[Towards] structure functions from lattice QCD

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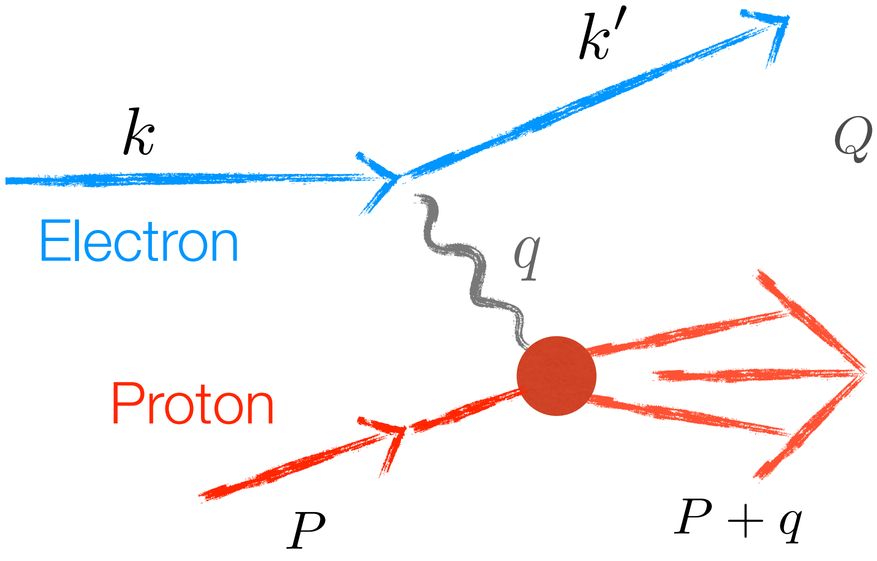
Deep inelastic structure of the proton

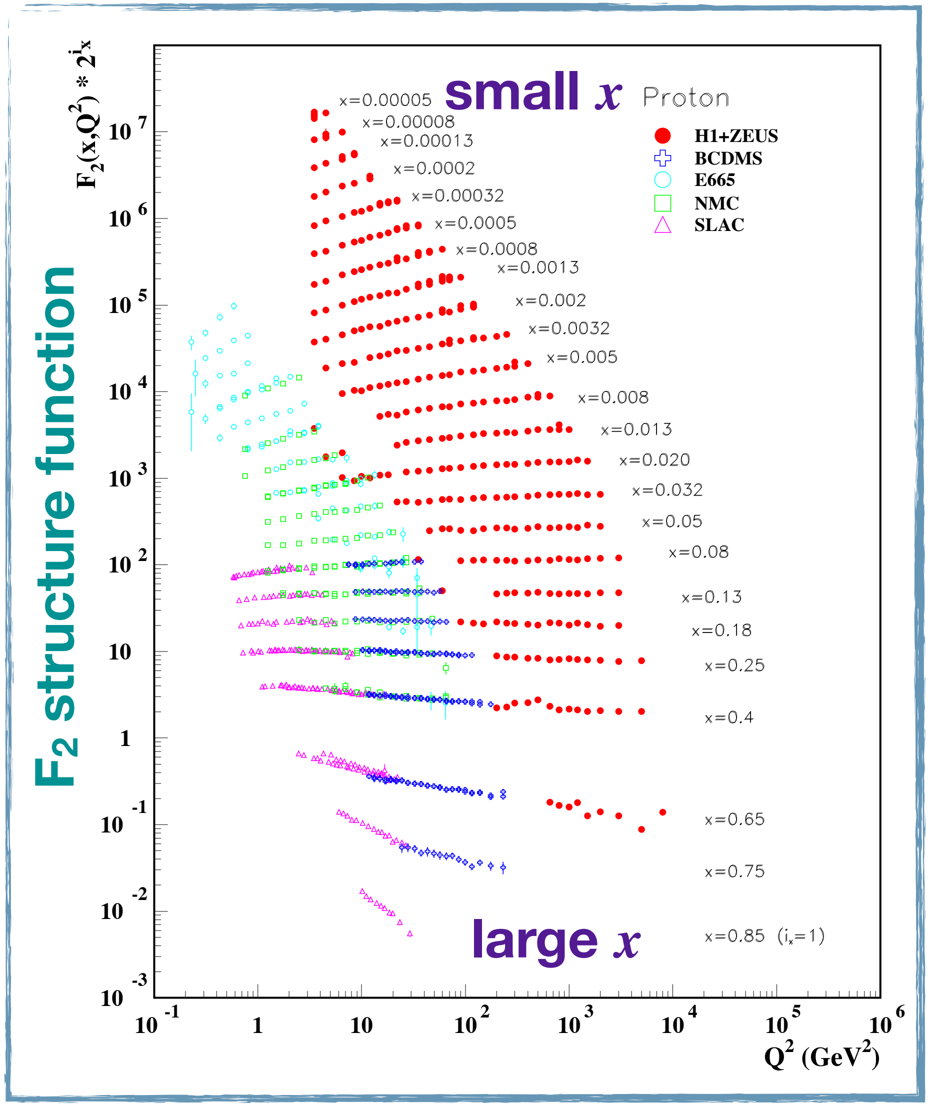


$$Q^2 = -q^2$$

Parton model

Scatter from non-interacting quarks
Bjorken scaling variable
[~longitudinal momentum fraction]

$$x = \frac{Q^2}{2P.q}$$



Almost Bjorken scaling

Deep-inelastic scattering

Slow deviations from scaling described by perturbative QCD

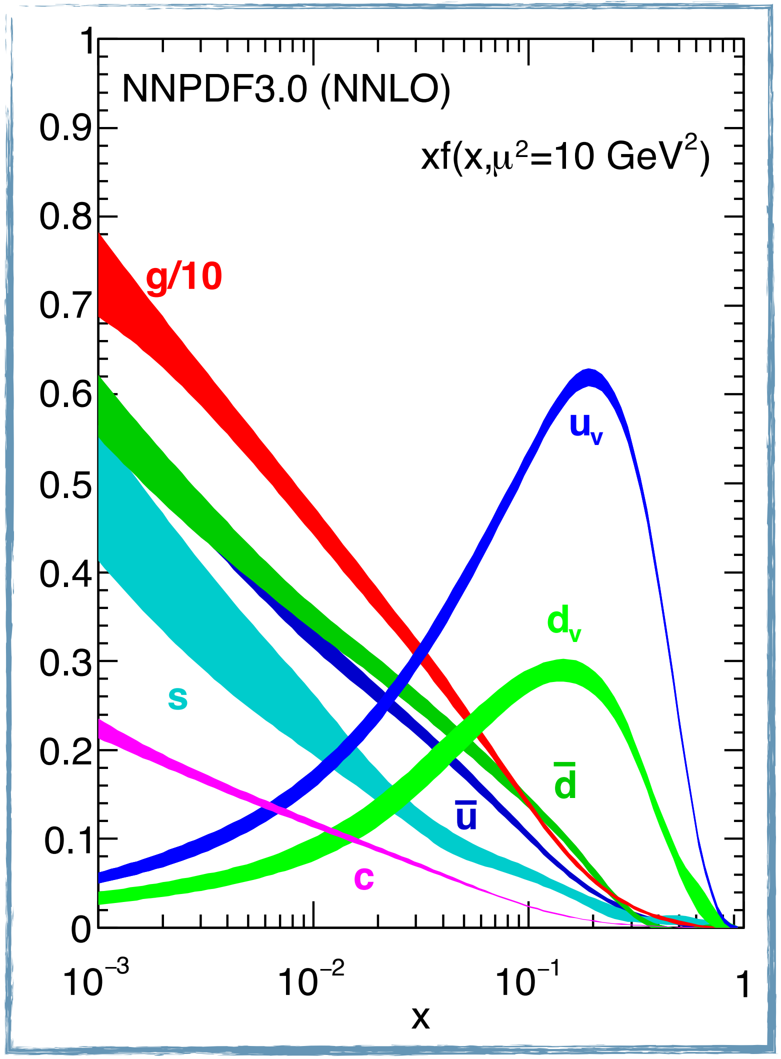
QCD: the densest matter in nature

for Zein-Eddine

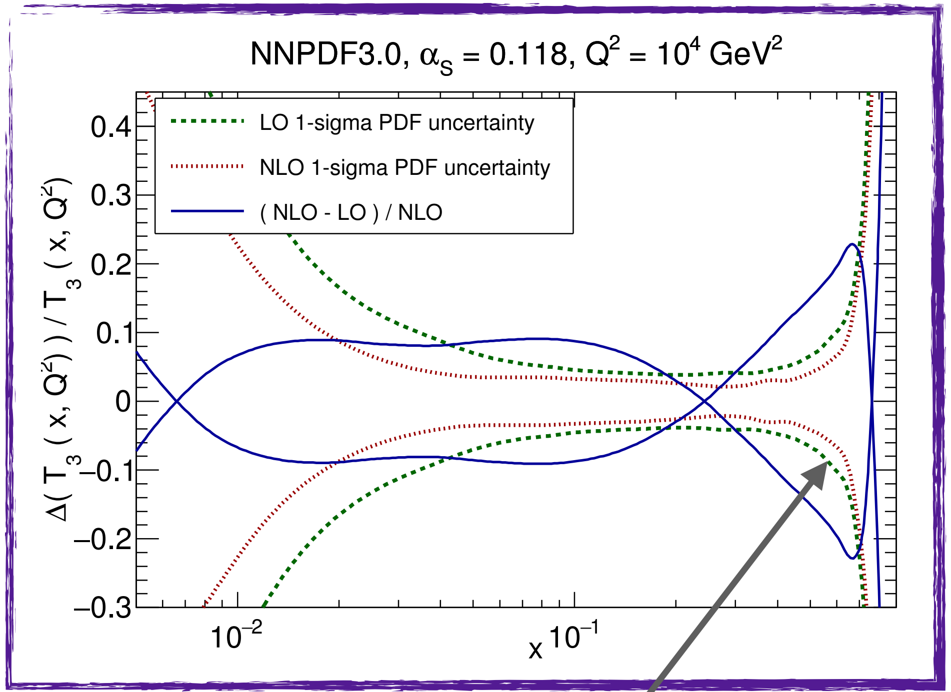
TABLE 10–1
Densities of Substances[†]

Substance	Density, ρ (kg/m^3)
Solids	
Aluminum	2.70×10^3
Iron and steel	7.8×10^3
Copper	8.9×10^3
Lead	11.3×10^3
Gold	19.3×10^3
Concrete	2.3×10^3
Granite	2.7×10^3
Wood (typical)	$0.3 - 0.9 \times 10^3$
Glass, common	$2.4 - 2.8 \times 10^3$
Ice (H_2O)	0.917×10^3
Bone	$1.7 - 2.0 \times 10^3$
Liquids	
Water (4°C)	1.00×10^3
Blood, plasma	1.03×10^3
Blood, whole	1.05×10^3
Sea water	1.025×10^3
Mercury	13.6×10^3
Alcohol, ethyl	0.79×10^3
Gasoline	0.68×10^3
Gases	
Air	1.29
Helium	0.179
Carbon dioxide	1.98
Water (steam) (100°C)	0.598

[†] Densities are given at 0°C and 1 atm pressure unless otherwise specified.



