Advances in Lattice Calculations of Nucleon Structure Functions

and PDFs

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With

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QCDSF Collaboration

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Classical Approach

- Moments

$$\mu_n(q^2) = f \int_0^1 dx\, x^n\, F_1(x, q^2)$$

Mellin transform

$$F_1(x, q^2) = \frac{1}{2\pi i f} \int_{c-i\infty}^{c+i\infty} ds\, x^{-s-1} \mu_s(q^2)$$

- OPE

$$\mu_1(q^2) = c_2(q^2 a^2)\langle N|\mathcal{O}_2(a)|N\rangle + \frac{c_4(q^2 a^2)}{q^2}\langle N|\mathcal{O}_4(a)|N\rangle + \cdots$$

Lattice: $q^2 \sim \frac{1}{a^2}$

$$\mu_2(q^2) = c_3(q^2 a^2)\langle N|\mathcal{O}_3(a)|N\rangle + \frac{c_5(q^2 a^2)}{q^2}\langle N|\mathcal{O}_5(a)|N\rangle + \cdots$$

$$\vdots$$

- The computations are limited to a few lower moments, due to issues of operator mixing and renormalization. Even so, the uncertainties are at least comparable to the magnitude of the power corrections

Martinelli & Sachrajda

OPE without OPE

Compton amplitude: mother of all

$$T_{\mu\nu}(p,q) = \int \mathrm{d}^4x\, \mathrm{e}^{iqx} \langle p, s | T J_\mu(x) J_\nu(0) | p, s\rangle$$

$$= \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \mathcal{F}_1(\omega, q^2) + \left(p_\mu - \frac{pq}{q^2} q_\mu\right) \left(p_\nu - \frac{pq}{q^2} q_\nu\right) \frac{1}{pq} \mathcal{F}_2(\omega, q^2)$$

$$+ \; \epsilon_{\mu\nu\lambda\sigma}\, q_\lambda\, s_\sigma \frac{1}{pq} \mathcal{G}_1(\omega, q^2) + \; \epsilon_{\mu\nu\lambda\sigma}\, q_\lambda \left(pq\, s_\sigma - sq\, p_\sigma\right) \frac{1}{(pq)^2} \mathcal{G}_2(\omega, q^2)$$

Crossing symmetry, $T_{\mu\nu}(p,q) = T_{\nu\mu}(p,-q)$, implies

$$\mathcal{F}_{1,2}(-\omega, q^2) = \pm\, \mathcal{F}_{1,2}(\omega, q^2)\,, \quad \mathcal{G}_{1,2}(-\omega, q^2) = -\, \mathcal{G}_{1,2}(\omega, q^2) \qquad \omega = \frac{1}{x} = \frac{2pq}{q^2}$$

In the physical region $1 \leq |\omega| \leq \infty$

$$\mathrm{Im}\, \mathcal{F}_{1,2}(\omega, q^2) = 2\pi F_{1,2}(\omega, q^2)\,, \quad \mathrm{Im}\, \mathcal{G}_{1,2}(\omega, q^2) = 2\pi g_{1,2}(\omega, q^2)$$

Dispersion relations

$$\mathcal{F}_1(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{F_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} - \frac{F_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right] + \mathcal{F}_1(0, q^2)$$

$$\mathcal{F}_2(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{F_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} + \frac{F_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right]$$

$$\mathcal{G}_1(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{g_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} + \frac{g_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right]$$

$$\mathcal{G}_2(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{g_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} + \frac{g_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right]$$

$\uparrow$

polarizability

For $p_3 = q_3 = q_0 = 0$, substituting $\bar{\omega}$ by $1/x$

$$T_{33}(p,q) = \mathcal{F}_1(\omega, q^2) = 4\omega \int_0^1 dx \, \frac{\omega x}{1-(\omega x)^2} \, F_1(x, q^2) + \mathcal{F}_1(0, q^2)$$

$$= \sum_{n=2,4,\cdots}^{\infty} 4\omega^n \int_0^1 dx \, x^{n-1} F_1(x, q^2) + \mathcal{F}_1(0, q^2)$$

$$T_{03}(p,q) \stackrel{\vec{s}\|\vec{p}}{=} \frac{(\vec{q}\times\vec{s})_3}{pq} \, \mathcal{G}_1(\omega, q^2) = \frac{(\vec{q}\times\vec{s})_3}{pq} \, 4\omega \int_0^1 dx \, \frac{1}{1-(\omega x)^2} \, g_1(x, q^2)$$

$$= \frac{(\vec{q}\times\vec{s})_3}{pq} \sum_{n=1,3,\cdots}^{\infty} 4\omega^n \int_0^1 dx \, x^{n-1} g_1(x, q^2)$$

$$T_{03}(p,q) \stackrel{\vec{s}\|\vec{q}}{=} -\frac{(\vec{p}\times\vec{q})_3 \, \vec{s}\vec{q}}{(pq)^2} \, \mathcal{G}_2(\omega, q^2) = -\frac{(\vec{p}\times\vec{q})_3 \, \vec{s}\vec{q}}{(pq)^2} \, 4\omega \int_0^1 dx \, \frac{1}{1-(\omega x)^2} \, g_2(x, q^2)$$

$$= -\frac{(\vec{p}\times\vec{q})_3 \, \vec{s}\vec{q}}{(pq)^2} \sum_{n=1,3,\cdots}^{\infty} 4\omega^n \int_0^1 dx \, x^{n-1} g_2(x, q^2)$$

includes power corrections

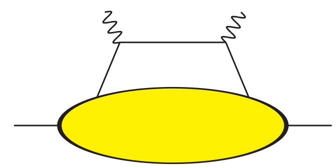
From T_{33} to μ_n and $F_1(x, q^2)$

The Compton amplitude can be computed most efficiently, including singlet (disconnected) matrix elements, by the Feynman-Hellmann technique. By introducing the perturbation to the Lagrangian

