

[Towards] structure functions from lattice QCD

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Deep inelastic structure of the proton

Parton model

Scatter from non-interacting quarks Bjorken scaling variable [~longitudinal momentum fraction]

$$
x = \frac{Q^2}{2P.q}
$$

Deep-inelastic scattering

 $\left| \begin{array}{c|c} \hline \end{array} \right|$ described by perturbative QCD been slightly offset in $\mathcal{L}_\mathcal{L}$ for clarity. The H1+ZEUS combined binning in this plot; all other plot; are in the shall shall shall shall be shall shall be purposed as a shall be placed by 2011 μ placed by 21 μ R bit, where R is the number of R in R and R \mathcal{P} are al., al., al., bending the setter in all \mathcal{P} are setter. By \mathcal{P} \mathbb{R}^n and do the algebra \mathbb{R}^n phys. Rev. D44, 3006 (1996); Nucl. \mathbb{R}^n

was within 10% of unity. The data are plotted as a function of \mathcal{L}_2 in bins of fixed \mathcal{L}_2

positrons on protons (collider experiments H1 and ZEUS for Q² ≥ 2 GeV2), in the kinematic domain of the HERA data (see Fig. 19.10 for data at 19.10 for data at smaller x and α Almost Bjorken scaling

(1997); SLAC — L.W. Whitlow et al., Phys. Lett. B282, 475 (1992). Lett. B282, 475 (1992). Lett. B282, 475 (199

QCD: the densest matter in nature

for Zein-Eddine

[†] Densities are given at 0° C and 1 atm pressure unless otherwise specified.

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Isovector quark distributions -0.2

−0.15

Relative uncertainties diverge beyond *x*~0.6: **Opportunity for lattice to contribute** x **Opportunity for lattice to contribute**

utions

Parton distributions | In principle, these could be Mutions contraction at at at an internative model in gluon-funcional model in gluon-funcion in gluon-f **Challenging so far!** In principle, these could be

Outline

• Structure functions and the **operator product expansion**

• **Feynman-Hellmann** (FH) approach to hadron structure on the lattice

- **Compton amplitude** on the lattice 1 (x) = 1
1 (x) = 1
(x) = 1
(x)
	- Toy model test *x*[u(x) − d(x)]
	- · Exploratory numerical study

Structure functions and the operator product expansion

Inelastic scattering

(Virtual) Compton amplitude

• Forward Compton amplitude

$$
T^{\mu\nu}(p,q) = \rho_{ss'} \int d^4x \, e^{iq.x} \langle p, s | T \{ J^{\mu}(x) J^{\nu}(0) \} | p, s \rangle
$$

= $\left(-g^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right) T_1(P,q,Q^2) + \frac{1}{P.q} \left(p^{\mu} - \frac{P.q}{q^2} q_{\mu} \right) \left(p^{\nu} - \frac{P.q}{q^2} q_{\nu} \right) T_2(P,q,Q^2)$

• Looking ahead to lattice results shown at end, consider simple case

$$
\mu = \nu = 3, q_3 = 0, P_3 = 0
$$

\n $\Rightarrow T_{33}(P, q) = T_1(P, q, Q^2)$

Analytic structure of Compton amplitude

Analytic structure of Compton amplitude

Analytic structure of Compton amplitude

Moments of structure functions

• Re-express integral over familiar Bjorken *x*:

$$
T_{1}(\omega, Q^{2}) - T_{1}(\omega, 0) = \frac{4\omega^{2}}{2\pi} \int_{1}^{\infty} d\omega' \frac{\text{Im} T_{1}(\omega', Q^{2})}{\omega'(\omega^{2} - \omega'^{2})} = 4\omega^{2} \int_{0}^{1} dx \, x \frac{F_{1}(x, Q^{2})}{1 - (\omega x)^{2}}
$$

Subtraction term:
Cottingham sum rule; Muonic hydrogen.
Recently, see also:
Agadjanov, Meißner & Russesky, PRD(2017),
Hill & Paz, PRD(2017), ...
Taylor
Taylor

• **Moments of structure functions**

$$
T_1(\omega, Q^2) - T_1(\omega, 0) = \sum_{j=1}^{\infty} 4\omega^{2j} \int_0^1 dx \, x^{2j-1} F_1(x, Q^2)
$$