Isovector quark distributions



Relative uncertainties diverge beyond $x \sim 0.6$:
Opportunity for lattice to contribute

Parton distributions

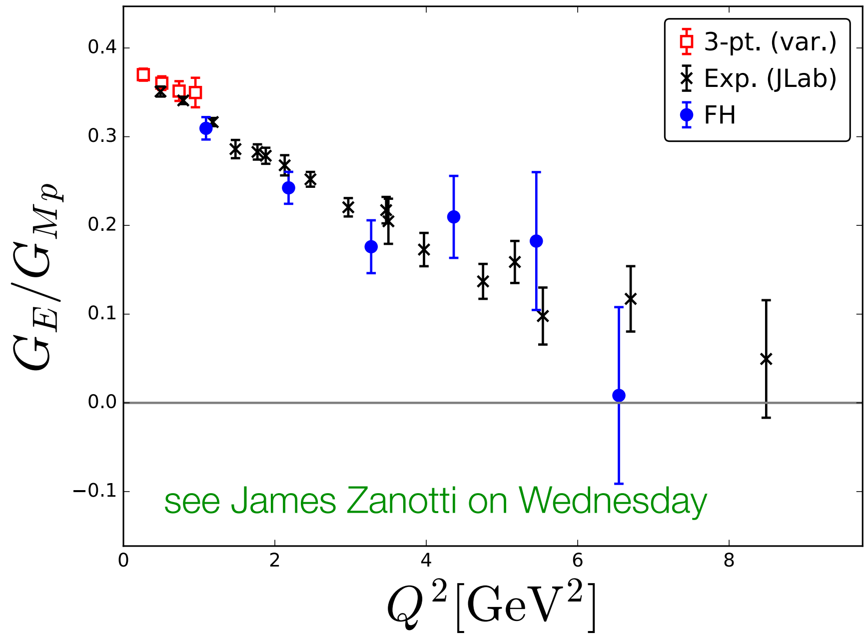
In principle, these could be determined from QCD
Challenging so far!

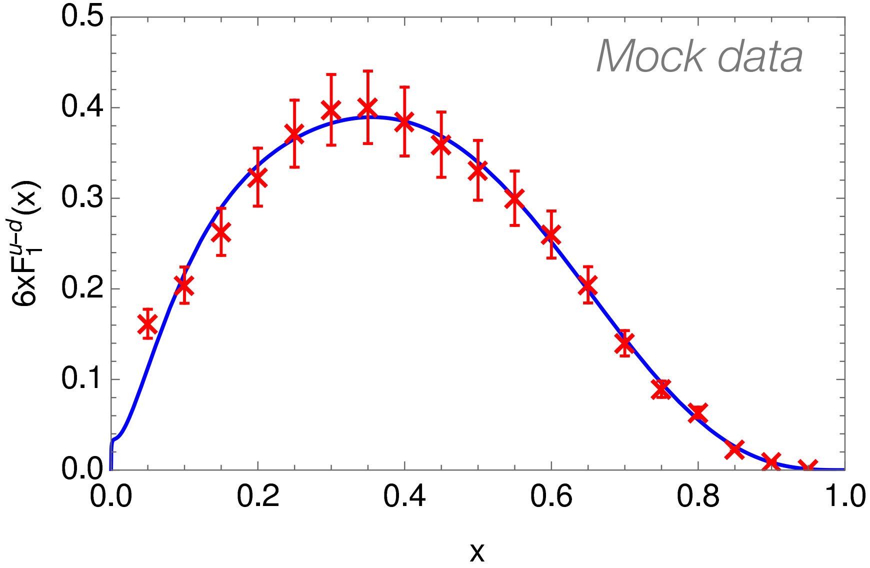
Outline

- Structure functions and the **operator product expansion**

- **Feynman-Hellmann** (FH) approach to hadron structure on the lattice

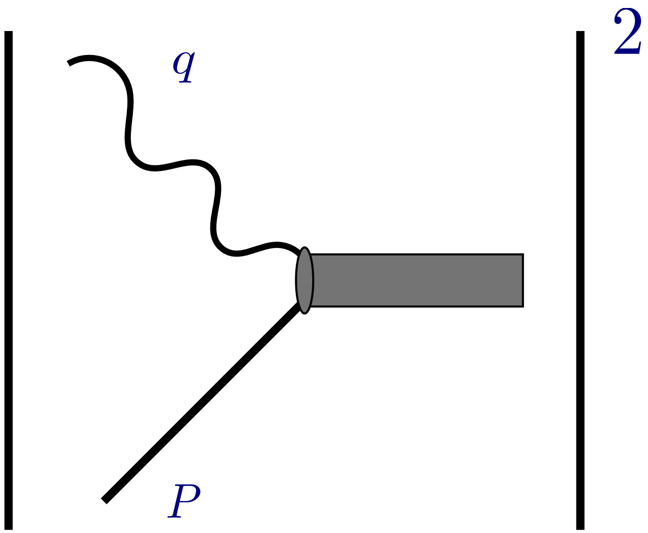
- **Compton amplitude** on the lattice
  - Toy model test
  - Exploratory numerical study





Structure functions and the operator product expansion

Inelastic scattering



Cross section ~ Hadron tensor

$$W_{\mu\nu} \sim \int d^4x\, e^{iq.x} \langle p|\, [J_\mu(x), J_\nu(0)]\, |p\rangle$$

Structure functions $F_{1,2}(P.q, Q^2)$

$$\boxed{F_i = \frac{1}{2\pi} \mathrm{Im}\, T_i}$$

Forward Compton amplitude

$$T_{\mu\nu} \sim \int d^4x\, e^{iq.x} \langle p| \mathrm{T} J_\mu(x) J_\nu(0) |p\rangle$$

Lorentz-scalar functions $T_{1,2}(P.q, Q^2)$

(Virtual) Compton amplitude

- Forward Compton amplitude

$$T^{\mu\nu}(p,q) = \rho_{ss'} \int d^4x \, e^{iq.x} \langle p,s| \mathrm{T}\left\{ J^\mu(x) J^\nu(0) \right\} |p,s\rangle$$

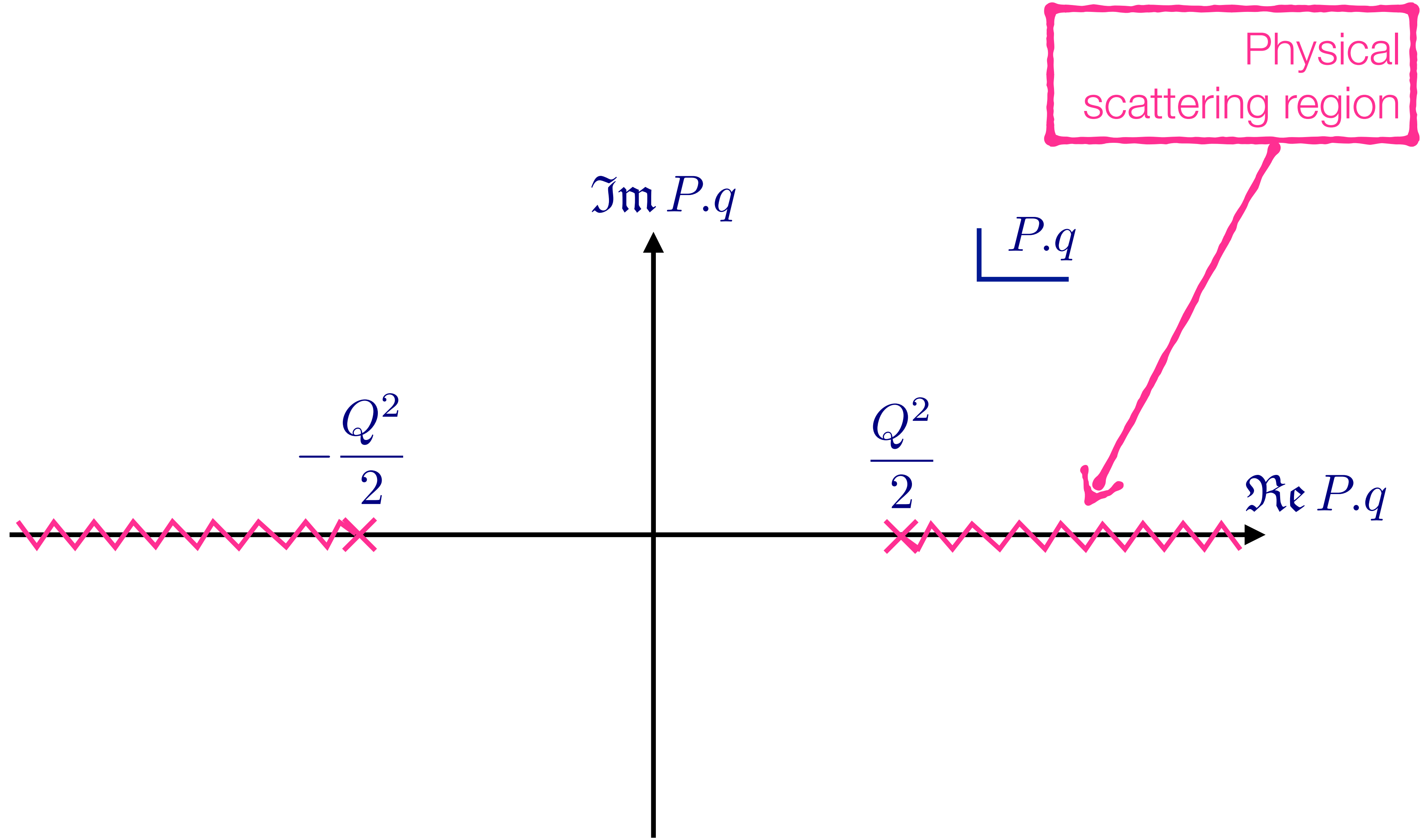
$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) T_1(P.q, Q^2) + \frac{1}{P.q} \left(p^\mu - \frac{P.q}{q^2} q_\mu \right) \left(p^\nu - \frac{P.q}{q^2} q_\nu \right) T_2(P.q, Q^2)$$

