# Factorization Theorem Relating Euclidean and Light-cone Parton Distributions

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INT Workshop 17-68W, Flavor Structure of the Nucleon Sea, University of Washington, Seattle 10/02-13, 2017

T. Izubuchi, X. Ji, L. Jin, I. Stewart and Y.Z., to be published.

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# Outline

 $\triangle$  1. Quasi and pseudo distribution approaches to calculating PDF from lattice QCD

ò 2. Equivalence between large *Pz* and small |*z|* factorizations

### $\div$  3. Hints on lattice calculations

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### $\triangle$  3. Hints on lattice calculations

## PDF from the Euclidean Lattice



PDF not directly accessible from the lattice!

### Parton model:

- Emerges in the infinite momentum frame
- Or, the proton seen by an observer moving at the speed of light (on the light-cone)
- Parton distribution function

$$
q(x,\mu) = \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \left\langle P \middle| \overline{\psi}(\xi^-) \gamma^+ U(\xi^-,0) \psi(0) \middle| P \right\rangle
$$
  

$$
\xi^{\pm} = (t \pm z) / \sqrt{2} \qquad U(\xi^-,0) = P \exp \left[ -ig \int_0^{\xi^-} d\eta^- A^+(\eta^-) \right]
$$

### Lattice QCD:

- Euclidean space
- Nucleon at finite momenta
- Cannot calculate time-dependent quantities with contribution from physical poles
- Light-cone separation  $\Delta s^2 = 0 \Rightarrow \Delta s^{\mu} = (0,0,0,0)$

## Methods based on operator product expansion (OPE)

### ò Hadronic Tensor (Unpolarized)

$$
W_{\mu\nu}(q, P) = \frac{1}{\pi} \text{Im} \, T_{\mu\nu} = \int \frac{d^4 z}{4\pi} e^{iq.z} \left\langle P \Big| \Big[ J_{\mu}(z), J_{\nu}(0) \Big] \Big| P \right\rangle
$$
  
=  $(g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2}) F_1(\omega, Q^2) + \hat{P}_{\mu} \hat{P}_{\nu} F_2(\omega, Q^2)$ 

$$
Q^{2} = -q^{2},
$$
  
\n
$$
\omega = 2P \cdot q / Q^{2},
$$
  
\n
$$
\hat{P}_{\mu} = P_{\mu} - \frac{P \cdot q}{q^{2}} q_{\mu}.
$$

In the Bjorken (DIS) limit (not directly accessible on the lattice),

$$
Q^2 \to \infty, \, q \cdot P \to \infty, \, \omega = \frac{1}{x} \in (1, \infty), \quad F_i(x, Q) = \int dy \sum_{f = q, g} C_{i, f} \left(\frac{x}{y}, \frac{Q}{\mu}\right) q_f(y, \mu)
$$

$$
\oint \text{ Euclidean OPE of the Compton Amplitude } Q^2 \rightarrow \infty, Q^2 \gg q \cdot P, \omega \rightarrow 0
$$
\n
$$
T_{\mu\nu}(q, P) = \int \frac{d^4 z}{4\pi} e^{iq \cdot z} \langle P | T J_{\mu}(z) J_{\nu}(0) | P \rangle = \sum_{i,n} C_{i,n} (q \cdot P, Q^2) A_{i,n} (Q^2) \omega^{n-1} + O(\frac{1}{Q})
$$
\n
$$
A_{i,n} (Q^2) \sim \int dx \ x^{n-1} F_i(x, Q^2)
$$
\n
$$
= \frac{A_{i,n} (Q^2) \sim \int dx \ x^{n-1} F_i(x, Q^2)}{10^{10/9/17}}
$$

 $\frac{1}{2}$  10/9/17 10/

## Methods based on operator product expansion (OPE)

 $\triangleleft$  Direct computation of PDF moments:

 $\int dx$   $x^{n-1}q(x,\mu)dx \sim n$  $\mu_{1}$ *n*  $\mu_{2}^{}$  $\cdots$ *n*  $\mu_{n}$  $P|\overline{\psi}(0)\gamma^{\mu_{1}}i$  $\ddot{ }$  $\ddot{D}^{\mu_2}\cdots \ddot{p}$  $\ddot{ }$  $\left| D^{\mu _{n}}\right| P$ 

- Moments are calculable as matrix elements of local gaugeinvariant and frame-independent operators;
- Fitting the PDF using the finite number of moments calculated;
- Operator mixing on the lattice limits computation for moments higher than 3.

**n≤3**, W. Detmold et al., EPJ 2001, PRD 2002; D. Dolgov et al. (LHPC, TXL), PRD 2002; **n>3**, Z. Davoudi and M. Savage, PRD 2012.

## Methods based on operator product expansion (OPE)

### $\triangleleft$  Fictitious heavy quark current

Auxiliary heavy quark mass sets the OPE scale Higher twist effects suppressed by heavy quark mass Capable of calculating higher moments

ò Direct use of OPE of the Compton amplitude

Utilizing full dependence of  $\omega$  (only for  $\omega$  <1?) Obtain many moments to fit the PDF

D. Lin and W. Detmold, PRD 2006. See D. Lin's talk

A. J. Chambers et al. (QCDSF), PRL 2017 See G. Schierholz's talk

Direct computation of the hadronic tensor

K.F. Liu (et al.), 1994, 1999, 1998, 2000, 2017. See K.F. Liu's talk

## Factorization approaches (not the moments)

 $\div$  Large momentum effective theory (LaMET)

Quasi-PDF approach

 $\triangle$  Lattice cross section

X. Ji, PRL 2013, Sci.China Phys.Mech.Astron., 2014. See H.-W. Lin and J.-W. Chen's talks

Y.-Q. Ma and J. Qiu, 2014, 2017.

ò Pseudo-PDF approach

A. Radyushkin, PRD 2017; K. Orginos, A. Radyushkin, J. Karpie and S. Zafeiropoulos, 2017

 $\triangleleft$  Factorization of Euclidean correlations

V. M. Braun and D. Mueller, EPJ.C. 2008 G. S. Bali, V. M. Braun, A. Schaefer, et al., 2017.

## LaMET approach

- Quasi-PDF  $\tilde{q}(x, P^z, \Lambda = a^{-1}) =$ *dz*  $\int \! \frac{dZ}{4\pi} e^{ixP^z z} \left\langle P \middle| \overline{\psi}(z) \gamma^z U(z,0) \psi(0) \middle| P \right\rangle \quad \textit{U}(z,0) = P \exp \left[ -ig \int_0^z dz' A^z(z') \right]$  $z^{\mu}$  = (0,0,0,*z*)
- Spatial correlation along the *z*  $e^{z}$   $e^{z}$ direction, calculable in lattice QCD;
- Under an infinite Lorentz boost along the *z* direction, the spatial gauge link approaches the lightcone direction;



## LaMET approach



- $\triangleleft$  (Light-cone) PDF  $P^z \gg \Lambda \gg M, \Lambda_{QCD}$
- $\triangle$  Taking the infinite momentum limit changes the UV physics, but not the IR physics:
- $\triangle$  The UV difference be calculated in perturbative QCD, so the quasi-PDF can be factorized into the light-cone PDF!