Lattice QCD: Traditional way

$$
T_1(\omega, Q^2) - T_1(\omega, 0) = \sum_{j=1}^{\infty} 4\omega^{2j} \int_0^1 dx \, x^{2j-1} F_1(x, Q^2)
$$

• Matrix elements of local twist-2 operators:

 $\langle P| \mathcal{O}^{\{\nu_1...\nu_n\}}|P\rangle = 2 a(n,\mu) P^{\nu_1} \dots P^{\nu_n} - \text{traces}$ $\mathcal{O}^{\{\nu_1...\nu_n\}} = \overline{\psi}(0)\gamma^{\nu_1}D^{\nu_2} \dots D^{\nu_n}\psi(0)$ $a(n,\mu) = \int_0^1$ 0 $dx x^{2n-1}F(x,\mu)$ Operator mixing on the lattice prohibits the study of operators with increasing numbers of derivatives: **Typically only access lowest moment (e.g. quark momentum fractions)**

Study Compton amplitude directly on lattice

$$
T_1(\omega, Q^2) - T_1(\omega, 0) = 4\omega^2 \int_0^1 dx \, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}
$$

4-point function on the lattice? Preferably not

Feynman-Hellmann theorem in lattice QCD

Matrix elements from "Feynman–Hellmann"

• Feynman–Hellmann in quantum mechanics:

$$
\frac{dE_n}{d\lambda} = \langle n|\frac{\partial H}{\partial \lambda}|n\rangle
$$

- matrix elements of the derivative of the Hamiltonian determined by derivative of corresponding energy eigenstates
- Lattice QCD: evaluate energy shifts with respect to weak external fields
- Modify action with external field:

$$
S \to S + \lambda \int d^4x \, \mathcal{O}(x)
$$

real parameter
local operator, e.g. $\bar{q}(x)\gamma_5\gamma_3q(x)$

• Calculation of matrix element = hadron spectroscopy [2-pt functions only]

$$
\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)} \langle H|\mathcal{O}|H\rangle
$$

Spin content [connected]

• Modify action

$$
S \to S + \lambda \sum_{x} \bar{q}(x) i \gamma_5 \gamma_3 q(x)
$$

• Nucleon energy shift isolates spin content

$$
\frac{\partial E_N(\lambda)}{\partial \lambda} = \frac{1}{2M_N} \langle N|\overline{q}i\gamma_5\gamma_3 q|N\rangle
$$

$$
= \Delta q
$$

Slope → matrix element

Strength of external field

[Chambers *et al*. PRD(2014)]

3-pt function → 2-pt function

the Feynman –Hellman –Hellman –Hellmann method, from a variational analysississimal analysissimal analysissima

[Chambers *et al*. arXiv:1702.01513]

Proton form factors | See James Zanotti on Wednesday \overline{P} three-point functions \overline{P} **is is not include by the magnetic moment of the magnetic moment of the moment of the PTOLOM** *I*O*p <i>p* this would require phenomenological fits to the low *Q*² data,

See James Zanotti on Wednesday

Feynman–Hellmann (2nd order): Study Compton amplitude directly

$$
T_1(\omega, Q^2) - T_1(\omega, 0) = 4\omega^2 \int_0^1 dx \, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}
$$

Feynman–Hellman (2nd order)

• Field theory version of 2nd order perturbation theory:

$$
E = E_0 + \lambda \langle N|V|N \rangle + \lambda^2 \sum_{X \neq N} \frac{\langle N|V|X\rangle \langle X|V|N \rangle}{E_0 - E_X} + \dots
$$

Only get a linear term
for elastic case $\omega = 1$
of $\omega = 1$

• Final result. We study second-order perturbation on the lattice

$$
\frac{\partial^2 E_{\mathbf{p}}}{\partial \lambda^2} = -\frac{1}{2E_{\mathbf{p}}} \int d^4 \xi \left(e^{iq.\xi} + e^{-iq.\xi} \right) \langle \mathbf{p} | T J(\xi) J(0) | \mathbf{p} \rangle
$$

see backup slides, or RDY, presentation @Lattice 2017; Somfleth et al. … soon

Test case: Compton amplitude → SFs

Taylor expansion

• Consider moments of structure function

$$
\mu_{2m-1} = \int_0^1 dx \, x^{2m-1} F_1(x)
$$

• Series expansion of Compton amplitude

"Inversion"

