Open Quantum Systems for Quarkonia in QGP

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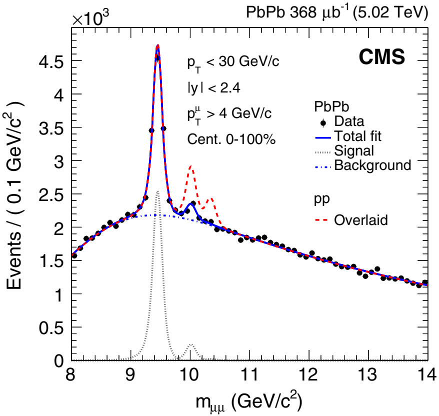
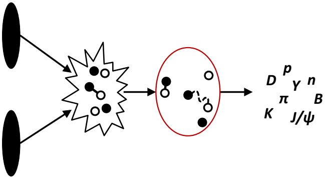
"Multi-Scale Problems Using Effective Field Theories"

References: Akamatsu-Asakawa-Kajimoto-Rothkopf 1805.00167,
Kajimoto-Akamatsu-Asakawa-Rothkopf (18),
Akamatsu (15,13), Akamatsu-Rothkopf (12)

Motivation and Outline



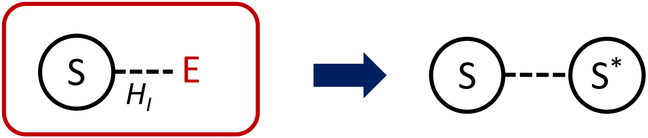


1. Basics of Open Quantum System
2. Application to Quarkonium in QGP
3. Quantum State Diffusion Simulation for a Heavy Quark

What do we learn from heavy-ion data?
Can we understand the data in terms of in-medium QCD forces at high T?

Basics of Open Quantum System

Open quantum systems



1. Total system consists of system (S) and environment (E)

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_S \otimes \mathcal{H}_E$$

2. Hamiltonian

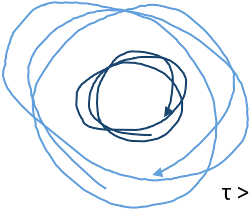
$$H_{\text{tot}} = H_S \otimes 1 + 1 \otimes H_E + H_I, \quad H_I = \sum H_I^{(S)} \otimes H_I^{(E)}$$

3. Reduced density matrix & Master equation

$$\rho_S(t) \equiv \text{Tr}_E \rho_{\text{tot}}(t), \quad i\frac{d}{dt}\rho_{\text{tot}} = [H_{\text{tot}}, \rho_{\text{tot}}] \quad \rightarrow \quad \underbrace{i\frac{d}{dt}\rho_S = ?}_{\text{Markovian limit}}$$

4. Theoretical methods
   - Influence functional – path integral representation for the master equation
   - Schwinger-Dyson equation – time evolution equation for the density matrix

Time scale hierarchies



Three basic time scales

- Environment correlation time τ_E
- System intrinsic time scale τ_S
- System relaxation time τ_R

Time scale hierarchies

- Quantum Brownian motion

$$\underbrace{\tau_E \ll \tau_R}_{\text{Markov approx.}}, \quad \underbrace{\tau_E \ll \tau_S}_{\text{derivative expansion}} \quad \to \text{good description in phase space}$$

- Quantum optical system

$$\underbrace{\tau_E \ll \tau_R}_{\text{Markov approx.}}, \quad \underbrace{\tau_S \ll \tau_R}_{\text{rotating wave approx.}} \quad \to \text{good description in eigenbasis}$$

It is very important to estimate the relevant time scales
We adopt QBM-type approximation scheme to study quarkonium

Time scales of a quarkonium quantum Brownian motion in QGP

- Environment (QGP) correlation time τ_E
  1. Time scales of QGP

Particle collision intervals	soft $\sim 1/g^2T$, hard $\sim 1/g^4T$
Field correlation times	electric $\sim 1/gT$, magnetic $\sim 1/g^4T\ln(1/g)$

  2. Heavy quarks mostly couple to electric field

$$\tau_E \sim \frac{1}{gT}$$

- System (Quarkonium) intrinsic time scale τ_S

  Orbital period = inverse energy gap = formation time

$$\tau_S \sim \frac{1}{\Delta E} \quad \underbrace{\sim \frac{1}{M\alpha^2}}_{\text{Coulomb bound states}} \quad , \quad \underbrace{\sim \infty}_{\text{above threshold}}$$

