Lattice QCD investigations
of quark transverse momentum in hadrons

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Fundamental TMD correlator

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\ldots) \equiv \frac{1}{2}\langle P,S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0,\ldots,b]\ q(b)\ |P,S\rangle$$

$$\Phi^{[\Gamma]}(x,k_T,P,S,\ldots) \equiv \int \frac{d^2b_T}{(2\pi)^2} \int \frac{d(b\cdot P)}{(2\pi)P^+} \exp\left(ix(b\cdot P) - ib_T\cdot k_T\right) \left. \frac{\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\ldots)}{\widetilde{\mathcal{S}}(b^2,\ldots)} \right|_{b^+=0}$$

- "Soft factor" $\widetilde{\mathcal{S}}$ required to subtract divergences of Wilson line $\mathcal{U}$
- $\widetilde{\mathcal{S}}$ is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel

Relation to physical processes

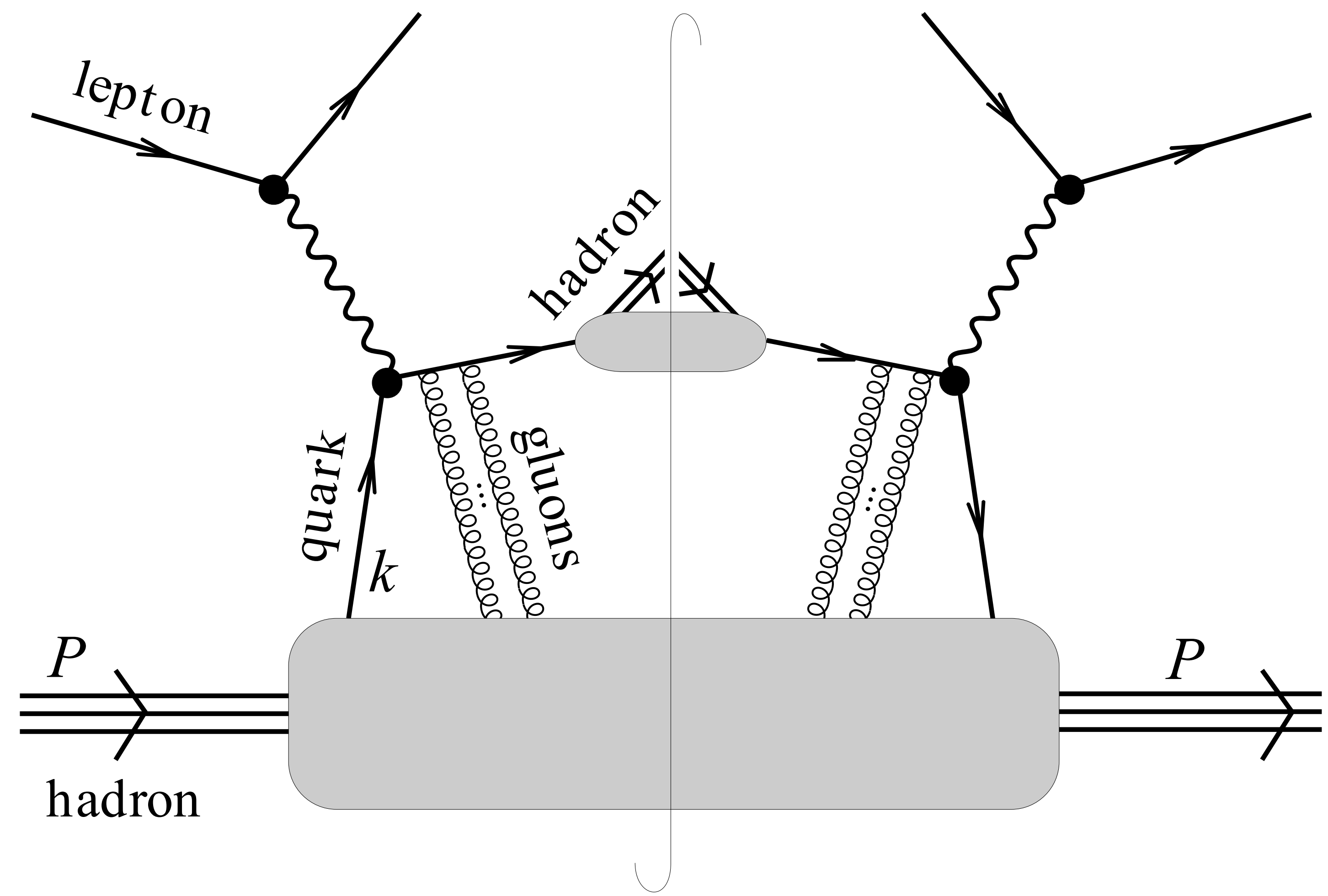
Context: All this is largely academic if we can't connect it to a physical measurement.