- Discrete approximation to structure function $F_1(x)$
- Consider discretised integral

$$
T_{33}(\omega_n) = \sum_{m=1}^{M} K_{nm} F_1(x_m), \quad x_m = \frac{m}{M} \qquad K_{nm} = \frac{4\omega_n^2 x_m}{1 - (\omega_n x_n)^2}
$$

$$
N < M
$$

• Use singular value decomposition to invert *N×M* matrix

$$
K = U \left[\text{diag}(w_1, \dots, w_{N'}, w_{N'+1}, \dots, w_N) \right] V^{\top}
$$

$$
N \times M \text{ "diag"}
$$

$$
w_{N'+1} \dots, w_N \simeq 0, N' \le N
$$

• Pseudoinverse

$$
K^{-1} = V \left[\mathrm{diag}(1/\omega_1, \ldots, 1/\omega_{N'}, 0, \ldots, 0) \right] U^{\top}
$$

$$
T_{33} = 4\omega \int_0^1 dx \frac{\omega x}{1 - (\omega x)^2} F_1^{u-d}(x)
$$

$$
2x F_1^{u-d}(x) = \frac{1}{3} x [u(x) - d(x)]
$$

Chambers et al., PRL(2017)

3 input PDFs: MSTW(LO)

Numerical investigation

Numerical set-up

Single external momenta

 $\vec{q} = (3, 5, 0) \frac{2\pi}{I}$ *L*

$$
\omega = \frac{2P.q}{Q^2} = \frac{2\vec{P}.\vec{q}}{\vec{q}^2}
$$

$$
q_4 = 0
$$

Lattice specs

SU(3) symmetric point: 323x64, a≈0.074 fm O(900) configs $m_{\pi} \simeq 400 \,\text{MeV}$

Blue dots: different nucleon Fourier momenta

Lattice kinematics $\left\| \begin{array}{c} \text{Broad coverage of } \omega \text{ from single} \\ \text{coluction (computation)} \end{array} \right\|$ calculation (computationally "cheap")

Numerical test: Lattice results There is the question with a control with real data. It is the second with real data. It i

New access to form factors at large momenta ω = 2, we were able to retrieve the singlet structure function *F^u*+*d*+*u*¯+*d*¯+*s*¯ ¹ (*x*) [13] down to fractional **x** \blacksquare . The structure in the structure functions of the structure functions in the structure functions in be obtained by also including the local axial vector current ψ¯ *^f*(*x*)γ3γ5ψ*f*(*x*) to (11) and studying the interference with the vector current. The vector current \mathbf{a} simple extension of the simple e procedure described above. The method can be generalized to nonforward Compton scattering

FIG. 4. Scaled pion form factor *Q*²*F*⇡ from the Feynman– Hellmann technique and from experiment [12]. The solid lines are the vector meson dominance at the relevant pion masses, and the dotted lines are the asymptotic values predicted by perturbative QCD (see [13] for a discussion of this value and

using resources awarded at the NCI National Facility in Canberra, Australia, and the iVEC facilities at the Pawsey Supercomputing Centre. These resources are provided through the National Computational Merit Allocation Scheme and the University of Adelaide Partner Share supported by the Australian Government. This work was supported in part through supercomputing resources provided by the Phoenix HPC service at the University of Adelaide. The BlueGene codes were optimised using Bagel [46]. The Chroma software library [47], was used in the data analysis. AC was supported by the Australian Government Research Training Program Scholarship. JD gratefully acknowledges support by the National Superconducting Cyclotron Laboratory (NSCL)/Facility GS was supported by DFG grant SCHI 179/8-1. HP was supported by DFG grant SCHI 422/10-1. for Rare Isotope Beams (FRIB) and Michigan State University (MSU) during the preparation of this work. This investigation has been supported by the Australian Research Council under grants FT120100821, FT100100005 and DP140103067 (RDY and JMZ).

its limitations).

−0.05

0.05

0.1

0.15

−0.1

−0.3

Figure 37. The shift in NNPDF3.0 PDFs with αs(MZ)=0.118 at Q² = 10⁴ GeV² when going

−0.2

−0.1

0.1

0.2

0.3

0.4

as well. That will allow us to derive generalized parton distribution functions (GPDs).

