss of conservation of transverse momentum in a background magnetic field

Absence of Transverse Conductivity

Suppose we apply a small transverse electric field $\boldsymbol{E}_\perp$. From $J^\mu = \sigma F^{\mu\nu} u_\nu$, we have

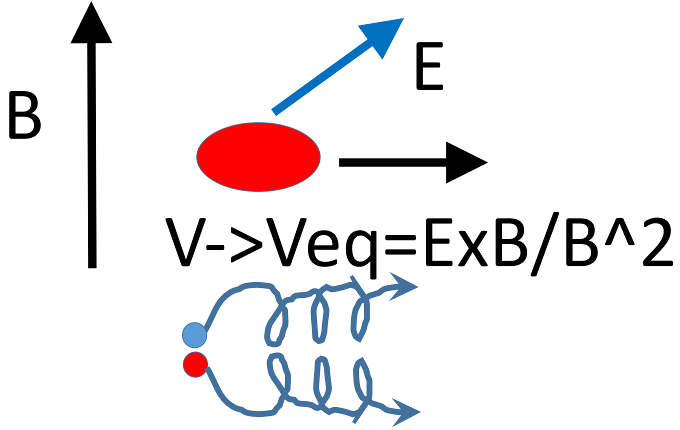
$$\boldsymbol{j} = \sigma(\boldsymbol{E}_\perp + \boldsymbol{v}_\perp \times \boldsymbol{B})$$

and the energy-momentum conservation gives

$$\begin{aligned} \partial_t \boldsymbol{v}_\perp &= \frac{1}{(\epsilon+p)} \boldsymbol{j} \times \boldsymbol{B} = \frac{\sigma}{(\epsilon+p)} \left(\boldsymbol{E}_\perp \times \boldsymbol{B} - B^2 \boldsymbol{v}_\perp\right) \\ &= -\frac{1}{\tau_R}(\boldsymbol{v}_{eq} - \boldsymbol{v}_\perp)\,, \quad \boldsymbol{v}_{eq} \equiv \frac{\boldsymbol{E}_\perp \times \boldsymbol{B}}{B^2} \end{aligned}$$

that is, $\boldsymbol{v}_\perp$ relaxes to $\boldsymbol{v}_{eq}$ with the same relaxation time τ_R. At equilibrium $\boldsymbol{v}_\perp = \boldsymbol{v}_{eq}$, the current vanishes

$$\boldsymbol{j} = \sigma(\boldsymbol{E}_\perp + \boldsymbol{v}_{eq} \times \boldsymbol{B}) = 0$$

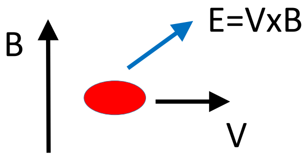

B
E
V->Veq=ExB/B^2

The charged particles get momentum kicks to the Poynting vector direction $\boldsymbol{E}_\perp \times \boldsymbol{B}$, and the fluid moves along that direction until it reaches $\boldsymbol{v}_{eq}$. In the frame with $\boldsymbol{v}_{eq}$, $\boldsymbol{E}' = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}_{eq} \times \boldsymbol{B})$ vanishes.

This is impossible when $|\boldsymbol{E}_\perp| > B$. In this case the fluid moves with $\boldsymbol{v}_{eq} = \frac{\boldsymbol{E}_\perp \times \boldsymbol{B}}{E_\perp^2}$ where the magnetic field vanishes in the fluid rest frame, and the electric field is $\boldsymbol{E}'_\perp = \boldsymbol{E}_\perp(1 - B^2/E_\perp^2)$. We have a current

$$\boldsymbol{j} = \sigma \boldsymbol{E}'_\perp = \sigma \boldsymbol{E}_\perp \left(1 - \frac{B^2}{E_\perp^2}\right) \Theta(E_\perp - B)$$

Dynamical Electromagnetism: Magnetohydrodynamics



The situation is different in magnetohydrodynamics. Since EM fields are dynamical, the E and $v_\perp$ can coordinate with each other to achieve an equilibrium

$$\boldsymbol{E} + \boldsymbol{v}_\perp \times \boldsymbol{B} = 0$$

$v_\perp$ is now a dynamical variable, and E is slaved to $v_\perp$.

Given $\boldsymbol{v}_\perp$, from Maxwell eq,

$$\partial_t \boldsymbol{E} = -\boldsymbol{j} = -\sigma(\boldsymbol{E} + \boldsymbol{v}_\perp \times B) = -\sigma(\boldsymbol{E} - \boldsymbol{E}_{eq})$$

where $\boldsymbol{E}_{eq} = -\boldsymbol{v}_\perp \times \boldsymbol{B}$. The $\boldsymbol{E}$ relaxes to the equilibrium value with the relaxation time $1/\sigma$.

