Hydrodynamic fluctuations and fluctuating bounds on transport coefficients

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BEST COLLABORATION

Multi-Scale Problems Using Effective Field Theories
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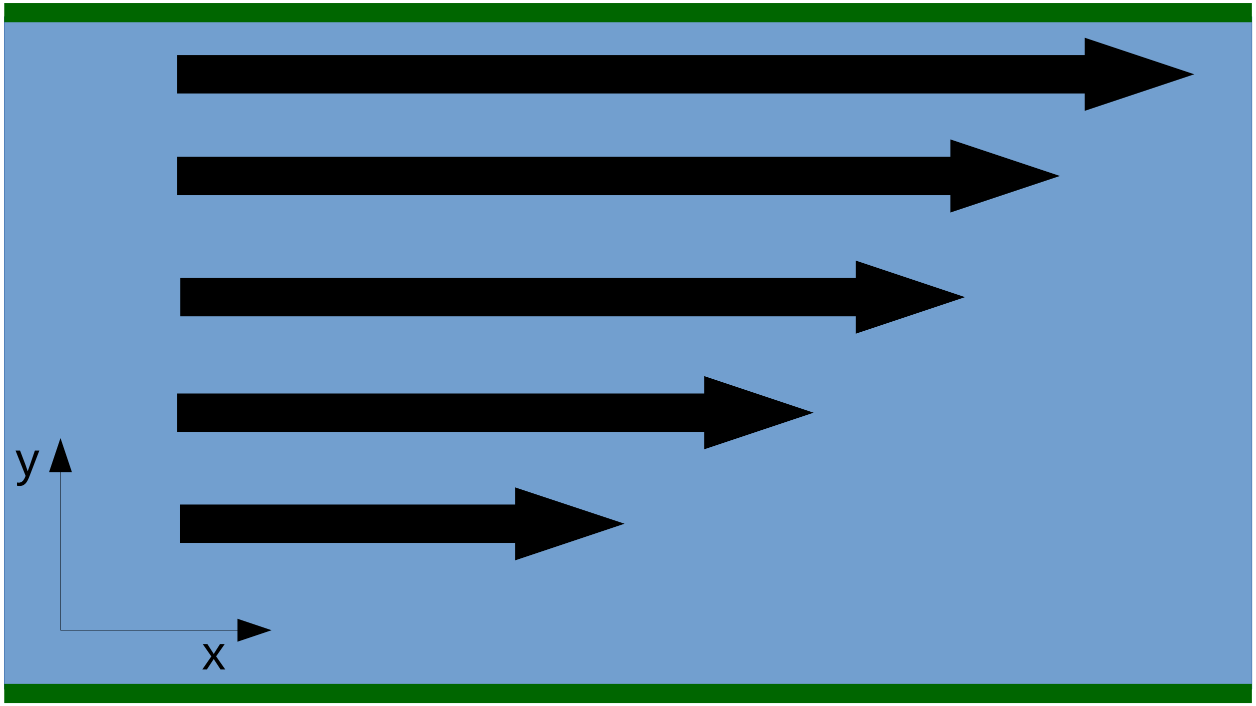
Work in collaboration with T. Schäfer
PRA 96 (2017) 063607