Not least, this should inform choice of gauge link $\mathcal{U}[0,\ldots,b]$ ...

Factorization theorem which allows one to separate cross section into hard amplitude, fragmentation function, TMD ?

For example, SIDIS:
$l + N(P) \longrightarrow l' + h(P_h) + X$

Note final state effects in SIDIS

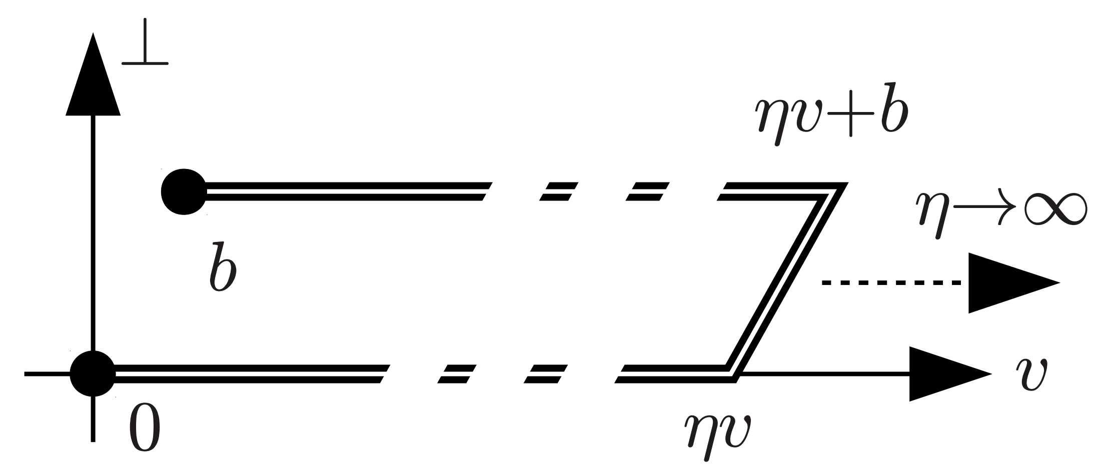


Relation to physical processes

In general, no factorization framework with well-defined TMDs exists (e.g., processes with multiple hadrons in both initial and final state)!

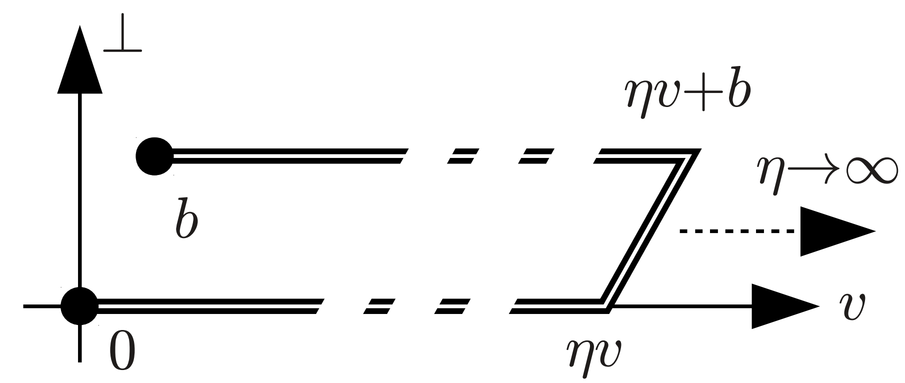
SIDIS and DY: Factorization framework has been given, which in particular includes:

- Specific form of the gauge link $\mathcal{U}[0, b]$
- Accounts for final state interactions
- Further regularization required!