### How matching works



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### LaMET approach The structure of the renormalization of the quasi-PDF in Eqs. (4) and (5) is similar to that of the quark beam-

 $\triangle$  The quasi-PDF is related to the PDF through a factorization formula: proof also implied that there is never parton mixing in this case. Since this lack of mixing has not yet been explored

$$
\tilde{q}_i(x, P^z, \tilde{\mu}) = \int_{-1}^{+1} \frac{dy}{|y|} C_{ij} \left( \frac{x}{y}, \frac{\tilde{\mu}}{P^z}, \frac{\mu}{P^z} \right) q_j(y, \mu) + \mathcal{O}\left( \frac{M^2}{P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{P_z^2} \right),
$$

- ò They have the same IR divergences; where *<sup>C</sup>ij* is the matching coecient, and the *<sup>O</sup>*(*M*<sup>2</sup>*/P*<sup>2</sup> *<sup>z</sup> ,*⇤<sup>2</sup> QCD*/P*<sup>2</sup> **by They have the same IR divergences;**  $\rightarrow$  They have the same IR divergences;
- ↑ *C* factor matches the difference in their UV divergence, and can be calculated in perturbative QCD; contribution. The power corrections are related to higher-twist contributions in the  $qD$  contributions in the quasi PDF. Note that it is in the  $qD$  $\forall$  C factor matches the difference in their  $\cup$  v  $\overline{OCD}$ . On the other hand the matching coefficients of matrix occurs occur in a relation between renormalized matrix of  $\overline{OCD}$ . elements of di↵erent operators. The ˜*q* and *q* have the same collinear and infrared (IR) divergences, so at perturbative
- **♦** Higher twist corrections suppressed by powers of  $P^z$ .  $\mathcal{L}$  proposal, the proposal, the proposal, the proposal, the proposal, the summarized as:

### Current status



#### Ioffe-time and pseudo distributions  $\overline{C}$   $\overline{$ rie-time and pseudo distributions offe-time and pseudo distributions the distribution of the distribution of the second state of the second state of the second state  $\frac{1}{2}$  $\overline{a}$  distribution,  $\overline{a}$ *Q*˜<sup>0</sup> (⇣ = *P<sup>z</sup>z, z*<sup>2</sup> *,* ✏) = <sup>1</sup>

ò Ioffe-time distribution:  $\mathbf{a}$  find distribution,  $^{\rm{M}}$ me (

$$
\tilde{Q}_{\gamma^0}(\zeta = P^z z, z^2, \epsilon) = \frac{1}{2P^0} \langle P|\tilde{O}_{\gamma^0}(z)|P\rangle ,
$$

$$
\tilde{O}_{\Gamma}(z) = \bar{\psi}(z)\Gamma U(z, 0)\psi(0) ,
$$

A. Radyushkin, PRD 2017

*z2*=0, reduces to the light-cone correlation. *O*  $\alpha$  *D*  $\beta$  *D*  $\$ Here *A*. Kaayusiikin, PRD 2017<br>O. reduces to the light-cone correlation.  $\frac{D}{\sqrt{D}}$  $i=0$ , reduces to the light-cone correlation.

 $\triangle$  Pseudo distribution man<br>Seudo distributio

$$
\mathcal{P}(x, z^2, \epsilon) = \int \frac{d\zeta}{2\pi} e^{ix\zeta} \tilde{Q}(\zeta, z^2, \epsilon).
$$

Support  $-1 \le x \le 1$ .  $z^2 = 0$ , reduces to the PDF.  $z^2 = 0$ , reduces  $\mathbf{e}$ *eix*⇣*Q*˜ ipport  $-1 \le x \le 1$ ,  $z^2 = 0$ , reduces to the PDF.

INT Workshop, UW, Seattle 14 and 10/9/17  $T$  is a special case of the pseudo distribution of the pseudo distribution when  $10/9/1$ 



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## Renormalization

- $\triangleleft$  Auxiliary field formulation of the Wilson line *DVDV* ∫ *<sup>V</sup>*(*z*)*V*(0)*<sup>e</sup>*  $\int d^4x \left[ L_{QCD}(x) + \overline{V}(x) i n_x \cdot DV(x) \right] = \left\langle U(z,0) \right\rangle$  H. Dorn,  $\tilde{O}_{\Gamma}(z) = \overline{\psi}(z)\Gamma U(z,0)\psi(0) = j_1(z)j_2(0), \ \ j_1 = \overline{\psi} \ \Gamma V, j_2 = \overline{V}\psi$ Fortsch. Phys. 1986
	- $\rightarrow$  *V(x)* is a one-dimensional Grassman field that only depends on the coordinate z, similar to a heavy quark;
	- $\phi$  *j*<sub>1</sub> and *j*<sub>2</sub> are proven to be multiplicatively renormalizable (like heavy-to-light current in HQET). No further renormalization for the nonlocal current-current correlator; X. Ji, J.-H. Zhang, and
	- ò Self-energy of *V* can induce a mass correction. J. Green et al., 2017 Y.Z., 2017

## Renormalization

 $\triangle$  The gauge-invariant quark Wilson line operator can be renormalized multiplicatively in the coordinate space:

$$
\tilde{O}(z) = \overline{\psi}(z)\Gamma U(z,0)\psi(0) = Z_{\psi,z}e^{-\delta m|z|}(\overline{\psi}(z)\Gamma U(z,0)\psi(0))^R
$$

X. Ji, J.-H. Zhang, and Y.Z., 2017 T. Ishikawa, Y.-Q. Ma, J.Qiu, S. Yoshida, 2017

 $\triangle$  Different renormalization schemes can be matched perturbatively in the coordinate space. Without loss of generality we can choose the MSbar scheme.