From $\partial_t n = -\nabla \cdot \boldsymbol{j} = -\sigma \nabla \cdot \boldsymbol{E} = -\sigma n$, the charge density n is also not a variable in magnetohydrodynamics.

The variables in magnetohydrodynamics below the scale of σ are $T(x)$, $\boldsymbol{v}(x)$ (or $u^\mu(x)$) and $\boldsymbol{B}(x)$, whereas the electric field is slaved to be

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \frac{1}{\sigma}\nabla \times \boldsymbol{B} + \cdots$$

Alfven Waves

In the presence of background $\boldsymbol{B}_0$, consider a transverse velocity fluctuation $\delta \boldsymbol{v}_\perp$, and the electric field is $\delta \boldsymbol{E}_\perp = \delta \boldsymbol{v}_\perp \times \boldsymbol{B}_0$. This implies a magnetic field fluctuation by $\partial_t \delta \boldsymbol{B} = \nabla \times \delta \boldsymbol{E}_\perp$ and the current by $\delta \boldsymbol{j} = \nabla \times \delta \boldsymbol{B}$. From the Lorentz force

$$\partial_t \boldsymbol{v}_\perp = \frac{1}{\epsilon + p} \delta \boldsymbol{j} \times \boldsymbol{B}_0$$

we have

$$\begin{aligned}
\partial_t^2 \boldsymbol{v}_\perp &= \frac{1}{\epsilon + p} (\nabla \times \nabla \times \delta \boldsymbol{E}_\perp) \times \boldsymbol{B}_0 \\
&= \frac{1}{\epsilon + p} (\nabla \times \nabla \times (\delta \boldsymbol{v}_\perp \times \boldsymbol{B}_0)) \times \boldsymbol{B}_0 \\
&= \frac{B_0^2}{\epsilon + p} \nabla_\parallel^2 \boldsymbol{v}_\perp \,, \quad v_{Alfven}^2 = \frac{B_0^2}{\epsilon + p}
\end{aligned}$$

Back to non-dynamical B case: Reduction of hydro variables in the new low energy hydrodynamics below τ_R

We have $T(x)$ and 1+1 dimensional longitudinal velocity $u_{||}^{\mu}(x)$, but note that x contains all directions

The most general constitutive relation for the energy-momentum tensor

$$
\begin{aligned}
T^{\mu\nu} &= (\epsilon + p_{||}) u_{||}^{\mu} u_{||}^{\nu} + p_{||} g_{||}^{\mu\nu} + p_{\perp} g_{\perp}^{\mu\nu} \\
&- \eta(\partial_{\perp}^{\mu} u_{||}^{\nu} + \partial_{\perp}^{\nu} u_{||}^{\mu}) - (\zeta(u_{||}^{\mu} u_{||}^{\nu} + g_{||}^{\mu\nu}) + \zeta' g_{\perp}^{\mu\nu})(\partial_{||\alpha} u_{||}^{\alpha})
\end{aligned}
$$

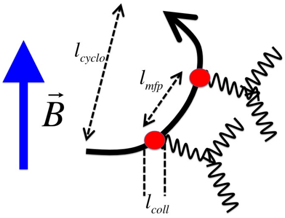
with one shear viscosity η and two bulk viscosities (ζ, ζ')

Computation of η in finite T perturbative QCD in leading log

The QCD Boltzmann eq. in leading log is well known (Monien-Pethick-Ravenhall-Baym, Arnold-Moore-Yaffe, Hong-Teaney).

$$\frac{\partial f}{\partial t} + \hat{\boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \mathcal{C}[f]$$

$$\dot{\boldsymbol{p}} = \pm q_F \hat{\boldsymbol{p}} \times (e\boldsymbol{B}) \,, \quad \mathcal{C}[f] \sim g^4 T \log(1/g)$$



We assume $eB \sim g^4 \log(1/g) T^2$ so that the cyclotron orbit size is comparable to the mean-free path

$$
\begin{aligned}
l_{cyclo} &\sim p/(eB) \sim 1/(g^4 \log(1/g) T) \\
l_{mfp} &\sim \mathcal{C}^{-1} \sim 1/(g^4 \log(1/g) T)
\end{aligned}
$$

The collision time is shorter than this $l_{coll} \sim 1/(gT)$, so there is no effect from the magnetic field to the collision term

The result depends on the dimensionless strength of magnetic field $\bar{B} = (eB)/(g^4 \log(1/g) T^2) \sim l_{mfp}/l_{cyclo}$