Staple-shaped gauge link $\mathcal{U}[0, \eta v, \eta v + b, b]$

Gauge link structure motivated by SIDIS



Beyond tree level: Rapidity divergences suggest taking staple direction slightly off the light cone. Approach of Aybat, Collins, Qiu, Rogers makes v space-like. Parametrize in terms of Collins-Soper parameter

$$\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$$

Light-like staple for $\hat{\zeta} \to \infty$. Perturbative evolution equations for large $\hat{\zeta}$.

"Modified universality", $f^{\text{T-odd, SIDIS}} = -f^{\text{T-odd, DY}}$

Fundamental TMD correlator

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\ldots) \equiv \frac{1}{2}\langle P,S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0,\eta v,\eta v+b,b]\ q(b)\ |P,S\rangle$$

$$\Phi^{[\Gamma]}(x,k_T,P,S,\ldots) \equiv \int \frac{d^2 b_T}{(2\pi)^2} \int \frac{d(b\cdot P)}{(2\pi)P^+} \exp\left(ix(b\cdot P) - ib_T\cdot k_T\right) \frac{\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\ldots)}{\widetilde{\mathcal{S}}(b^2,\ldots)}\Bigg|_{b^+=0}$$

- "Soft factor" $\widetilde{\mathcal{S}}$ required to subtract divergences of Wilson line $\mathcal{U}$
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- Here, will consider only ratios in which soft factors cancel

Decomposition of Φ into TMDs

All leading twist structures:

$$\Phi^{[\gamma^+]} = f_1 - \left[\frac{\epsilon_{ij} k_i S_j}{m_H} f_{1T}^{\perp}\right] \text{odd}$$

$$\Phi^{[\gamma^+\gamma^5]} = \Lambda g_1 + \frac{k_T \cdot S_T}{m_H} g_{1T}$$

$$\Phi^{[i\sigma^{i+}\gamma^5]} = S_i h_1 + \frac{(2k_i k_j - k_T^2 \delta_{ij}) S_j}{2m_H^2} h_{1T}^{\perp} + \frac{\Lambda k_i}{m_H} h_{1L}^{\perp} + \left[\frac{\epsilon_{ij} k_j}{m_H} h_1^{\perp}\right] \text{odd}$$

TMD Classification

All leading twist structures:

$q \rightarrow$ H $\downarrow$	U	L	T
U	f_1		$h_1^{\perp}$
L		g_1	$h_{1L}^{\perp}$
T	$f_{1T}^{\perp}$	g_{1T}	h_1 $h_{1T}^{\perp}$

$\longleftarrow$ Boer-Mulders (T-odd) ($h_1^{\perp}$)

$\uparrow$ Sivers (T-odd) ($f_{1T}^{\perp}$)

Decomposition of $\widetilde{\Phi}$ into amplitudes

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P, S\rangle$$

Decompose in terms of invariant amplitudes; at leading twist,

$$\begin{aligned}
\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} &= \widetilde{A}_{2B} + i m_H \epsilon_{ij} b_i S_j \widetilde{A}_{12B} \\
\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} &= -\Lambda \widetilde{A}_{6B} + i[(b \cdot P)\Lambda - m_H (b_T \cdot S_T)]\widetilde{A}_{7B} \\
\frac{1}{2P^+}\widetilde{\Phi}^{[i\sigma^{i+}\gamma^5]}_{\text{unsubtr.}} &= i m_H \epsilon_{ij} b_j \widetilde{A}_{4B} - S_i \widetilde{A}_{9B} \\
&\quad - i m_H \Lambda b_i \widetilde{A}_{10B} + m_H[(b \cdot P)\Lambda - m_H (b_T \cdot S_T)] b_i \widetilde{A}_{11B}
\end{aligned}$$

(Decompositions analogous to work by Metz et al. in momentum space)

Fourier-transformed TMDs

$$\tilde{f}(x, b_T^2, \ldots) \equiv \int d^2k_T \, \exp(ib_T \cdot k_T) f(x, k_T^2, \ldots)$$

$$\tilde{f}^{(n)}(x, b_T^2, \ldots) \equiv n! \left(-\frac{2}{m_H^2} \partial_{b_T^2} \right)^n \tilde{f}(x, b_T^2, \ldots)$$

In limit $|b_T| \rightarrow 0$, recover k_T-moments:

$$\tilde{f}^{(n)}(x, 0, \ldots) \equiv \int d^2k_T \left(\frac{k_T^2}{2m_H^2} \right)^n f(x, k_T^2, \ldots) \equiv f^{(n)}(x)$$

ill-defined for large k_T, so will not attempt to extrapolate to $b_T = 0$, but give results at finite $|b_T|$.