- Looking ahead to lattice results shown at end, consider simple case

$$\mu = \nu = 3, \;\; q_3 = 0, \;\; P_3 = 0$$

$$\Rightarrow T_{33}(P,q) = T_1(P.q, Q^2)$$

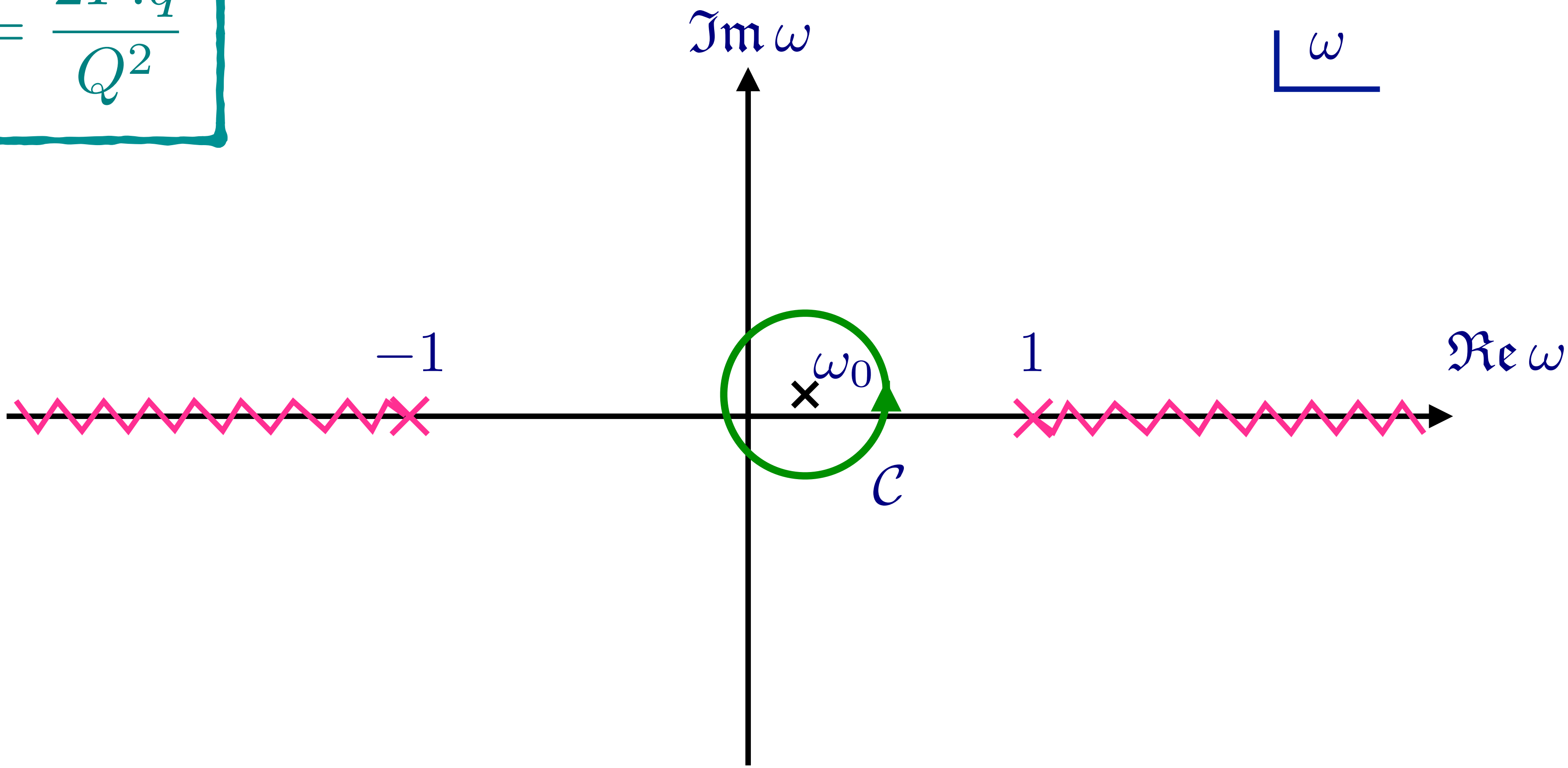
Analytic structure of Compton amplitude



Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$

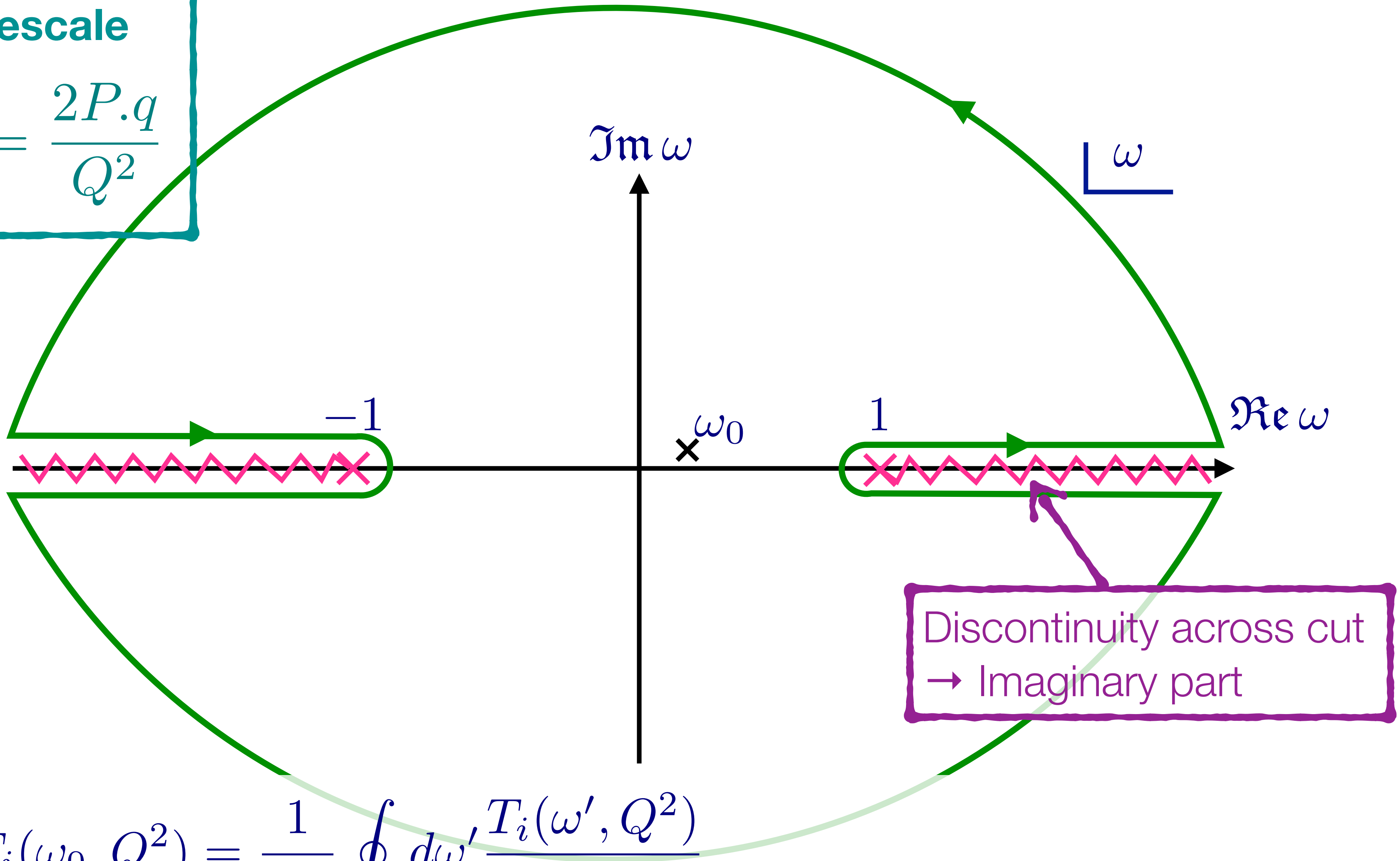


$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i} \oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$

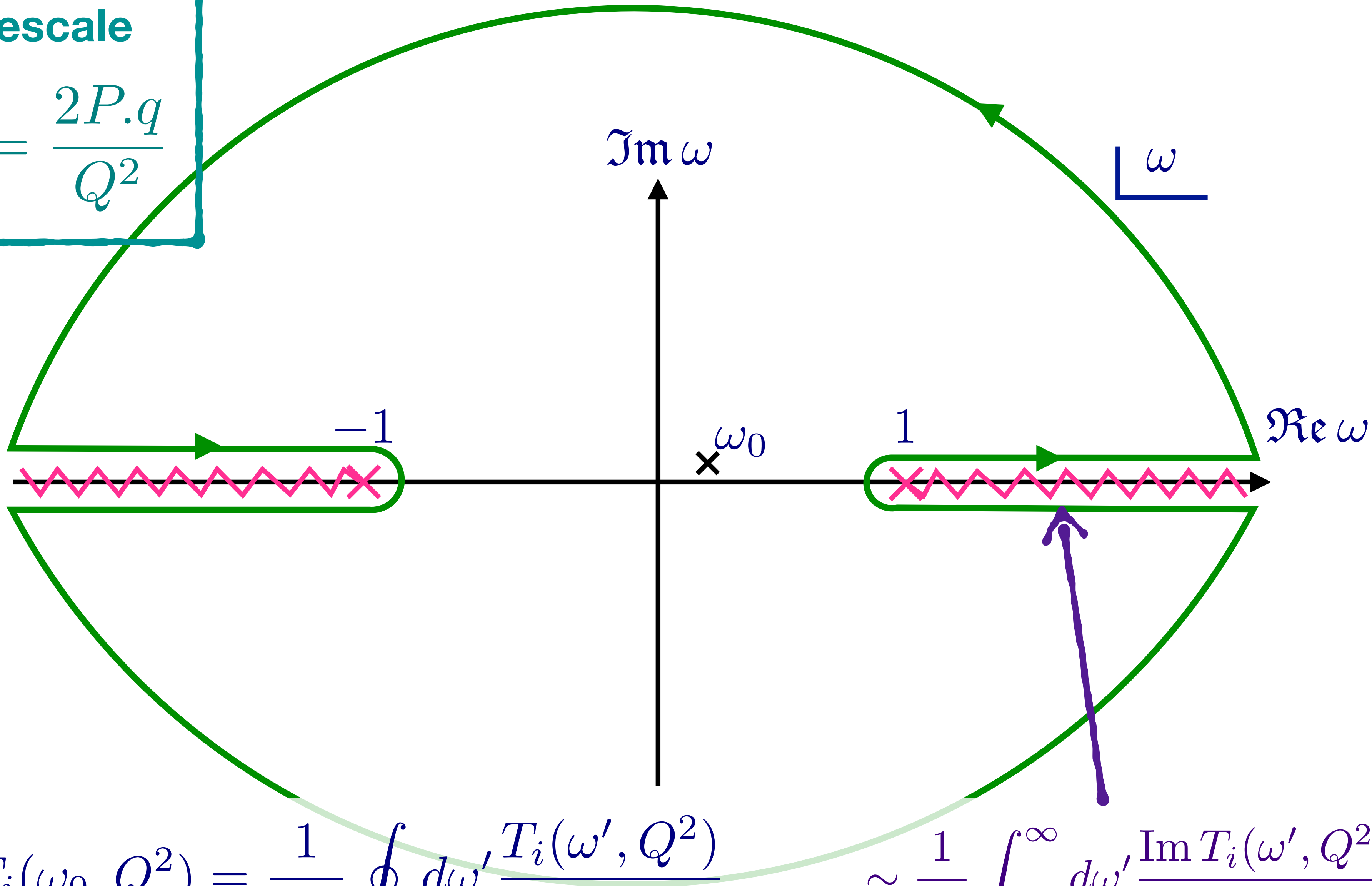


$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Analytic structure of Compton amplitude

Rescale

$$\omega = \frac{2P.q}{Q^2}$$



$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0} \sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\, T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Moments of structure functions

- Re-express integral over familiar Bjorken x:

$$T_1(\omega, Q^2) - T_1(\omega, 0) = \frac{4\omega^2}{2\pi} \int_1^\infty d\omega' \frac{\mathrm{Im}\, T_1(\omega', Q^2)}{\omega'(\omega^2 - \omega'^2)} = 4\omega^2 \int_0^1 dx\, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}$$

Subtraction term:
Cottingham sum rule; Muonic hydrogen.
Recently, see also:
Agadjanov, Meißner & Rusetsky, PRD(2017),
Hill & Paz, PRD(2017), …

$x = 1/\omega'$

Taylor expansion

- **Moments of structure functions**

$$T_1(\omega, Q^2) - T_1(\omega, 0) = \sum_{j=1}^{\infty} 4\omega^{2j} \int_0^1 dx\, x^{2j-1} F_1(x, Q^2)$$

Lattice QCD: Traditional way

$$T_1(\omega, Q^2) - T_1(\omega, 0) = \sum_{j=1}^{\infty} 4\omega^{2j} \int_0^1 dx\, x^{2j-1} F_1(x, Q^2)$$