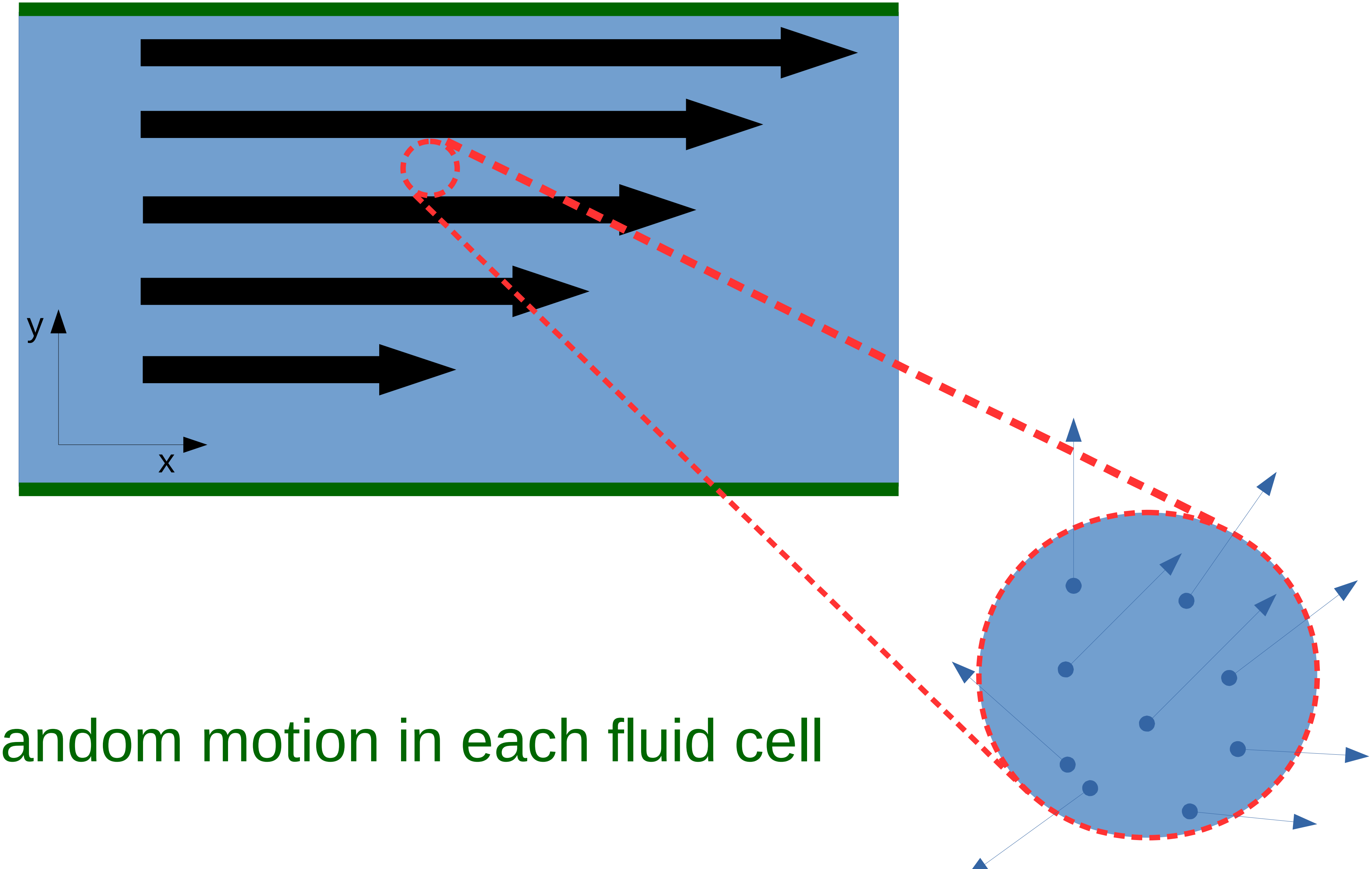
Hydrodynamic fluctuations



$$\pi_{yx} = -\eta\,\partial_y v_x$$

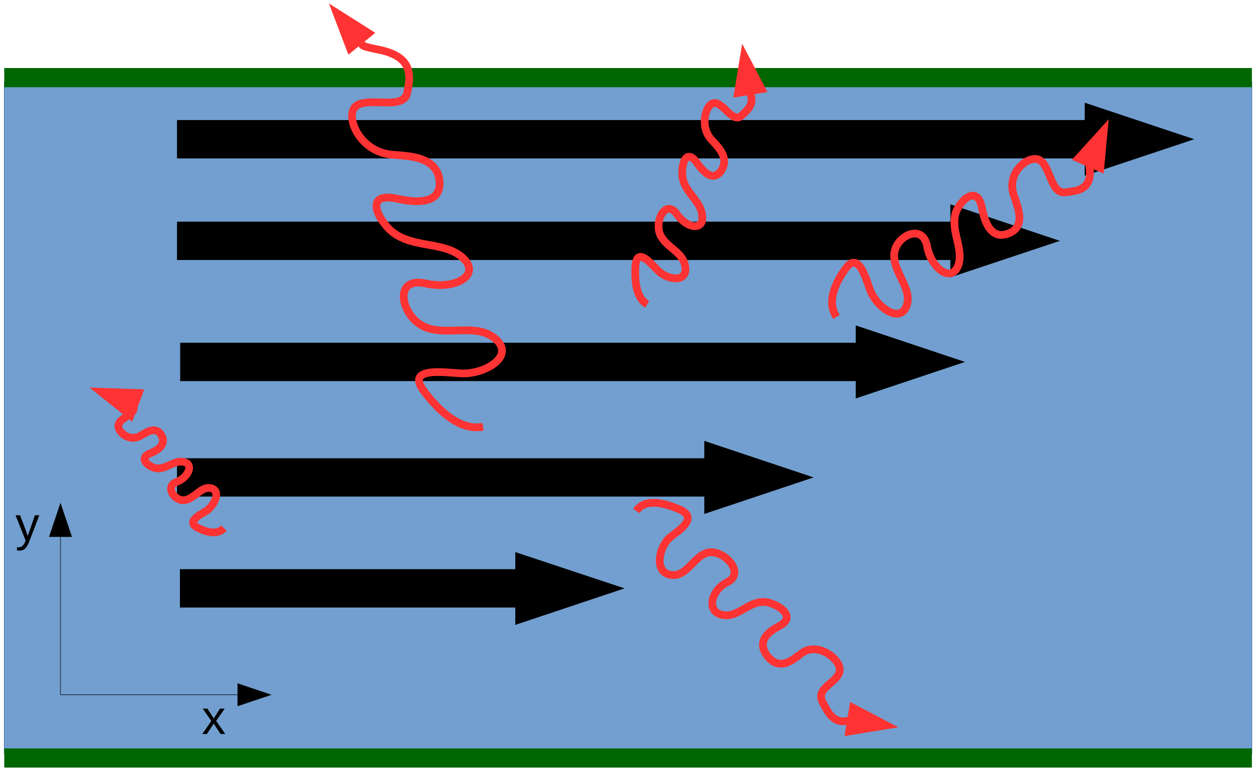
Navier Stokes

Hydrodynamic fluctuations



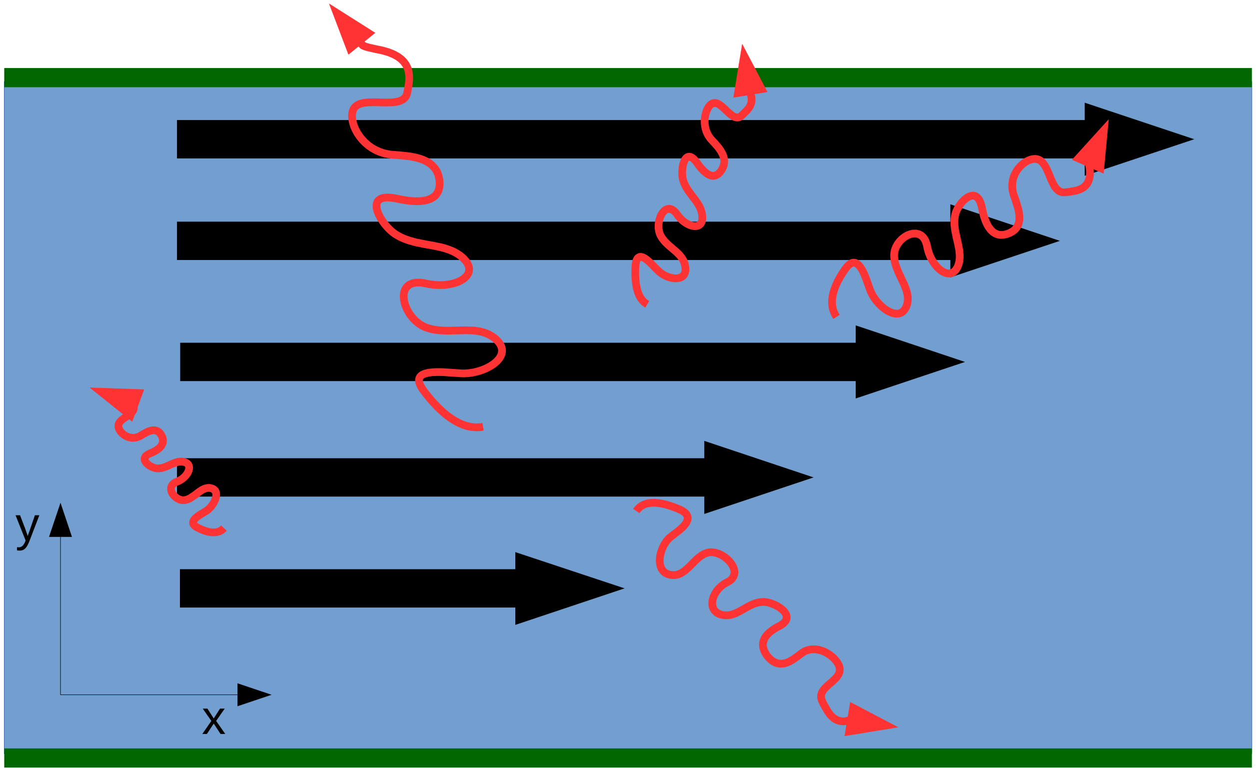
Random motion in each fluid cell

Hydrodynamic fluctuations



Microscopic random motion can effectively transfer momentum between fluid cells

Hydrodynamic fluctuations



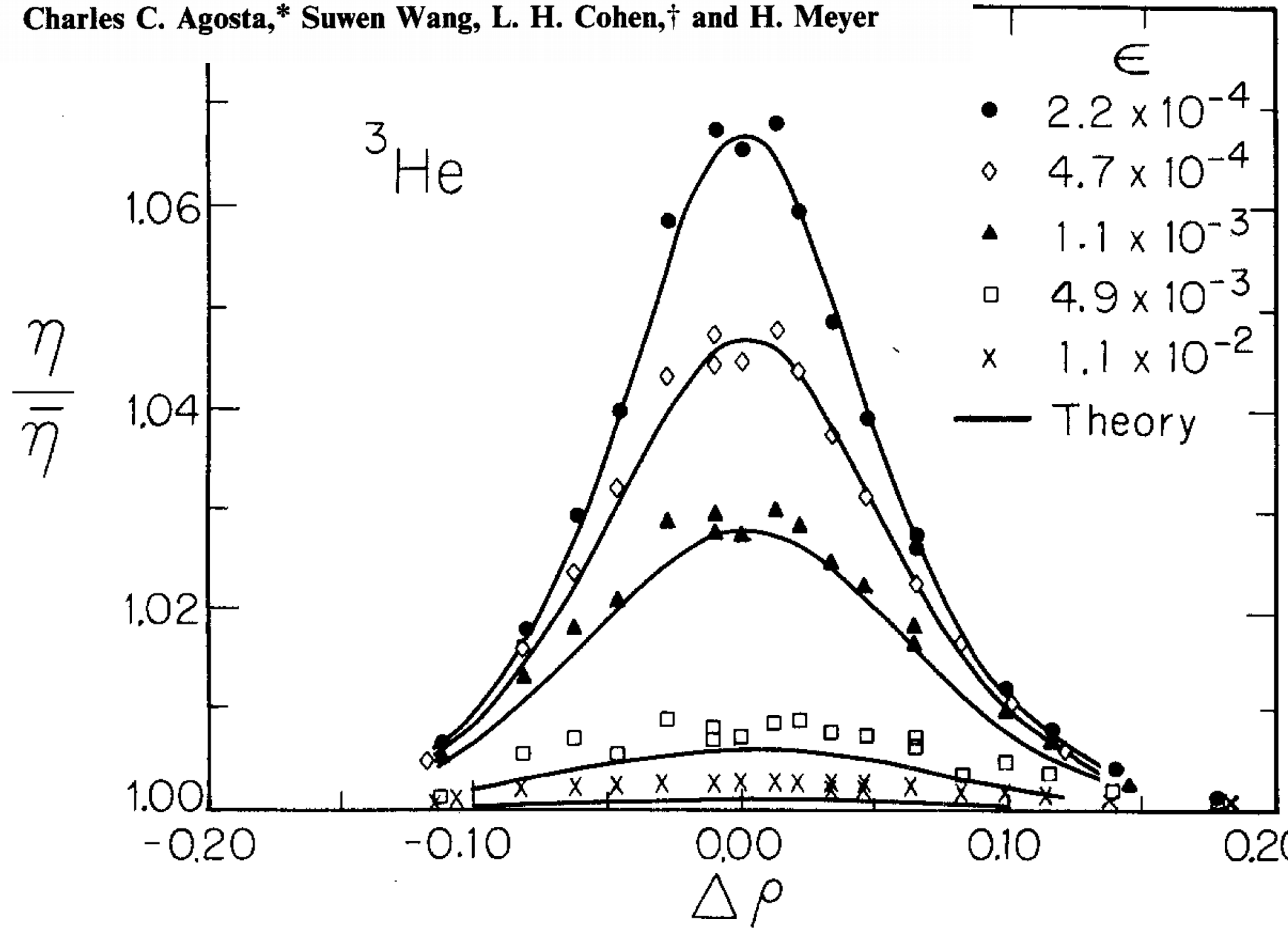
$$\pi_{yx} \sim (\eta + \delta\eta)\, \partial_y v_x$$