#### OPE of the "heavy-to-light" currentcurrent correlator momentum *P<sup>z</sup>*. Nevertheless, the matching coecient of the form in  $\mathbf{C}$ OPE is a technique to expand nonlocal operators with time distribution, *Q*˜<sup>0</sup> (⇣ = *P<sup>z</sup>z, z*<sup>2</sup> <sup>2</sup>*P*<sup>0</sup> <sup>h</sup>*P|O*˜<sup>0</sup> (*z*)*|P*i*,* (7) from lattice QCD has been proposed in addition to the  $m<sup>3</sup>$  continued in the MS scheme can be expanded in the MS scheme can be expanded in the  $m<sup>4</sup>$ culicillcalculation of PDF  $\alpha$  $\mathcal{L}$  and  $\mathcal{L}$  and  $\mathcal{L}$  and  $\mathcal{L}$  $\overline{\phantom{a}}$  is a technique to expand nonlocal operators with  $\overline{\phantom{a}}$  $\text{correlator}$ momentum *P<sup>z</sup>*. Nevertheless, the matching coecient of the form in  $\mathbf{C}$ II. FACTORIZATION FROM OPERATORIZATION FROM OPERATORIZ ODE of the "heavy to light" OF *L* Or the spatial correlation of  $\alpha$ malized in the MS scheme can be expanded in terms of local gauge-invariant operators as  $\epsilon$  during the set of  $\epsilon$

 $\triangleleft$  For the gauge-invariant Wilson line operator, it can have For the gauge-invariant wilson line opera<br>an OPE in the Euclidean limit of  $z^2$  ->0: <sup>2</sup>*P*<sup>0</sup> <sup>h</sup>*P|O*˜<sup>0</sup> (*z*)*|P*i*,* (7) of *<sup>z</sup>*<sup>2</sup> ! 0. The spatial correlation operator *<sup>O</sup>*˜(*z*) renorthe gauge-invariant Wilson line operator, in the ↑ For the gauge-invariant Wilson line operator, it can have *<sup>n</sup>*(*µ*<sup>2</sup>*z*<sup>2</sup>)  $\cdot$  $\overline{P}$  $\overline{P}$  in the Euclidean limit of  $z^2 \rightarrow 0$ *n*=2019 *n n*<sub>1</sub> *or*<sub>*n*</sub> |
|
| *l* in the E  $\overline{\mathbf{r}}$ *<sup>n</sup>*! *<sup>n</sup><sup>µ</sup>*<sup>1</sup> *··· <sup>n</sup><sup>µ</sup><sup>n</sup> <sup>O</sup><sup>µ</sup>*0*µ*1*···µ<sup>n</sup>* 2

$$
\tilde{O}_{\gamma^{z}}(z) = \sum_{n=0} \left[ C_{n}(\mu^{2} z^{2}) \frac{(iz)^{n}}{n!} n_{\mu_{1}} \cdots n_{\mu_{n}} O_{1}^{\mu_{0}\mu_{1}\cdots\mu_{n}} + C'_{n}(\mu^{2} z^{2}) \frac{(iz)^{n}}{n!} n_{\mu_{1}} \cdots n_{\mu_{n}} O_{2}^{\mu_{0}\mu_{1}\cdots\mu_{n}} \mu_{0} = z, \n+ \text{higher-twist operators} \right] \frac{(13)}{O(z^{2} \Lambda_{QCD}^{2})} \sigma_{1}^{\mu_{0}\mu_{1}\ldots\mu_{n}} = \bar{\psi}\gamma^{(\mu_{0}} i D^{\mu_{1}} \ldots i D^{\mu_{n}}) \psi - \text{trace}, \qquad C_{n} = 1 + O(\alpha_{s}) \nO_{2}^{\mu_{0}\mu_{1}\ldots\mu_{n}} = F^{(\mu_{0}\rho} i D^{\mu_{1}} \ldots i D^{\mu_{n-1}} F_{\rho}^{\mu_{n}} - \text{trace}, \qquad C'_{n} = O(\alpha_{s})
$$

We only consider the iso-vector case, so  $O_2$  can be dropped.<br>INT Workshop, UW, Seattle with  $\mathbf{p}$  standing for the symmetrization of the symmetrization of the symmetrization of the  $\mathbf{p}$ *eix*⇣*Q*˜ ⇣*, z*<sup>2</sup> *.* (9) operators in the OPE, We only consider the iso-vector case, so  $O_2$  can be dropped.

INT Workshop, UW, Seattle 10/9/17 *O<sup>µ</sup>*0*µ*1*...µ<sup>n</sup> .* (9) operators in the OPE, The above OPE is valid for the operator itself. In the  $\frac{19}{2}$ *O*(⇠) = ¯(⇠)*U*(⇠*,* 0) (0) *.* (11) INT WORKSHOP, UW, SEATTIE  $\mathbf{r}_1$ , we can neglect the mixing with the mixing with the gluon wit

#### *OPE* of the "heavy-to-light" currentcurrent correlator <sup>2</sup>*<sup>P</sup>* <sup>+</sup> <sup>h</sup>*P|O*<sup>+</sup> (⇠)*|P*<sup>i</sup> *,* (10) *O*(⇠) = ¯(⇠)*U*(⇠*,* 0) (0) *.* (11) with (*···*) standing for the symmetrization of the Lorentz  $\cap$ f tl Thus nearly to hand cancel operator, which we will stick to for the rest of discussion. contribution is the leading approximation of the nucleon E of the "heavy-to-light" curre now on we will drop all the higher-twist contributions for  $\overline{C}11$  $C$  *L UISLE COLL COLL COLL COLL COLL COLL COLL COLL*  $\mathbf{L}$  as diagonal understood if we con-

 $\triangle$  In the large momentum limit, all (kinematic) higher twist ob *O*<br>contributions are suppressed:  $\leftrightarrow$ enough information for the Io↵e-time distribution with small *z*<sup>2</sup>, one has to do lattice calculations with large  $M_{\odot}$  is the same requirement of the same requirement for the same requirement for the same requirement for the quasi- $\frac{1}{2}$  **p** = 0.000 *p* = 0.000 *p* = 1.000 *p* = 1.000 *p* = 1.000 *p* = 1.000 *p = 1.000 p = 1.00* large momentum limit, all (kinematic) higherefore, the momentum limit, all (kinematic) higher MS scheme, they are singular near *z*<sup>2</sup> = 0, and so is  $P$ **Public diverse process.**<br>Polological p  $t_{\rm max}$  $w$  1St $\hskip1cm$ 

$$
\langle P|O_1^{\mu_0\mu_1\cdots\mu_n}|P\rangle = 2a_{n+1}(\mu)\left(P^{\mu_0}P^{\mu_1}\dots P^{\mu_n} - \text{trace}\right), \quad O(M^2/P_z^2)
$$

$$
a_{n+1}(\mu) = \int_{-1}^1 dx \, x^n q(x,\mu) ,
$$

↑ Leading twist approximation:  $T_{\text{max}}$  rest of the rest organized as  $T_{\text{max}}$ and the explicit expression of the trace term has been  $ng$  twist approximation:

$$
\tilde{Q}_{\gamma^z} (\zeta, \mu^2 z^2) \n= \sum_{n=0} C_n (\mu^2 z^2) \frac{(-i\zeta)^n}{n!} a_{n+1} (\mu) \n= \sum_{n=0} C_n (\mu^2 z^2) \frac{(-i\zeta)^n}{n!} \int_{-1}^1 dy y^n q(y, \mu) .
$$