- System relaxation time τ_R

  Kinetic equilibration / color relxation (for a single HQ / longer for a quarkonium)

$$\tau_R^{\text{kin}} \sim \frac{M}{T}\frac{1}{g^4T\ln(1/g)}, \quad \tau_R^{\text{color}} \sim \frac{1}{g^2T}$$

⇒Time scale hierarchy for quarkonium quantum Brownian motion

$$\tau_E \ll \tau_R, \quad \tau_E \ll \tau_S \to g \underbrace{\ll}_{\text{color}} 1, \quad g^3\ln(1/g) \underbrace{\ll}_{\text{kinetic}} \frac{M}{T} \underbrace{\ll}_{\text{potential}} \frac{g}{\alpha^2} \sim \frac{100}{g^3}$$

Scale hierarchy satisfied/challenged at weak/strong coupling

Open quantum system by path integral

1. Path integral

$$\rho_{\text{tot}}(t, \underbrace{x, y}_{\in S}, \underbrace{X, Y}_{\in E}) = \int dx_0 dy_0 dX_0 dY_0 \int_{x_0,y_0,X_0,Y_0}^{x,y,X,Y} \mathcal{D}[\bar{x}, \bar{y}, \bar{X}, \bar{Y}]$$
$$\times \underbrace{\rho_{\text{tot}}(0, x_0, y_0, X_0, Y_0)}_{\text{factorizable } \rho_S(0) \otimes \rho_E^{\text{eq}}} e^{iS_{\text{tot}}[\bar{x},\bar{X}] - iS_{\text{tot}}[\bar{y},\bar{Y}]}$$

2. Influence functional S_{IF} [Feynman-Vernon (63)]

$$\rho_S(t, x, y) = \underbrace{\int dX dY \delta(X - Y)}_{\text{trace out } E \,=\, \text{path closed at } t} \rho_{\text{tot}}(t, x, y, X, Y)$$
$$= \int dx_0 dy_0 \rho_S(0, x_0, y_0) \int_{x_0,y_0}^{x,y} \mathcal{D}[\bar{x}, \bar{y}] e^{iS_S[\bar{x}] - iS_S[\bar{y}] + iS_{\text{IF}}[\bar{x},\bar{y}]}$$

Influence functional contains all the information of the open system

Coarse graining for quantum Brownian motion

1. Influence functional up to quadratic order

$$iS_{\mathsf{IF}}[x,y] = -\frac{1}{2}\int_0^t dt_1 dt_2 (x,y)_{(t_1)} \underbrace{\begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix}_{(t_1,t_2)}}_{\text{correlation function of } E} \begin{pmatrix} x \\ y \end{pmatrix}_{(t_2)}$$

2. Choice of time after coarse graining

$$t^{>} = \max(t_1, t_2), \quad s = |t_1 - t_2|$$

3. Derivative expansion in s when $\tau_S \gg \tau_E$

$$iS_{\mathsf{IF}}[x,y] = \underbrace{2\gamma m T \int_0^t dt^{>} (x,y) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{momentum diffusion (fluctuation)}}$$

$$+ \underbrace{i\gamma m \int_0^t dt^{>} (x,y) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}}_{\text{drag force (dissipation)}} + \cdots$$

Influence functional is single time integral after coarse graining

Caldeira-Leggett master equation

1. From path integral to differential equation

$$\rho_S(t,x,y) = \int dx_0 dy_0 \rho_S(0,x_0,y_0) \int_{x_0,y_0}^{x,y} \mathcal{D}[\bar{x},\bar{y}] e^{iS_S[\bar{x}] - iS_S[\bar{y}] + iS_{\mathsf{IF}}[\bar{x},\bar{y}]}$$

$$\to i\frac{\partial}{\partial t}\rho_S(t,x,y) = H(x)\rho_S(t,x,y) - H(y)\rho_S(t,x,y)$$
$$- i\gamma\Big[\underbrace{2mT(x-y)^2}_{\text{fluctuation}} + \underbrace{(x-y)(\partial_x - \partial_y)}_{\text{dissipation}}\Big]\rho_S(t,x,y)$$