The shear viscosity is obtained from

$$T^{\perp z} = -\eta \partial_\perp v_z$$

in the presence of slowly varying $v_z(x_\perp)$. The distribution functions are

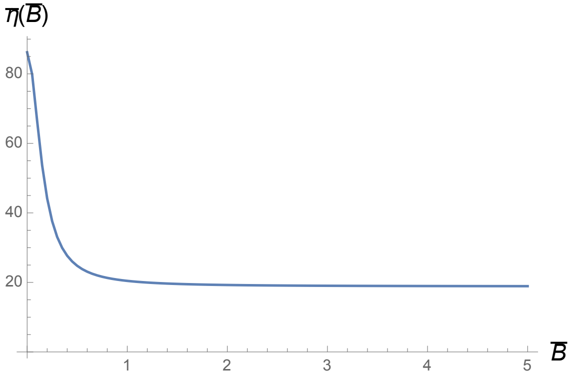
$$f(x,p) = f_{eq}(u_\parallel^\mu(x)) + \delta f$$

where $\mathcal{C}[f_{eq}] = 0$ and we obtain the disturbance δf from solving the Boltzmann equation that is linear in the gradient of $u_\parallel^\mu(x)$

$$\delta f \sim \mathcal{C}^{-1} \cdot (\partial_\perp v_z)$$

Then

$$T^{\perp z} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{\boldsymbol{p}_\perp p_z}{E_p} \left(\nu_q \sum_F (\delta f_q + \delta f_{\bar{q}}) + \nu_g \delta f_g \right)$$



$$\eta = \eta(\bar{B}) \frac{T^3}{g^4 \log(1/g)}$$

QCD Critical Point in Magnetic Field: Dynamic Universality Class

The static universality class of QCD critical point is the 3D Ising class (or "$\lambda\phi^4$ theory", Wilson-Fisher)

This is only about the thermal equilibrium distribution of order parameter fluctuations averaged in time.

How fast the system approaches to this equilibrium distribution when it is perturbed, say relaxation rates, or how in real-time this distribution is realized is a completely independent "dynamics" questions.

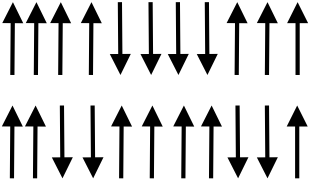
In the critical regime $k \gg \xi^{-1}$ (ξ: correlation length), many systems show "dynamical critical behaviors" with the scaling law for the relaxation frequency $\omega \sim k^z$, with z: dynamic critical exponent

This dynamical evolution is heavily dependent of conservation laws: for example, the order parameter may or may not be conserved. In the former case, we will have a diffusion-type relaxation with $\omega \sim D(\xi)k^2$ in the hydro-regime of $k \ll \xi^{-1}$, while in the latter case, $\omega \sim f(\xi, k \to 0)$ is a finite constant even in $k \to 0$ limit. The static universality class has no information about this: it is an additional information on the system

Two systems with the same static universality class can belong to the different dynamic universality classes

Eg. 1) The 3D Ising model:

The order parameter is the net spin density. Since the spin density is conserved locally, spin fluctuations can only diffuse.
However, there is no "fluid motion" since there is no notion of conserved momentum. This gives the Model C



Eg. 2) Gas-Liquid Critical Point:

The order parameter is the relative portion of gas-liquid, equivalently the mass density. Since the mass is locally conserved, its fluctuations can only diffuse. In addition, there are fluctuations of "fluid motions" due to momentum conservation which also can only diffuse. The fluid motions can deliver the mass fluctuations from one place to another place, enhancing the mass diffusion. This mode-coupling dynamics becomes important in the critical regime, mutually enhancing the mass and the momentum diffusion constants. This gives the Model H

The QCD critical point belongs to the dynamic universality class of "model H" (Son-Stephanov)

The model H :
1) the order parameter is conserved,
2) the total momentum of the fluid is conserved

From 1), the order parameter relaxes by a diffusion-type $\omega \sim \frac{\lambda}{\chi} k^2$ where λ is the conductivity and χ is the susceptibility
From 2), the shear component of velocity fluctuations relax with the shear diffusion-type $\omega \sim \eta k^2$

But, all these coefficients are "running" with the scales k itself, due to non-linear couplings in the dynamics

$$\lambda(k)\,,\quad \eta(k)$$

With $\lambda(k) \sim k^{z-4}$ and $\chi(k) \sim k^{-2}$ in the scaling regime $k \gg \xi^{-1}$

$$\omega \sim \frac{\lambda(k)}{\chi(k)} k^2 \sim k^z$$

The RG running of these transport coefficients are obtained by doing Wilsonian momentum shell integration between $b\Lambda < k < \Lambda$ with $b < 1$. For $z > 1$, the ω range is elongated quickly to $[-\infty, +\infty]$

