Factorization Theorem Relating Euclidean and Light-cone Parton Distributions

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T. Izubuchi, X. Ji, L. Jin, I. Stewart and Y.Z., to be published.

Outline

- 1. Quasi and pseudo distribution approaches to calculating PDF from lattice QCD
- 2. Equivalence between large P^z and small $|z|$ factorizations
- 3. Hints on lattice calculations

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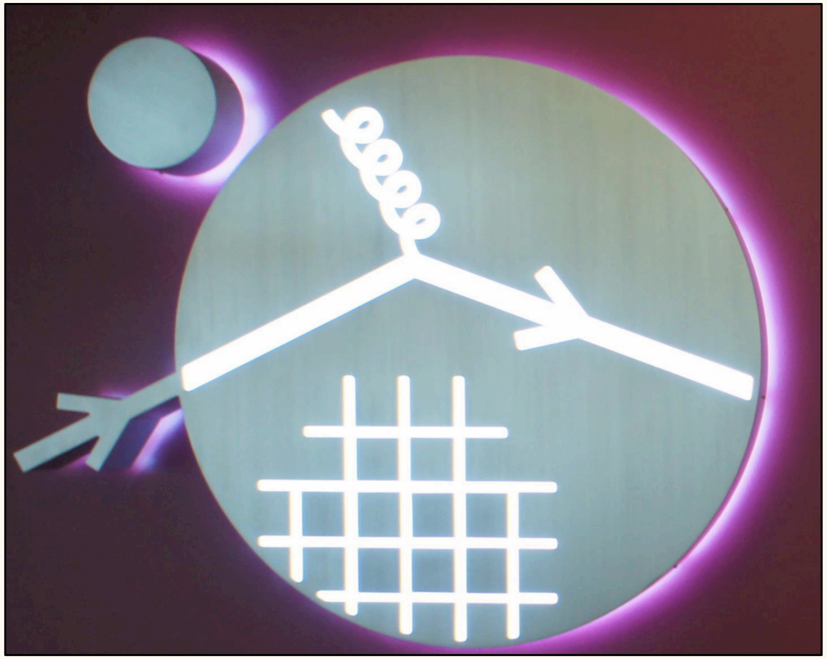
PDF from the Euclidean Lattice



PDF not directly accessible from the lattice!

Parton model:

- Emerges in the infinite momentum frame
- Or, the proton seen by an observer moving at the speed of light (on the light-cone)
- Parton distribution function

$$q(x,\mu) = \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P | \bar{\psi}(\xi^-)\gamma^+ U(\xi^-,0)\psi(0) | P \rangle$$

$$\xi^\pm = (t \pm z)/\sqrt{2} \qquad U(\xi^-,0) = P\exp\left[-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)\right]$$

Lattice QCD:

- Euclidean space
- Nucleon at finite momenta
- Cannot calculate time-dependent quantities with contribution from physical poles
- Light-cone separation $\Delta s^2=0 \Rightarrow \Delta s^\mu=(0,0,0,0)$

Methods based on operator product expansion (OPE)

- **Hadronic Tensor (Unpolarized)**

$$W_{\mu\nu}(q,P) = \frac{1}{\pi}\mathrm{Im}\,T_{\mu\nu} = \int \frac{d^4z}{4\pi} e^{iq.z} \left\langle P \middle| \left[J_\mu(z), J_\nu(0) \right] \middle| P \right\rangle$$

$$= (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) F_1(\omega, Q^2) + \hat{P}_\mu \hat{P}_\nu F_2(\omega, Q^2)$$

$$\boxed{\begin{aligned} &Q^2 = -q^2, \\ &\omega = 2P\cdot q / Q^2, \\ &\hat{P}_\mu = P_\mu - \frac{P\cdot q}{q^2} q_\mu. \end{aligned}}$$

In the Bjorken (DIS) limit (not directly accessible on the lattice),

$$Q^2 \to \infty,\ q\cdot P \to \infty,\ \omega = \frac{1}{x} \in (1,\infty), \quad F_i(x,Q) = \int dy \sum_{f=q,g} C_{i,f}(\frac{x}{y}, \frac{Q}{\mu}) q_f(y,\mu)$$

- **Euclidean OPE of the Compton Amplitude** $Q^2 \to \infty,\ Q^2 >> q\cdot P,\ \omega \to 0$

$$T_{\mu\nu}(q,P) = \int \frac{d^4z}{4\pi} e^{iq.z} \left\langle P \middle| T J_\mu(z) J_\nu(0) \middle| P \right\rangle = \sum_{i,n} C_{i,n}(q\cdot P, Q^2) A_{i,n}(Q^2) \omega^{n-1} + O(\frac{1}{Q})$$

$$\boxed{A_{i,n}(Q^2) \sim \int dx\ x^{n-1} F_i(x, Q^2)}$$

Methods based on operator product expansion (OPE)

- Direct computation of PDF moments:

$$\int dx\, x^{n-1} q(x,\mu) dx \sim n_{\mu_1} n_{\mu_2} \cdots n_{\mu_n} \left\langle P \middle| \bar{\psi}(0) \gamma^{\mu_1} i\overleftrightarrow{D}^{\mu_2} \cdots i\overleftrightarrow{D}^{\mu_n} \middle| P \right\rangle$$

  - Moments are calculable as matrix elements of local gauge-invariant and frame-independent operators;
  - Fitting the PDF using the finite number of moments calculated;
  - Operator mixing on the lattice limits computation for moments higher than 3.

n≤3, W. Detmold et al., EPJ 2001, PRD 2002;
D. Dolgov et al. (LHPC, TXL), PRD 2002;
n>3, Z. Davoudi and M. Savage, PRD 2012.

Methods based on operator product expansion (OPE)

- Fictitious heavy quark current

  D. Lin and W. Detmold, PRD 2006.
  See D. Lin's talk

  Auxiliary heavy quark mass sets the OPE scale

  Higher twist effects suppressed by heavy quark mass

  Capable of calculating higher moments

- Direct use of OPE of the Compton amplitude

  Utilizing full dependence of ω (only for $\omega<1$?)

  Obtain many moments to fit the PDF

  A. J. Chambers et al. (QCDSF), PRL 2017
  See G. Schierholz's talk

- Direct computation of the hadronic tensor

  K.F. Liu (et al.), 1994, 1999, 1998, 2000, 2017. See K.F. Liu's talk

Factorization approaches (not the moments)

- **Large momentum effective theory (LaMET)**

  Quasi-PDF approach

  X. Ji, PRL 2013, Sci.China Phys.Mech.Astron., 2014.
  See H.-W. Lin and J.-W. Chen's talks

- Lattice cross section

  Y.-Q. Ma and J. Qiu, 2014, 2017.