(Virtual) Compton amplitude accessible on the lattice FIG. 3. Ratio *GE/G^M* for the proton from application of an Gormolon annollude, (and a compton amplitude Cessible on the −0.15 $-\sqrt{2}$ 0 0.05

Wonperturbative constraint $\mathbf{F}^{\circ 2}$ **Microsoft and include the structure in the shows the error.** functions \rightarrow PDFs + higher twist $\mathcal{L}^{\text{max}}_{\text{max}}$, and $\mathcal{L}^{\text{max}}_{\text{max}}$, and $\mathcal{L}^{\text{max}}_{\text{max}}$, and $\mathcal{L}^{\text{max}}_{\text{max}}$, and $\mathcal{L}^{\text{max}}_{\text{max}}$ bag' diagram in Fig. 1. *ZV has been to convent the solid line* shows a sixth order political line shows a sixth order political lines of ≥ 0.1

Back-up slides

Feynman–Hellman with momentum transfer

Warm up: Periodic potential, 1-D QM

- Almost free particle $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$
- Subject to weak external periodic potential $V(x) = 2\lambda V_0 \cos(qx)$

$$
\hat{V}|p\rangle = \lambda V_0|p+q\rangle + \lambda V_0|p-q\rangle
$$

Warm up: Periodic potential, 1-D QM

Degenerate perturbation theory

• Exact degeneracy: $p = q/2$

$$
H = \begin{pmatrix} \frac{p^2}{2m} & \lambda V_0 \\ \lambda V_0 & \frac{p^2}{2m} \end{pmatrix} \qquad H
$$

$$
H\left\{|q/2\rangle\pm|-q/2\rangle\right\} = \left(E_{q/2} \pm \lambda V_0\right)\left\{|q/2\rangle\pm|-q/2\rangle\right\}
$$

• Consider mixing on almost-degenerate states $p \sim q/2$

External momentum field on the lattice

• Modify Lagrangian with external field containing a spatial Fourier transform [constant in time]

 $\mathcal{L}(y) \to \mathcal{L}_0(y) + \lambda 2 \cos(\vec{q} \cdot \vec{y}) \overline{q}(y) \gamma_\mu q(y)$

• Project onto "back-to-back" momentum state: $|\vec{q}/2\rangle + |-\vec{q}/2\rangle$

• E.g. pion form factor

"Breit frame" kinematics

$$
\langle \pi(\vec{p}')|\overline{q}(0)\gamma_{\mu}q(0)|\pi(\vec{p})\rangle = (p+p')_{\mu} F_{\pi}(q^2)
$$

• "Feynman-Hellmann"

$$
\frac{\partial E}{\partial \lambda}\bigg|_{\lambda=0} = \frac{(p+p')_{\mu}}{2E} F_{\pi}(q^2) \qquad \frac{\mu=4}{\lambda} \qquad \frac{\partial E}{\partial \lambda}\bigg|_{\lambda=0} = F_{\pi}(q^2)
$$

3-pt functions **Proton Form Factors**

Phenomenologicallyinteresting region. Domain dominated by model calculations… previously prohibitive to study in lattice QCD.

[Chambers *et al*. arXiv:1702.01513]

Proton form factors

[my comments]

- One volume
	- Not worried (yet)
- One quark mass
	- Surprised that we see a similar trend as experiment
- One lattice spacing
	- We should investigate further

this would require phenomenological fits to the low *Q*² data, [Chambers *et al.* arXiv:1702.01513]

Second-order "Feynman-Hellmann" (with external momentum)

Feynman–Hellmann (2nd order)

• Two-point correlator

$$
\int d^3x \, e^{-i\mathbf{p} \cdot \mathbf{x}} \frac{1}{\mathcal{Z}(\lambda)} \int \mathcal{D}\phi \, \chi(x) \chi^{\dagger}(0) e^{-S(\lambda)} = \sum_{N} \frac{|\lambda \langle \Omega | \chi | N, \mathbf{p} \rangle_{\lambda}|^2}{2E_{N, \mathbf{p}}(\lambda)} e^{-E_{N, \mathbf{p}}(\lambda)x_0}
$$
\nIntegral over all fields
\nIntegral over all fields
\nshift of ground-state energy
\n
$$
\simeq A_{\mathbf{p}}(\lambda) e^{-E_{\mathbf{p}}(\lambda)x_0}
$$
\n"Momentum" quantum# at finite field
\n
$$
|N, \mathbf{p} \rangle_{\lambda}
$$
\n
$$
\mathbf{p} \equiv \mathbf{p} + n\mathbf{q}, \ n \in \mathbb{Z}
$$