INT Workshop, UW, Seattle 10/9/17 It should be noted that the only approximation  $\mathcal{L}_\mathbf{p}$  $20$  and  $20$ 

#### Large P<sup>z</sup> factorization formula suppressed by the large momentum *P<sup>z</sup>* of the nucleon.  $\blacksquare$ ation formula $\blacksquare$ suppressed by the large momentum *P<sup>z</sup>* of the nucleon.

#### $\triangleleft$  Fourier transform to get the quasi PDF *q*˜ י<br>ג *x, µ*2  $\mathsf{T}$

 $\tilde{q}$ 

$$
\tilde{q}\left(x, \frac{\mu^2}{P_z^2}\right)
$$
\n
$$
= \int \frac{d\zeta}{2\pi} e^{ix\zeta} \tilde{Q}\left(\zeta, \frac{\mu^2 \zeta^2}{P_z^2}\right)
$$
\n
$$
= \int_{-1}^1 dy \left[ \int \frac{d\zeta}{2\pi} e^{ix\zeta} \sum_{n=0} C_n \left(\frac{\mu^2 \zeta^2}{P_z^2}\right) \frac{(-i\zeta)^n}{n!} y^n \right] q(y, \mu)
$$
\n
$$
= \int_{-1}^1 \frac{dy}{|y|} \left[ \int \frac{d\zeta}{2\pi} e^{i\frac{x}{y}\zeta} \sum_{n=0} C_n \left(\frac{\mu^2 \zeta^2}{(yP^z)^2}\right) \frac{(-i\zeta)^n}{n!} q(y, \mu) \right],
$$
\n
$$
\tilde{q}\left(x, \frac{\mu^2}{P_z^2}\right) = \int_{-1}^1 \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu^2}{(yP^z)^2}\right) q(y, \mu) \qquad C\left(\frac{x}{y}, \frac{\mu^2}{(yP^z)^2}\right) = \int \frac{d\zeta}{2\pi} e^{i\frac{x}{y}\zeta} \sum_{n=0} C_n \left(\frac{\mu^2 \zeta^2}{(yP^z)^2}\right) \frac{(-i\zeta)^n}{n!},
$$

INT Workshop, UW, Seattle 10/9/17 and used in the early papers on the early papers ready, not the nucleon momentum  $\mathcal{P}$ <sup>z</sup>l Ji, Schaefer, Xio mentum, not the nucleon moniemum  $\overrightarrow{r}$  if  $\overrightarrow{PRD}$  2015 define the this kernel to be the top the this kernel to be the this control to be th I Ji, Schaefer, Xiong, and Zhang, *P*(*z*<sub>2</sub>*y*<sub>2</sub>) = *Z z*<sub>2</sub>*y*<sub>2</sub>  $wP^z$  the narton momentum not the nucleon me  $t^2$ , the parton measurements, not the motion has *yPz* , the parton momentum, not the nucleon momentum *Pz* PRD 2015 21

#### Small |*z*| factorization *<sup>q</sup>*(*y, µ*) = <sup>Z</sup> <sup>1</sup> *d*⇣  $I = \bigcup_{i=1}^n I_i$ made so far is ignoring the higher twist e $\mathcal{L}_{\mathcal{A}}$  is ignoring that are twist e $\mathcal{L}_{\mathcal{A}}$  $\overline{\phantom{a}}$ **Small** |z| factorization **11 12**, the leading approximation of  $\boldsymbol{\mu}$  $\int$ *C*(*C*(*C*) *C*(*C*)  $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$ sional regularization. In the MS scheme, *q*(*x, µ*) does not  $\lfloor |z| \rfloor$  factorization  $\overline{u}$   $\overline{$ at the reference studies in the reference set in the set of the reference in the 1 GeV. The set is a set of th corresponding Fourier-transformed position space  $\mathbf{F}$  $d$  data can be directly confronted with the theory since  $\mathcal{L}$  $\mathbf{z}$ ations based on model parameterizations of  $\mathbf{z}$ the DAs can easily be transformed to position space.

↑ Ioffe-time distribution: *<sup>z</sup> <sup>M</sup>*<sup>2</sup>, we have *<sup>P</sup>*<sup>0</sup> ⇠ *<sup>P</sup><sup>z</sup>*, so even if lattice for a range of values of *p* ·*z* = p · z and *z*<sup>2</sup> = z<sup>2</sup>  $\epsilon$  distribution space (3) that con-

$$
\Phi \text{Iotte-time distribution:}
$$
\n
$$
\tilde{Q}(\zeta, z^2 \mu^2)
$$
\n
$$
= \int_{-1}^1 dy \int_{-\infty}^{\infty} d\alpha \ e^{-i\alpha(y\zeta)} \mathcal{C}(\alpha, \mu^2 z^2) q(y, \mu)
$$
\n
$$
= \int_{-\infty}^1 dy \int_{-\infty}^{\infty} d\alpha \ e^{-i\alpha(y\zeta)} \mathcal{C}(\alpha, \mu^2 z^2) q(y, \mu)
$$
\n
$$
= \int_{-\infty}^{\infty} d\alpha \ \mathcal{C}(\alpha, \mu^2 z^2) Q(\alpha\zeta, \mu).
$$
\nSimilarly, the final solution is:\n
$$
\text{Similar to}
$$
\n
$$
\text{Value}^{i(u-1/2)(p\cdot z)} H(u, z^2, \mu) \phi_{\pi}(u, \mu) + T^{\text{HT}},
$$
\n
$$
\text{Value}^{\text{I}} H(u, z^2, \mu) \phi_{\pi}(u, \mu) + T^{\text{HT}},
$$
\n
$$
\text{Value}^{\text{I}} H(u, z^2, \mu) \phi_{\pi}(u, \mu) + T^{\text{HT}},
$$
\n
$$
\text{Value}^{\text{I}} H(u, z^2, \mu) \phi_{\pi}(u, \mu) + T^{\text{HT}},
$$

↑ Ratio function?  $\mathbf{v}$ *dy* atio *ei x* **↑** Rat ↑ Ratio function?