Shear viscosity receives contributions from these microscopic collective motion of the constituents

Hydrodynamic fluctuations

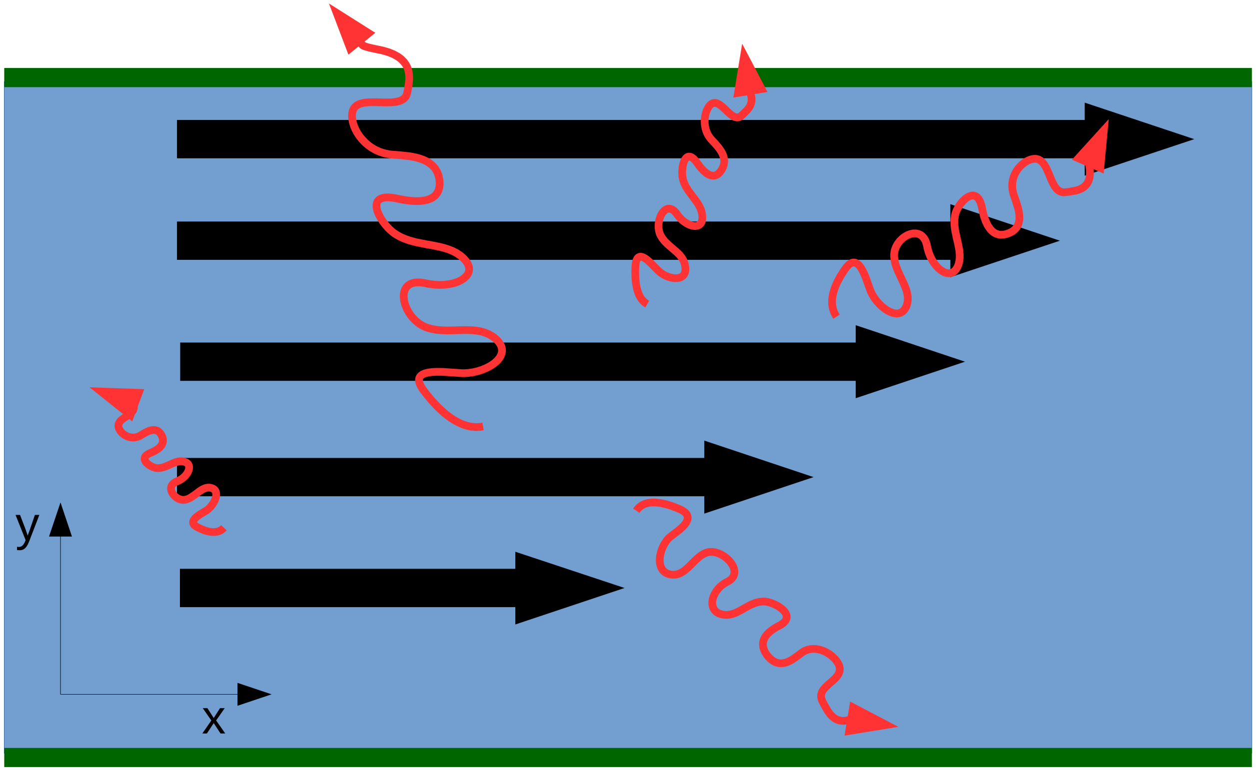
Transport Properties of Helium near the Liquid–Vapor Critical Point. IV. The Shear Viscosity of ^3He and ^4He

Charles C. Agosta,* Suwen Wang, L. H. Cohen,† and H. Meyer



Critical behaviour of the shear viscosity is small but these effects are larger for the bulk viscosity and heat conductivity