- **Pseudo-PDF approach**

  A. Radyushkin, PRD 2017;
  K. Orginos, A. Radyushkin, J. Karpie and S. Zafeiropoulos, 2017

- Factorization of Euclidean correlations

  V. M. Braun and D. Mueller, EPJ.C. 2008
  G. S. Bali, V. M. Braun, A. Schaefer, et al., 2017.

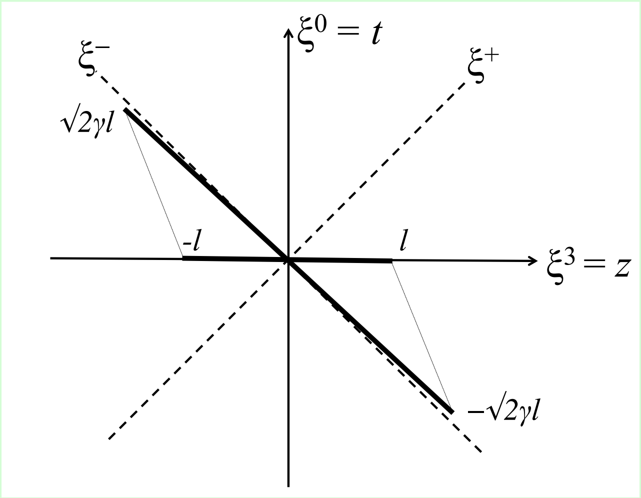
LaMET approach

✧ Quasi-PDF

$$z^{\mu} = (0,0,0,z)$$

$$\tilde{q}(x, P^z, \Lambda = a^{-1}) = \int \frac{dz}{4\pi} e^{ixP^z z} \left\langle P \middle| \bar{\psi}(z) \gamma^z U(z,0) \psi(0) \middle| P \right\rangle \quad U(z,0) = P\exp\left[-ig\int_0^z dz' A^z(z')\right]$$

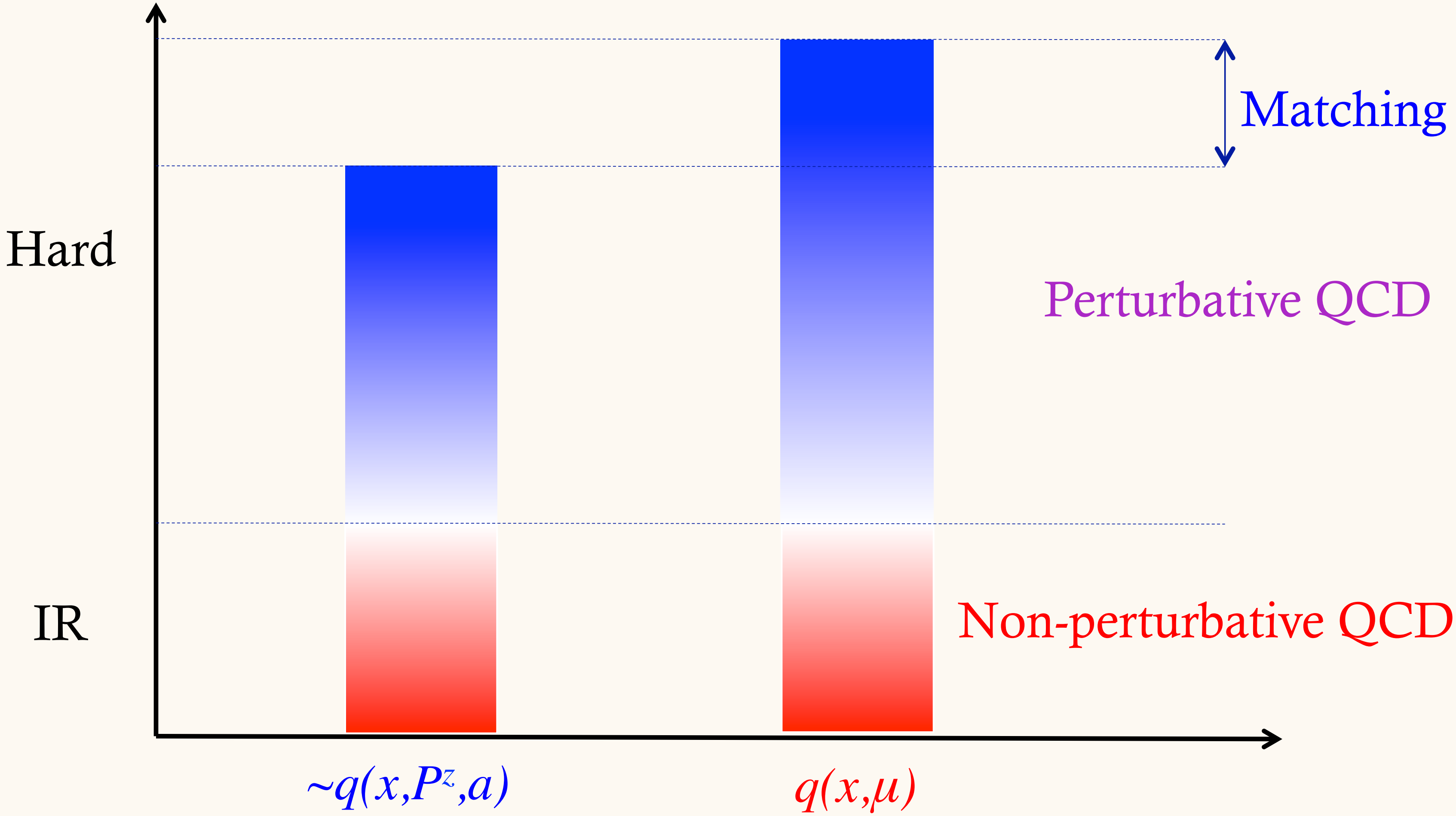
- Spatial correlation along the *z* direction, calculable in lattice QCD;
- Under an infinite Lorentz boost along the *z* direction, the spatial gauge link approaches the light-cone direction;



LaMET approach

- Hierarchy of scales
  UV cut-off of the theory
  Soft scales, related to IR physics
  - Quasi PDF $\Lambda >> P^z >> M, \Lambda_{\text{QCD}}$
  - (Light-cone) PDF $P^z >> \Lambda >> M, \Lambda_{\text{QCD}}$
- Taking the infinite momentum limit changes the UV physics, but not the IR physics:
- The UV difference be calculated in perturbative QCD, so the quasi-PDF can be factorized into the light-cone PDF!