- Matrix elements of local twist-2 operators:

$$\langle P|\mathcal{O}^{\{\nu_1 \ldots \nu_n\}}|P\rangle = 2\, a(n,\mu) P^{\nu_1} \ldots P^{\nu_n} - \text{traces}$$

$$a(n,\mu) = \int_0^1 dx\, x^{2n-1} F(x,\mu)$$

$$\mathcal{O}^{\{\nu_1 \ldots \nu_n\}} = \overline{\psi}(0)\gamma^{\nu_1} D^{\nu_2} \ldots D^{\nu_n} \psi(0)$$

Operator mixing on the lattice prohibits the study of operators with increasing numbers of derivatives:

Typically only access lowest moment (e.g. quark momentum fractions)

Study Compton amplitude directly on lattice

$$T_1(\omega, Q^2) - T_1(\omega, 0) = 4\omega^2 \int_0^1 dx\, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}$$

4-point function on the lattice?
Preferably not

Feynman-Hellmann theorem in lattice QCD

Matrix elements from “Feynman–Hellmann”

- Feynman–Hellmann in quantum mechanics:

$$\frac{dE_n}{d\lambda} = \langle n|\frac{\partial H}{\partial \lambda}|n\rangle$$

  - matrix elements of the derivative of the Hamiltonian determined by derivative of corresponding energy eigenstates

- Lattice QCD: evaluate energy shifts with respect to weak external fields

- Modify action with external field:

$$S \to S + \lambda \int \mathrm{d}^4x\, \mathcal{O}(x)$$

real parameter

local operator, e.g. $\bar{q}(x)\gamma_5\gamma_3 q(x)$

- Calculation of matrix element ≡ hadron spectroscopy [2-pt functions only]

$$\frac{\partial E_H(\lambda)}{\partial \lambda} = \frac{1}{2E_H(\lambda)}\langle H|\mathcal{O}|H\rangle$$

Spin content [connected]

- Modify action

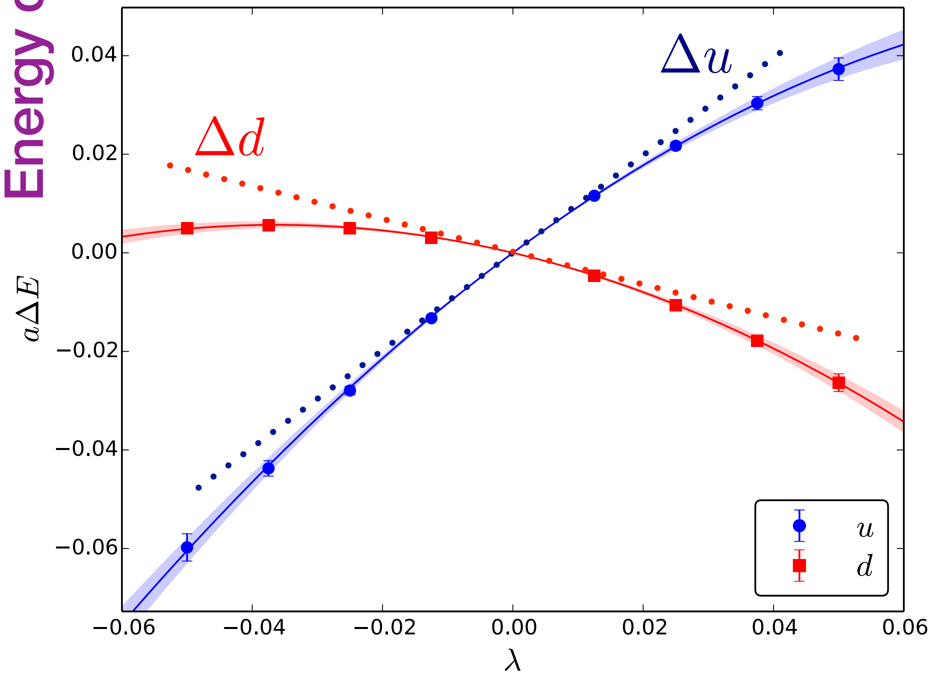
$$S \to S + \lambda \sum_x \bar{q}(x) i\gamma_5\gamma_3 q(x)$$

- Nucleon energy shift isolates spin content

$$\frac{\partial E_N(\lambda)}{\partial \lambda} = \frac{1}{2M_N}\langle N|\bar{q}i\gamma_5\gamma_3 q|N\rangle$$
$$= \Delta q$$

Slope → matrix element

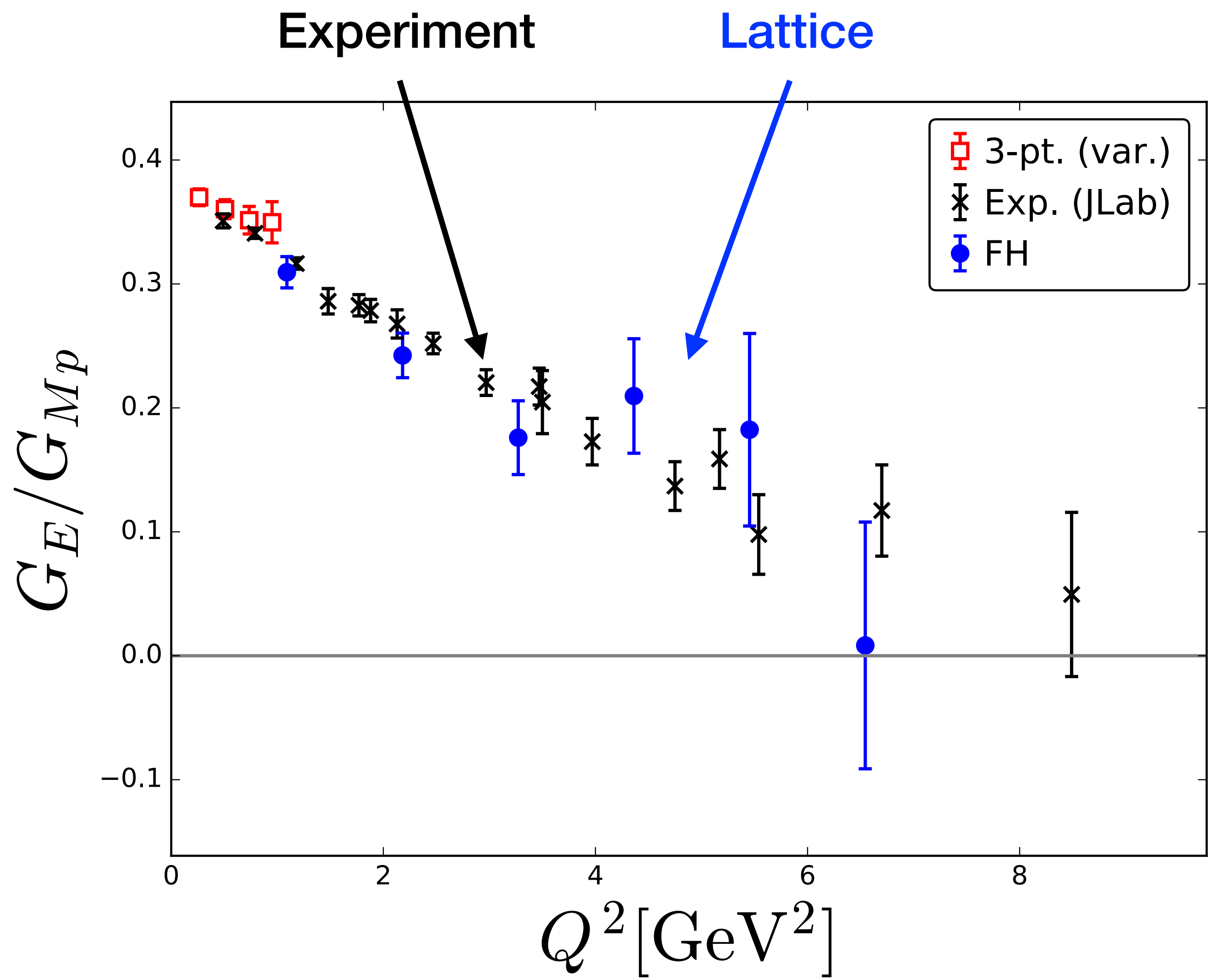
Energy difference



Strength of external field

[Chambers *et al*. PRD(2014)]

3-pt function → 2-pt function



[Chambers *et al*. arXiv:1702.01513]

Proton form factors

See James Zanotti on Wednesday

Feynman–Hellmann (2nd order):
Study Compton amplitude directly

$$T_1(\omega, Q^2) - T_1(\omega, 0) = 4\omega^2 \int_0^1 dx\, x \frac{F_1(x, Q^2)}{1 - (\omega x)^2}$$

Feynman–Hellman (2nd order)

- Field theory version of 2nd order perturbation theory:

$$E = E_0 + \lambda \langle N|V|N\rangle + \lambda^2 \sum_{X\neq N} \frac{\langle N|V|X\rangle\langle X|V|N\rangle}{E_0 - E_X} + \ldots$$

Only get a linear term for elastic case $\boldsymbol{\omega}$=1

$E_0 < E_X$
Intermediate states cannot go on-shell for $\boldsymbol{\omega}$<1

- Final result. We study second-order perturbation on the lattice

$$\frac{\partial^2 E_{\mathbf{p}}}{\partial \lambda^2} = -\frac{1}{2E_{\mathbf{p}}} \int d^4\xi \left(e^{iq.\xi} + e^{-iq.\xi}\right) \langle \mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$

see backup slides, or
RDY, presentation @Lattice 2017;
Somfleth et al. … soon

Test case:
Compton amplitude → SFs

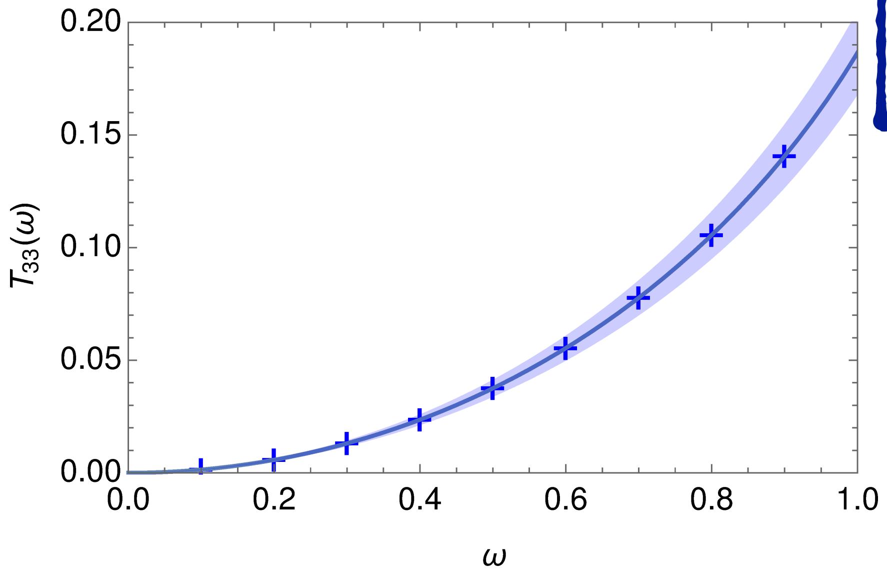
Taylor expansion

- Consider moments of structure function

$$\mu_{2m-1} = \int_0^1 dx\, x^{2m-1} F_1(x)$$

- Series expansion of Compton amplitude

$$T_{33}(\omega)/4 = \omega^2 \mu_1 + \omega^4 \mu_3 + \omega^6 \mu_5 + \ldots$$



Compton amplitude in unphysical region

input PDFs: MSTW(LO)