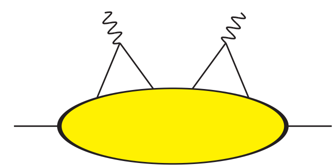
$$\mathcal{L}(x) \to \mathcal{L}(x) + \lambda \mathcal{J}_3(x)\,, \quad \mathcal{J}_3(x) = Z_V \cos(\vec{q}\vec{x})\; e_q\, \bar{q}(x)\gamma_3 q(x)$$

and taking the second derivative of $\langle N(\vec{p}, t)\bar{N}(\vec{p}, 0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\, t}$ with respect to λ on both sides, we obtain

$$-2E_\lambda(p,q)\,\frac{\partial^2}{\partial\lambda^2} E_\lambda(p,q)\,\Big|_{\lambda=0} = T_{33}(p,q)$$



The amplitude encompasses the dominating 'handbag' diagram as well as the power-suppressed 'cats ears' diagram. Varying q^2 will allow to test the twist expansion. No further renormalization is needed



Implementation

All we need to compute are nucleon two-point functions, from which we derive the energy levels E_λ. If that has been done successfully, we can resort to continuum language

Valence quark distribution functions

- Computationally cheap. No extra background (vacuum) gauge field configurations have to be generated
- The electromagnetic current needs to be inserted in quark propagators of nucleon two-point function only
- Propagators can be used to compute a variety of other observables, including form factors and Compton amplitudes of all stable particles

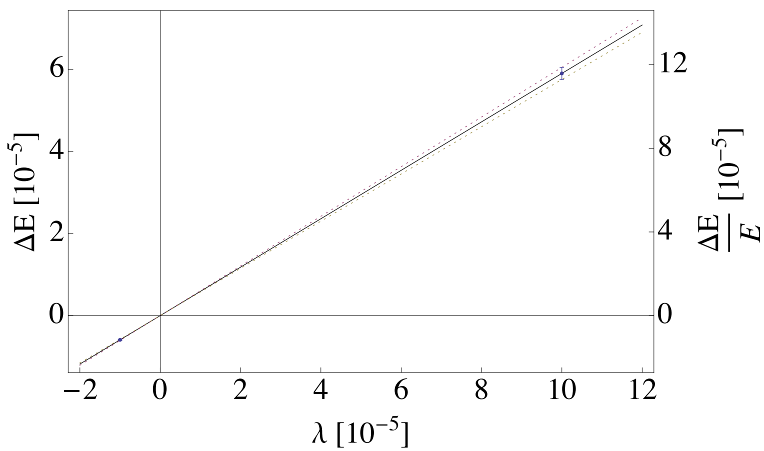
Sea quark and gluon distribution functions

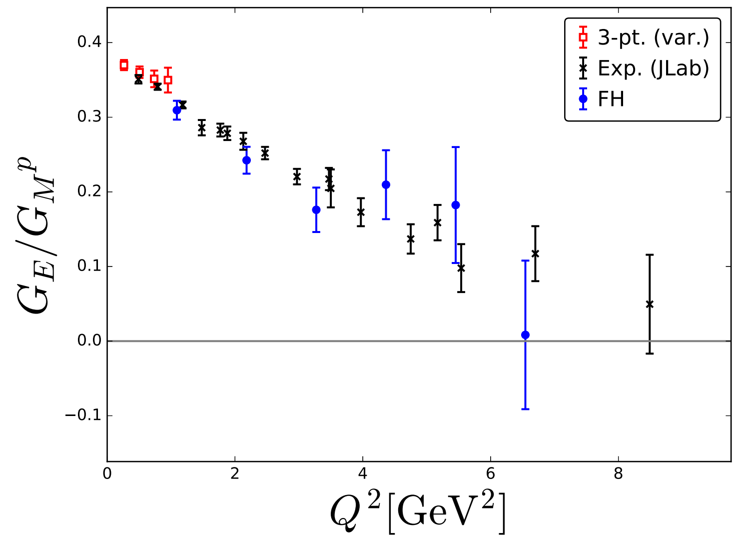
- Need to generate new background gauge field configurations with the electromagnetic current being attached to the sea quarks
- As before, the new configurations lend themselves to the calculation of many other observables, besides the nucleon Compton amplitude

Example: Nucleon form factor at large q^2

$$F_{1,2}(q^2) \propto \partial E_\lambda(p,p')/\partial\lambda\big|_{\lambda=0}$$

$$\vec{p} + \vec{p}' = 0$$ Breit frame





Powerful method

arXiv:1702.01513

Moments

Task: Compute the lowest M moments [odd moments need $\langle p,\ s|T J_\mu(x) J_\nu^5(0)|p,\ s\rangle$]

$$\mu_{2m-1} = \int_0^1 dx\, x^{2m-1} F_1(x)$$

from a finite number of sampled points

$$t_n = T_{33}(\omega_n)\,,\ n = 1, \cdots, N$$

Compton amplitude and moments are connected by the set of equations

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{pmatrix} = \begin{pmatrix} 4\omega_1^2 & 4\omega_1^4 & \cdots & 4\omega_1^{2M} \\ 4\omega_2^2 & 4\omega_2^4 & \cdots & 4\omega_2^{2M} \\ \vdots & \vdots & \vdots & \vdots \\ 4\omega_N^2 & 4\omega_N^4 & \cdots & 4\omega_N^{2M} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_3 \\ \vdots \\ \mu_{2M-1} \end{pmatrix}$$ Vandermonde M