In this study, only consider first x-moments (accessible at $b \cdot P = 0$), rather than scanning range of $b \cdot P$:

$$f^{[1]}(k_T^2, \ldots) \equiv \int_{-1}^{1} dx \, f(x, k_T^2, \ldots)$$

$\longrightarrow$ Bessel-weighted asymmetries (Boer, Gamberg, Musch, Prokudin, JHEP 1110 (2011) 021)

Relation between Fourier-transformed TMDs and invariant amplitudes $\widetilde{A}_i$

Invariant amplitudes directly give selected x-integrated TMDs in Fourier (b_T) space (showing just the ones relevant for Sivers, Boer-Mulders shifts), up to soft factors:

$$\tilde{f}_1^{[1](0)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\widetilde{S}(b^2, \ldots)$$

$$\tilde{f}_{1T}^{\perp[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = -2\widetilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\widetilde{S}(b^2, \ldots)$$

$$\tilde{h}_1^{\perp[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\widetilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\widetilde{S}(b^2, \ldots)$$

Generalized shifts

Form ratios in which soft factors, (Γ-independent) multiplicative renormalization factors cancel

Boer-Mulders shift:

$$\langle k_y \rangle_{UT} \equiv m_H \frac{\tilde{h}_1^{\perp [1](1)}}{\tilde{f}_1^{[1](0)}} = \left. \frac{\int dx \int d^2 k_T \, k_y \Phi^{[\gamma^+ + s^j i\sigma^{j+}\gamma^5]}(x, k_T, P, \ldots)}{\int dx \int d^2 k_T \, \Phi^{[\gamma^+ + s^j i\sigma^{j+}\gamma^5]}(x, k_T, P, \ldots)} \right|_{s_T = (1,0)}$$

Average transverse momentum of quarks polarized in the orthogonal transverse ("T") direction in an unpolarized ("U") hadron; normalized to the number of valence quarks. "Dipole moment" in $b_T^2 = 0$ limit, "shift".

Issue: k_T-moments in this ratio singular; generalize to ratio of Fourier-transformed TMDs at *nonzero* b_T^2,

$$\langle k_y \rangle_{UT}(b_T^2, \ldots) \equiv m_H \frac{\tilde{h}_1^{\perp [1](1)}(b_T^2, \ldots)}{\tilde{f}_1^{[1](0)}(b_T^2, \ldots)}$$

(remember singular $b_T \rightarrow 0$ limit corresponds to taking k_T-moment). "Generalized shift".

Generalized shifts from amplitudes

Now, can also express this in terms of invariant amplitudes:

$$\langle k_y \rangle_{UT}(b_T^2, \ldots) \equiv m_H \frac{\tilde{h}_1^{\perp [1](1)}(b_T^2, \ldots)}{\tilde{f}_1^{[1](0)}(b_T^2, \ldots)} = m_H \frac{\widetilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

Analogously, Sivers shift (in a polarized hadron):