*P*<sup>2</sup>

 $\frac{1}{\cdot}$ *d*<sub>*d*</sub></sub> $\frac{1}{2}$ *d*<sub>*d*</sub> $\frac{1}{2}$ µ۱.<br>∢ *d*<sub>*d*</sub> *d*<sub>*d*</sub><sup>*d*</sup> *d*<sub>*d*</sub><sup>*d*</sup><sub>*e*</sub>*d*<sub>*d*</sub><sup>*d*</sup><sub>*e*</sub>*d*<sub>*d*</sub><sup>*d*</sup>*d*<sup>*d*</sup>*d*</sub>*d*<sub>*d*</sub><sup>*d*</sup>*d*<sup>*d*</sup>*d*</sub>*d*<sub>*d*</sub>*dd*<sub>*d*</sub>*ddd*</sub>*ddddd*</sub>*dddddddd*</del>*dddddddd*</del>*dddddd C*(*C.* 2008;<br>*C*(*C*) *Z*(*C*)*d*(*C. 2017*).  $\mathbf{e}^{\dagger}$  $\frac{a_{11}}{a_{12}}$ ✓*x* G. S. Bali, V. M. Braun, A. Schaefer, et al., 2017. $F_{\rm{201D}}$  and D. Mueller, EPI  $C_{\rm{2008}}$ V.M. Braun and D. Mueller, EPJ. C. 2008;

 $\int_{a}^{a} 1/x$  and  $\int_{a}^{a} 2/x$ 

$$
\tilde{Q}(0, \mu^2 z^2) = C_0(\mu^2 z^2), \qquad \frac{\tilde{Q}(\zeta, \mu^2 z^2)}{\tilde{Q}(0, \mu^2 z^2)} = \sum_n \frac{C_n(\mu^2 z^2) (-i\zeta)^n}{C_0(\mu^2 z^2)} a_{n+1}(\mu).
$$
\nThe ratio still has a mild degrees over  $z^2$ .

- **••** The ratio still has a mild dependence over  $z^2$ ; ready, one can see that the matching kernel is a function If  $z^2$ ; ready, one can see that the matching kernel is a function
- The ratio is not the light-cone correlation, rather, it can be understood as a "renormalized" Ioffe-time distribution; + *<sup>|</sup>y<sup>|</sup> <sup>C</sup> y z*<sub>2</sub> *ne* ratio is not the light-cone correlation, rather "renormalized" Ioffe-time distribution;  $t$  the pseudo distribution, the pseudo-*<sup>P</sup>*(*x, z*<sup>2</sup>*µ*<sup>2</sup>) = <sup>Z</sup> <sup>1</sup> of *x/y* and *µ*<sup>2</sup>*/*(*yP<sup>z</sup>*)<sup>2</sup> if the series in *n* converges. We  $t$  relation, rather it can be und *P*(*p*)<br>*P*(*p*)<br>*P*(*x* 22*i*<sub>0</sub>) = *p*(*x* 2*i*<sub>0</sub>) = *p*(*x* 2*i*<sub>0</sub>) ⇣*, µ*<sup>2</sup>*z*<sup>2</sup> rmalized" **n** *n a a***n** *a<sub>n</sub> a* **<b>***a a* mo topentence over 2),<br>white one correlation rather it can be understood as a  $\mathcal{F}_{\mathcal{D}}$ ght-cone correlation, rather, it can be understood as a<br>*ztime distribution*:
	- One still needs the small  $|z|$  factorization formula to extract the PDF. .<br>الم <sup>11</sup>11 م *y y y µ , <sup>µ</sup>*<sup>2</sup>  $\mathbf{e}$ *e ds* the s **m**2<sup>11</sup> ⊥∤ (*yP<sup>z</sup>*)<sup>2</sup>  $factori$ *n*<sub>|</sub><br>*n***<sub>|</sub><br>***|***</sup>***x***<sup>***|***</sup></sup>|<br>***|* One still needs the small  $|z|$  factorization formula to extract the PDF. Z *d*⇣ *|x| <sup>|</sup>y<sup>|</sup> <sup>C</sup> y normalized* fore ante electrocies,<br>*e* still needs the small |z| factorization formula to extract the PDF  $\ln \alpha$  to extract the  $\Gamma$ DF.  $\text{p}$  and  $|z|$  factorization formula to extract the PDF.

INT Workshop, UW, Seattle 10/9/17 T V *e*<br>*F*<br>*C*<sub>n</sub><br>*C*<sub>n</sub><br>*C*<sub>n</sub><br>*C*<sub>n</sub>  $\overline{I}$  *INT* Workshop, UW, Seattle  $\overline{22}$ 

*, µ*<sup>2</sup>*z*<sup>2</sup>

#### Small |*z|* factorization 1  $\frac{d}{dx}$  $1d$  $11 |y|$  factoriza;

#### $\triangleleft$  Pseudo distribution ready, one can see that the matching kernel is a function

$$
\mathcal{P}(x, z^2 \mu^2) = \int_{|x|}^1 \frac{dy}{|y|} C\left(\frac{x}{y}, z^2 \mu^2\right) q(y, \mu) \qquad \text{X. Ji, J.}
$$
  
-1 \le x \le 1 \qquad \qquad + \int\_{-1}^{-|x|} \frac{dy}{|y|} C\left(\frac{x}{y}, z^2 \mu^2\right) q(y, \mu),

X. Ji, J.-H. Zhang, and Y.Z., arXiv: 1706.07416

 $\triangleleft$  One loop matching coefficient

$$
C(\alpha, z^2 \mu^2) = 1 + \frac{\alpha_s C_F}{2\pi} \left( \frac{3}{2} \ln(z^2 \mu^2) + \frac{3}{2} \right) \delta(1 - \alpha)
$$
  
+ 
$$
\frac{\alpha_s C_F}{2\pi} \left[ -\left( \frac{1 + \alpha^2}{1 - \alpha} \right)_+ \left( \ln(z^2 \mu^2) + 1 \right) - \left( \frac{4 \ln(1 - \alpha)}{1 - \alpha} \right)_+ + 2(1 - \alpha) \right] \qquad 0 \le \alpha \le 1
$$