Solutions are well documented in the literature. Alternatively, we can fit the Compton amplitude by the interpolating polynomial

$$T_{33}(\omega) = 4\left(\omega^2 \mu_1 + \omega^4 \mu_3 + \cdots + \omega^{2M} \mu_{2M-1}\right)$$

Structure function

Ultimate goal: Compute $F_1(x)$ from $T_{33}(\omega)$. Therefor we discretize the integral

$$t_n = \epsilon \sum_{m=1}^{M} K_{nm} f_m \,, \quad n = 1, \cdots, N$$

[here: points equidistant with step size ϵ]

with

$$f_m = F_1(x_m)\,, \quad K_{nm} = \frac{4\,\omega_n^2 x_m}{1-(\omega_n x_m)^2}\,, \quad N < M$$

The $N \times M$ matrix K is written

$$K = U\ [\mathrm{diag}(w_1, \cdots, w_N)]\ V^T$$

where W is singular: $w_k \approx 0, K < k \leq N$. Solution by singular value decomposition (SVD)

$$f_m = \sum_{n=1}^{N} K^{-1}_{mn} \epsilon^{-1}\, t_n$$

with K^{-1} being the pseudoinverse

$$K^{-1} = V\ [\mathrm{diag}(1/w_1, \cdots, 1/w_K, 0, \cdots, 0)]\ U^T$$

`Mathematica`

Conceivable alternative:

Educated fit

$$x\,q(x) = A_q\,x^{\alpha}(1-x)^{\beta} \qquad\qquad q = u, d, S, \cdots$$

$$\mu_n = f\int_0^1 dx\,x^{n-1}A_q\,x^{\alpha}(1-x)^{\beta} = f\,A_q\,\frac{\Gamma(\alpha+n)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta+n)}$$

$$T_{33}(\omega) = 4f\,A_q\,\Gamma(1+\beta)\sum_n \frac{\Gamma(\alpha+n)}{\Gamma(1+\alpha+\beta+n)}\,\omega^{n+1}$$

$\leftarrow$ Fit to data on $T_{33}(\omega)$

3 Overview of theoretical framework

In this section we first give a brief overview of the standard theoretical formalism used, and then present a summary of the theoretical improvements and changes in methodology in the global analysis. A more detailed discussion of the various items is given later in separate sections.

We work within the standard framework of leading-twist fixed-order collinear factorisation in the $\overline{\mathrm{MS}}$ scheme, where structure functions in DIS, $F_i(x, Q^2)$, can be written as a convolution of coefficient functions, $C_{i,a}$, with PDFs of flavour a in a hadron of type A, $f_{a/A}(x, Q^2)$, i.e.

$$F_i(x, Q^2) = \sum_{a=q,g} C_{i,a} \otimes f_{a/A}(x, Q^2). \tag{2}$$

Similarly, in hadron–hadron collisions, hadronic cross sections can be written as process-dependent partonic cross sections convoluted with the same universal PDFs, i.e.

$$\sigma_{AB} = \sum_{a,b=q,g} \hat{\sigma}_{ab} \otimes f_{a/A}(x_1, Q^2) \otimes f_{b/B}(x_2, Q^2). \tag{3}$$

The scale dependence of the PDFs is given by the DGLAP evolution equation in terms of the calculable splitting functions, $P_{aa'}$, i.e.

$$\frac{\partial f_{a/A}}{\partial \ln Q^2} = \sum_{a'=q,g} P_{aa'} \otimes f_{a'/A}. \tag{4}$$

The DIS coefficient functions, $C_{i,a}$, the partonic cross sections, $\hat{\sigma}_{ab}$, and the splitting functions, $P_{aa'}$, can each be expanded as perturbative series in the running strong coupling, $\alpha_S(Q^2)$. The strong coupling satisfies the renormalisation group equation, which up to NNLO reads

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}\left(\frac{\alpha_S}{4\pi}\right) = -\beta_0\left(\frac{\alpha_S}{4\pi}\right)^2 - \beta_1\left(\frac{\alpha_S}{4\pi}\right)^3 - \beta_2\left(\frac{\alpha_S}{4\pi}\right)^4 - \ldots. \tag{5}$$

The input for the evolution equations, (4) and (5), $f_{a/A}(x, Q_0^2)$ and $\alpha_S(Q_0^2)$, at a reference input scale, taken to be $Q_0^2 = 1$ GeV^2, must be determined from a global analysis of data. In the present study we use a slightly extended form, compared to previous MRST fits, of the parameterisation of the parton distributions at the input scale $Q_0^2 = 1$ GeV^2:

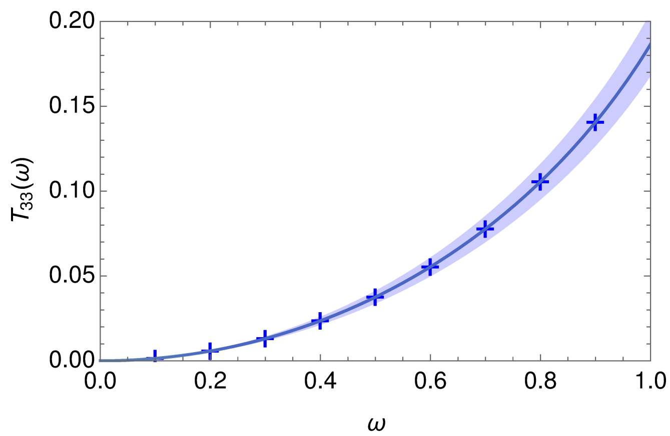
$$xu_v(x, Q_0^2) = A_u\, x^{\eta_1}(1-x)^{\eta_2}(1 + \epsilon_u\sqrt{x} + \gamma_u\, x), \tag{6}$$
$$xd_v(x, Q_0^2) = A_d\, x^{\eta_3}(1-x)^{\eta_4}(1 + \epsilon_d\sqrt{x} + \gamma_d\, x), \tag{7}$$
$$xS(x, Q_0^2) = A_S\, x^{\delta_S}(1-x)^{\eta_S}(1 + \epsilon_S\sqrt{x} + \gamma_S\, x), \tag{8}$$
$$x\Delta(x, Q_0^2) = A_\Delta\, x^{\eta_\Delta}(1-x)^{\eta_S+2}(1 + \gamma_\Delta\, x + \delta_\Delta\, x^2), \tag{9}$$
$$xg(x, Q_0^2) = A_g\, x^{\delta_g}(1-x)^{\eta_g}(1 + \epsilon_g\sqrt{x} + \gamma_g\, x) + A_{g'}\, x^{\delta_{g'}}(1-x)^{\eta_{g'}}, \tag{10}$$
$$x(s+\bar{s})(x, Q_0^2) = A_+\, x^{\delta_S}\,(1-x)^{\eta_+}(1 + \epsilon_S\sqrt{x} + \gamma_S\, x), \tag{11}$$
$$x(s-\bar{s})(x, Q_0^2) = A_-\, x^{\delta_-}(1-x)^{\eta_-}(1 - x/x_0), \tag{12}$$

Parameter	LO	NLO	NNLO
$\alpha_S(Q_0^2)$	0.68183	0.49128	0.45077
$\alpha_S(M_Z^2)$	0.13939	0.12018	0.11707
A_u	1.4335	0.25871	0.22250
η_1	$0.45232^{+0.022}_{-0.018}$	$0.29065^{+0.019}_{-0.013}$	$0.27871^{+0.018}_{-0.014}$
η_2	$3.0409^{+0.079}_{-0.067}$	$3.2432^{+0.062}_{-0.039}$	$3.3627^{+0.061}_{-0.044}$
ϵ_u	$-2.3737^{+0.54}_{-0.48}$	$4.0603^{+1.6}_{-2.3}$	$4.4343^{+2.4}_{-2.7}$
γ_u	8.9924	30.687	38.599
A_d	5.0903	12.288	17.938
η_3	$0.71978^{+0.057}_{-0.082}$	$0.96809^{+0.11}_{-0.11}$	$1.0839^{+0.12}_{-0.11}$
$\eta_4 - \eta_2$	$2.0835^{+0.32}_{-0.45}$	$2.7003^{+0.50}_{-0.52}$	$2.7865^{+0.50}_{-0.44}$
ϵ_d	$-4.3654^{+0.28}_{-0.22}$	$-3.8911^{+0.31}_{-0.29}$	$-3.6387^{+0.27}_{-0.28}$
γ_d	7.4730	6.0542	5.2577
A_S	$0.59964^{+0.036}_{-0.030}$	$0.31620^{+0.030}_{-0.021}$	$0.64942^{+0.047}_{-0.041}$
δ_S	−0.16276	−0.21515	−0.11912
η_S	$8.8801^{+0.33}_{-0.33}$	$9.2726^{+0.23}_{-0.33}$	$9.4189^{+0.25}_{-0.33}$
ϵ_S	$-2.9012^{+0.33}_{-0.37}$	$-2.6022^{+0.71}_{-0.96}$	$-2.6287^{+0.49}_{-0.51}$
γ_S	16.865	30.785	18.065
$\int_0^1 \mathrm{d}x\, \Delta(x, Q_0^2)$	$0.091031^{+0.012}_{-0.009}$	$0.087673^{+0.013}_{-0.011}$	$0.078167^{+0.012}_{-0.0091}$
A_Δ	8.9413	8.1084	16.244
η_Δ	$1.8760^{+0.24}_{-0.30}$	$1.8691^{+0.23}_{-0.32}$	$2.0741^{+0.18}_{-0.35}$
γ_Δ	$8.4703^{+2.0}_{-0.3}$	$13.609^{+1.1}_{-0.6}$	$6.7640^{+0.77}_{-0.41}$
δ_Δ	−36.507	−59.289	−36.090
A_g	0.0012216	1.0805	3.4055
δ_g	$-0.83657^{+0.15}_{-0.14}$	$-0.42848^{+0.066}_{-0.057}$	$-0.12178^{+0.23}_{-0.16}$
η_g	$2.3882^{+0.51}_{-0.50}$	$3.0225^{+0.43}_{-0.36}$	$2.9278^{+0.68}_{-0.41}$
ϵ_g	-38.997^{+36}_{-35}	−2.2922	−2.3210
γ_g	1445.5^{+880}_{-750}	3.4894	1.9233
$A_{g'}$	—	−1.1168	−1.6189
$\delta_{g'}$	—	$-0.42776^{+0.053}_{-0.047}$	$-0.23999^{+0.14}_{-0.10}$
$\eta_{g'}$	—	$32.869^{+6.5}_{-5.9}$	$24.792^{+6.5}_{-5.2}$
A_+	$0.10302^{+0.029}_{-0.017}$	$0.047915^{+0.0095}_{-0.0076}$	$0.10455^{+0.019}_{-0.016}$
η_+	$13.242^{+2.9}_{-1.4}$	$9.7466^{+1.0}_{-0.8}$	$9.8689^{+1.0}_{-0.6}$
A_-	$-0.011523^{+0.009}_{-0.018}$	$-0.011629^{+0.009}_{-0.023}$	$-0.0093692^{+0.006}_{-0.024}$
η_-	10.285^{+16}_{-6}	11.261^{+22}_{-6}	9.5783^{+26}_{-5}
x_0	0.017414	0.016050	0.018556
r_1	−0.39484	−0.57631	−0.80834
r_2	−1.0719	0.81878	1.2669
r_3	−0.28973	−0.083208	0.15098