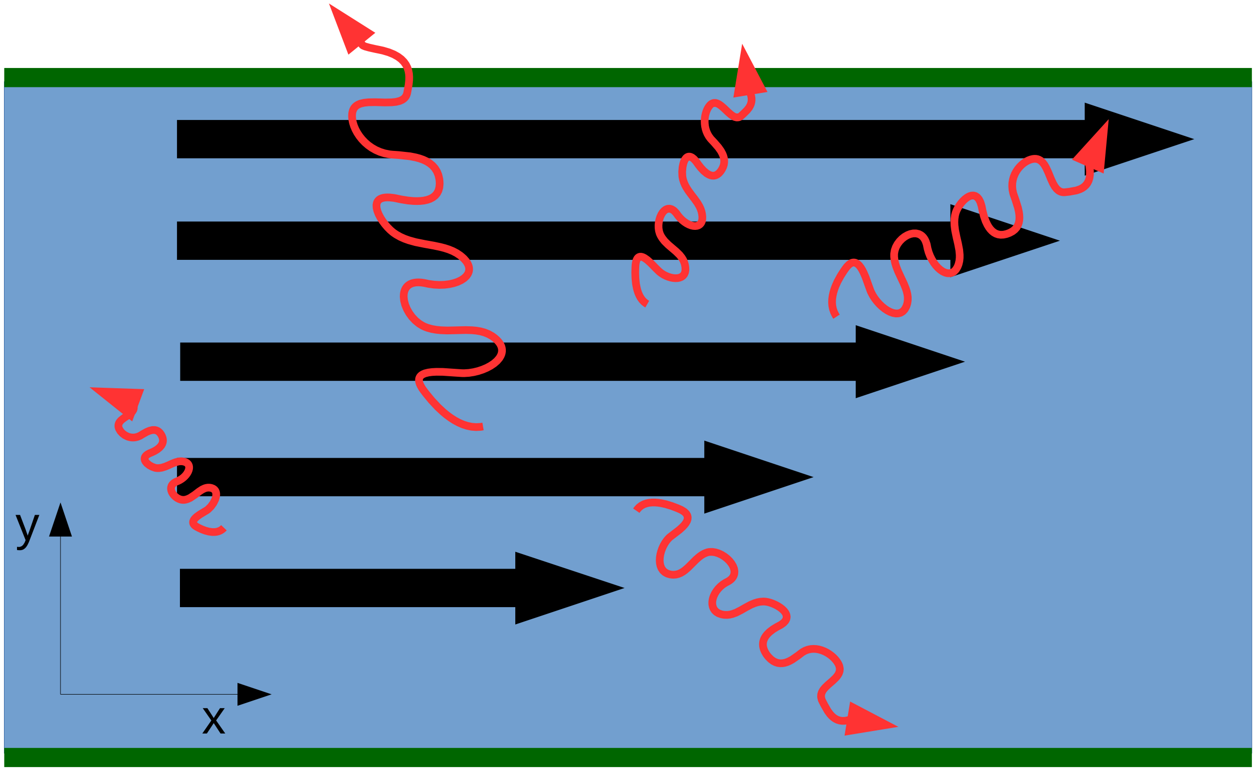
Hydrodynamic fluctuations



$$\pi_{yx} \sim \left(\eta + \delta\eta\right) \partial_y v_x$$

For dilute Fermi gas at unitarity $\delta\eta$ was calculated and the bounds on the shear viscosity were determined
Chafin and Schäfer PRA 87 (2013) 023629
In this talk:
How to calculate $\delta\zeta$? What are the bounds on the values of the bulk viscosity due to hydrodynamic fluctuations?
Martinez and Schäfer PRA 96 (2017) 063607

Hydrodynamic fluctuations



$$\pi_{yx} \sim \left(\eta + \delta\eta\right) \partial_y v_x$$

The leading non-analytic contribution to the hydrodynamic correlators are due to hydrodynamic fluctuations

Outline

- Short review of hydrodynamics
- Kubo formula for bulk viscosity
- Hydrodynamic tails: formalism
- Fluctuation bound of the bulk viscosity
- Conclusions and Outlook

Short review of hydrodynamics

Hydrodynamics

- **Hydrodynamics:** Effective field theory of long wavelength, low frequency dynamics of **ANY** many body system

$$\partial_t \rho + \partial_i (\rho v_i) = 0 \,,$$ → Continuity equation conservation of mass

$$\partial_t (\rho v_i) + \partial_j \Pi_{ij} = 0 \,,$$ → Euler equation Momentum conservation

$$\partial_t \left(\epsilon + \frac{\rho v^2}{2} \right) + \partial_i j_i^\epsilon = 0 \,,$$ → Energy conservation

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$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j$$

$$j_i^{\epsilon} = \left(w + \frac{\rho v^2}{2}\right) v_i$$

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Ideal fluid

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Momentum conservation

$$\partial_t\left(\epsilon + \frac{\rho v^2}{2}\right) + \partial_i j_i^\epsilon = 0\,,$$
→ Energy conservation

Ideal fluid

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \Sigma_{ij}$$

Dissipative

$$\Sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\,\partial_k v_k\right) + \zeta \delta_{ij}\partial_k v_k$$

$$j_i^\epsilon = \left(w + \frac{\rho v^2}{2}\right) v_i - \Sigma_{ij} v_j - \kappa\,\partial_i T$$

Hydrodynamics

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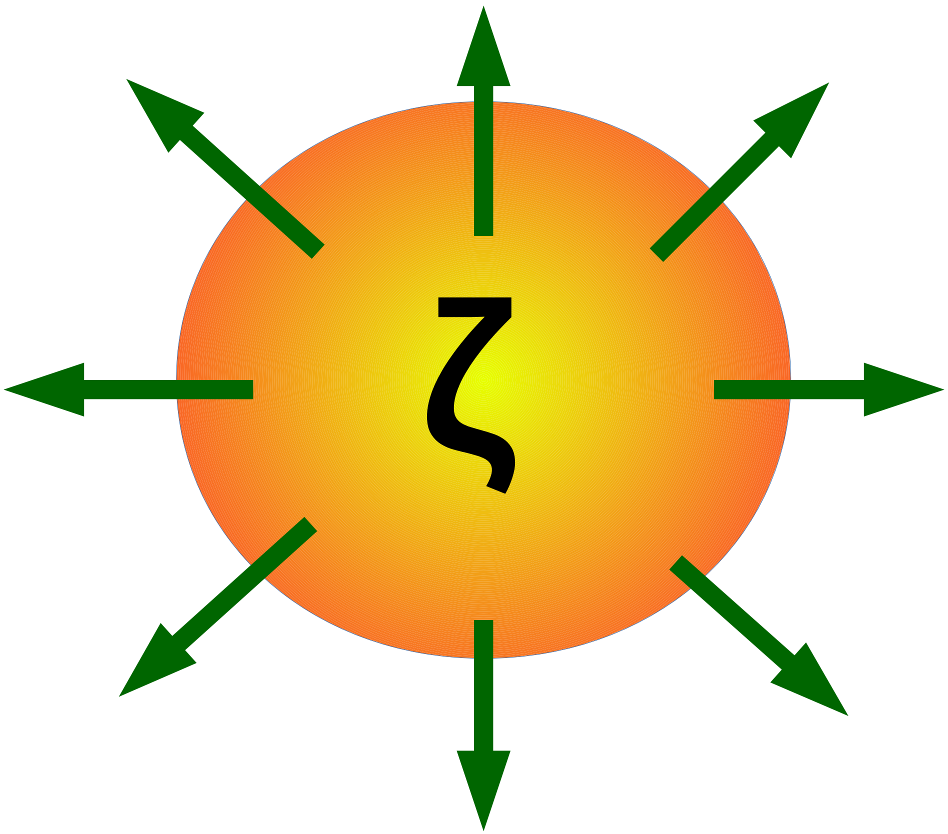
Dissipation and fluctuation are intrinsically related (Fluctuation-dissipation theorem)
Where are the fluctuations in the hydrodynamical equations?
⇛Stochastic fluid dynamics (Landau & Lifshitz)

Kubo formula for the bulk viscosity



Kubo formula for ζ

The retarded correlator of the stress tensor Π_{ij} is

$$G_R^{ijkl}(\omega,\mathbf{k}) = -i\int dt\int d\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\Theta(t)\langle[\Pi^{ij}(t,\mathbf{x}),\Pi^{kl}(0,\mathbf{0})]\rangle\,.$$

Kubo formula for bulk viscosity is obtained by considering the trace of the stress tensor

$$\zeta = -\lim_{\omega\to 0}\frac{1}{9\omega}\,\mathrm{Im}\,G_R^{iijj}(\omega,\mathbf{0}).$$

However, bulk viscosity is usually related with breaking of scale invariance and thus, it would be nice to see this explicitly in the Kubo formula

Kubo formula for ζ

The retarded correlator of the stress tensor Π_{ij} is

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Kubo formula for bulk viscosity is obtained by considering the trace of the stress tensor

$$\zeta = -\lim_{\omega \to 0} \frac{1}{9\omega} \operatorname{Im} G_R^{iijj}(\omega, \mathbf{0}).$$