How matching works



LaMET approach

- The quasi PDF is related to the PDF through a factorization formula:

$$\tilde{q}_i(x, P^z, \tilde{\mu}) = \int_{-1}^{+1} \frac{dy}{|y|}\, C_{ij}\left(\frac{x}{y}, \frac{\tilde{\mu}}{P^z}, \frac{\mu}{P^z}\right) q_j(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}\right)\,,$$

  - They have the same IR divergences;
  - C factor matches the difference in their UV divergence, and can be calculated in perturbative QCD;
  - Higher twist corrections suppressed by powers of P^z.

Current status

Lattice simulation of the bare quasi PDF	✔: Iso-vector quark distributions H. W. Lin et al., 2015; J.-W. Chen et al., 2016; C. Alexandrou et al., 2015, 2016 O(a) improvement: M. Constantinou and Panagopoulos, 2017; Ishikawa et al (LP3)., 2017
Renormalization of the quasi PDF on the lattice	✔: nonperturbative renormalization and O(a) improve Ishikawa et al., 2016, 2017; J.-W. Chen et al., 2016; X. Xiong, 2017; M. Constantinou et al., 2017; J.W. Chen, Y.Z., et al., 2017; I. Stewart and Y. Z., 2017; Ji, Zhang, and Y.Z., 2017; J. Green et al., 2017; Ishikawa et al (LP3)., 2017
Subtraction of the higher twist corrections	✔: All orders of mass correction $M^2/P_z^{\,2}$ exactly calculated; $O(\Lambda^2_{QCD}/P_z^{\,2})$ correction fitted. H. W. Lin et al., 2015; J.-W. Chen et al., 2016; C. Alexandrou et al., 2015, 2016
Matching the quasi PDF to PDF in the MSbar scheme.	✔: One-loop matching coefficient obtained in the continuum theory Xiong, Ji, Zhang and Y.Z., 2014; Y. Ma and J. Qiu, 2014.

Ioffe-time and pseudo distributions

- Ioffe-time distribution:

$$\tilde{Q}_{\gamma^0}(\zeta = P^z z, z^2, \epsilon) = \frac{1}{2P^0}\langle P|\tilde{O}_{\gamma^0}(z)|P\rangle \, ,$$

$$\tilde{O}_\Gamma(z) = \bar{\psi}(z)\Gamma U(z,0)\psi(0) \, ,$$

A. Radyushkin, PRD 2017

z^2=0, reduces to the light-cone correlation.

- Pseudo distribution

$$\mathcal{P}\left(x, z^2, \epsilon\right) = \int \frac{d\zeta}{2\pi} e^{ix\zeta} \tilde{Q}\left(\zeta, z^2, \epsilon\right) .$$

Support $-1<x<1$. z^2=0, reduces to the PDF.

Factorization theorem?

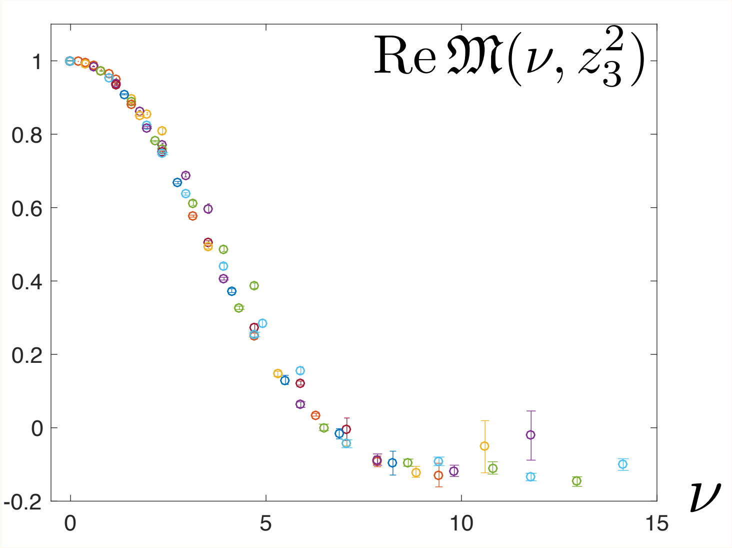
$$\frac{\tilde{Q}(\zeta, z^2, a^{-1})}{\tilde{Q}(0, z^2, a^{-1})} \xrightarrow{?} Q(\zeta, 0)$$

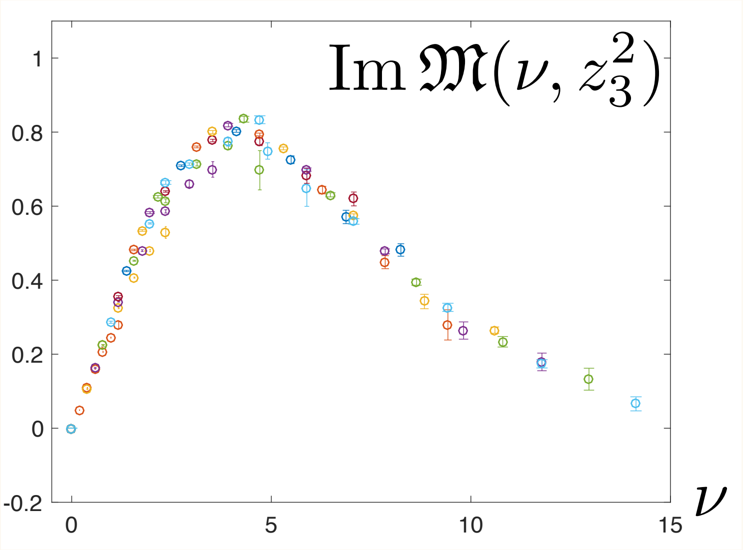
A. Radyushkin, PRD 2017

- Scaling at small z^2, higher twist corrections suppressed by $(z\Lambda_{\mathrm{QCD}})^2$;

Orginos, Radyushkin, Karpie, and Zafeiropoulos, 2017

- Can be tested on the lattice.