Table 4: The optimal values of α_S and the input PDF parameters at $Q_0^2 = 1$ GeV^2 determined from the global analysis. The one-sigma errors are calculated using (51) and (52) using the 68% C.L. tolerance discussed in Section 6, and are shown only for the 20 parameters allowed to go free when determining the eigenvector PDF sets. The parameters A_u, A_d, A_g and x_0 are determined from sum rules and are not fitted parameters. Similarly, A_Δ is determined from $\int_0^1 \mathrm{d}x\, \Delta(x, Q_0^2)$. The three parameters r_i, defined in (73), are associated with the nuclear corrections to the neutrino data; see Section 7.3. The parameter values are given to five significant figures solely for accuracy in the case of reproduction of the PDFs.

Proof of Concept

In

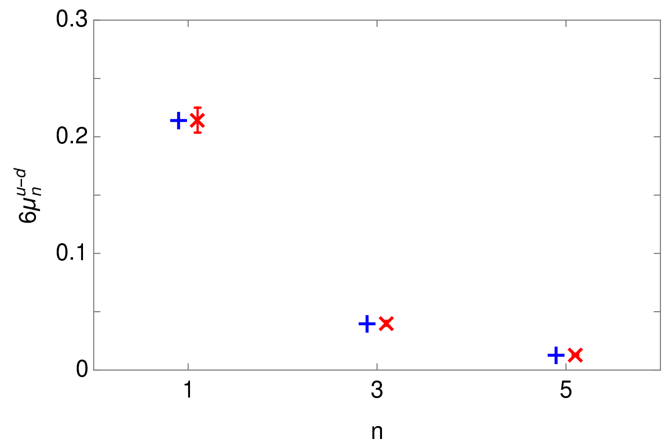


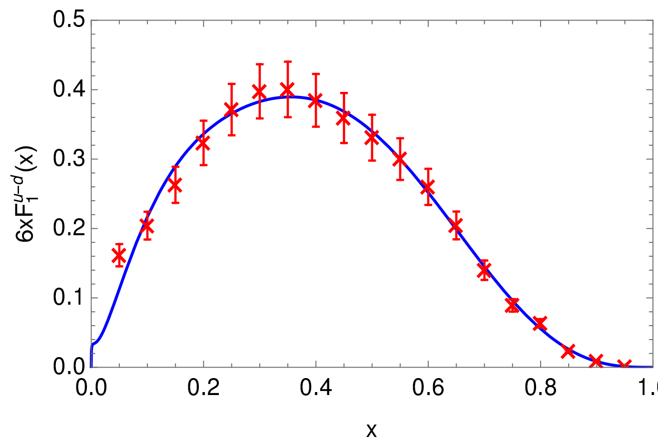
$$T_{33}(\omega) = 4\omega \int_0^1 dx \, \frac{\omega x}{1-(\omega x)^2} F_1^{u-d}(x)$$

$$2xF_1^{u-d}(x) = \frac{1}{3} x[u(x) - d(x)]$$

MSTW-lo

Out





$F_1(x)$ at very small x: needs $\omega > 1$

Not accessible via moments

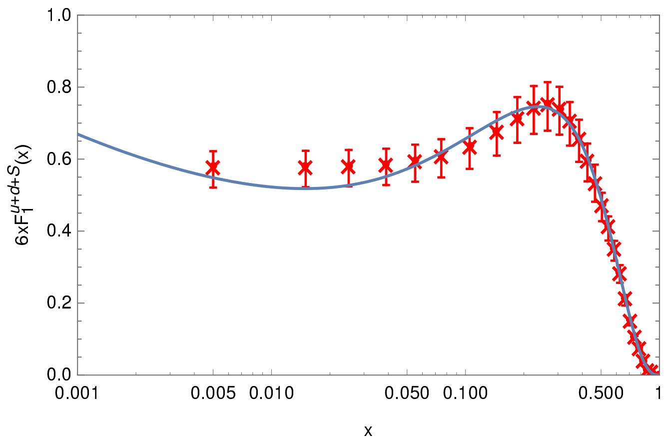
In

$$T_{33}(\omega) = 4\omega \, \mathrm{P}\int_0^1 dx \, \frac{\omega x}{1-(\omega x)^2} F_1^{u+d+S}(x)$$