X. Ji, J.-H. Zhang, and Y.Z., arXiv: 1706.07416

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#### Equivalence of two factorizations  $24$ UIVAICHICC OI TWO IACIOITZALIO  $s_{\text{univ}}$  lange of two footomization <u>quivalence</u> of two iactorizations

ò Quasi- and pseudo PDFs are different representations of the Ioffe-time distribution:  $\mathbf{a}$  and  $\mathbf{b}$  factor  $\mathbf{b}$  factor theorem they are definition they are definition they are definition to  $\mathbf{b}$ asi- and pseudo PDFs are different representation  $\mathbf{f}$  $\sum_{i=1}^n$ asi- and pseudo PDFs are different representare Ioffe-time distributions tone and  $\alpha$  $\mathop{\mathrm{as}}\nolimits$  of

$$
\tilde{q}\left(x,\frac{\mu^2}{P_z^2}\right) = \int_{-1}^1 dy \int \frac{d\zeta}{2\pi} e^{i(x-y)\zeta} \mathcal{P}\left(y,\frac{\mu^2 \zeta^2}{P_z^2}\right)
$$

 $\triangleleft$  The factorizations are related by Fourier transforms

$$
C\left(\frac{x}{y}, \frac{\mu^2}{(yP^z)^2}\right) = \int_{-1}^1 d\alpha \int \frac{d\zeta}{2\pi} e^{i(\frac{x}{y}-\alpha)\zeta} C\left(\alpha, \frac{\mu^2 \zeta^2}{(yP^z)^2}\right)
$$

INT Workshop, UW, Seattle 10/9/17  $HWW\text{ Seattle}$  and  $24$ time distribution satisfies instead a di↵erent factoriza $top$ , UW, Seattle  $24$ 

*.*

### Equivalence at one-loop order



Feynman rules for one-loop diagrams take the form:

$$
\int \frac{d^d k}{(2\pi)^d} L(k) \bigg( e^{-ip^z z} - e^{-ik^z z} \bigg)
$$

- 1. For Ioffe-time distribution, first integrate over  $d^d k$ , and then Fourier transform the Ioffe-time into x to obtain the pseudo-PDF;
- 2. For the quasi-PDF, first Fourier transform *z* into  $zp^z$ , so one obtains  $\delta(p^z z^z)$  $xp^z$ )- $\delta(k^z \text{-} xp^z)$ , and then carry out the loop integration.

Two methods could be different as one exchanges the order of UV regularization and Fourier transform.

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#### Equivalence at one-loop order t, **a** 1 *du eiu*⇣(✏IR)4✏IR *<sup>z</sup>*<sup>2</sup>✏IR <sup>+</sup>  $100p$  ( *ei*⇣ ĺ  $\bullet$   $\bullet$   $\bullet$   $\bullet$  $1$  $\alpha$ iche al Un e-100}<br>} *ei*⇣ )

### **Preliminary Results:**

$$
\tilde{Q}^{(1)}(\zeta, z^2, \epsilon) = \frac{\alpha_s C_F}{2\pi} (4\pi\mu^2)^{\epsilon} \left\{ \int_0^1 du \left[ (1 - \epsilon_{\text{IR}})(1 - u) - 1 \right] e^{-iu\zeta} \Gamma(-\epsilon_{\text{IR}}) 4^{-\epsilon_{\text{IR}}} z^{2\epsilon_{\text{IR}}} \n+ (i\zeta) \int_0^1 du \int_0^1 dt (2 - u) e^{-i(1 - ut)\zeta} \Gamma(-\epsilon_{\text{IR}}) 4^{-\epsilon_{\text{IR}}}(t^2 z^2)^{\epsilon_{\text{IR}}} + \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) e^{-i\zeta} \n- \frac{\Gamma(-\epsilon_{\text{UV}})}{1 - 2\epsilon_{\text{UV}}} 4^{-\epsilon_{\text{UV}}} z^{2\epsilon_{\text{UV}}} e^{-i\zeta} + \delta Z_{\psi} e^{-i\zeta} \right\}.
$$

$$
\mathcal{P}^{(1)}(x, z^2, \mu, \epsilon)
$$
\n
$$
= \frac{\alpha_s C_F}{2\pi} \left[ (1 - \epsilon_{\text{IR}})(1 - x) - 1 - \left( \frac{1}{1 - 2\epsilon_{\text{IR}}} - \frac{2(1 - \epsilon_{\text{IR}})}{1 - 2\epsilon_{\text{IR}}} \frac{1}{(1 - x)^{1 - 2\epsilon_{\text{IR}}}} \right)_+ \right] \Gamma(-\epsilon_{\text{IR}})(\pi z^2 \mu^2)^{\epsilon_{\text{IR}}} \theta(x) \theta(1 - x)
$$
\n
$$
+ \frac{\alpha_s C_F}{2\pi} \left[ -\frac{\Gamma(-\epsilon_{\text{UV}})}{1 - 2\epsilon_{\text{UV}}} (\pi z^2 \mu^2)^{\epsilon_{\text{UV}}} + \frac{1}{2} \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \delta(1 - x) ,
$$
\nT. Izubuchi, X. Ji, L. Jin, I. Stewart and Y.Z., to be published.

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Equivalence at one-loop order wave function in the set of the s where  $\Gamma_{\alpha_{11}}$ 

#### $\mathbf{r}$  is subtle and the details and the details are provided in  $\mathbf{r}$ Fourier transform is exactly the same as the quasi-PDF!  $\tilde{q}^{(1)}(x,p^z,\epsilon)$ straightforwardness, the Fourier transform is subtle and the details are provided in App. A. Here we show the result