   - Equivalent to Fokker-Planck equation through Wigner transform

2. Ehrenfest equations

$$\frac{d}{dt}\langle p\rangle = -2\gamma\langle p\rangle, \quad \frac{d}{dt}\langle H\rangle = -4\gamma\left(\langle H\rangle - \frac{T}{2}\right)$$

Quantum mechanical description for Brownian motion

Caldeira-Leggett master equation is NOT Lindblad

1. Positivity of the density matrix

$$\forall|\alpha\rangle \to \langle\alpha|\rho_S|\alpha\rangle \geq 0$$

2. Any Markovian positive map is written by the Lindblad equation [Lindblad (76)]

$$\frac{d}{dt}\rho_S(t) = -i[H,\rho_S] + \sum_{i=1}^{N}\gamma_i\left(L_i\rho_S L_i^\dagger - \frac{1}{2}L_i^\dagger L_i\rho_S - \frac{1}{2}\rho_S L_i^\dagger L_i\right)$$

3. Lindblad form is obtained when higher order expansion is included [Diosi (93)]

$$S_{\mathrm{IF}} = \underbrace{\underbrace{S_{\mathrm{fluct}}}_{\propto\, xx} + \underbrace{S_{\mathrm{diss}}}_{\propto\, x\dot{x}}}_{\text{Caldeira-Leggett}} + \underbrace{S_{(2)}}_{\propto\, \dot{x}\dot{x}}$$

If $L \sim x + \dot{x}$, then $L^\dagger L \ni \dot{x}\dot{x}$

Lindblad equation is not a must, but theoretically more complete

Application to Quarkonium in QGP

Influence functional for heavy quarks

1. Heavy quarks in the non relativistic limit

$$\mathcal{L}_I = -gA_0^a \left[Q^\dagger t^a Q + Q_c t^a Q_c^\dagger \right] = -gA_0^a \rho^a$$

2. Influence functional: $-gA_0^a \rho^a$ is a source term for QGP

$$e^{iS_{\mathsf{IF}}[\rho]} \simeq \int \mathcal{D}[A,q] \rho_{\mathsf{QGP}}^{\mathsf{eq}}[A,q] \exp\left[i \int_{x \in \mathsf{CTP}} \{ \mathcal{L}_{\mathsf{QGP}}(A,q) - gA_0^a \rho^a \} \right]$$

3. Perturbative expansion in terms of gluon correlators in QGP
   - Choose $t^> = \max(t_1, t_2)$ as a single time variable in S_{IF}

$$iS_{\mathsf{IF}} = -g^2 \int_{t^>} \int_{\boldsymbol{xy}} (\rho_1^a,\ \rho_2^a)_{(t,\boldsymbol{x})} \int_{s>0} \begin{bmatrix} G^F & -G^< \\ -G^> & G^{\tilde{F}} \end{bmatrix}_{(s,\boldsymbol{x}-\boldsymbol{y})} \begin{pmatrix} \rho_1^a \\ \rho_2^a \end{pmatrix}_{(t-s,\boldsymbol{y})}$$

4. Derivative expansion based on hierarchy of time scales between G and ρ
   - Expand in s

$$S_{\mathsf{IF}} = \underbrace{S_{\mathsf{pot}} + S_{\mathsf{fluct}}}_{\propto\ \rho\rho} + \underbrace{S_{\mathsf{diss}}}_{\propto\ \rho\dot{\rho}} + \underbrace{S_{(2)}}_{\propto\ \dot{\rho}\dot{\rho}} + \cdots$$

More on influence functional for heavy quarks

1. Gluon correlators at low frequencies

$$V(r) = g^2 G_R(\omega = 0, r), \quad D(r) = g^2 T \frac{\partial}{\partial \omega} \underbrace{\sigma(\omega = 0, r)}_{\text{spectral function}}$$

2. Using the ra-basis: $\rho_r = (\rho_1 + \rho_2)/2, \rho_a = \rho_1 - \rho_2$
   - potenital

   $$S_{\text{pot}} = \int_t \int_{\boldsymbol{xy}} V(\boldsymbol{x} - \boldsymbol{y}) \rho_a(x) \rho_r(y)$$

   - fluctuation

   $$S_{\text{fluct}} = \frac{i}{2} \int_t \int_{\boldsymbol{xy}} D(\boldsymbol{x} - \boldsymbol{y}) \rho_a(x) \rho_a(y) \Leftrightarrow S^{CL}_{\text{fluct}} = 2i\gamma m T x_a^2$$