$$2xF_1^{u+d+S}(x) = \frac{1}{3}\, x[u(x) + d(x) + S(x)]$$

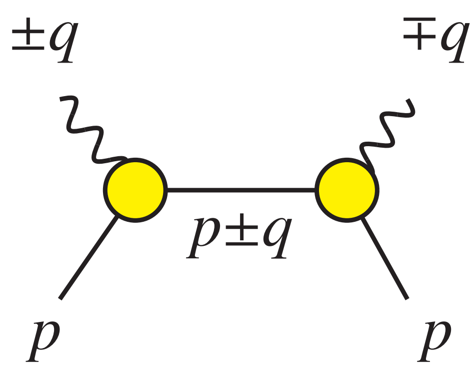
MSTW-lo

Out



$\omega \in [0, 2]$

Note that intermediate states of the (semi-)elastic Compton amplitude $T_{\mu\nu}(\omega, q^2)$ can go on-shell for $\omega \geq 1$



For example: nucleon pole

$$p^2 = -E^2 + \vec{p}^2 = -m_N^2$$

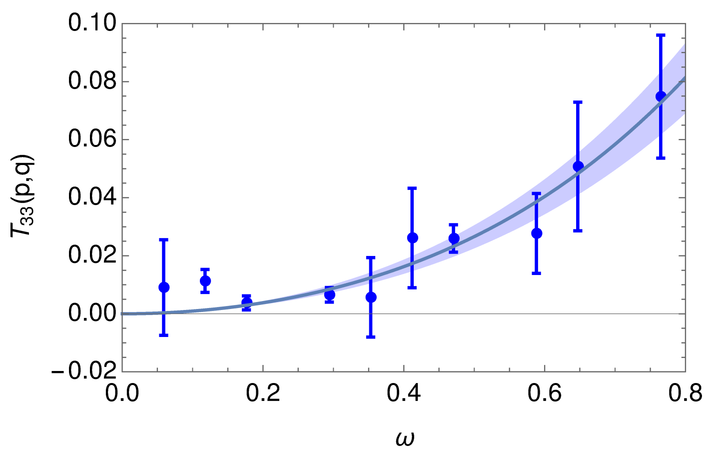
$$(p \pm q)^2 = p^2 \pm 2pq + q^2 = -m_N^2 + q^2(1 \pm \omega)$$
$$= -m_N^2 \quad \text{for } \omega = \mp 1$$

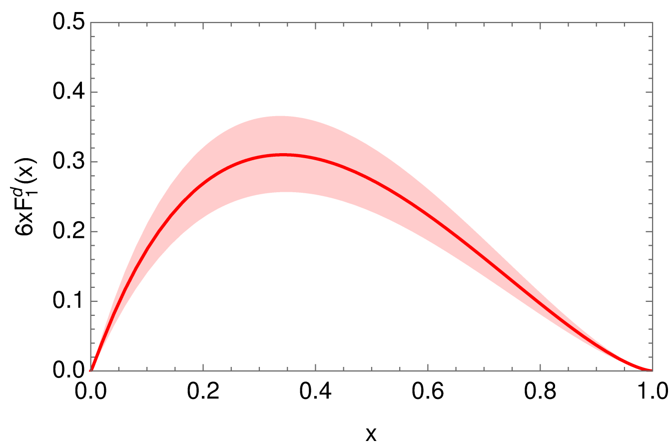
However, this contribution is power suppressed by the product of nucleon form factors, $(F_1(q^2))^2$. In our example (see next slide) $q^2 \approx 9\,\text{GeV}^2$, which leads to a suppression factor of $\approx 1/10.000$

Lattice Study

SU(3) symmetric point

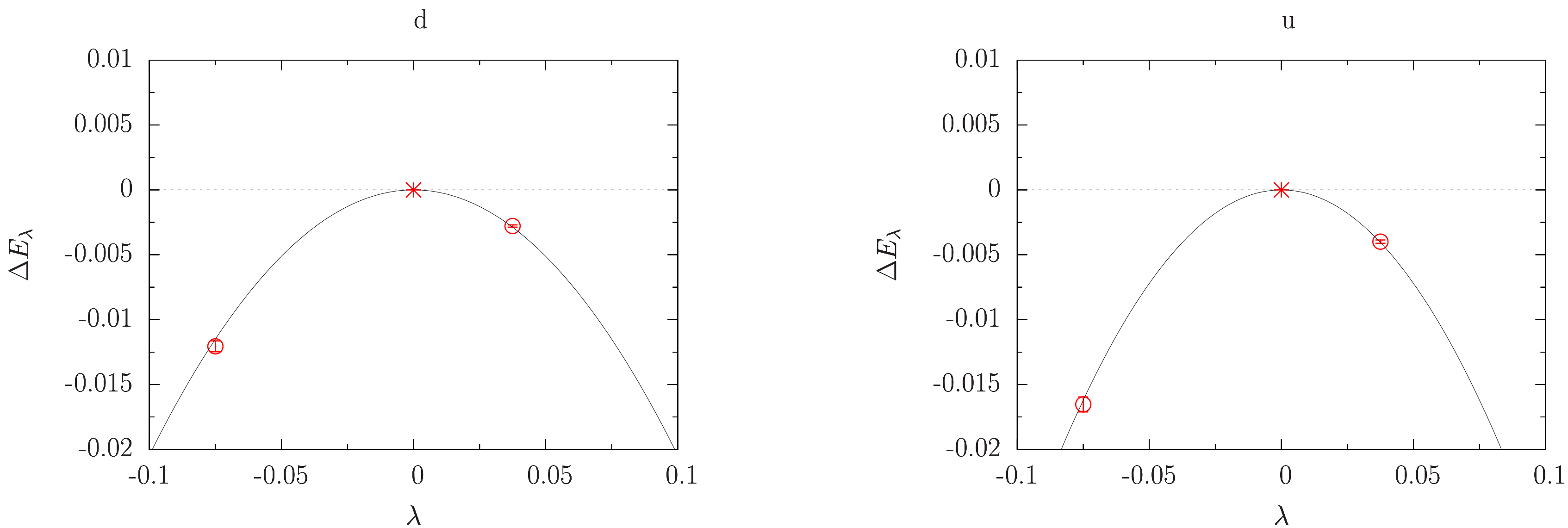
V	M_π	M_K	a [fm]	q^2 [GeV^2]
$32^3 \times 64$	420	420	0.075	9.2