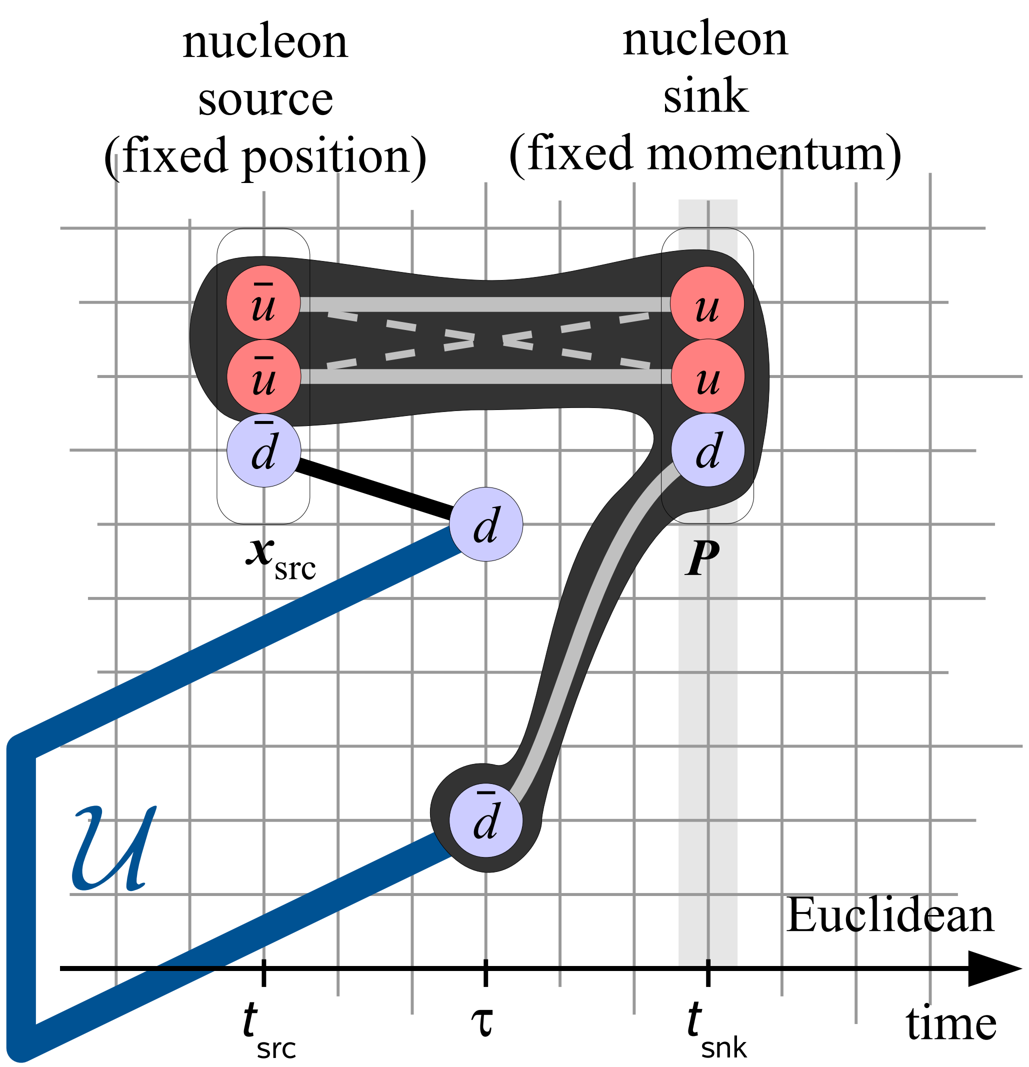
$$\langle k_y \rangle_{TU}(b_T^2, \ldots) = -m_H \frac{\widetilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

Worm-gear (g_{1T}) shift:

$$\langle k_x \rangle_{TL}(b_T^2, \ldots) = -m_N \frac{\widetilde{A}_{7B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

Generalized tensor charge (no k-weighting) :

$$\frac{\tilde{h}_1^{[1](0)}}{\tilde{f}_1^{[1](0)}} = -\frac{\widetilde{A}_{9B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P) - (m_N^2 b^2/2)\widetilde{A}_{11B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$



Lattice setup

- Evaluate directly $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu)$

  $\equiv \frac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P, S\rangle$

- Euclidean time: Place entire operator at one time slice, i.e., b, ηv purely spatial
- Since generic b, v space-like, no obstacle to boosting system to such a frame!
- Parametrization of correlator in terms of $\widetilde{A}_i$ invariants permits direct translation of results back to original frame; form desired $\widetilde{A}_i$ ratios.
- Use variety of P, b, ηv; here $b \perp P$, $b \perp v$ (lowest x-moment, kinematical choices/constraints)
- Extrapolate $\eta \to \infty$, $\hat{\zeta} \to \infty$ numerically.