   - dissipation

   $$S_{\text{diss}} = -\frac{1}{2T} \int_t \int_{\boldsymbol{xy}} D(\boldsymbol{x} - \boldsymbol{y}) \rho_a(x) \dot{\rho}_r(y) \Leftrightarrow S^{CL}_{\text{diss}} = -2\gamma m x_a \dot{x}_r$$

   - 2nd order

   $$S_{(2)} \simeq \frac{i}{4} \int_t \int_{\boldsymbol{xy}} \frac{D(\boldsymbol{x} - \boldsymbol{y})}{8T^2} \dot{\rho}_a(x) \dot{\rho}_a(y)$$

Fluctuation-dissipation theorem in QGP sector relates S_{fluct} and S_{diss}

Master equation from influence functional

THIS IS THE MOST DIRTY PART

1. From path integral to functional differential equation
   - Analogous to "Schrödinger equation from path integral"

$$\underbrace{\rho_S[t, Q_1^{\text{fin}}, Q_2^{\text{fin}}]}_{\text{“wave function” at } t} = \int dQ_{1,2}^{\text{ini}} \underbrace{\rho_S[0, Q_1^{\text{init}}, Q_2^{\text{init}}]}_{\text{initial “wave function”}} \int_{Q_{1,2}^{\text{init}}}^{Q_{1,2}^{\text{fin}}} \mathcal{D}[Q_{1,2}] e^{iS_S[Q_1] - iS_S[Q_2] + iS_{\text{IF}}[Q_1, Q_2]}$$

$$\to \frac{\partial}{\partial t}\rho_S[t, Q_1, Q_2] = \mathcal{L}[Q_1, Q_2]\rho_S[t, Q_1, Q_2]$$

2. From functional density matrix to density matrix
   (i) Recall that the basis of the functional space is the coherent state

$$|Q\rangle \sim e^{-\int_{\boldsymbol{x}} Q(\boldsymbol{x})\hat{Q}^\dagger(\boldsymbol{x})}|\Omega\rangle$$

   (ii) Introduce a heavy quark by functional differentiation

$$\rho_Q(t, \boldsymbol{x}, \boldsymbol{y}) \sim \frac{\delta}{\delta Q_1(\boldsymbol{x})} \frac{\delta}{\delta Q_2(\boldsymbol{y})} \rho_S[t, Q_1, Q_2]|_{Q=0}$$

There must be several ways to derive the master equation from S_{IF}

Lindblad equation for a quarkonium in QGP

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i[H, \rho_{Q\bar{Q}}] + \sum_k \left(L_k \rho_{Q\bar{Q}} L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_{Q\bar{Q}} - \frac{1}{2} \rho_{Q\bar{Q}} L_k^\dagger L_k \right)$$

$$L_k = \sqrt{D(k)} e^{ikx/2} \Big[1 + \underbrace{\frac{ik \cdot \nabla_x}{4MT}}_{\Delta x_Q \sim k/MT} \Big] e^{ikx/2} \quad + \text{heavy antiquark}$$

- Scattering $Qg \to Qg$ with momentum transfer k with rate $D(k)$
- Momentum transfer without recoil $=$ stochastic potential (no dissipation)

$$L_k = \underbrace{\sqrt{D(k)} e^{ikx}}_{\Delta p_Q = k} \quad + \text{heavy antiquark}$$

- Quantum dissipation from heavy quark recoil during a collision
- Coefficient $1/4MT$ fixed by fluctuation-dissipation theorem for QGP correlators

Quantum State Diffusion Simulation for a Heavy Quark

Quantum State Diffusion simulation for Lindblad equation

1. Lindblad equation

$$\frac{d}{dt}\rho_S(t) = -i[H, \rho_S] + \sum_{i=1}^{N} \gamma_i \left(L_i \rho_S L_i^\dagger - \frac{1}{2} L_i^\dagger L_i \rho_S - \frac{1}{2} \rho_S L_i^\dagger L_i \right)$$