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Renormalization

- Auxiliary field formulation of the Wilson line

$$\int DVD\bar{V}\ V(z)\bar{V}(0)e^{i\int d^4x\left[L_{QCD}(x)+\bar{V}(x)in_z\cdot DV(x)\right]}=\left\langle U(z,0)\right\rangle$$

H. Dorn, Fortsch. Phys. 1986

$$\tilde{O}_\Gamma(z)=\bar{\psi}(z)\Gamma U(z,0)\psi(0)=j_1(z)j_2(0),\ \ j_1=\bar{\psi}\ \Gamma V,\ j_2=\bar{V}\psi$$

  - *V(x)* is a one-dimensional Grassman field that only depends on the coordinate z, similar to a heavy quark;
  - j_1 and j_2 are proven to be multiplicatively renormalizable (like heavy-to-light current in HQET). No further renormalization for the nonlocal current-current correlator;
  - Self-energy of *V* can induce a mass correction.

X. Ji, J.-H. Zhang, and Y.Z., 2017
J. Green et al., 2017

Renormalization

- The gauge-invariant quark Wilson line operator can be renormalized multiplicatively in the coordinate space:

$$\tilde{O}(z) = \bar{\psi}(z)\Gamma U(z,0)\psi(0) = Z_{\psi,z} e^{-\delta m|z|} \left(\bar{\psi}(z)\Gamma U(z,0)\psi(0)\right)^R$$

X. Ji, J.-H. Zhang, and Y.Z., 2017
T. Ishikawa, Y.-Q. Ma, J.Qiu, S. Yoshida, 2017

- Different renormalization schemes can be matched perturbatively in the coordinate space. Without loss of generality we can choose the MSbar scheme.

OPE of the "heavy-to-light" current-current correlator

- For the gauge-invariant Wilson line operator, it can have an OPE in the Euclidean limit of z^2—>0:

$$\tilde{O}_{\gamma^z}(z) = \sum_{n=0} \Big[C_n(\mu^2 z^2) \frac{(iz)^n}{n!} n_{\mu_1} \cdots n_{\mu_n} O_1^{\mu_0\mu_1\cdots\mu_n} + C'_n(\mu^2 z^2) \frac{(iz)^n}{n!} n_{\mu_1} \cdots n_{\mu_n} O_2^{\mu_0\mu_1\cdots\mu_n} + \cancel{\text{higher-twist operators}} \Big] , \qquad (13)$$

$$\mu_0 = z,$$

$$\boxed{O(z^2\Lambda^2_{\mathrm{QCD}})}$$

$$O_1^{\mu_0\mu_1\ldots\mu_n} = \bar{\psi}\gamma^{(\mu_0} iD^{\mu_1} \ldots iD^{\mu_n)}\psi - \text{trace} ,$$

$$O_2^{\mu_0\mu_1\ldots\mu_n} = F^{(\mu_0\rho} iD^{\mu_1} \ldots iD^{\mu_{n-1}} F_\rho{}^{\mu_n)} - \text{trace} ,$$

$$C_n = 1 + O(\alpha_s)$$

$$C'_n = O(\alpha_s)$$

We only consider the iso-vector case, so O_2 can be dropped.

OPE of the "heavy-to-light" current-current correlator

- In the large momentum limit, all (kinematic) higher twist contributions are suppressed:

$$\langle P|O_1^{\mu_0\mu_1\cdots\mu_n}|P\rangle = 2a_{n+1}(\mu)\left(P^{\mu_0}P^{\mu_1}\ldots P^{\mu_n} - \cancel{\text{trace}}\right), \quad \boxed{O(M^2/P_z^2)}$$

$$a_{n+1}(\mu) = \int_{-1}^{1} dx\, x^n q(x,\mu) \ ,$$

- Leading twist approximation:

$$\begin{aligned}
&\tilde{Q}_{\gamma^z}\left(\zeta, \mu^2 z^2\right) \\
&= \sum_{n=0} C_n(\mu^2 z^2)\frac{(-i\zeta)^n}{n!} a_{n+1}(\mu) \\
&= \sum_{n=0} C_n(\mu^2 z^2)\frac{(-i\zeta)^n}{n!} \int_{-1}^{1} dy\, y^n q(y,\mu) \ .
\end{aligned}$$

Large P^z factorization formula

- Fourier transform to get the quasi PDF

$$
\begin{aligned}
&\tilde{q}\left(x, \frac{\mu^2}{P_z^2}\right) \\
&= \int \frac{d\zeta}{2\pi} e^{ix\zeta} \tilde{Q}\left(\zeta, \frac{\mu^2\zeta^2}{P_z^2}\right) \\
&= \int_{-1}^{1} dy \left[\int \frac{d\zeta}{2\pi} e^{ix\zeta} \sum_{n=0} C_n\left(\frac{\mu^2\zeta^2}{P_z^2}\right) \frac{(-i\zeta)^n}{n!} y^n\right] q(y,\mu) \\
&= \int_{-1}^{1} \frac{dy}{|y|} \left[\int \frac{d\zeta}{2\pi} e^{i\frac{x}{y}\zeta} \sum_{n=0} C_n\left(\frac{\mu^2\zeta^2}{(yP^z)^2}\right) \frac{(-i\zeta)^n}{n!}\right] q(y,\mu) \ ,
\end{aligned}
$$

$$
\tilde{q}\left(x, \frac{\mu^2}{P_z^2}\right) = \int_{-1}^{1} \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu^2}{(yP^z)^2}\right) q(y,\mu)
$$

$$
C\left(\frac{x}{y}, \frac{\mu^2}{(yP^z)^2}\right) = \int \frac{d\zeta}{2\pi} e^{i\frac{x}{y}\zeta} \sum_{n=0} C_n\left(\frac{\mu^2\zeta^2}{(yP^z)^2}\right) \frac{(-i\zeta)^n}{n!} \ ,
$$

yP^z, the parton momentum, not the nucleon momentum P^z!

Ji, Schaefer, Xiong, and Zhang, PRD 2015

Small $|z|$ factorization

- Ioffe-time distribution:

$$\tilde{Q}(\zeta, z^2\mu^2)$$
$$= \int_{-1}^{1} dy \int_{-\infty}^{\infty} d\alpha \; e^{-i\alpha(y\zeta)} \mathcal{C}(\alpha, \mu^2 z^2) q(y, \mu)$$
$$= \int_{-\infty}^{\infty} d\alpha \; \mathcal{C}(\alpha, \mu^2 z^2) Q(\alpha\zeta, \mu) \, .$$

$$\mathcal{C}(\alpha, \mu^2 z^2) = \int \frac{d\zeta}{2\pi} e^{i\alpha\zeta} \sum_n C_n(\mu^2 z^2) \frac{(-i\zeta)^n}{n!} \, .$$

$$q(y, \mu) = \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \; e^{iy\zeta} Q(\zeta, \mu) \, ,$$

Similar to

$$T(p\cdot z, z^2) = F_\pi \frac{p\cdot z}{2\pi^2 z^4} \int_0^1 du \, e^{i(u-1/2)(p\cdot z)} H(u, z^2, \mu)\phi_\pi(u,\mu) + T^{\mathrm{HT}} \, , \qquad (5)$$

V.M. Braun and D. Mueller, EPJ. C. 2008;
G. S. Bali, V. M. Braun, A. Schaefer, et al., 2017.