Initial Numerical Investigation

Use three MILC 2+1-flavor gauge ensembles with $a \approx 0.12\,\mathrm{fm}$:

$m_\pi = 369\,\mathrm{MeV}$; $28^3 \times 64$; 2184 samples

$m_\pi = 369\,\mathrm{MeV}$; $20^3 \times 64$; 5264 samples

$m_\pi = 518\,\mathrm{MeV}$; $20^3 \times 64$; 3888 samples

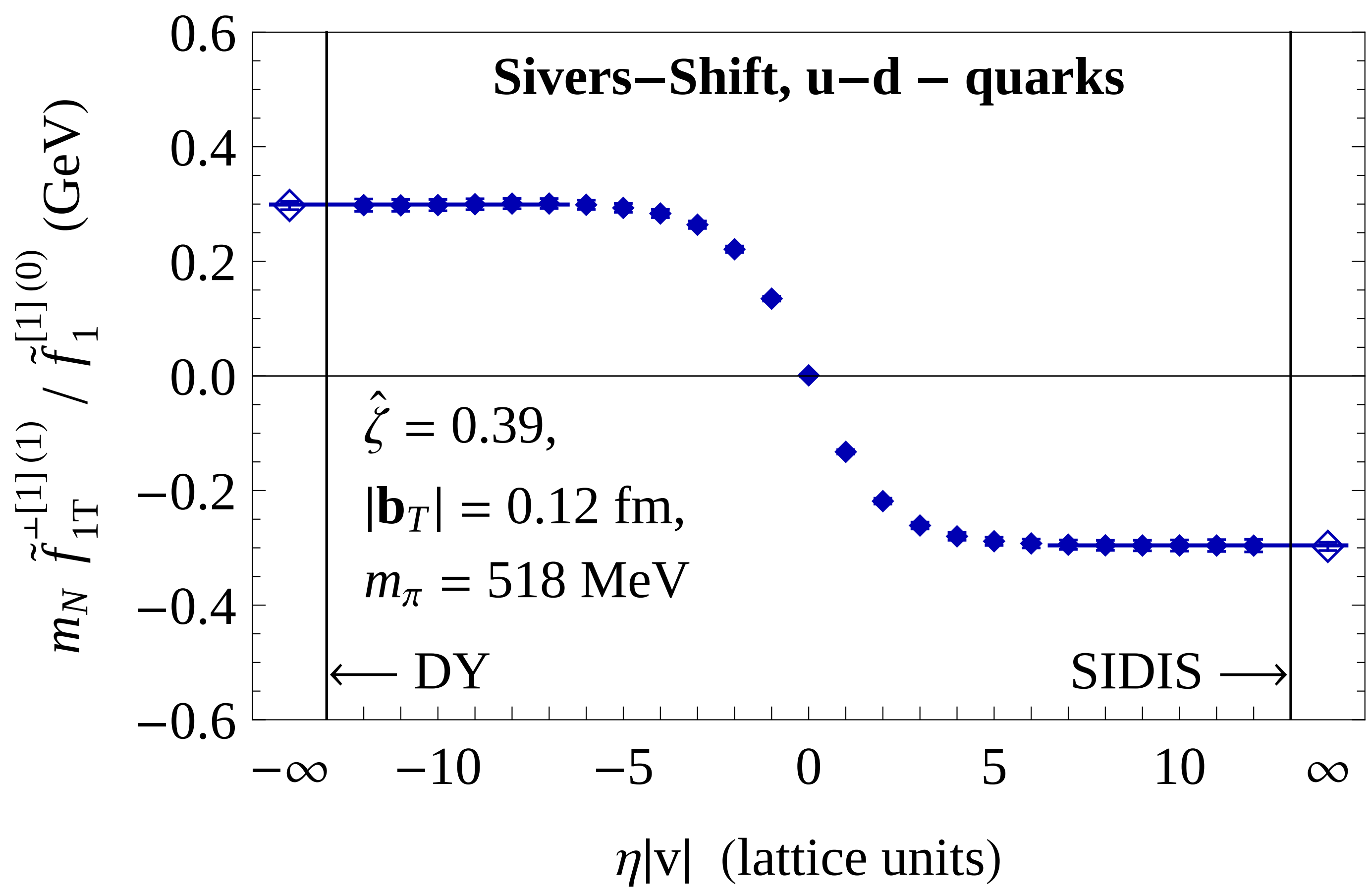
Sink momenta P: $(0,0,0)$, $(-1,0,0)$, $(-2,0,0)$, $(1,-1,0)$