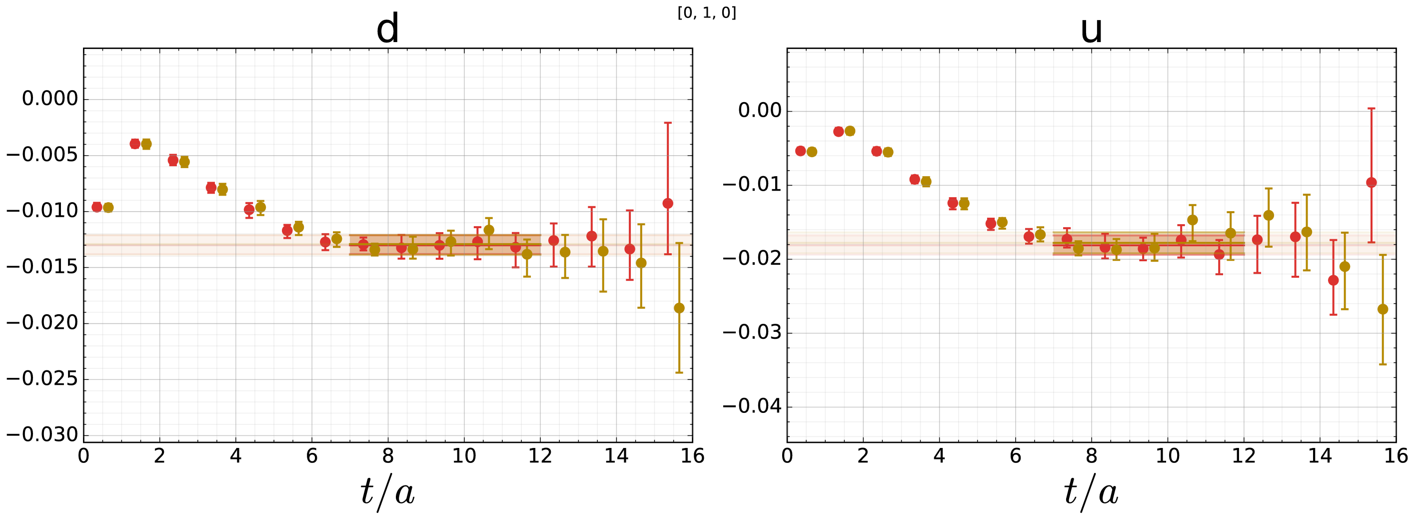
$$\mathcal{J}_3(x) = Z_V \cos(\vec{q}\vec{x})\; e_d\, \bar{d}(x)\gamma_3 d(x)$$

$\Delta E_\lambda = E_\lambda - E_0 \propto \lambda^2$ $\omega = 0.3$



$\partial^2 E_\lambda / \partial \lambda^2 \big|_{\lambda=0}$

$\omega = 0.3$



From T_{03} to $g_1(x, q^2)$ and $g_2(x, q^2)$

The Compton amplitude $T_{03}(\omega, q^2)$ needs to be antisymmetric in the Lorentz indices, $T_{03}(\omega, q^2) = -T_{30}(\omega, q^2)$, in this case. That can be achieved by introducing the perturbation to the Lagrangian

$$\mathcal{L}(x) \to \mathcal{L}(x) + \lambda \mathcal{J}_{0+3}(x)\,,\ \mathcal{J}_{0+3}(x) = Z_V\, e_q\, \bar{q}(x)(\gamma_0 \cos(\vec{q}\vec{x}) + \gamma_3 \sin(\vec{q}\vec{x}))q(x)$$

and taking the second derivative of $\langle N(\vec{p}, t)\bar{N}(\vec{p}, 0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\, t}$ with respect to λ as before, giving

$$-2E_\lambda(p, q)\, \frac{\partial^2}{\partial\lambda^2} E_\lambda(p, q)\,\big|_{\lambda=0} = T_{03}(p, q) - T_{30}(p, q)$$