- Ratio function?

$$\tilde{Q}(0, \mu^2 z^2) = C_0(\mu^2 z^2) \, , \qquad \frac{\tilde{Q}\,(\zeta, \mu^2 z^2)}{\tilde{Q}\,(0, \mu^2 z^2)} = \sum_n \frac{C_n(\mu^2 z^2)}{C_0(\mu^2 z^2)} \frac{(-i\zeta)^n}{n!} a_{n+1}(\mu) \, .$$

  - The ratio still has a mild dependence over z^2;
  - The ratio is not the light-cone correlation, rather, it can be understood as a "renormalized" Ioffe-time distribution;
  - One still needs the small $|z|$ factorization formula to extract the PDF.

Small $|z|$ factorization

- Pseudo distribution

$$\mathcal{P}(x, z^2\mu^2) = \int_{|x|}^{1} \frac{dy}{|y|}\, \mathcal{C}\left(\frac{x}{y}, z^2\mu^2\right) q(y,\mu) + \int_{-1}^{-|x|} \frac{dy}{|y|}\, \mathcal{C}\left(\frac{x}{y}, z^2\mu^2\right) q(y,\mu)\,,$$

X. Ji, J.-H. Zhang, and Y.Z., arXiv: 1706.07416

$-1 \le x \le 1$

- One loop matching coefficient

$$C(\alpha, z^2\mu^2) = 1 + \frac{\alpha_S C_F}{2\pi}\left(\frac{3}{2}\ln(z^2\mu^2) + \frac{3}{2}\right)\delta(1-\alpha) + \frac{\alpha_S C_F}{2\pi}\left[-\left(\frac{1+\alpha^2}{1-\alpha}\right)_+ \left(\ln(z^2\mu^2)+1\right) - \left(\frac{4\ln(1-\alpha)}{1-\alpha}\right)_+ + 2(1-\alpha)\right]$$

$0 \le \alpha \le 1$

X. Ji, J.-H. Zhang, and Y.Z., arXiv: 1706.07416

Equivalence of two factorizations

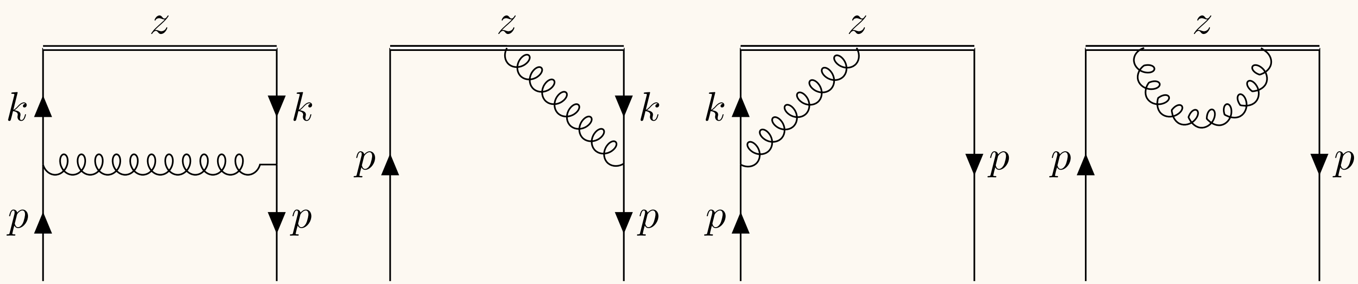
- Quasi- and pseudo PDFs are different representations of the Ioffe-time distribution:

$$\tilde{q}\left(x, \frac{\mu^2}{P_z^2}\right) = \int_{-1}^{1} dy \int \frac{d\zeta}{2\pi}\; e^{i(x-y)\zeta} \mathcal{P}\left(y, \frac{\mu^2\zeta^2}{P_z^2}\right)$$

- The factorizations are related by Fourier transforms

$$C\left(\frac{x}{y}, \frac{\mu^2}{(yP^z)^2}\right) = \int_{-1}^{1} d\alpha \int \frac{d\zeta}{2\pi}\; e^{i(\frac{x}{y}-\alpha)\zeta} \mathcal{C}\left(\alpha, \frac{\mu^2\zeta^2}{(yP^z)^2}\right).$$

Equivalence at one-loop order



Feynman rules for one-loop diagrams take the form:

$$\int \frac{d^d k}{(2\pi)^d} L(k)\left(e^{-ip^z z} - e^{-ik^z z}\right)$$

1. For Ioffe-time distribution, first integrate over $d^d k$, and then Fourier transform the Ioffe-time into x to obtain the pseudo-PDF;

2. For the quasi-PDF, first Fourier transform z into zp^z, so one obtains $\delta(p^z - xp^z) - \delta(k^z - xp^z)$, and then carry out the loop integration.

Two methods could be different as one exchanges the order of UV regularization and Fourier transform.

Equivalence at one-loop order

Preliminary Results:

$$\tilde{Q}^{(1)}(\zeta, z^2, \epsilon) = \frac{\alpha_s C_F}{2\pi}(4\pi\mu^2)^{\epsilon}\left\{\int_0^1 du[(1-\epsilon_{\rm IR})(1-u)-1]e^{-iu\zeta}\Gamma(-\epsilon_{\rm IR})4^{-\epsilon_{\rm IR}}z^{2\epsilon_{\rm IR}}\right.$$
$$+(i\zeta)\int_0^1 du\int_0^1 dt(2-u)e^{-i(1-ut)\zeta}\Gamma(-\epsilon_{\rm IR})4^{-\epsilon_{\rm IR}}(t^2z^2)^{\epsilon_{\rm IR}}+\left(\frac{1}{\epsilon_{\rm UV}}-\frac{1}{\epsilon_{\rm IR}}\right)e^{-i\zeta}$$
$$\left.-\frac{\Gamma(-\epsilon_{\rm UV})}{1-2\epsilon_{\rm UV}}4^{-\epsilon_{\rm UV}}z^{2\epsilon_{\rm UV}}e^{-i\zeta}+\delta Z_{\psi}e^{-i\zeta}\right\} .$$