Variety of b, ηv; note $b \perp P$, $b \perp v$ (lowest x-moment, kinematical choices/constraints)

Largest $\hat{\zeta} = 0.78$

Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$



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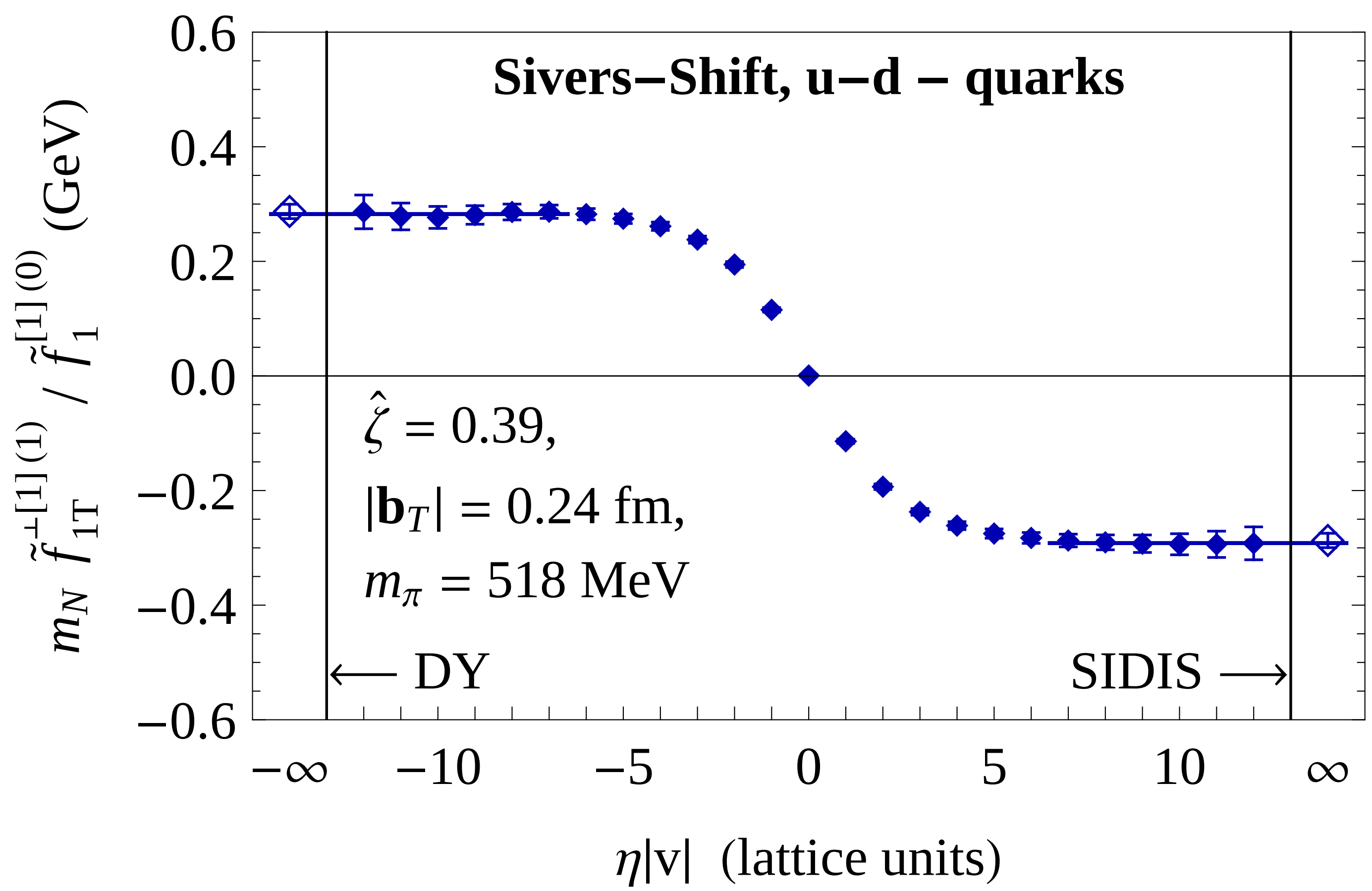