T_{00}, T_{33} drop out

PDFs

$$F_1(x) = \sum_{i=u,d,\cdots,g} \int_x^1 \frac{dy}{y}\, c_{1,i}(x/y,\mu^2)\, f_i(y,\mu^2)$$

$$g_1(x) = \sum_{i=u,d,\cdots,g} \int_x^1 \frac{dy}{y}\, e_{1,i}(x/y,\mu^2)\, \Delta f_i(y,\mu^2)$$

↑

perturbatively known

$$f_u(x) = u(x) \qquad \Delta f_u(x) = \Delta u(x)$$
$$f_d(x) = d(x) \qquad \Delta f_d(x) = \Delta d(x)$$
$$f_{\bar{u}}(x) = \bar{u}(x) \qquad \Delta f_{\bar{u}}(x) = \Delta \bar{u}(x)$$
$$f_{\bar{d}}(x) = \bar{d}(x) \qquad \Delta f_{\bar{d}}(x) = \Delta \bar{d}(x)$$

Solely need to replace

$$K_{nm} = \frac{4\,\omega_n^2 x_m}{1-(\omega_n x_m)^2} \quad \rightarrow \quad K_{nm} = 2\,\omega_n^2 \int_0^1 dy\, y\, x_m \frac{c_1(y,\mu^2)}{1-(y\,\omega_n\,x_m)^2}$$

Check factorization

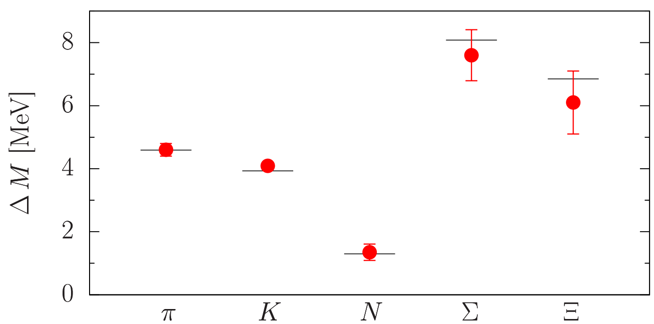
Flavor Structure of the Nucleon Sea

Need to consider QCD + QED

ΔM	QCD + QED	QED
$M_{\pi^+} - M_{\pi^0}$	$4.60(20)$	$4.60(20)$
$M_{K^0} - M_{K^+}$	$4.09(10)$	$-1.66(6)$

$$\frac{m_u}{m_d} = \left.\frac{2M_{\pi^0}^2 - M_{\pi^+}^2 + M_{K^+}^2 - M_{K^0}^2}{M_{\pi^+}^2 - M_{K^+}^2 + M_{K^0}^2}\right|_{\rm QCD} \approx 1.1$$

QCD+QED: ≈ 0.5



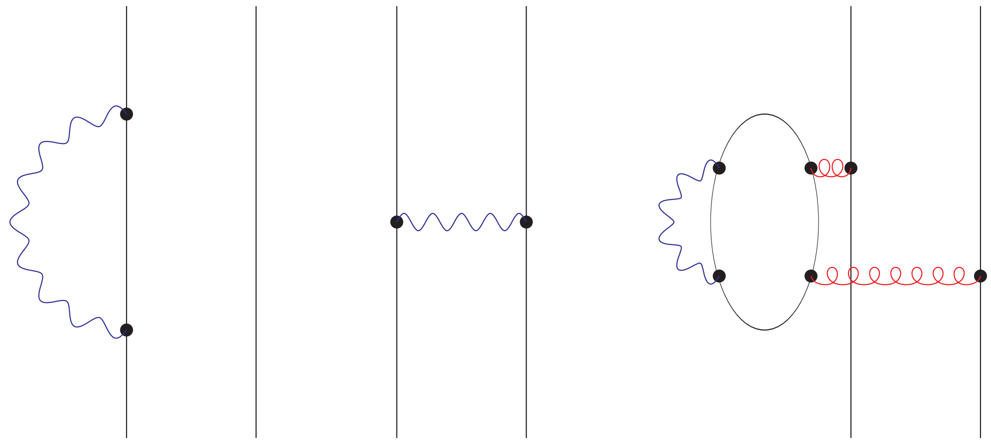
QCDSF

Requires

$$\mathcal{L}(x) \to \mathcal{L}_{\rm QCD+QED}(x) + \lambda \mathcal{J}_3(x)$$

in simulations of background field configurations

Diagrams contributing to the pseudoscalar meson masses, which determine the up, down and strange quark masses



$$M^2(a\bar{b}) = M_0^2 + \alpha\,(\delta m_a + \delta m_b) + \beta_0^{\rm EM}\,(e_u^2 + e_d^2 + e_s^2) + \beta_1^{\rm EM}\,(e_a^2 + e_b^2) + \beta_2^{\rm EM}\,(e_a - e_b)^2 + \cdots$$

Outlook

- Computations can be improved in many respects
  - Apply Bayesian regression with SVD to alleviate overfitting
  - Employ momentum smearing techniques for larger values of ω
- With gradual improvements, we should be able to compute the structure functions $F_1(x, q^2)$ and $F_2(x, q^2)$, as well as $g_1(x, q^2)$ and $g_2(x, q^2)$, including contributions of higher twist, from the Compton amplitude with unprecedented accuracy
- This is possible, because the calculation skirts the issue of renormalization and operator mixing
- The method can easily be generalized to generalized parton distribution functions (GPDs) $H(x, \xi, q^2)$ and $E(x, \xi, q^2)$