$$\mathcal{P}^{(1)}(x, z^2, \mu, \epsilon)$$
$$=\frac{\alpha_s C_F}{2\pi}\left[(1-\epsilon_{\rm IR})(1-x)-1-\left(\frac{1}{1-2\epsilon_{\rm IR}}-\frac{2(1-\epsilon_{\rm IR})}{1-2\epsilon_{\rm IR}}\frac{1}{(1-x)^{1-2\epsilon_{\rm IR}}}\right)_+\right]\Gamma(-\epsilon_{\rm IR})(\pi z^2\mu^2)^{\epsilon_{\rm IR}}\theta(x)\theta(1-x)$$
$$+\frac{\alpha_s C_F}{2\pi}\left[-\frac{\Gamma(-\epsilon_{\rm UV})}{1-2\epsilon_{\rm UV}}(\pi z^2\mu^2)^{\epsilon_{\rm UV}}+\frac{1}{2}\left(\frac{1}{\epsilon_{\rm UV}}-\frac{1}{\epsilon_{\rm IR}}\right)\right]\delta(1-x)\ ,$$

T. Izubuchi, X. Ji, L. Jin, I. Stewart and Y.Z., to be published.

Equivalence at one-loop order

Fourier transform is exactly the same as the quasi-PDF!

T. Izubuchi, X. Ji, L. Jin, I. Stewart and Y.Z., to be published.

$$
\begin{aligned}
&\tilde{q}^{(1)}(x, p^z, \epsilon) \\
&= \frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{p_z^2}\right)^{\epsilon}\frac{\Gamma[\epsilon+\frac{1}{2}]}{\sqrt{\pi}} \\
&\times \begin{cases}
-\frac{1+x^2}{1-x}\frac{(1-\epsilon_{\mathrm{IR}})x^{-2\epsilon_{\mathrm{IR}}}-(x-1)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})}+\frac{(1-\epsilon_{\mathrm{IR}})x^{-2\epsilon_{\mathrm{IR}}}}{1-2\epsilon_{\mathrm{IR}}}+\frac{x^2+1}{2(1-2\epsilon_{\mathrm{IR}})(x-1)^{1+2\epsilon_{\mathrm{IR}}}}, & x>1 \\
-\frac{1+x^2}{1-x}\frac{(1-\epsilon_{\mathrm{IR}})x^{-2\epsilon_{\mathrm{IR}}}+(1-x)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})}+\frac{(1-\epsilon_{\mathrm{IR}})x^{-2\epsilon_{\mathrm{IR}}}}{1-2\epsilon_{\mathrm{IR}}}+\frac{x^2+1}{2(1-2\epsilon_{\mathrm{IR}})(1-x)^{1+2\epsilon_{\mathrm{IR}}}}, & 0<x<1 \\
\frac{1+x^2}{1-x}\frac{(1-\epsilon_{\mathrm{IR}})(-x)^{-2\epsilon_{\mathrm{IR}}}-(1-x)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})}-\frac{(1-\epsilon_{\mathrm{IR}})(-x)^{-2\epsilon_{\mathrm{IR}}}}{1-2\epsilon_{\mathrm{IR}}}+\frac{x^2+1}{2(1-2\epsilon_{\mathrm{IR}})(1-x)^{1+2\epsilon_{\mathrm{IR}}}}, & x<0
\end{cases} \\
&-\frac{\alpha_s C_F}{2\pi}\left(\frac{4\pi\mu^2}{p_z^2}\right)^{\epsilon}\frac{\Gamma[\epsilon+\frac{1}{2}]}{\sqrt{\pi}}\left[D_0(\epsilon_{\mathrm{IR}})+\frac{3}{2}\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right)\right]\delta(1-x),
\end{aligned}
$$

Preliminary!

$$
\begin{aligned}
D_0(\epsilon_{\mathrm{IR}}) &= \int_1^{\infty} dy\left[-\frac{1+y}{1-y}\frac{y^{-2\epsilon_{\mathrm{IR}}}-(y-1)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})}+\frac{y^{-2\epsilon_{\mathrm{IR}}}-2(y-1)^{-2\epsilon_{\mathrm{IR}}}}{(1-2\epsilon_{\mathrm{IR}})(1-y)}\right] \\
&+\int_0^{1} dy\left[-\frac{1+y}{1-y}\frac{y^{-2\epsilon_{\mathrm{IR}}}+(1-y)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})}+\frac{y^{-2\epsilon_{\mathrm{IR}}}+2(1-y)^{-2\epsilon_{\mathrm{IR}}}}{(1-2\epsilon_{\mathrm{IR}})(1-y)}\right] \\
&+\int_{-\infty}^{0} dy\left[\frac{1+y}{1-y}\frac{y^{-2\epsilon_{\mathrm{IR}}}-(1-y)^{-2\epsilon_{\mathrm{IR}}}}{2\epsilon_{\mathrm{IR}}(1-2\epsilon_{\mathrm{IR}})}-\frac{y^{-2\epsilon_{\mathrm{IR}}}-2(1-y)^{-2\epsilon_{\mathrm{IR}}}}{(1-2\epsilon_{\mathrm{IR}})(1-y)}\right]
\end{aligned}
$$

Different renormalization schemes

- Conversion between different renormalization schemes:

$$\tilde{O}_\Gamma(z) = Z_{\overline{\mathrm{MS}}}(z^2,\mu)\tilde{O}_\Gamma^{\overline{\mathrm{MS}}}(z,\mu) = Z_X(z^2,\mu)\tilde{O}_\Gamma^X(z,\mu)$$

$$\tilde{O}_\Gamma^X(z,\mu) = \frac{Z_{\overline{\mathrm{MS}}}(z^2,\mu)}{Z_X(z^2,\mu)}\tilde{O}_\Gamma^{\overline{\mathrm{MS}}}(z,\mu)$$

$$\begin{aligned}&\tilde{Q}(\zeta, z^2\mu^2)\\ &= \int_{-1}^{1} dy \int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha(y\zeta)}\mathcal{C}(\alpha,\mu^2 z^2)q(y,\mu)\\ &= \int_{-\infty}^{\infty} d\alpha\ \mathcal{C}(\alpha,\mu^2 z^2)Q(\alpha\zeta,\mu)\,.\end{aligned}$$

$$\begin{aligned}\tilde{Q}^X(\zeta,z^2\mu^2) &= \frac{Z_{\overline{\mathrm{MS}}}(z,\mu)}{Z_X(z,\mu)}\tilde{Q}(\zeta,z^2\mu^2)\\ &= \int_{-\infty}^{\infty} d\alpha\left[\frac{Z_{\overline{\mathrm{MS}}}(z,\mu)}{Z_X(z,\mu)}C(\alpha,\mu^2 z^2)\right]Q(\alpha\zeta,\mu)\end{aligned}$$