“Inversion”

- Discrete approximation to structure function $F_1(x)$

- Consider discretised integral

$$T_{33}(\omega_n) = \sum_{m=1}^{M} K_{nm} F_1(x_m), \quad x_m = \frac{m}{M} \qquad K_{nm} = \frac{4\omega_n^2 x_m}{1-(\omega_n x_n)^2}$$

$$N < M$$

- Use singular value decomposition to invert $N{\times}M$ matrix

$$K = U\left[\mathrm{diag}(w_1, \ldots, w_{N'}, w_{N'+1} \ldots, w_N)\right] V^\top$$

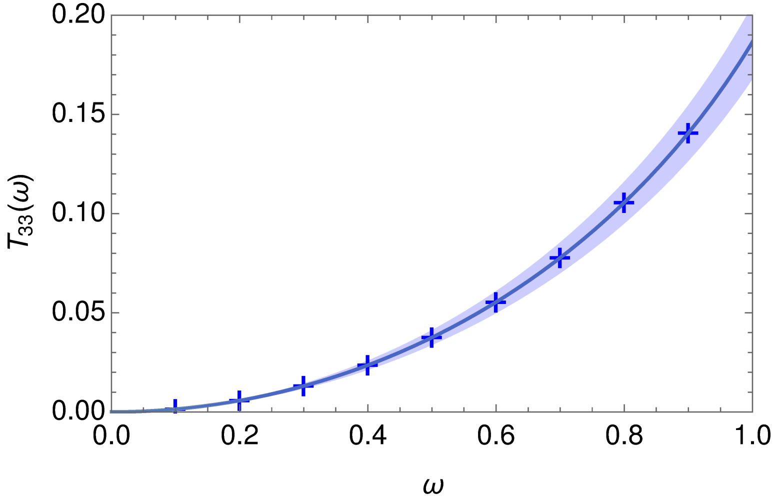
$N{\times}M$ “diag”

$$w_{N'+1} \ldots, w_N \simeq 0,\ N' \leq N$$

  - Pseudoinverse

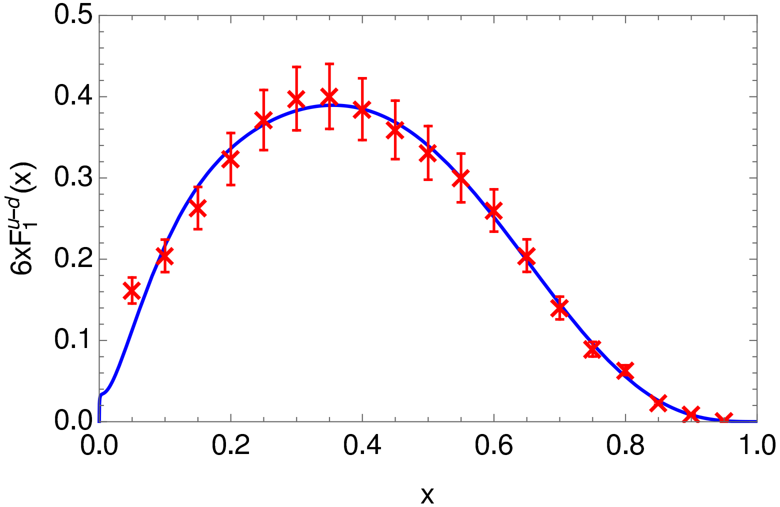
$$K^{-1} = V\left[\mathrm{diag}(1/\omega_1, \ldots, 1/\omega_{N'}, 0, \ldots, 0)\right] U^\top$$

Input



"Pseudo-inverse"

Output



$$T_{33} = 4\omega \int_0^1 dx \frac{\omega x}{1-(\omega x)^2} F_1^{u-d}(x)$$

$$2xF_1^{u-d}(x) = \frac{1}{3} x \left[u(x) - d(x)\right]$$

input PDFs: MSTW(LO)

Chambers et al., PRL(2017)

Toy model test

Numerical investigation

Numerical set-up

Single external momenta

$$\vec{q} = (3, 5, 0)\,\frac{2\pi}{L}$$

$$\omega = \frac{2P.q}{Q^2} = \frac{2\vec{P}.\vec{q}}{\vec{q}^2}$$

$q_4 = 0$

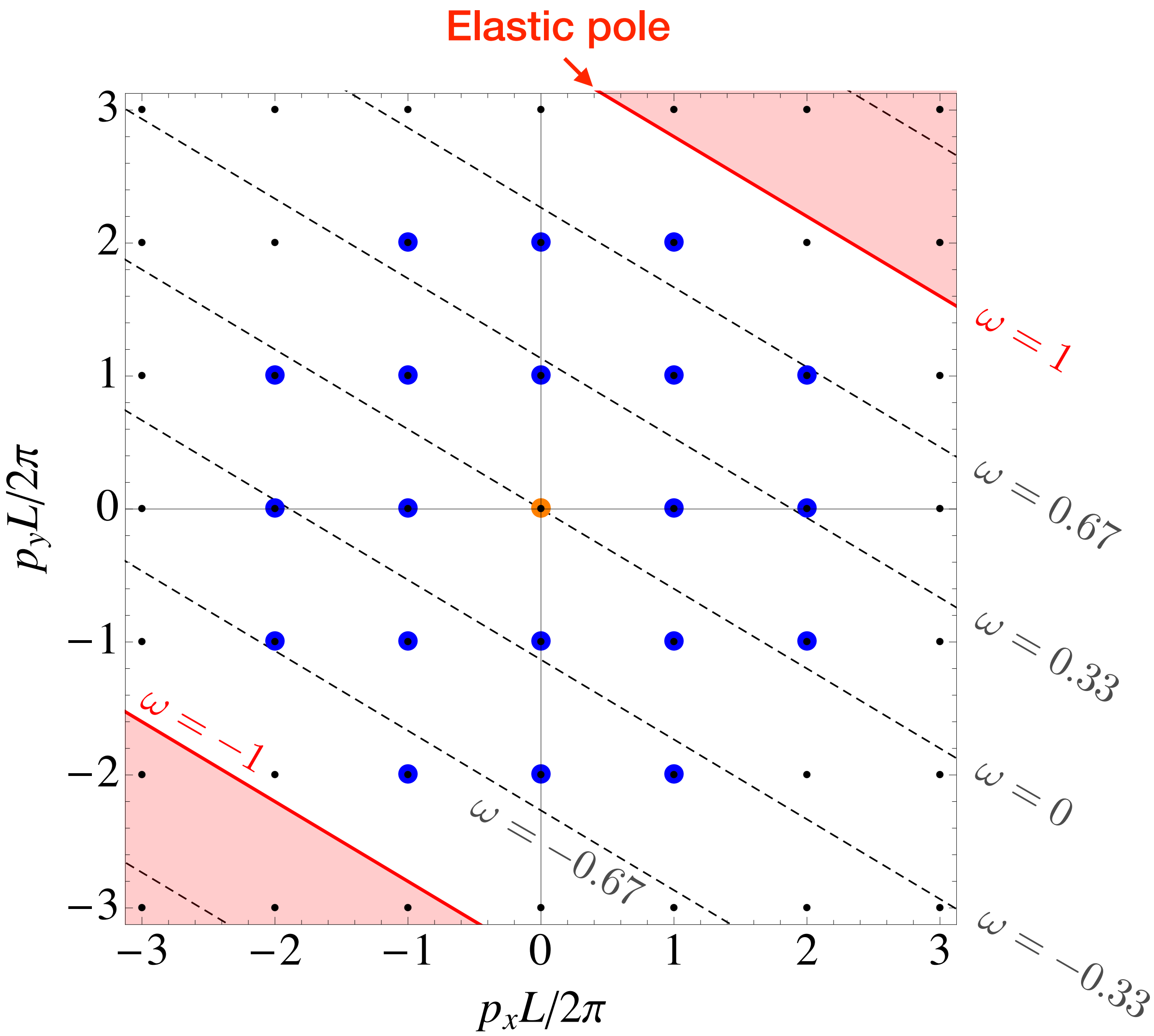
Lattice specs

SU(3) symmetric point:

$m_\pi \simeq 400\,\mathrm{MeV}$

32^3x64, a≈0.074 fm

O(900) configs



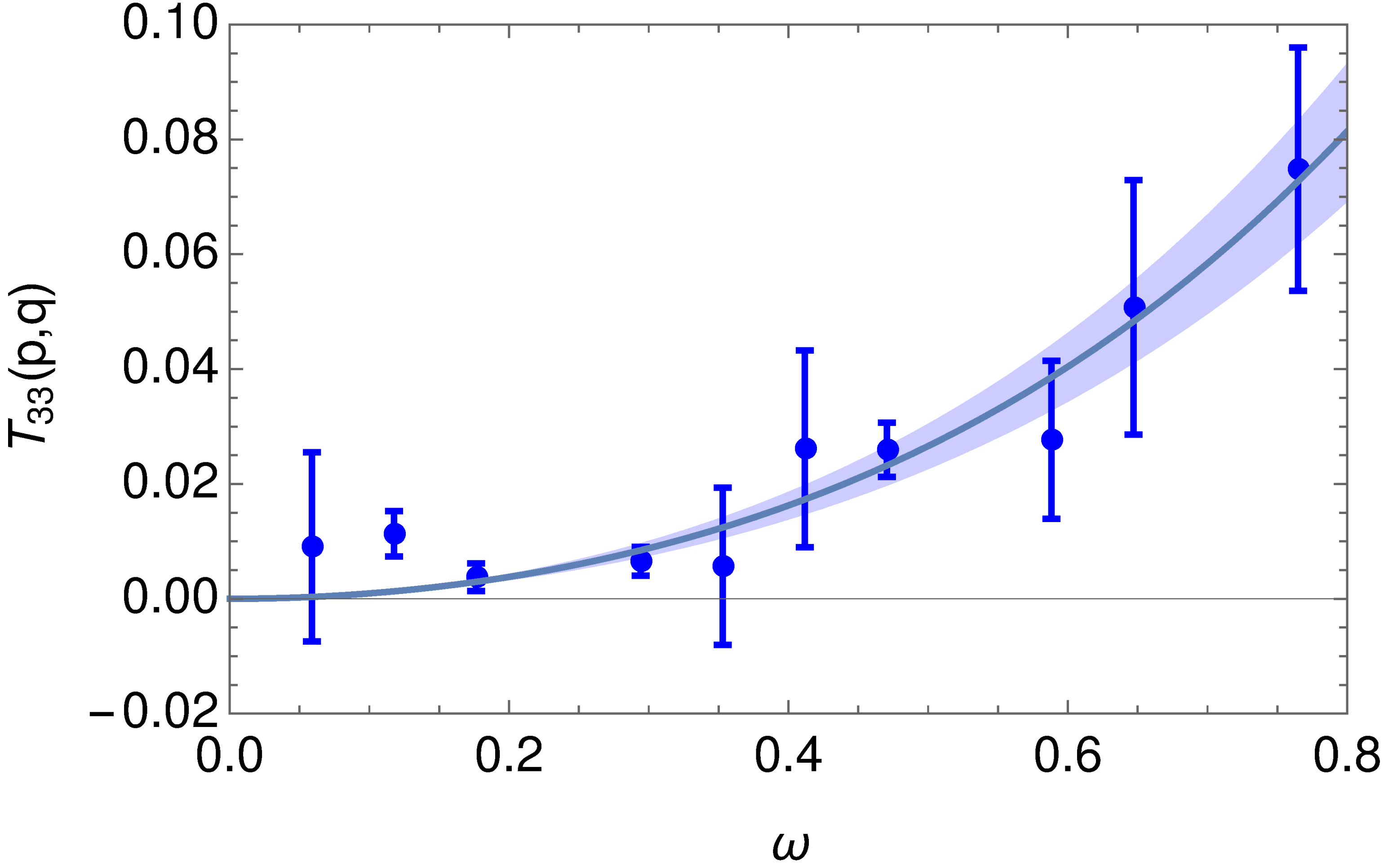
Blue dots: different nucleon Fourier momenta