Smaller scale fluctuations affect the larger scale properties and renormalize the transport coefficients at larger scales. The same physics as in the static RG, but applied to dynamical equations

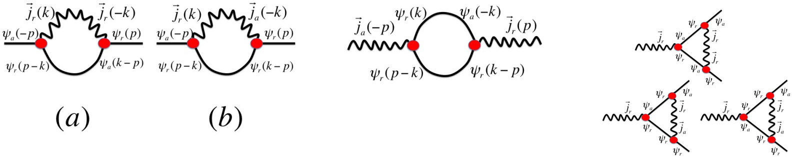
The detailed model equations are diffusion-type with Langevin thermal noise to satisfy fluctuation-dissipation to reproduce the static universality class thermal probability $P = e^{-F}$

$$
\begin{aligned}
\frac{\partial \psi}{\partial t} &= \lambda \Lambda^{z-4} \nabla^2 \frac{\delta F}{\delta \psi} - g \Lambda^{z-3+\epsilon/2} \boldsymbol{\nabla} \psi \cdot \frac{\delta F}{\delta \boldsymbol{j}} + \theta \\
\frac{\partial \boldsymbol{j}}{\partial t} &= \mathcal{P} \left(\bar{\eta} \Lambda^{z-2} \nabla^2 \boldsymbol{j} + g \Lambda^{z-3+\epsilon/2} \boldsymbol{\nabla} \psi \frac{\delta F}{\delta \psi} + \boldsymbol{\xi} \right)
\end{aligned}
$$

$$
F = \int d^{4-\epsilon} x \left(\frac{1}{2} (\partial \psi)^2 + \frac{r \Lambda^2}{2} \psi^2 + \frac{u \Lambda^\epsilon}{4!} \psi^4 + \frac{1}{2} \boldsymbol{j}_\parallel \cdot \boldsymbol{j}_\parallel \right)
$$

We use the ϵ-expansion as in the static case, since it turns out that the dynamic fixed point constant $f = \frac{1}{8\pi^2} \frac{g^2}{\lambda \bar{\eta}}$ is order ϵ and we can do perturbation theory on this small coupling

It is easiest to do momentum-shell integration after going to the path integral formulation in the Schwinger-Keldysh



After all, we get $z \approx 4 - C\epsilon = 3.05$ with $C = 0.947$ and $\epsilon = 1$

The conductivity diverges as $\lambda \sim \xi^{C\epsilon}$, the shear viscosity $\bar{\eta} \sim \xi^{(1-C)\epsilon}$

Back to non-dynamical B

We observe that the transverse component of velocity is not a critical variable any more due to a finite relaxation time $\tau_R \sim 1/(\lambda_\perp B_0^2)$ ($\lambda_\perp$: transverse conductivity)

This implies that there is no non-linear dynamics between $\lambda_\perp$ and $\boldsymbol{v}_\perp$ fluctuations to give rise to the previous diverging λ, and $\lambda_\perp$ is expected to remain constant as in the model C. Since $\lambda_\parallel$ is expected to diverge, we can neglect $\lambda_\perp$ in the scaling analysis.

We consider a modification of model H without any diffusion along $d_T = 2$ transverse dimensions: this implies a reduction of phase space volume for the critical non-linear dynamics

$$
\begin{aligned}
\frac{\partial \psi}{\partial t} &= \lambda_{\parallel} \Lambda^{z-4} \nabla_{\parallel}^2 \frac{\delta F}{\delta \psi} - g \Lambda^{z-3+\epsilon/2} \boldsymbol{\nabla}_{\parallel} \psi \cdot \frac{\delta F}{\delta \boldsymbol{j}_{\parallel}} + \theta \\
\frac{\partial \boldsymbol{j}_{\parallel}}{\partial t} &= \mathcal{P} \left(\bar{\eta}_{\perp} \Lambda^{z-2} \nabla_{\perp}^2 \boldsymbol{j}_{\parallel} + \bar{\eta}_{\parallel} \Lambda^{z-2} \nabla_{\parallel}^2 \boldsymbol{j}_{\parallel} + g \Lambda^{z-3+\epsilon/2} \boldsymbol{\nabla}_{\parallel} \psi \frac{\delta F}{\delta \psi} + \boldsymbol{\xi} \right)
\end{aligned}
$$

This is also obvious since we loose the momentum conservation in the transverse dimensions.

After all computations, we get $z \approx 4 - C\epsilon = 3.15$ with $C = 0.847$ and $\epsilon = 1$

The C gets smaller than the original model H value of 0.947 due to the reduced phase space volume for the critical non-linear coupling dynamics

Thank you!