Outline

- 1. Quasi and pseudo distribution approaches to calculating PDF from lattice QCD

- 2. Equivalence between large P^z and small $|z|$ factorizations

- 3. Hints on lattice calculations

Requirements on the lattice

- Ioffe-time distribution factorization:

  Small z^2, large P^z, zP^z~1 to obtain enough information on the lattice;

$$\Lambda_{\mathrm{QCD}} \sim 0.3\mathrm{GeV},\ a^{-1} \sim 3\mathrm{GeV},$$

$$P^z = \frac{n}{L}\frac{2\pi}{a} << \frac{2\pi}{a} \Rightarrow n << L, \quad z = ma,\ z\Lambda_{\mathrm{QCD}} << 1 \Rightarrow m << 10,$$

$$zP^z \sim 1 \Rightarrow mn \sim \frac{L}{2\pi} \Rightarrow n >> \frac{L}{20\pi}, \quad P^z >> 1\mathrm{GeV} \Rightarrow n >> \frac{L}{6\pi}$$

For L=48, $2\pi \approx 6$, then $m \sim \{0,1,2,3,?\}$, $n \sim \{2,3,4,?\}$

Hints on lattice calculations

- For the quasi PDF, large P^z means a contracted proton, so most wavefunction information is centered around small z values;
- When z is large, the higher twist corrections are not suppressed. This cannot be saved even if the proton momentum is large;
- Useful lattice data is restricted to small finite range of z.

Hints on lattice calculations

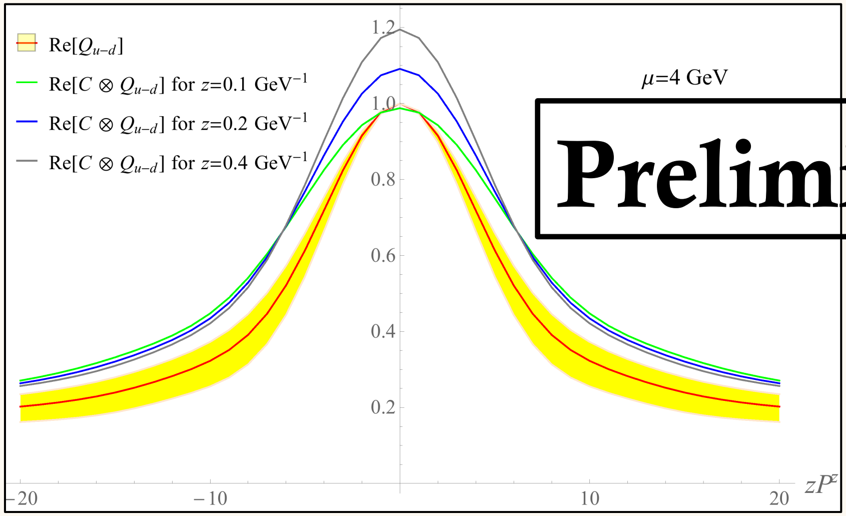
- For each small $z=1a$, $2a$, $3a$, there is a finite number of momenta that can be used, so the total number of useful data points are limited for us to use the factorization formula. We can use RG equation in $\ln z^2$ to evolve all data points to the same z^2; A. Radyushkin, PRD 2017
- This is also similar to DIS, where one can only extract a finite number of data points of (x, Q^2) for the PDF. So maybe with current data we should fit the moments instead of matching the full PDF?

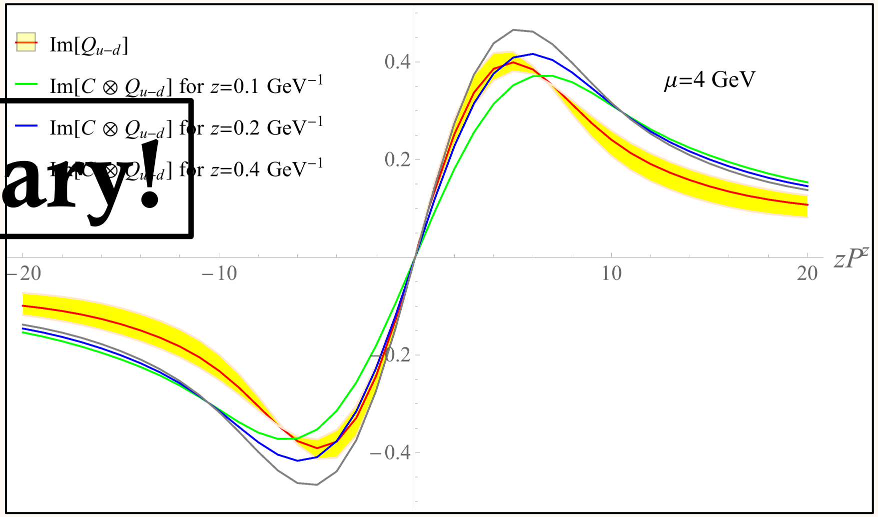
Direct Calculation of the hadronic tensor, See K.F. Liu's talk
Lattice cross section, Y.-Q. Ma and J. Qiu, 2014, 2017.

Hints on lattice calculations

- Ioffe-time distribution from the convolution:

$$\tilde{Q}(\zeta, z^2\mu^2) = \int d\alpha\ C(\alpha, \mu^2 z^2) Q(\alpha\zeta, \mu)$$