We can use any combination of the form

$$\mathcal{O} = \frac{1}{3}\Pi^{ii} + c\mathcal{E}$$

For simplicity c= -2/3

Kubo formula for ζ

The retarded correlator of the stress tensor Π_{ij} is

$$G_R^{ijkl}(\omega, \mathbf{k}) = -i \int dt \int d\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \Theta(t) \langle [\Pi^{ij}(t, \mathbf{x}), \Pi^{kl}(0, \mathbf{0})] \rangle \,.$$

Kubo formula for bulk viscosity is obtained by considering the trace of the stress tensor

$$\Rightarrow \mathcal{O} = \Delta_{Tr} P = P - \frac{2}{3}\mathcal{E}^0$$

$$\zeta = -\lim_{\omega \to 0} \frac{1}{9\omega} \operatorname{Im} G_R^{iijj}(\omega, \mathbf{0}) \Rightarrow \zeta = -\lim_{\omega \to 0} \frac{1}{9\omega} \operatorname{Im} G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0})$$

Hydrodynamic tails: formalism

Statistical field theory method

It is convenient to calculate the symmetric correlation function

$$G_S^{\mathcal{O}\mathcal{O}}(\omega,\mathbf{k}) = \int dt \int d\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \left\langle \frac{1}{2}\{\mathcal{O}(t,\mathbf{x}),\mathcal{O}(0,\mathbf{0})\} \right\rangle .$$

Fluctuation-dissipation theorem gives us the retarded correlation function

$$G_S(\omega,\mathbf{k}) \simeq -\frac{2T}{\omega}\,\mathrm{Im}\, G_R(\omega,\mathbf{k}) .$$

Statistical field theory method

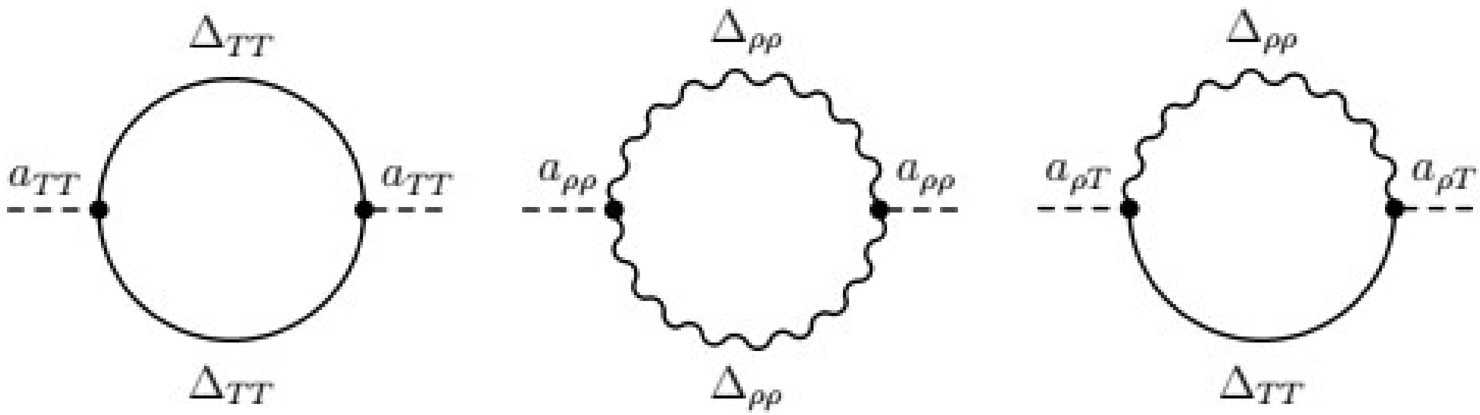
$$G_S^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{k}) = \int dt \int d\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \left\langle \frac{1}{2}\{\mathcal{O}(t, \mathbf{x}), \mathcal{O}(0, \mathbf{0})\} \right\rangle .$$

Expand operator $\mathcal{O}$ to second order in (ΔT,$\Delta\rho$)

$$\mathcal{O} = \mathcal{O}_0 + a_\rho \Delta\rho + a_T \Delta T + a_{\rho\rho}(\Delta\rho)^2 + a_{\rho T}\Delta\rho\Delta T + a_{TT}(\Delta T)^2 + \ldots .$$

- $a_{\rho\rho}$, $a_{\rho T}$, a_{TT} are thermodynamical quantities
- Linear terms in the perturbations do not matter
- Second-order terms contribute to the non-singular terms (see NN-slides)

Statistical field theory method

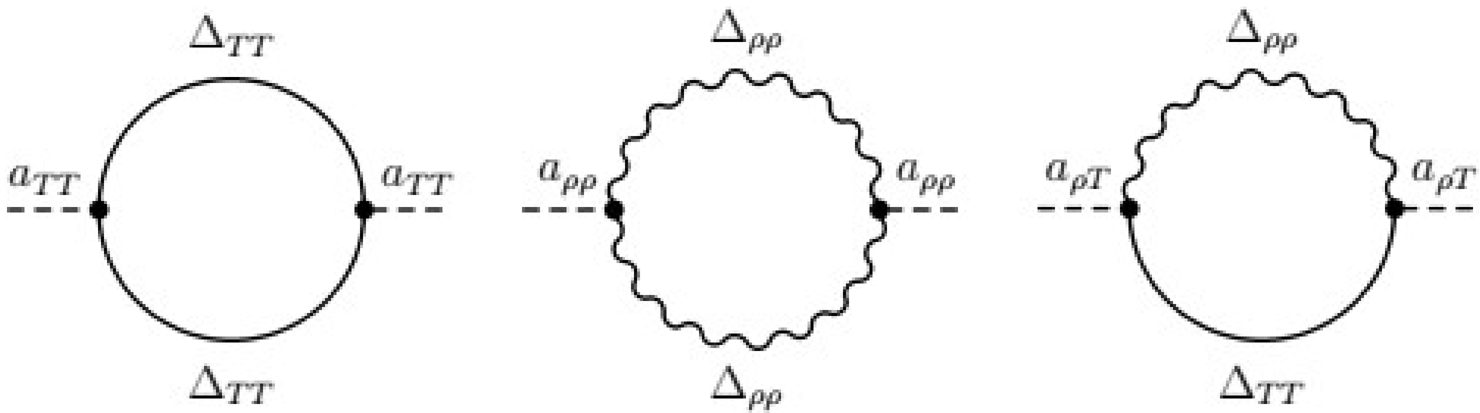


In the Gaussian approximation (white random noise)

$$G_S^{\mathcal{O}\mathcal{O}}(\omega,0) = \int \frac{d\omega'}{2\pi}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[2a_{\rho\rho}^2\Delta_S^{\rho\rho}(\omega',\mathbf{k})\Delta_S^{\rho\rho}(\omega-\omega',\mathbf{k}) + a_{\rho T}^2\Delta_S^{\rho\rho}(\omega',\mathbf{k})\Delta_S^{TT}(\omega-\omega',\mathbf{k}) + 2a_{TT}^2\Delta_S^{TT}(\omega',\mathbf{k})\Delta_S^{TT}(\omega-\omega',\mathbf{k})\Big]\,.$$

- Symmetrized correlation functions
- These correlators are obtained from the solutions to the linearized hydro equations!!!
  (see Kovtun & Yaffe, Kadanoff-Martin, etc)