2. Stochastic unravelling
   - Equivalent to a nonlinear stochastic Schrödinger equation [Gisin-Percival (92)]

$$\rho_S(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{|\phi_i(t)\rangle\langle\phi_i(t)|}{\underbrace{||\phi_i(t)||^2}_{\phi(t)\ \text{unnormalized}}} = \mathsf{M}\left[\frac{|\phi(t)\rangle\langle\phi(t)|}{||\phi(t)||^2}\right],$$

$$|d\phi\rangle = -iH|\phi(t)\rangle dt + \sum_n \Big(\underbrace{2\langle L_n^\dagger\rangle_\phi L_n}_{\text{nonlinear in } \phi} - L_n^\dagger L_n \Big)|\phi(t)\rangle dt + \sum_n L_n|\phi(t)\rangle d\xi_n,$$

$$\underbrace{\langle d\xi_n d\xi_m^* \rangle}_{\text{complex noise}} = 2\delta_{nm} dt$$

Apply this technique to heavy quark Lindblad equation

Nonlinear stochastic Schrödinger equation for a heavy quark

- Nonlinear stochastic Schrödnger equation

$$
\begin{aligned}
&d\phi(x,t) = \phi(x,t+dt) - \phi(x,t) \\
&\simeq \left(i\frac{\nabla^2}{2M} - \frac{1}{2}D(0) \right) \phi(x)dt + d\xi(x)\phi(x) \\
&\quad + \frac{dt}{||\phi(t)||^2} \int d^3y D(x-y)\phi^*(y)\phi(y)\phi(x) + \mathcal{O}(T/M)
\end{aligned}
$$

- Correlation of complex noise field

$$
\langle d\xi(x) d\xi^*(y) \rangle = D(x-y)dt, \quad \langle d\xi(x) d\xi(y) \rangle = \langle d\xi^*(x) d\xi^*(y) \rangle = 0
$$

- Density matrix for a heavy quark

$$
\rho_Q(x,y,t) = \mathsf{M}\left[\frac{\phi(x,t)\phi^*(y,t)}{||\phi(t)||^2} \right]
$$

What is the equilibrium solution of the Lindblad equation?
How does a heavy quark approach equilibrium?

QSD simulation for a single heavy quark in an external potential

Numerical setups

$$V_{\text{ext}}(x) = 0, \quad \frac{1}{2}M\omega^2 x^2, \quad -\frac{\alpha}{\sqrt{x^2 + r_c^2}}$$

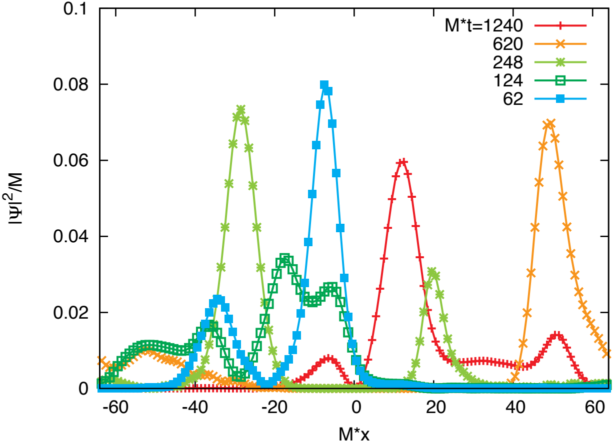
$$D(x) = \gamma \exp\left[-x^2/l_{\text{corr}}^2\right]$$

Δx	Δt	N_x	T	γ	l_{corr}	ω	α	r_c
$1/M$	$0.1M(\Delta x)^2$	128, 127	$0.1M$	T/π	$1/T$	$0.04M$	0.3	$1/M$

$$\Delta x = \frac{1}{M} \ll l_{\text{corr}} = \frac{10}{M} \ll N_x \Delta x = \frac{128}{M}$$

Do the density matrix approach $\propto \exp(-H/T)$?