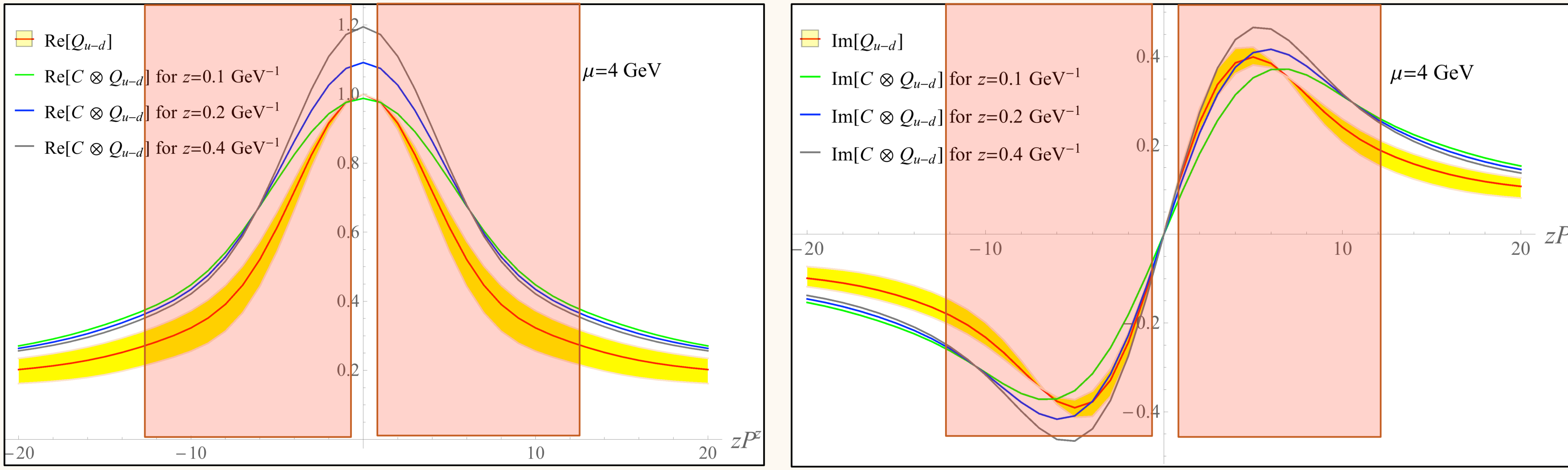




Preliminary!

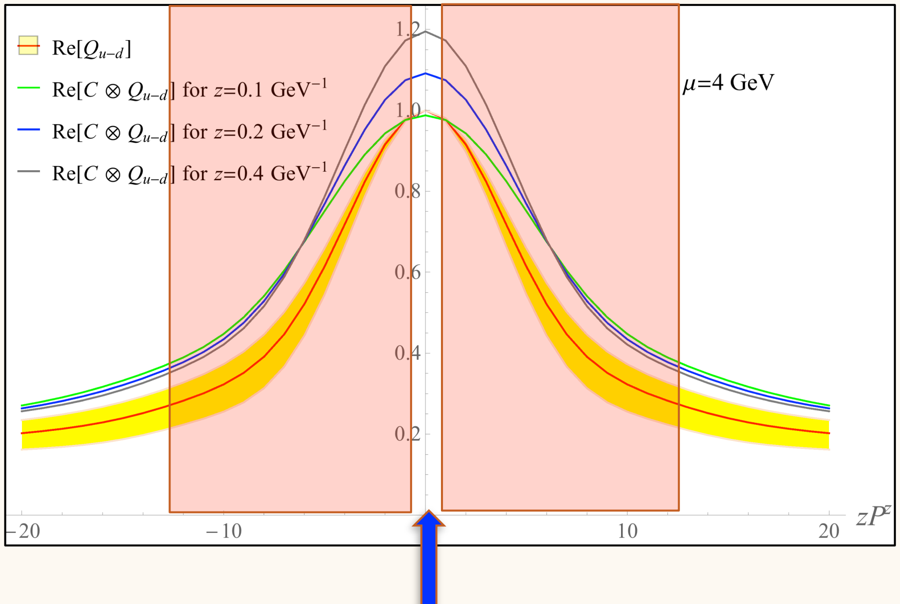
MSTW2008, A.D. Martin et al., PRD 2009

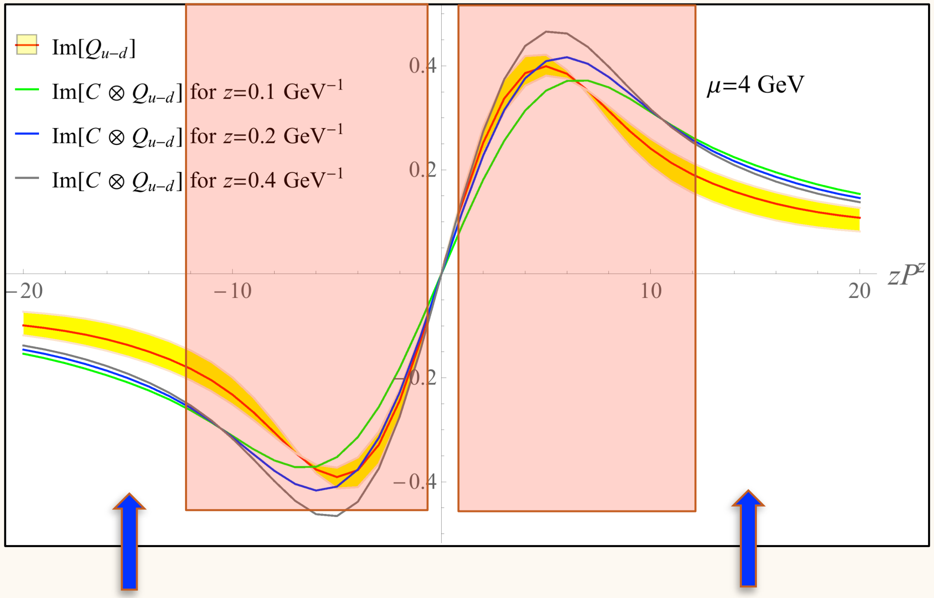
Hints on lattice calculations



- P^z must be large, so zP^z cannot be too small;
- Largest P^z is limited by $1/a$, so zP^z has to be truncated at a a cutoff;
- With finer lattice spacing, the number of useful data points in the shaded region will grow geometrically.

Improvements





- Smallest nonzero Ioffe-time is a^*P^z <<1, the size of this region is small;
- $O(M^2/P_z^2)$ corrections, results known;
- Behavior at z^2=0 is singular (ln z^2) for the MSbar scheme, but regular for other schemes like the RI/MOM (I. Stewart and Y.Z., 2017) on the lattice.

- Low-pass filter method;
- Fourier transform the derivative of the Ioffe-time distribution;

  H.-W. Lin et al. (LP3), 2017
- Gaussian re-weighting method

  J.-H. Zhang et al., in preparation; See A. Schaefer's talk.

Summary

- The quasi, Ioffe-time, and pseudo distributions are just different representations of the same observable. Factorizations in different representations are equivalent;
- The ratio is not a factorization, instead, the Ioffe-time and pseudo distribution satisfy a different small distance factorization;
- The matching coefficient should depend on the parton momentum, not the nucleon momentum;
- The requirements for the lattice are the same for all the different approaches. And the difficulty is also the same.