Statistical field theory method



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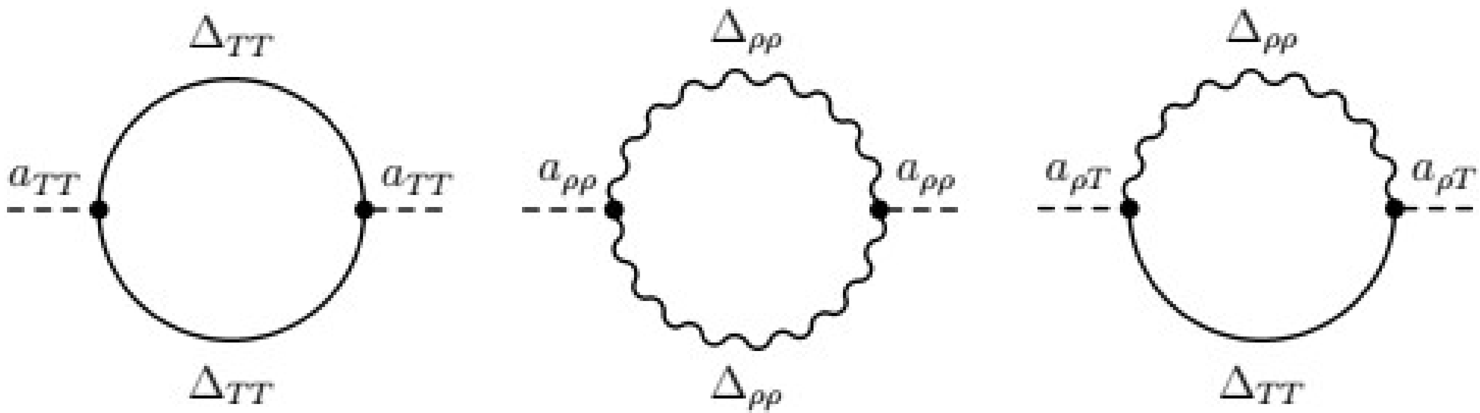
$$\Delta_S^{TT}(\omega,\mathbf{k}) = \frac{2T^2}{c_P} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T \mathbf{k}^2)^2}$$

Dominated by the diffusive heat wave

$$\Delta_S^{\rho\rho}(\omega,\mathbf{k}) = 2\rho T \left\{ \frac{\Gamma k^4}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} + \frac{\Delta c_P}{c_s^2} \frac{D_T \mathbf{k}^2}{\omega^2 + (D_T k^2)^2} - \frac{\Delta c_P}{c_s^2} \frac{(\omega^2 - c_s^2 k^2) D_T \mathbf{k}^2}{(\omega^2 - c_s^2 k^2)^2 + (\Gamma \omega k^2)^2} \right\} ,$$

Mix of sound and diffusive modes

Statistical field theory method



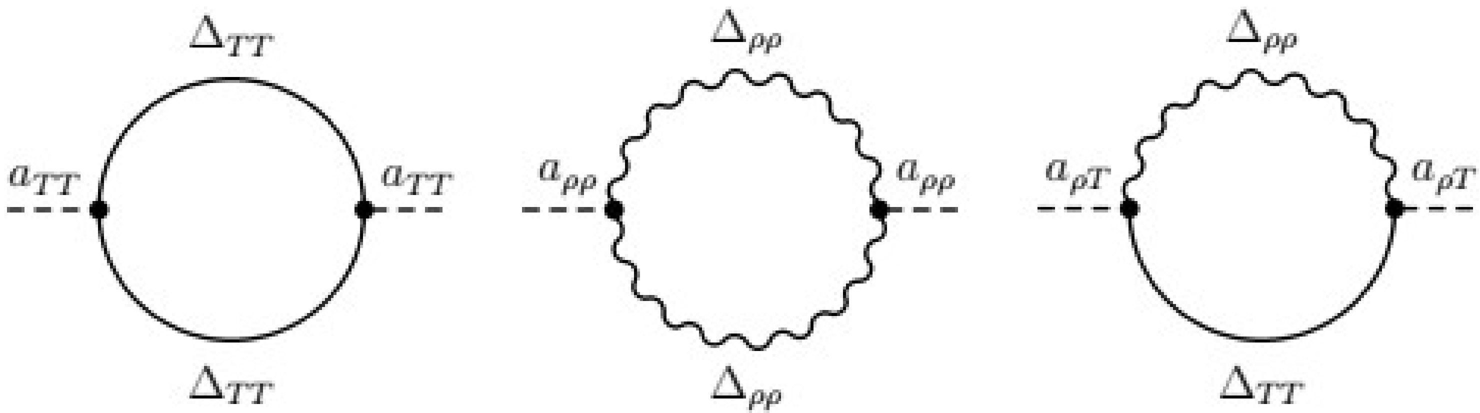
After a long algebra plus pole analysis of propagators

$$G_R^{\mathcal{O}\mathcal{O}}(\omega, \mathbf{0}) = -A_T L(\omega, \Lambda, 2D_T) - A_\Gamma L(\omega, \Lambda, \Gamma)$$

$$L(\omega, \Lambda, D_i) = \frac{1}{2\pi^2}\left\{\frac{\Lambda^3}{3} + \frac{i\omega\Lambda}{D_i} - \frac{\pi}{2\sqrt{2}}(1+i)\left(\frac{\omega}{D_i}\right)^{3/2} + \ldots\right\}.$$

Contributes to the compressibility.
Renormalizes the pressure

Statistical field theory method



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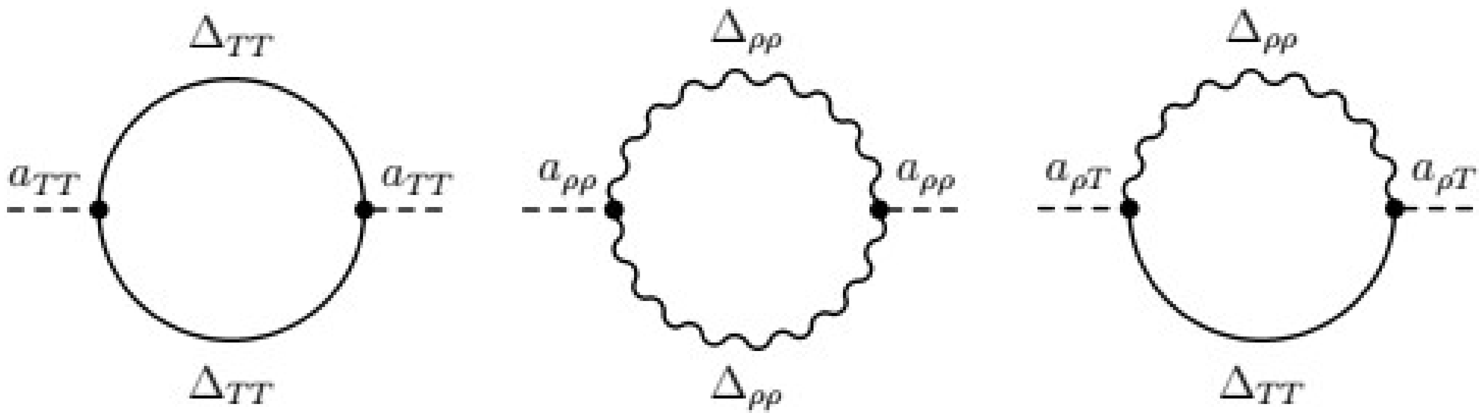
Contributes to the bulk viscosity
(cut-off dependence)