Solitonic wave function in one sampling

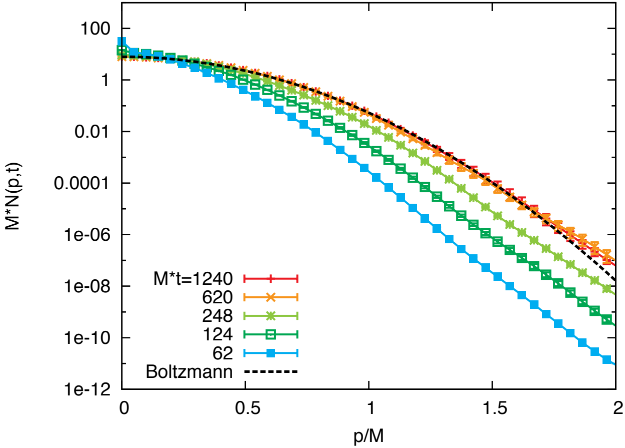


Wave function is localized because of the nonlinear evolution equation

Equilibration of a heavy quark: $V_{\text{ext}} = 0$

Time evolution of momentum distribution

- Relaxation time of corresponding classical system $M\tau_{\text{relax}} \sim 300$

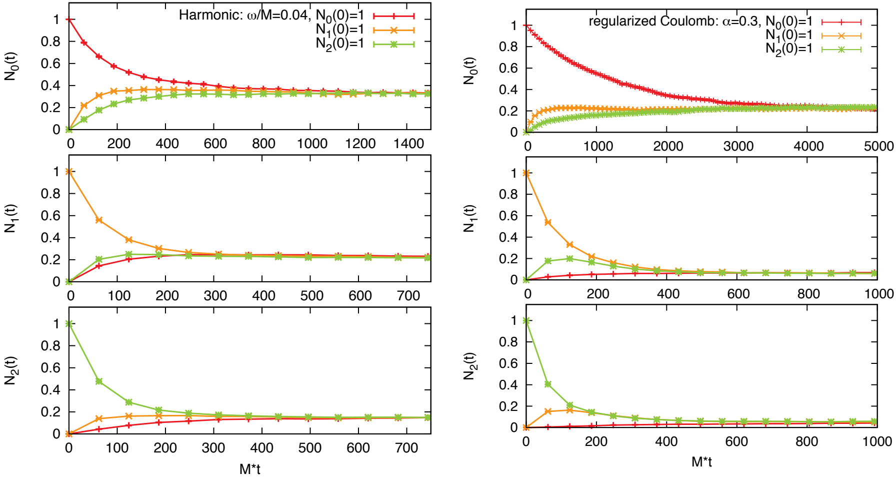


Equilibrium momentum distribution is the Boltzmann distribution!

Equilibration of a heavy quark: $V_{\text{ext}} = V_{\text{HO/Coulomb}}$

Time evolution of eigenstate occupation (lowest 3 levels)

- Harmonic potential (left), regularized Coulomb potential (right)

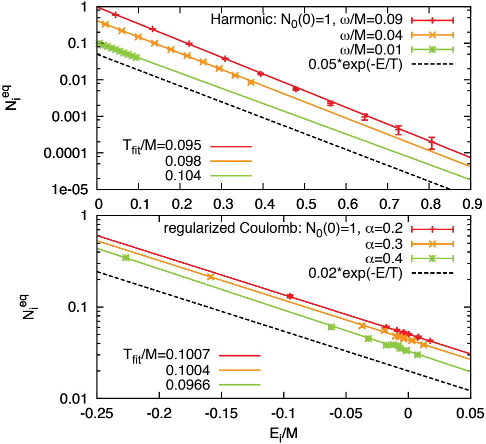


Eigenstate occupation relaxes to a static state
Relaxation time depends on the initial state and rate equation is inapplicable

Equilibrium distribution of a heavy quark: $V_{\rm ext} = V_{\rm HO/Coulomb}$

Equilibrium distribution of eigenstates (lowest 10 levels)

- Harmonic potential (top), regularized Coulomb potential (bottom)



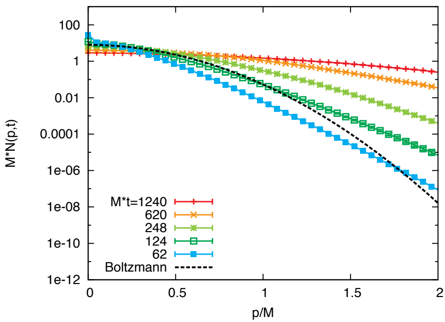
We also checked that off-diagonal part is 0 within statistical fluctuation