 $-\int \frac{4\pi\mu^2}{2\pi}$  $\overline{n^2}$ *∖*<sup> $\epsilon$ </sup> *Γ*[ $\epsilon$ *p*2 *z*  $+\frac{1}{2}$ ]  $=\frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{n^2}\right)^{\epsilon}\frac{\Gamma[\epsilon+\frac{1}{2}]}{\sqrt{\pi}}$  $\overline{\phantom{a}}$ 8  $\frac{1}{2}$  $\frac{1}{2}$  $\frac{2}{1} (1 - \epsilon)$ 1 *x*  $\frac{(\text{R})x^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})} + \frac{(1-\epsilon_{\text{IR}})x^{-2\epsilon_{\text{IR}}}}{1-2\epsilon_{\text{IR}}}$  $\mathbf{r}$  $x^2 +$  $\frac{x^2+1}{(2)(1-2\epsilon_{\text{F}})(x-1)^{1+2\epsilon_{\text{IR}}}},$  *x* :<br> $\frac{2}{2}(1 - \epsilon)$  $\frac{1-\epsilon}{\epsilon}$  $\frac{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})}{8\pi r^{2\epsilon_{\text{IR}}}+(1-x)^{-2\epsilon_{\text{IR}}}} + \frac{(1-\epsilon_{\text{IR}})x^{-2\epsilon_{\text{IR}}}}{1-2\epsilon_{\text{IR}}}}$  $_{\rm IR}\,$ <sub>2</sub> $(x$  +  $x^2+1$ <br>(1 0 \times \times 1,1+2cm } , 0 < x  $\frac{1-\epsilon}{\epsilon}$  $\frac{2\epsilon_{\text{\tiny IR}}(1-2\epsilon_{\text{\tiny IR}})}{(\lambda)(-x)^{-2\epsilon_{\text{\tiny IR}}}- (1-x)^{-2\epsilon_{\text{\tiny IR}}}} - \frac{1-2\epsilon_{\text{\tiny IR}}}{(1-\epsilon_{\text{\tiny IR}})(-x)^{-2\epsilon_{\text{\tiny IR}}}}$  $(x-x)^{1}$  $\frac{x^2+1}{x^2+1}$  *x*  $\frac{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})}{\left(4\pi u^2\right)^{\epsilon} \Gamma[\epsilon+\frac{1}{2}|\Gamma_{\text{max}}]}$  $\pi\mu^2$ <sup>c</sup>  $\Gamma$  $\epsilon$  +  $\frac{1}{2}$  $\Gamma$ <sub>p</sub>  $\left\{ \mathbf{D}_{\mathbf{Y}}\Delta$ 1:  $-\frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{p_z^2}\right)^\epsilon \frac{\Gamma[\epsilon+\frac{1}{2}]}{\sqrt{\pi}} \left[D_0(\epsilon_{\text{IR}})\mathbf{R}_{\text{Z}}\left(\mathbf{Q}_{\text{UV}}\right)\mathbf{R}_{\text{IR}}\right]$  $D_0(\epsilon_{\text{IR}}) = \int^{\infty}$ 1  $dy$   $\left[-\frac{1+y}{1-y}\right]$  $1 - y$  $\frac{y^{-2\epsilon_{\text{IR}}}- (y-1)^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})} + \frac{y^{-2\epsilon_{\text{IR}}}-2(y-1)^{-2\epsilon_{\text{IR}}}}{(1-2\epsilon_{\text{IR}})(1-y)}$ 1  $+$  $\int_0^1$ 0  $dy$   $\left[-\frac{1+y}{1-y}\right]$  $1 - y$  $\frac{y^{-2\epsilon_{\text{IR}}} + (1-y)^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})} + \frac{y^{-2\epsilon_{\text{IR}}} + 2(1-y)^{-2\epsilon_{\text{IR}}}}{(1-2\epsilon_{\text{IR}})(1-y)}$  $+\int_0^1 dy \left[ -\frac{1+y}{4} \frac{y^{-2\epsilon_{\text{IR}}}(1-y)^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}} + \frac{y^{-2\epsilon_{\text{IR}}}(1-y)^{-2\epsilon_{\text{IR}}}}{4\epsilon_{\text{IR}}} + \frac{y^{-2\epsilon_{\text{IR}}}(1-y)^{-2\epsilon_{\text{IR}}}}{4\epsilon_{\text{IR}}} \right]$  $+$  $\int_0^0$  $-\infty$  $dy\left[\frac{1+y}{1}\right]$  $1 - y$  $\frac{y^{-2\epsilon_{\rm IR}}-(1-y)^{-2\epsilon_{\rm IR}}}{2\epsilon_{\rm IR}(1-2\epsilon_{\rm IR})} - \frac{y^{-2\epsilon_{\rm IR}}-2(1-y)^{-2\epsilon_{\rm IR}}}{(1-2\epsilon_{\rm IR})(1-y)}$  $+\int_0^0 dy \left[ \frac{1+y}{1-y} \frac{y^{-2\epsilon_{\text{IR}}}- (1-y)^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{L}}(1-2\epsilon_{\text{L}})} - \frac{y^{-2\epsilon_{\text{IR}}}- 2(1-y)^{-2\epsilon_{\text{IR}}}}{(1-2\epsilon_{\text{L}})(1-y)} \right]$  $2\pi$  $(4\pi\mu^2)$  $p_z^2$ *z*  $\int^{\epsilon} \frac{\Gamma[\epsilon + \frac{1}{2}]}{2}$  $\sqrt{\pi}$ ⇥  $\sqrt{ }$  $\int$  $\overline{\phantom{a}}$  $-\frac{1+x^2}{1-x}$  $1 - x$  $\frac{(1-\epsilon_{\text{IR}})x^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})} + \frac{(1-\epsilon_{\text{IR}})x^{-2\epsilon_{\text{IR}}}}{1-2\epsilon_{\text{IR}}}}$  $+$  $x^2 + 1$  $\frac{x}{2(1-2\epsilon_{\text{IR}})(x-1)^{1+2\epsilon_{\text{IR}}}}, \qquad x>1$  $-\frac{1+x^2}{1-x}$  $1-x$  $\frac{(1-\epsilon_{\rm IR})x^{-2\epsilon_{\rm IR}}+(1-x)^{-2\epsilon_{\rm IR}}}{2\epsilon_{\rm IR}(1-2\epsilon_{\rm IR})}+\frac{(1-\epsilon_{\rm IR})x^{-2\epsilon_{\rm IR}}}{1-2\epsilon_{\rm IR}}}$  $+$  $x^2 + 1$  $\frac{2(1-2\epsilon_{\text{IR}})(1-x)^{1+2\epsilon_{\text{IR}}}}{2}, \qquad 0 < x < 1$  $1 + x^2$  $1 - x$  $\frac{(1-\epsilon_{\mathrm{IR}})(-x)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})} - \frac{(1-\epsilon_{\mathrm{IR}})(-x)^{-2\epsilon_{\mathrm{IR}}}}{1-2\epsilon_{\mathrm{IR}}}$  $\pm$  $x^2 + 1$  $\frac{x}{2(1-2\epsilon_{\text{IR}})(1-x)^{1+2\epsilon_{\text{IR}}}}, \ x<0$  $(4\pi\mu^2)$ *p*2 *z*  $\int^{\epsilon} \frac{\Gamma[\epsilon + \frac{1}{2}]}{2}$  $\sqrt{\pi}$  $\left[D_{0}(\epsilon_{\text{\tiny IP}})\mathbf{\sum}\frac{\mathbf{B}}{2}\right]$ r⊿<br>r eliminary!  $\left| \int \right|$  - $\frac{1+y}{1-y} \frac{y^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}(1-2\epsilon_{\text{IR}})} + \frac{y^{-2\epsilon_{\text{IR}}}+2(1-y)^{-2\epsilon_{\text{IR}}}}{(1-2\epsilon_{\text{IR}})(1-y)}.$  $\binom{y}{1}$  $\frac{y}{1 + y} \frac{y^{-2\epsilon_{\text{IR}}}}{2\epsilon_{\text{IR}}(1 - 2\epsilon_{\text{IR}})} - \frac{y^{-2\epsilon_{\text{IR}}}}{(1 - 2\epsilon_{\text{IR}})(1 - y)} \bigg]$ T. Izubuchi, X. Ji, L. Jin, I. Stewart and Y.Z., to be published.