Lattice kinematics

Broad coverage of ω from single calculation (computationally "cheap")

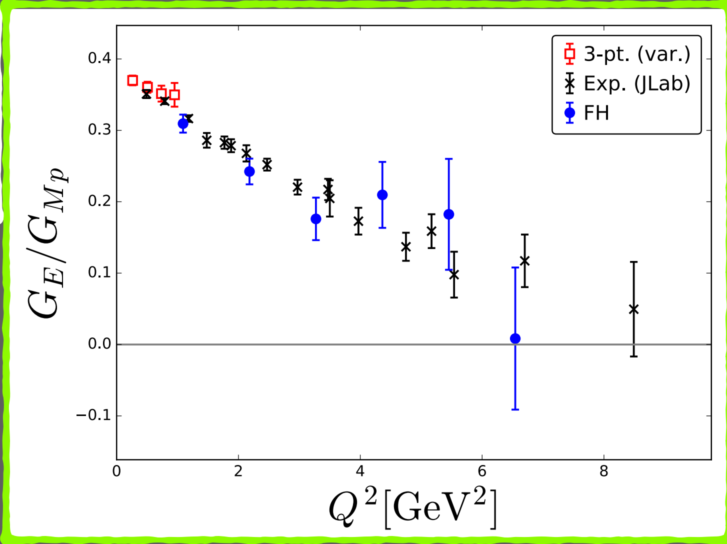
Numerical test: Lattice results

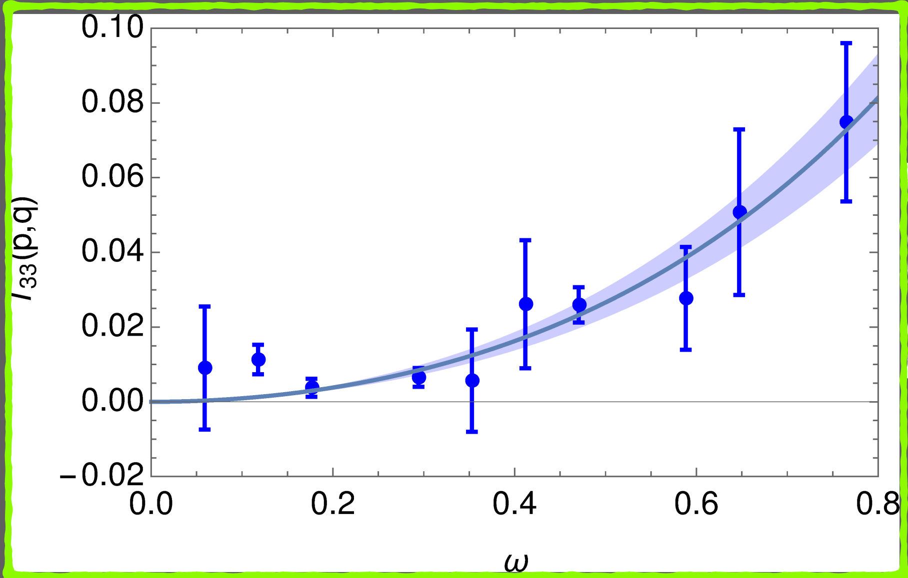
Compton amplitude from quadratic energy shift
(subtraction term removed)



Chambers et al., PRL(2017)

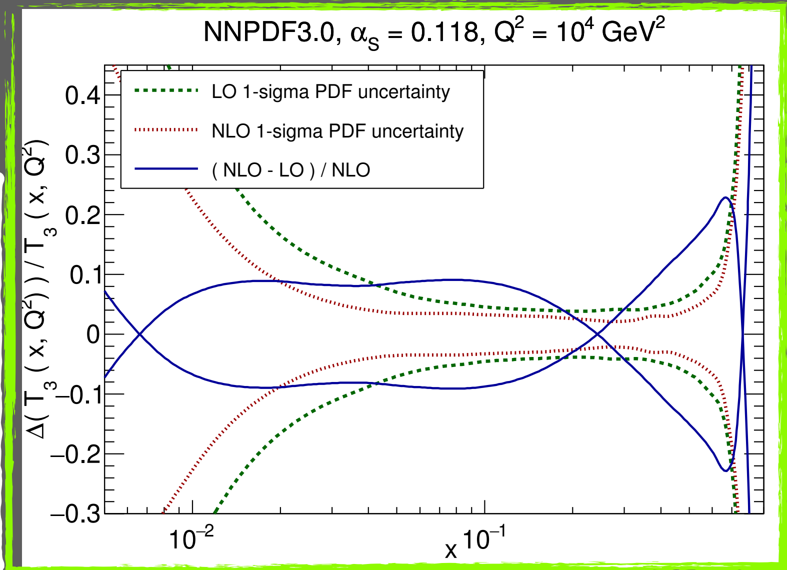
New access to form factors at large momenta





(Virtual) Compton amplitude accessible on the lattice

Nonperturbative constraint on hadronic structure functions
→ PDFs + higher twist

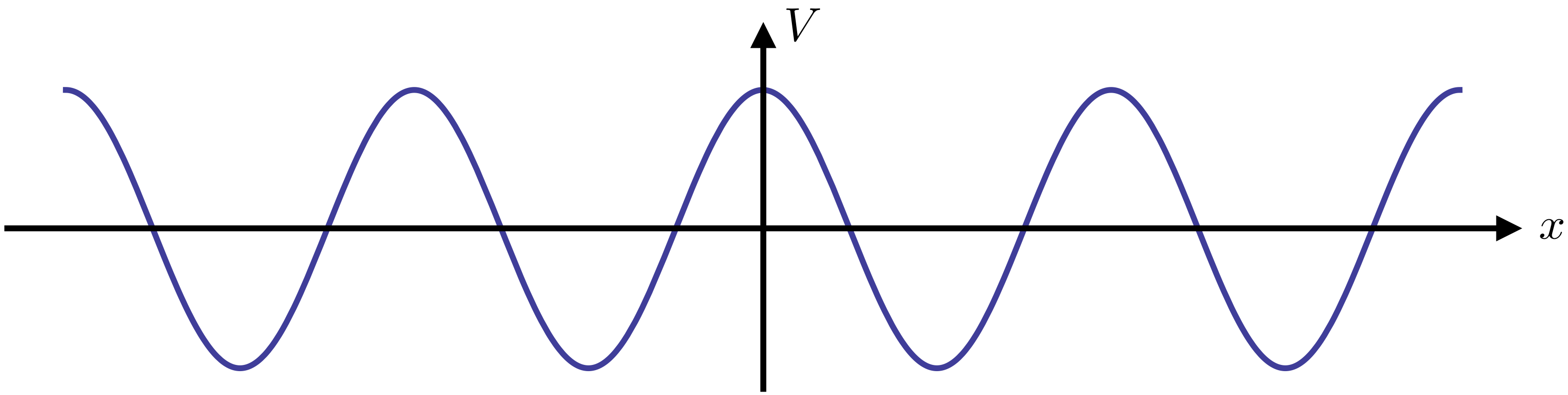


Back-up slides

Feynman–Hellman with momentum transfer

Warm up: Periodic potential, 1-D QM

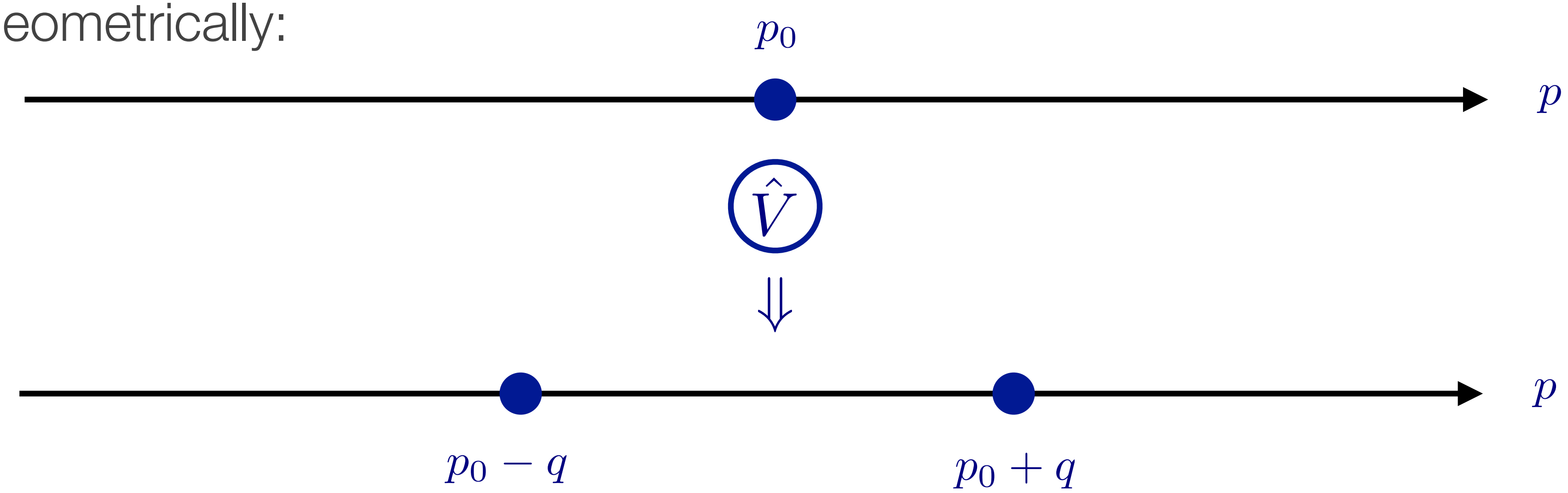
- Almost free particle $H_0|p\rangle = \frac{p^2}{2m}|p\rangle$
- Subject to weak external periodic potential $V(x) = 2\lambda V_0 \cos(qx)$



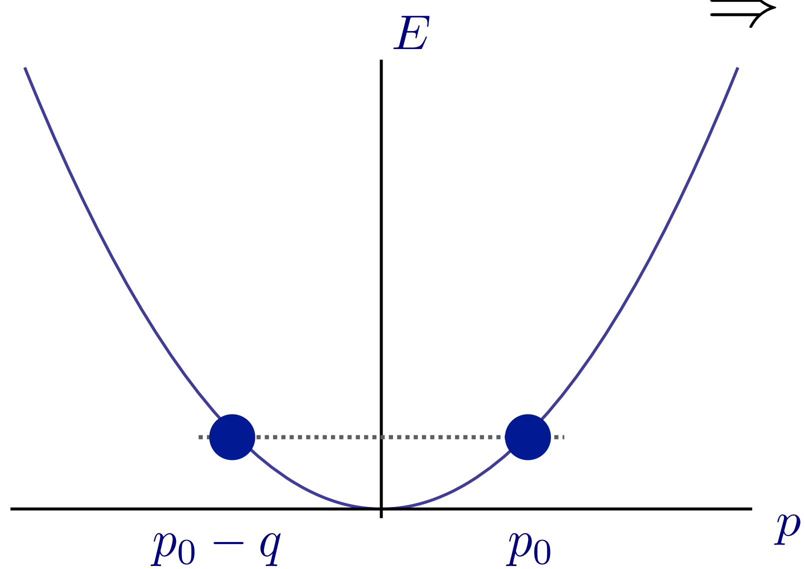
$$\hat{V}|p\rangle = \lambda V_0|p+q\rangle + \lambda V_0|p-q\rangle$$

Warm up: Periodic potential, 1-D QM

- Geometrically:



$$\Rightarrow \langle p|\hat{V}|p\rangle = 0$$



No first order energy shifts?