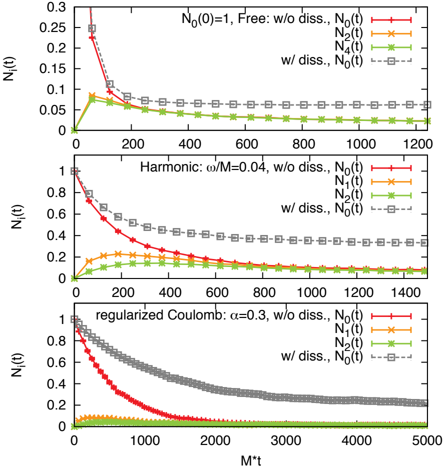
Eigenstate distribution in the external potential is also the Boltzmann distribution

QSD simulation without quantum dissipation ($=$ stochastic potential)

Heavy quark is overheated because energy increases without dissipation

- Neglect $\mathcal{O}(T/M)$ terms in the nonlinear stochastic Schrödinger equation





Dissipation is more important for smaller bound state because decoherence is ineffective

Summary and outlook

Quantum State Diffusion simulation for Lindblad equation

- Equivalent to nonlinear stochastic Schrödinger equation (integro-differential equation)
- Numerically confirm the equilibration of a heavy quark $\rightarrow$ Can be shown analytically?

Possible application

- Quarkonium evolution in heavy-ion collisions [Akamatsu et al, in progress]
- Dark matter bound state in early universe? [Kim-Laine (17)]
- Cold atomic gases? [Braaten-Hammer-Lepage (16)]

Back Up

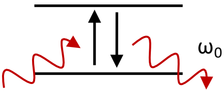
Explicit form of gluon correlators in HTL approximation

$$G_R(\omega = 0, r) = -\frac{e^{-m_D r}}{4\pi r},$$

$$\frac{\partial}{\partial\omega}\sigma_{ab,00}(0,\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\frac{\pi m_D^2 e^{i\vec{k}\cdot\vec{r}}}{k(k^2+m_D^2)^2},$$

$$m_D^2 = \frac{g^2T^2}{3}\left(N_c + \frac{N_f}{2}\right)$$

Example 1 – Quantum optical master equation



- A two-level atom in a photon gas

$$
\begin{aligned}
i\frac{d}{dt}\rho_A =& \gamma \underbrace{(N(\omega_0)+1)}_{\text{emission}} \left[\sigma_-\rho_A\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho_A - \frac{1}{2}\rho_A\sigma_+\sigma_-\right] \\
&+ \gamma \underbrace{N(\omega_0)}_{\text{absorption}} \left[\sigma_+\rho_A\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho_A - \frac{1}{2}\rho_A\sigma_-\sigma_+\right]
\end{aligned}
$$

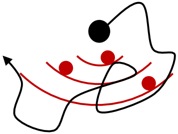
- Approximations

$$
\underbrace{\rho_{\text{tot}}(t) \simeq \rho_A(t) \otimes \rho_B^{\text{eq}},}_{\text{Born approx. (weak coupling)}} \quad \underbrace{\tau_B \ll \tau_R \equiv 1/\gamma,}_{\text{Markov approx.}} \quad \underbrace{\tau_A \equiv 1/\omega_0 \ll \tau_R}_{\text{rotating wave approx.}}
$$

  - Environment correlation time τ_B
  - System intrinsic time scale τ_A, system relaxation time τ_R

Master equation is an effective description at $\tau_R \gg \tau_B$ for $\tau_A \ll \tau_R$

Example 2 – Quantum Brownian motion



Caldeira-Leggett model [Caldeira-Leggett (83)]

- Brownian particle linearly coupled to harmonic oscillators

$$i\frac{d}{dt}\rho_A = [H_A, \rho_A] + \underbrace{\gamma[x, \{p, \rho_A\}]}_{\text{drag force}} - \underbrace{2i\gamma m T[x, [x, \rho_A]]}_{\text{momentum diffusion}}$$

- Approximations

$$\underbrace{\rho_{\text{tot}}(t) \simeq \rho_A(t) \otimes \rho_B^{\text{eq}},}_{\text{Born approx. (weak coupling)}} \quad \underbrace{\tau_B \ll \tau_R \equiv 1/\gamma,}_{\text{Markov approx.}} \quad \underbrace{\tau_B \ll \tau_A}_{\text{derivative expansion}}$$

  - Environment correlation time τ_B
  - System intrinsic time scale τ_A, system relaxation time τ_R

Master equation is an effective description at $\tau_R \gg \tau_B$ for $\tau_A \gg \tau_B$