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### Different renormalization schemes

↑ Conversion between different renormalization schemes:  $\tilde{O}_{\Gamma}(z) = Z_{\overline{\rm MS}}(z^2,\mu)\tilde{O}_{\Gamma}^{\rm M}$ <sup>MS</sup>(z,μ)= Z<sub>x</sub>(z<sup>2</sup>,μ)Õ<sub>Γ</sub>  $\tilde{O}_r(z) = Z_{\overline{MS}}(z^2, \mu) \tilde{O}_r^{MS}(z, \mu) = Z_{\overline{X}}(z^2, \mu) \tilde{O}_r^{X}(z, \mu)$  $\tilde O_\Gamma^\lambda$  $\int_{\Gamma}^X (z,\mu)=$  $Z_{_{\overline{\rm MS}}}(z^2,\mu)$  $Z_{\textstyle{\overline{X}}} (z^{\textstyle 2}, \mu)$  $\tilde{O}_{\Gamma}^{\scriptscriptstyle{{\rm {N}}}}$  $\tilde{O}_{\Gamma}^{X}(z,\mu) = \frac{Z_{\text{MS}}(z^2, \mu)}{Z_{\Gamma}^{2}(\mu)} \tilde{O}_{\Gamma}^{\text{MS}}(z,\mu)$  $\vec{r}$ *n Cn*(*µ*<sup>2</sup>*z*<sup>2</sup>) (*i*⇣)*<sup>n</sup>*  $1$  differe:  $\frac{1}{\sqrt{2}}$ 



# Outline

 $\triangle$  1. Quasi and pseudo distribution approaches to calculating PDF from lattice QCD

ò 2. Equivalence between large *Pz* and small |*z|* factorizations

### $\div$  3. Hints on lattice calculations

### Requirements on the lattice

 $\triangle$  Ioffe-time distribution factorization:

Small  $z^2$ , large  $P^z$ ,  $zP^z \sim 1$  to obtain enough information on the lattice;

$$
\Lambda_{\text{QCD}} \sim 0.3 \text{GeV}, \ a^{-1} \sim 3 \text{GeV},
$$
\n
$$
P^{z} = \frac{n}{L} \frac{2\pi}{a} \ll \frac{2\pi}{a} \Rightarrow n \ll L, \quad z = ma, \ z \Lambda_{\text{QCD}} \ll 1 \Rightarrow m \ll 10,
$$
\n
$$
zP^{z} \sim 1 \Rightarrow mn \sim \frac{L}{2\pi} \Rightarrow n \gg \frac{L}{20\pi}, \quad P^{z} \gg 1 \text{GeV} \Rightarrow n \gg \frac{L}{6\pi}
$$
\nFor  $L=48$ ,  $2\pi \approx 6$ , then  $m \sim \{0,1,2,3,?\}$ ,  $n \sim \{2,3,4,?\}$ 

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- $\triangleleft$  For the quasi PDF, large  $P^z$  means a contracted proton, so most wavefunction information is centered around small *z* values;
- ò When *z* is large, the higher twist corrections are not suppressed. This cannot be saved even if the proton momentum is large;
- ò Useful lattice data is restricted to small finite range of *z*.

- $\triangle$  For each small  $z=1a$ , 2a, 3a, there is a finite number of momenta that can be used, so the total number of useful data points are limited for us to use the factorization formula. We can use RG equation in ln*z2* to evolve all data points to the same *z2*; A. Radyushkin, PRD 2017
- $\triangleleft$  This is also similar to DIS, where one can only extract a finite number of data points of (*x, Q2*) for the PDF. So maybe with current data we should fit the moments instead of matching the full PDF?

Direct Calculation of the hadronic tensor, See K.F. Liu's talk Lattice cross section, Y.-Q. Ma and J. Qiu, 2014, 2017.

### $\triangleleft$  Ioffe-time distribution from the convolution:

 $\tilde{Q}(\zeta, z^2\mu^2)$ =  $\int d\alpha C(\alpha, \mu^2 z^2)Q(\alpha \zeta, \mu)$ 



MSTW2008, A.D. Martin et al., PRD 2009

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- $\rightarrow$  *Pz* must be large, so *zPz* cannot be too small;
- $\triangleleft$  Largest *Pz* is limited by  $1/a$ , so  $zP$ *z* has to be truncated at a a cutoff;
- $\triangleleft$  With finer lattice spacing, the number of useful data points in the shaded region will grow geometrically.

## Improvements



- Smallest nonzero Ioffe-time is  $a^*P^z \ll 1$ , the size of this region is small;
- O( $M^2/P_z^2$ ) corrections, results known;
- Behavior at  $z^2=0$  is singular (ln  $z^2$ ) for the MSbar scheme, but regular for other schemes like the RI/MOM (I. Stewart and Y.Z., 2017) on the lattice.



- Low-pass filter method;
- Fourier transform the derivative of the Ioffe-time distribution;

H.-W. Lin et al. (LP3), 2017

• Gaussian re-weighting method

INT Workshop, UW, Seattle 1978 10/9/17 J.-H. Zhang et al., in preparation; See A. Schaefer's talk.

# Summary

- $\triangle$  The quasi, Ioffe-time, and pseudo distributions are just different representations of the same observable. Factorizations in different representations are equivalent;
- $\triangle$  The ratio is not a factorization, instead, the Ioffe-time and pseudo distribution satisfy a different small distance factorization;
- $\triangle$  The matching coefficient should depend on the parton momentum, not the nucleon momentum;
- $\triangle$  The requirements for the lattice are the same for all the different approaches. And the difficulty is also the same.