Statistical field theory method



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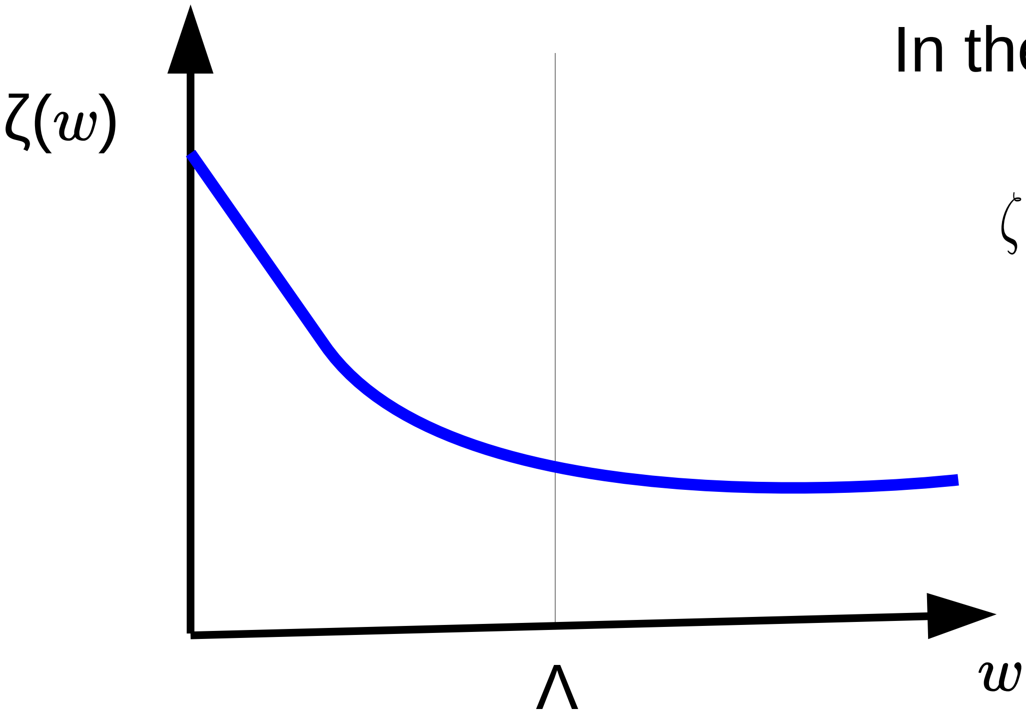
Hydrodynamic tail:

- **Imaginary part:** contribution to the frequency dependent ζ(w)
- **Real part:** non-analytic contribution to the bulk relaxation time which signals breaking of hydrodynamics

Fluctuation bound of the bulk viscosity

RG argument



In the short wavelenght regime $w < \Lambda$

$$\zeta(\omega) = \zeta_{phys}(0) - \left(\frac{A_T}{(2D_T)^{3/2}} + \frac{A_\Gamma}{\Gamma^{3/2}}\right)\frac{\sqrt{\omega}}{36\sqrt{2}\pi}\,.$$

$$\zeta_{phys} = \zeta_{bar} + \zeta_\Lambda$$

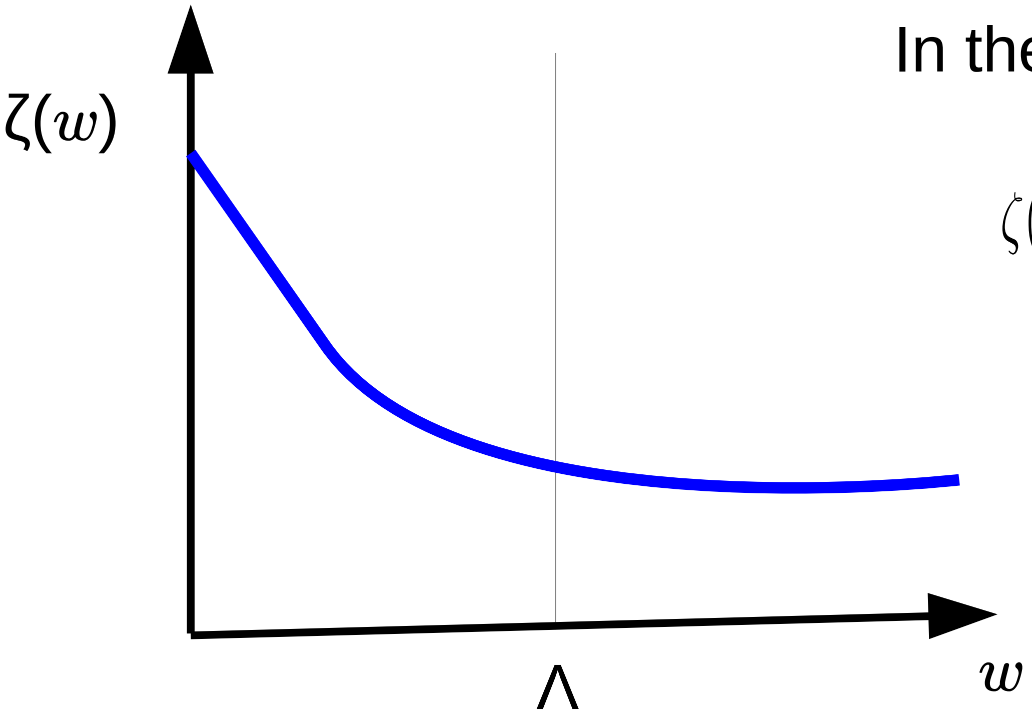
$$\zeta_\Lambda = \frac{1}{18\pi^2}\left(\frac{A_T\Lambda}{2D_T} + \frac{A_\Gamma\Lambda}{\Gamma}\right)$$

- $w < \Lambda$: Stochastic hydrodynamics
- $w \gtrsim \Lambda$: Matched to some microscopic theory
- ζ_{phys} is cut off independent

$$\Lambda\frac{d\zeta_{phys}}{d\Lambda} = 0$$

$$\Rightarrow \zeta_{bar} = \zeta_{bar}(\Lambda)$$

RG argument



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$$\zeta_{phys} = \zeta_{bar} + \zeta_\Lambda$$

$$\zeta_\Lambda = \frac{1}{18\pi^2}\left(\frac{A_T\Lambda}{2D_T} + \frac{A_\Gamma\Lambda}{\Gamma}\right)$$

The largest Λ is determined by the breaking scale of hydrodynamics
$\Rightarrow$ determined by the dispersion relation of the sound and diffusive modes

Estimation of the bound

1. Diffusive mode $\omega \sim D_T k^2 \ll (\tau_\kappa)^{-1}$

Using kinetic theory $\tau_\kappa = (m\kappa)/(c_P T)$

$$\Lambda_T \simeq \frac{1}{D_T}\left(\frac{T}{m}\right)^{1/2} .$$

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2. Sound mode $\omega \sim c_s k \ll \Gamma k^2$

For a weakly interacting gas $c_s^2 = (\partial P)/(\partial\rho)_{s/n} \simeq (5T)/(3m)$

$$\Lambda_\Gamma \simeq \frac{1}{\Gamma}\left(\frac{\partial P}{\partial \rho}\right)_{s/n}^{1/2} .$$