If $p_0 = \pm q/2$
$\Rightarrow$ transition between degenerate states

Degenerate perturbation theory

- Exact degeneracy: $p = q/2$

$$H = \begin{pmatrix} \frac{p^2}{2m} & \lambda V_0 \\ \lambda V_0 & \frac{p^2}{2m} \end{pmatrix}$$

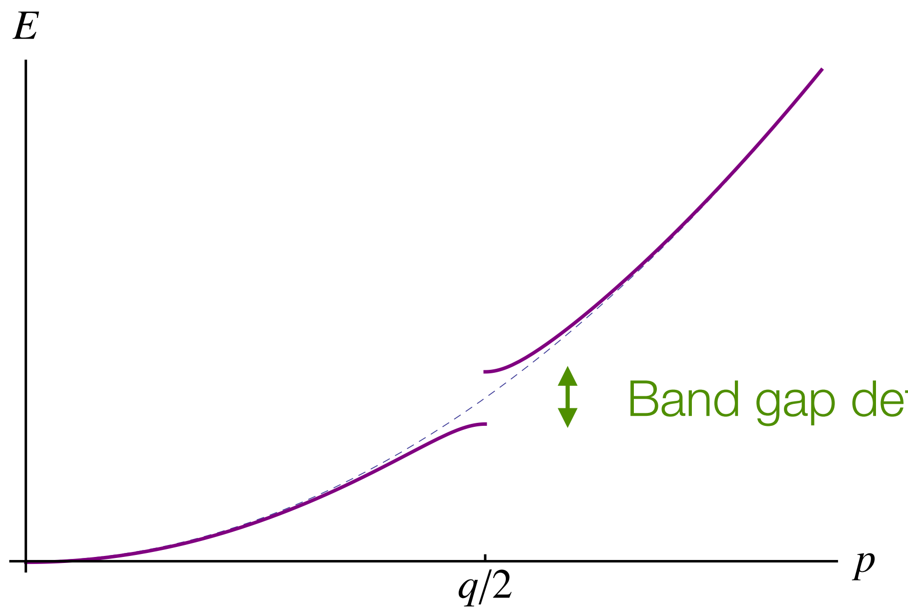
$$H\left\{|q/2\rangle \pm |-q/2\rangle\right\} = \left(E_{q/2} \pm \lambda V_0\right)\left\{|q/2\rangle \pm |-q/2\rangle\right\}$$

- Consider mixing on almost-degenerate states $p \sim q/2$

$$H = \begin{pmatrix} \frac{p^2}{2m} & \lambda V_0 \\ \lambda V_0 & \frac{(p-q)^2}{2m} \end{pmatrix}$$

Eigenvalues

$$\frac{p^2 + (p-q)^2}{4m} \pm \sqrt{\frac{q^2(q-2p)^2}{16m^2} + \lambda^2 V_0^2}$$



Band gap determined by strength of perturbation

$$\Delta E = 2\lambda V_0$$

External momentum field on the lattice

- Modify Lagrangian with external field containing a spatial Fourier transform [constant in time]

$$\mathcal{L}(y) \to \mathcal{L}_0(y) + \lambda 2 \cos(\vec{q}.\vec{y})\overline{q}(y)\gamma_\mu q(y)$$

- Project onto "back-to-back" momentum state: $|\vec{q}/2\rangle + |-\vec{q}/2\rangle$

"Breit frame" kinematics

- E.g. pion form factor

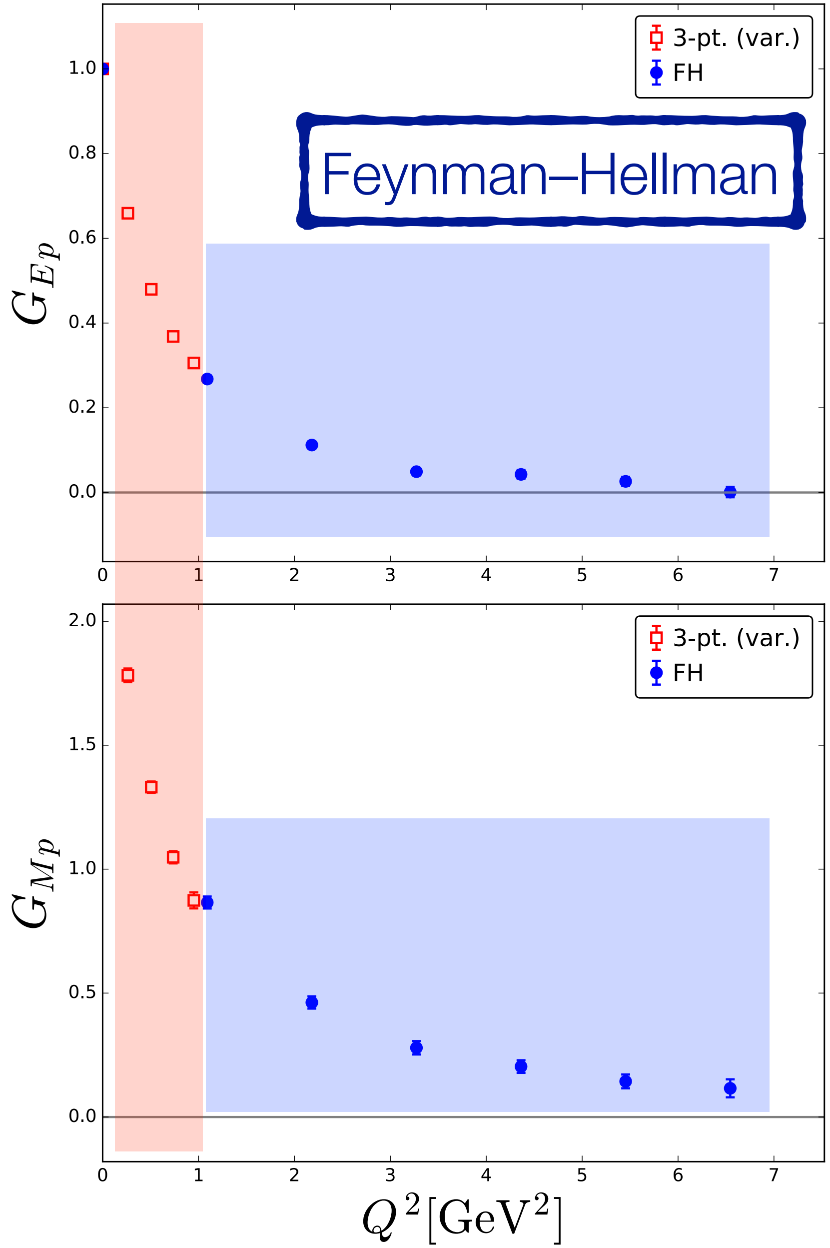
$$\langle \pi(\vec{p}')|\overline{q}(0)\gamma_\mu q(0)|\pi(\vec{p})\rangle = (p+p')_\mu F_\pi(q^2)$$

- "Feynman-Hellmann"

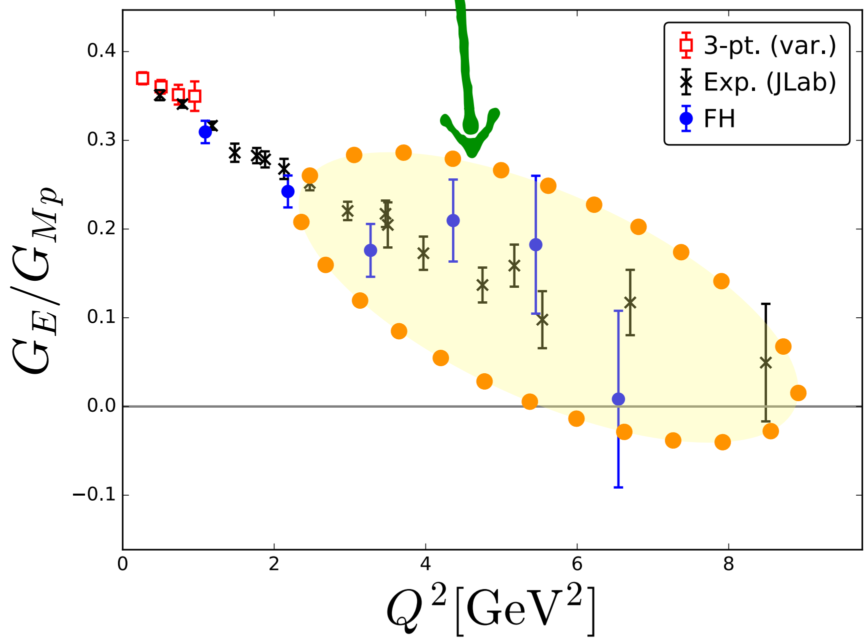
$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \frac{(p+p')_\mu}{2E} F_\pi(q^2) \quad \xrightarrow{\mu=4} \quad \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = F_\pi(q^2)$$

Proton Form Factors

3-pt functions



Phenomenologically-interesting region.
Domain dominated by model calculations…
previously prohibitive to study in lattice QCD.

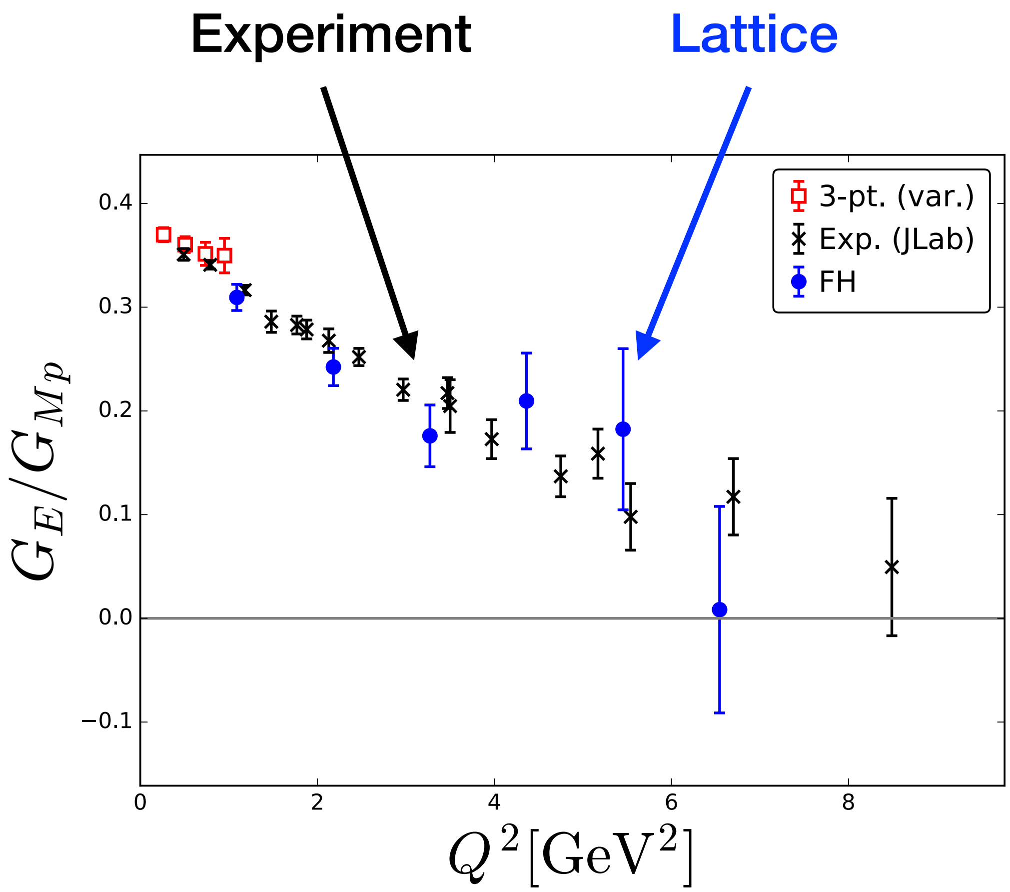


[Chambers *et al*. arXiv:1702.01513]