Feynman–Hellmann (2nd order)

• Differentiate spectral sum

$$
\frac{\partial}{\partial \lambda} \sum_{N} \frac{|\lambda \langle \Omega | \chi | N, \mathbf{p} \rangle_{\lambda}|^{2}}{2E_{N}(\mathbf{p}, \lambda)} e^{-E_{N, \mathbf{p}}(\lambda) x_{4}} = \sum_{N} \left[\frac{\partial A_{N, \mathbf{p}}(\lambda)}{\partial \lambda} - A_{N, \mathbf{p}}(\lambda) x_{4} \frac{\partial E_{N, \mathbf{p}}}{\partial \lambda} \right] e^{-E_{N, \mathbf{p}}(\lambda) x_{4}}
$$

$$
\rightarrow \left[\frac{\partial A_{\mathbf{p}}(\lambda)}{\partial \lambda} - A_{\mathbf{p}}(\lambda) x_{4} \frac{\partial E_{\mathbf{p}}}{\partial \lambda} \right] e^{-E_{\mathbf{p}}(\lambda) x_{4}}
$$

• And again **Not Breit frame,** ω **<1** \Rightarrow **0** $\sqrt{2}$

$$
\frac{\partial^2}{\partial \lambda^2} \left[\cdots \right] = \sum_{N} \left[\frac{\partial^2 A_{N,\mathbf{p}}(\lambda)}{\partial \lambda^2} - 2 \frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial \lambda} x_4 \frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial \lambda} - A_{N,\mathbf{p}}(\lambda) x_4 \frac{\partial^2 E_{N,\mathbf{p}}(\lambda)}{\partial \lambda^2} + A_{N,\mathbf{p}}(\lambda) x_4^2 \left(\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial \lambda} \right)^2 \right]
$$

$$
\rightarrow \left[\frac{\partial^2 A_{\mathbf{p}}(\lambda)}{\partial \lambda^2} - A_{\mathbf{p}}(\lambda) x_4 \frac{\partial^2 E_{\mathbf{p}}}{\partial \lambda^2} \right] e^{-E_{\mathbf{p}}(\lambda) x_4}
$$

Quadratic energy shift
Watch for temporal enhancement $\sim x_4 e^{-E_{\mathbf{p}} x_4}$

 $\overline{}$

Feynman–Hellmann (2nd order)

• **Differentiate path integral**

$$
\frac{\partial}{\partial \lambda} \int d^3 x \, e^{-i \mathbf{p} \cdot \mathbf{x}} \frac{1}{\mathcal{Z}(\lambda)} \int \mathcal{D} \phi \, \chi(x) \chi^{\dagger}(0) e^{-S(\lambda)} \n= \int d^3 x \, e^{-i \mathbf{p} \cdot \mathbf{x}} \frac{1}{\mathcal{Z}(\lambda)} \int \mathcal{D} \phi \, \chi(x) \chi^{\dagger}(0) \left[-\frac{\partial S}{\partial \lambda} - \frac{1}{\mathcal{Z}(\lambda)} \frac{\partial \mathcal{Z}}{\partial \lambda} \right] e^{-S(\lambda)},
$$

"Disconnected" operator insertions; drop for simplicity

• Differentiate again, take zero-field limit and note: $\frac{\partial^2 S}{\partial x^2}$ $\frac{\partial}{\partial \lambda^2} = 0$

$$
\frac{\partial^2}{\partial \lambda^2} \left[\cdots \right] \bigg|_{\lambda=0} = \int d^3x \, e^{-i \mathbf{p} \cdot \mathbf{x}} \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi \, \chi(x) \chi^{\dagger}(0) \left(\frac{\partial S}{\partial \lambda} \right)^2 e^{-S_0}
$$

*^d*⁴*^y* 2 cos(q*.*y)*q*(*y*)*µq*(*y*) Current insertions integrated over 4-volume

$$
\frac{\partial S}{\partial \lambda} = \int d^4 y \, 2 \cos(\mathbf{q}.\mathbf{y}) \overline{q}(y) \gamma_\mu q(y)
$$