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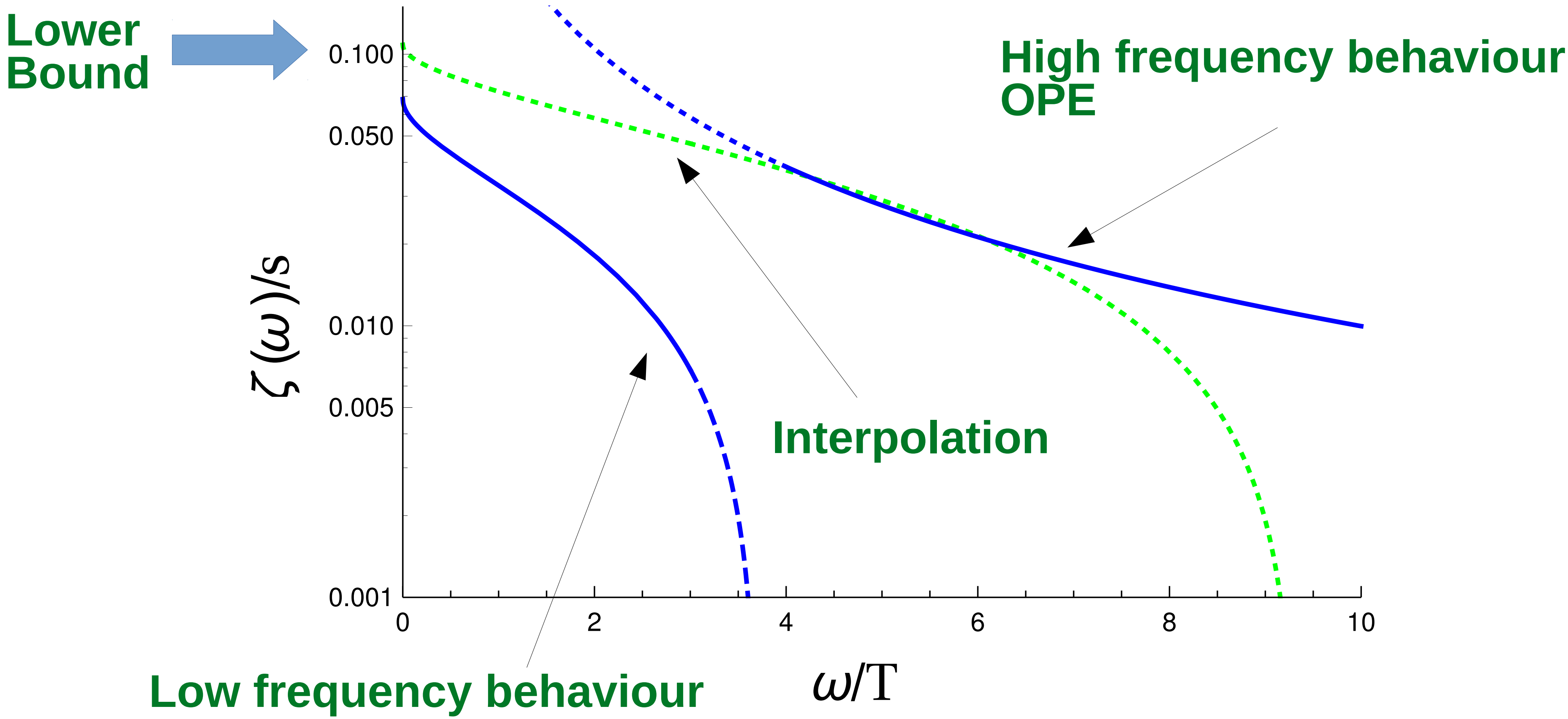
$$\Lambda_\Gamma \simeq \frac{1}{\Gamma}\left(\frac{\partial P}{\partial \rho}\right)_{s/n}^{1/2} .$$

The combination of these two estimates gives us

$$\zeta_{min} = \left(\frac{A_T}{2D_T^2} + \frac{\sqrt{5}A_\Gamma}{\sqrt{3}\Gamma^2}\right)\sqrt{\frac{T}{m}} .$$

Estimation of the bound: cold Fermi gas near unitarity



$$\zeta(\omega) = \zeta_{min} - c\sqrt{w}$$

$$\zeta_{min} = \left(\frac{A_T}{2D_T^2} + \frac{\sqrt{5}A_\Gamma}{\sqrt{3}\Gamma^2}\right)\sqrt{\frac{T}{m}}.$$

Conclusions

- We studied the role of hydrodynamic fluctuations in the bulk stress correlation function
- Hydro fluctuations provide a lower bound on the bulk viscosity. The bound depends on shear viscosity and thermal conductivity and scale breaking of the EOS

$$\zeta_{min} = \left(\mathcal{P} - \frac{2}{3}\mathcal{E}\right)^2 \sum_i D_i^{-2}$$

- For cold Fermi gas near unitarity

$$\frac{\zeta}{s} \gtrsim 0.1$$

Outlook

- Relativistic expanding fluids
- Role of critical fluctuations in the vecinity of a second order phase transition: enhancement of bulk viscosity near critical point

Backup slides

Kubo formula for ζ

The retarded correlator determines the stress induced by a small strain $g_{ij} = \delta_{ij} + h_{ij}(t,\mathbf{x})$

$$\delta\Pi_{ij}(\omega,\mathbf{k}) = -\frac{1}{2} G_R^{ijkl}(\omega,\mathbf{k}) h_{kl}(\omega,\mathbf{k})$$

Stress tensor is expanded in gradients in the presence of a background metric

$$\Pi_{ij} = \Pi^0_{ij} + \Pi^1_{ij} + \ldots$$

$$\Pi^0_{ij} = \rho v_i v_j + P g_{ij}$$ Ideal

$$\Pi^1_{ij} = -\eta\sigma_{ij} - \zeta g_{ij}\langle\sigma\rangle$$ Navier-Stokes

$$\sigma_{ij} = \nabla_i v_j + \nabla_j v_i + \dot{g}_{ij} - \frac{2}{3} g_{ij}\langle\sigma\rangle \,,$$

$$\langle\sigma\rangle = \nabla\cdot v + \frac{\dot{g}}{2g} \,,$$

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$$\delta\Pi_{ij}(\omega,\mathbf{k}) = -\frac{1}{2} G_R^{ijkl}(\omega,\mathbf{k}) h_{kl}(\omega,\mathbf{k})$$

The response to a bulk strain $h_{ij}(t,\mathbf{x}) \sim h\, e^{-i\omega t}\, \delta_{ij}$ gives us to $\mathcal{O}(w)$

$$\frac{1}{9} G_R^{iijj}(\omega,\mathbf{0}) = -\left(\frac{2}{3}P - \left(\frac{\partial P}{\partial \rho}\right)_s \rho\right) - i\omega\zeta\,,$$

Thus $$\zeta = -\lim_{\omega\to 0} \frac{1}{9\omega} \,\mathrm{Im}\, G_R^{iijj}(\omega,\mathbf{0}).$$ Kubo formula

Usually bulk viscosity is related with breaking of scale invariance and thus, it would be nice to see this explicitly in the Kubo formula

Some thermodynamical quantities

$$a_{\rho\rho} = \frac{1}{2}\frac{\partial}{\partial\rho}\left[c_T^2 - \frac{2}{3}\left(\frac{h}{m} - \frac{T\alpha\kappa_T}{\rho}\right)\right]_T ,$$

$$a_{\rho T} = \left.\frac{\partial c_T^2}{\partial T}\right|_\rho - \frac{2}{3}\left.\frac{\partial c_V}{\partial\rho}\right|_T ,$$

$$a_{TT} = \frac{1}{2}\left[\frac{1}{T}\left(1 - \rho\frac{\partial}{\partial\rho}\right)_T - \frac{2}{3}\left.\frac{\partial}{\partial T}\right|_\rho\right] c_V .$$

$$\Gamma = \frac{4}{3}\frac{\eta}{\rho} + \frac{\zeta}{\rho} + \kappa\left(\frac{1}{c_V} - \frac{1}{c_P}\right) = \frac{4}{3}\frac{\eta}{\rho}\left[1 + \frac{3}{4}\frac{\zeta}{\eta} + \frac{3}{4}\frac{\Delta c_P}{Pr}\right]$$

$$D_T = \kappa/c_P$$

$$A_T = \frac{2a_{TT}^2 T^3}{c_P^2} + \frac{2a_{\rho\rho}^2\rho^2 T(\Delta c_P)^2}{c_s^4} + \frac{a_{\rho T}^2\rho T^2\Delta c_P}{c_P c_s^2}, \qquad A_\Gamma = \frac{a_{\rho\rho}^2\rho^2 T}{c_s^4}$$