Proton form factors

[my comments]

- One volume
  - Not worried (yet)
- One quark mass
  - Surprised that we see a similar trend as experiment
- One lattice spacing
  - We should investigate further



[Chambers *et al*. arXiv:1702.01513]

Second-order “Feynman-Hellmann”
(with external momentum)

Feynman–Hellmann (2nd order)

- Two-point correlator

$$\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}} \frac{1}{\mathcal{Z}(\lambda)} \int \mathcal{D}\phi\, \chi(x)\chi^\dagger(0)e^{-S(\lambda)} = \sum_N \frac{|{}_\lambda\langle\Omega|\chi|N,\mathbf{p}\rangle_\lambda|^2}{2E_{N,\mathbf{p}}(\lambda)} e^{-E_{N,\mathbf{p}}(\lambda)x_0}$$

Integral over all fields

only interested in perturbative shift of ground-state energy

$$\simeq A_{\mathbf{p}}(\lambda)e^{-E_{\mathbf{p}}(\lambda)x_0}$$

"Momentum" quantum# at finite field

$$|N,\mathbf{p}\rangle_\lambda$$

$$\mathbf{p} \equiv \mathbf{p} + n\mathbf{q}, \;\; n \in \mathbb{Z}$$

Feynman–Hellmann (2nd order)

- Differentiate spectral sum

$$\frac{\partial}{\partial\lambda}\sum_N \frac{|{}_\lambda\langle\Omega|\chi|N,\mathbf{p}\rangle_\lambda|^2}{2E_N(\mathbf{p},\lambda)} e^{-E_{N,\mathbf{p}}(\lambda)x_4} = \sum_N \left[\frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial\lambda} - A_{N,\mathbf{p}}(\lambda)x_4\frac{\partial E_{N,\mathbf{p}}}{\partial\lambda}\right] e^{-E_{N,\mathbf{p}}(\lambda)x_4}$$

$$\to \left[\frac{\partial A_{\mathbf{p}}(\lambda)}{\partial\lambda} - A_{\mathbf{p}}(\lambda)x_4\frac{\partial E_{\mathbf{p}}}{\partial\lambda}\right] e^{-E_{\mathbf{p}}(\lambda)x_4}$$

- And again

Not Breit frame, ω<1 ⇒ 0

$$\frac{\partial^2}{\partial\lambda^2}\left[\cdots\right] = \sum_N\left[\frac{\partial^2 A_{N,\mathbf{p}}(\lambda)}{\partial\lambda^2} - 2\frac{\partial A_{N,\mathbf{p}}(\lambda)}{\partial\lambda}x_4\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial\lambda} - A_{N,\mathbf{p}}(\lambda)x_4\frac{\partial^2 E_{N,\mathbf{p}}(\lambda)}{\partial\lambda^2} + A_{N,\mathbf{p}}(\lambda)x_4^2\left(\frac{\partial E_{N,\mathbf{p}}(\lambda)}{\partial\lambda}\right)^2\right]$$

$$\to \left[\frac{\partial^2 A_{\mathbf{p}}(\lambda)}{\partial\lambda^2} - A_{\mathbf{p}}(\lambda)x_4\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2}\right] e^{-E_{\mathbf{p}}(\lambda)x_4}$$

Quadratic energy shift

Watch for temporal enhancement $\sim x_4\, e^{-E_{\mathbf{p}}x_4}$

Feynman–Hellmann (2nd order)

- **Differentiate path integral**

$$\frac{\partial}{\partial\lambda}\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}(\lambda)}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)e^{-S(\lambda)}$$

$$=\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}(\lambda)}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)\left[-\frac{\partial S}{\partial\lambda}-\frac{1}{\mathcal{Z}(\lambda)}\frac{\partial\mathcal{Z}}{\partial\lambda}\right]e^{-S(\lambda)},$$

"Disconnected" operator insertions;
drop for simplicity

- Differentiate again, take zero-field limit and note: $\frac{\partial^2 S}{\partial\lambda^2}=0$

$$\frac{\partial^2}{\partial\lambda^2}[\cdots]\Big|_{\lambda=0}=\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}}\frac{1}{\mathcal{Z}_0}\int\mathcal{D}\phi\,\chi(x)\chi^\dagger(0)\left(\frac{\partial S}{\partial\lambda}\right)^2 e^{-S_0}$$

Current insertions integrated over 4-volume

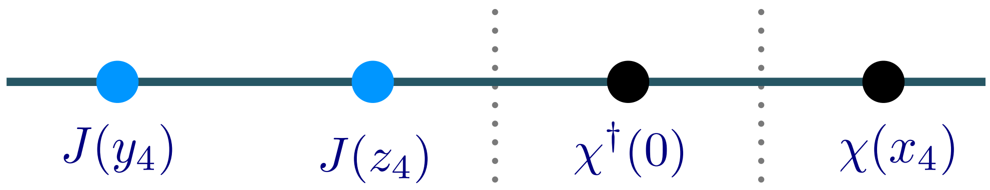
$$\frac{\partial S}{\partial\lambda}=\int d^4y\, 2\cos(\mathbf{q}.\mathbf{y})\overline{q}(y)\gamma_\mu q(y)$$

Field time orderings

ignore finite T

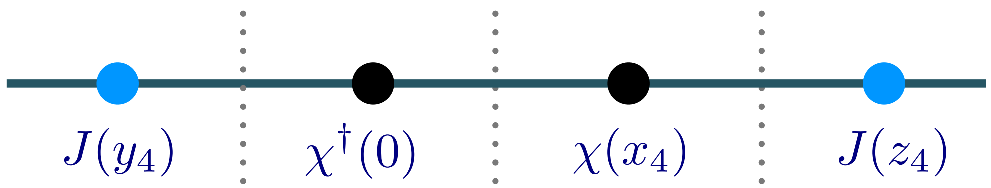
- Current insertion possibilities

  - Both currents “outside” (together)



$$\langle \chi(x)\chi^\dagger(0)\mathrm{T}(J(y)J(z))\rangle, \quad y_4, z_4 < 0 < x_4$$
$$\sim e^{-E_X x_4}, \quad E_X \gtrsim E_{\mathbf{p}}$$

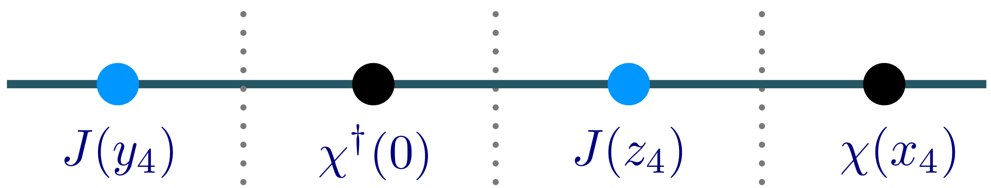
  - Both currents “outside” (opposite)



$$\langle J(z)\chi(x)\chi^\dagger(0)J(y)\rangle, \quad y_4 < 0 < x_4 < z_4$$
$$\sim e^{-E_X x_4}, \quad E_X \gtrsim E_{\mathbf{p}}$$

$E_X = E_{\mathbf{p}} \Rightarrow$ changes amplitudes

  - One current “inside”

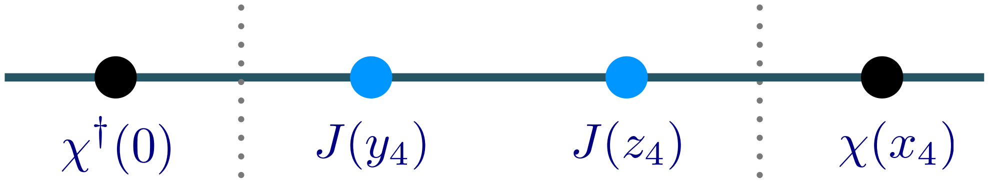


$$\langle \chi(x)J(z)\chi^\dagger(0)J(y)\rangle, \quad y_4 < 0 < z_4 < x_4$$
$$\sim \frac{\partial E_{\mathbf{p}}}{\partial \lambda} x_4 \, e^{-E_{\mathbf{p}} x_4} \to 0$$

linear energy shift
(and changed amplitude)

Field time orderings

- Both currents between creation/annihilation



$$\int d^3x\, e^{-i\mathbf{p}.\mathbf{x}} \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi\, \chi(x)\chi^\dagger(0) \left(\frac{\partial S}{\partial \lambda}\right)^2 e^{-S_0}$$

$$= \sum_{N,N'} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{N,\mathbf{k}}} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2E_{N',\mathbf{k}'}} \int d^3x \int d^4z \int d^4y\, e^{-i\mathbf{p}.\mathbf{x}} \left(e^{i\mathbf{q}.\mathbf{z}} + e^{-i\mathbf{q}.\mathbf{z}}\right) \left(e^{i\mathbf{q}.\mathbf{y}} + e^{-i\mathbf{q}.\mathbf{y}}\right)$$
$$\times \langle\Omega|\chi(x)|N,\mathbf{k}\rangle\langle\mathbf{k}|\mathrm{T}J(z)J(y)|\mathbf{k}'\rangle\langle N',\mathbf{k}'|\chi^\dagger(0)|\Omega\rangle,$$

$$\vdots$$

$$\to \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}} x_4 e^{-E_{\mathbf{p}}x_4} \int d^4\xi \left(e^{iq.\xi} + e^{-iq.\xi}\right) \langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$

Note $q_4 = 0 \quad \Rightarrow \mathbf{q}.\boldsymbol{\xi} = q.\xi$

Final steps

- Equate spectral sum and path integral representation
  - Asymptotically, we have

$$-A_{\mathbf{p}}\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2}x_4 e^{-E_{\mathbf{p}}x_4} = \frac{A_{\mathbf{p}}}{2E_{\mathbf{p}}}x_4 e^{-E_{\mathbf{p}}x_4}\int d^4\xi\,\left(e^{iq.\xi}+e^{-iq.\xi}\right)\langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$

$$\frac{\partial^2 E_{\mathbf{p}}}{\partial\lambda^2} = -\frac{1}{2E_{\mathbf{p}}}\int d^4\xi\,\left(e^{iq.\xi}+e^{-iq.\xi}\right)\langle\mathbf{p}|\mathrm{T}J(\xi)J(0)|\mathbf{p}\rangle$$