Field time orderings

• Current insertion possibilities • Both currents "outside" (together) • Both currents "outside" (opposite) • One current "inside" *t* $\langle \chi(x)\chi^{\dagger}(0)T(J(y)J(z))\rangle$, $y_4, z_4 < 0 < x_4$ $\sim e^{-E_X x_4}, \quad E_X \gtrsim E_p$ $J(y_4)$ $J(z_4)$ \vdots $\chi^{\dagger}(0)$ \vdots $\chi(x_4)$ $E_X = E_p \Rightarrow$ changes amplitudes $J(y_4)$ \vdots $\chi^{\dagger}(0)$ \vdots $J(z_4)$ \vdots $\chi(x_4)$ linear energy shift $\langle \chi(x)J(z)\chi^{\dagger}(0)J(y)\rangle$, $y_4 < 0 < z_4 < x_4$ $\sim \frac{\partial E_{\mathbf{p}}}{\partial \lambda} x_4 e^{-E_{\mathbf{p}} x_4} \to 0$ $J(y_4)$: $\chi^{\dagger}(0)$: $\chi(x_4)$: $J(z_4)$ $\langle J(z)\chi(x)\chi^{\dagger}(0)J(y)\rangle$, $y_4 < 0 < x_4 < z_4$ $\sim e^{-E_X x_4}, \quad E_X \gtrsim E_p$

(and changed amplitude)

ignore finite T

Field time orderings

• Both currents between creation/annihilation

$$
\begin{array}{c}\n\vdots \\
\chi^{\dagger}(0) \\
\vdots \\
\chi^{\dagger}(0)\n\end{array}\n\qquad\n\begin{array}{c}\nJ(y_4) \\
J(z_4) \\
\vdots \\
\chi(x_4)\n\end{array}
$$

$$
\int d^3x \, e^{-i\mathbf{p} \cdot \mathbf{x}} \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi \, \chi(x) \chi^{\dagger}(0) \left(\frac{\partial S}{\partial \lambda}\right)^2 e^{-S_0}
$$
\n
$$
= \sum_{N,N'} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{N,\mathbf{k}}} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2E_{N',\mathbf{k}'}} \int d^3x \int d^4z \int d^4y \, e^{-i\mathbf{p} \cdot \mathbf{x}} \left(e^{i\mathbf{q} \cdot \mathbf{z}} + e^{-i\mathbf{q} \cdot \mathbf{z}}\right) \left(e^{i\mathbf{q} \cdot \mathbf{y}} + e^{-i\mathbf{q} \cdot \mathbf{y}}\right)
$$
\n
$$
\times \langle \Omega | \chi(x) | N, \mathbf{k} \rangle \langle \mathbf{k} | T J(z) J(y) | \mathbf{k}' \rangle \langle N', \mathbf{k}' | \chi^{\dagger}(0) | \Omega \rangle,
$$
\n
$$
\vdots
$$
\n
$$
\to \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}} x_4 e^{-E_{\mathbf{p}} x_4} \int d^4\xi \left(e^{i\mathbf{q} \cdot \xi} + e^{-i\mathbf{q} \cdot \xi}\right) \langle \mathbf{p} | T J(\xi) J(0) | \mathbf{p} \rangle
$$

Note $q_4 = 0 \Rightarrow \mathbf{q}.\boldsymbol{\xi} = q.\boldsymbol{\xi}$

Final steps

- Equate spectral sum and path integral representation
	- Asymptotically, we have

$$
-A_{\mathbf{p}}\frac{\partial^2 E_{\mathbf{p}}}{\partial \lambda^2}x_4e^{-E_{\mathbf{p}}x_4} = \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}}x_4e^{-E_{\mathbf{p}}x_4}\int d^4\xi \left(e^{iq.\xi} + e^{-iq.\xi}\right)\langle \mathbf{p}|TJ(\xi)J(0)|\mathbf{p}\rangle
$$

$$
\frac{\partial^2 E_{\mathbf{p}}}{\partial \lambda^2} = -\frac{1}{2E_{\mathbf{p}}} \int d^4 \xi \left(e^{iq.\xi} + e^{-iq.\xi} \right) \langle \mathbf{p} | T J(\xi) J(0) | \mathbf{p} \rangle
$$