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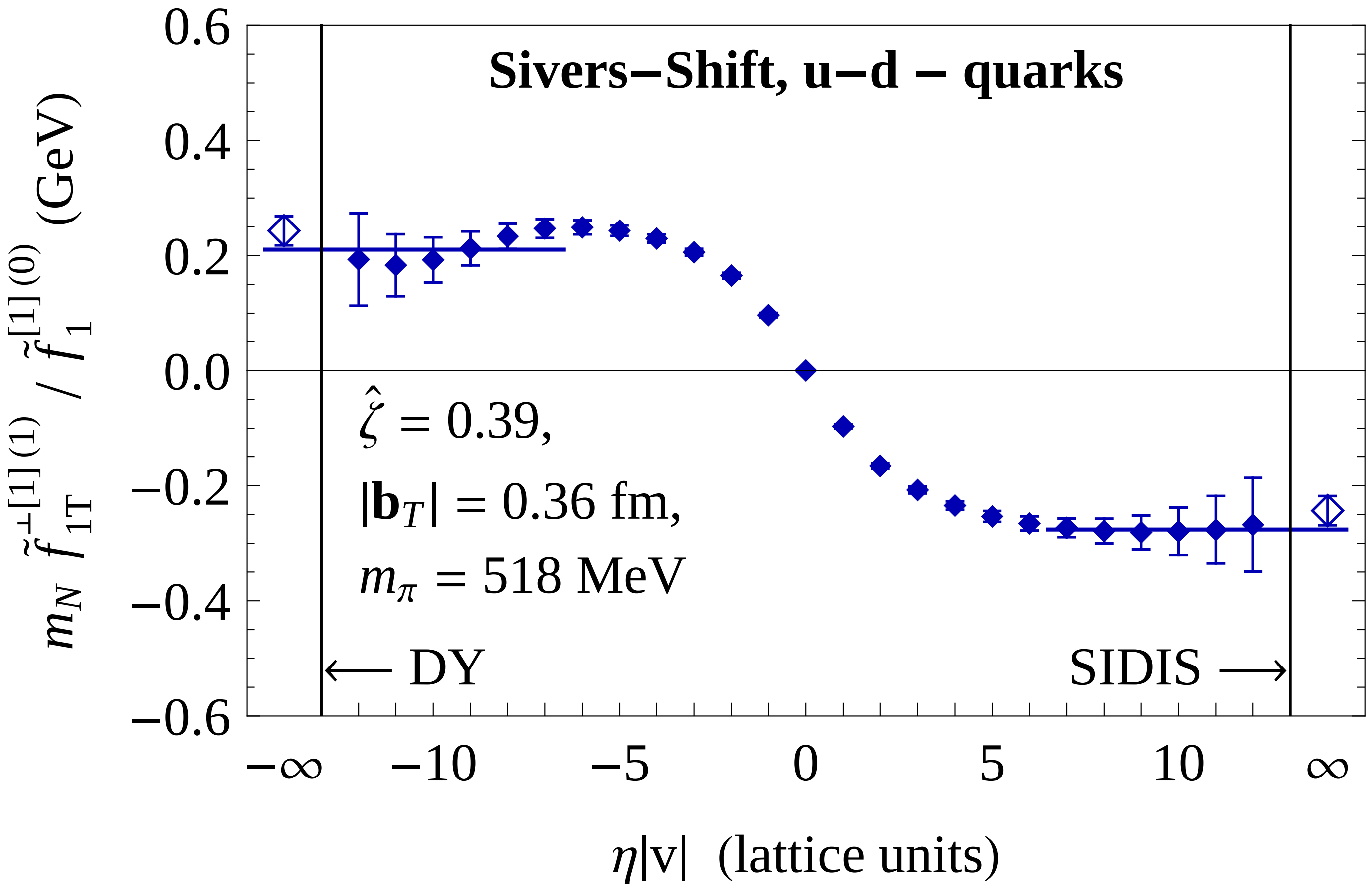

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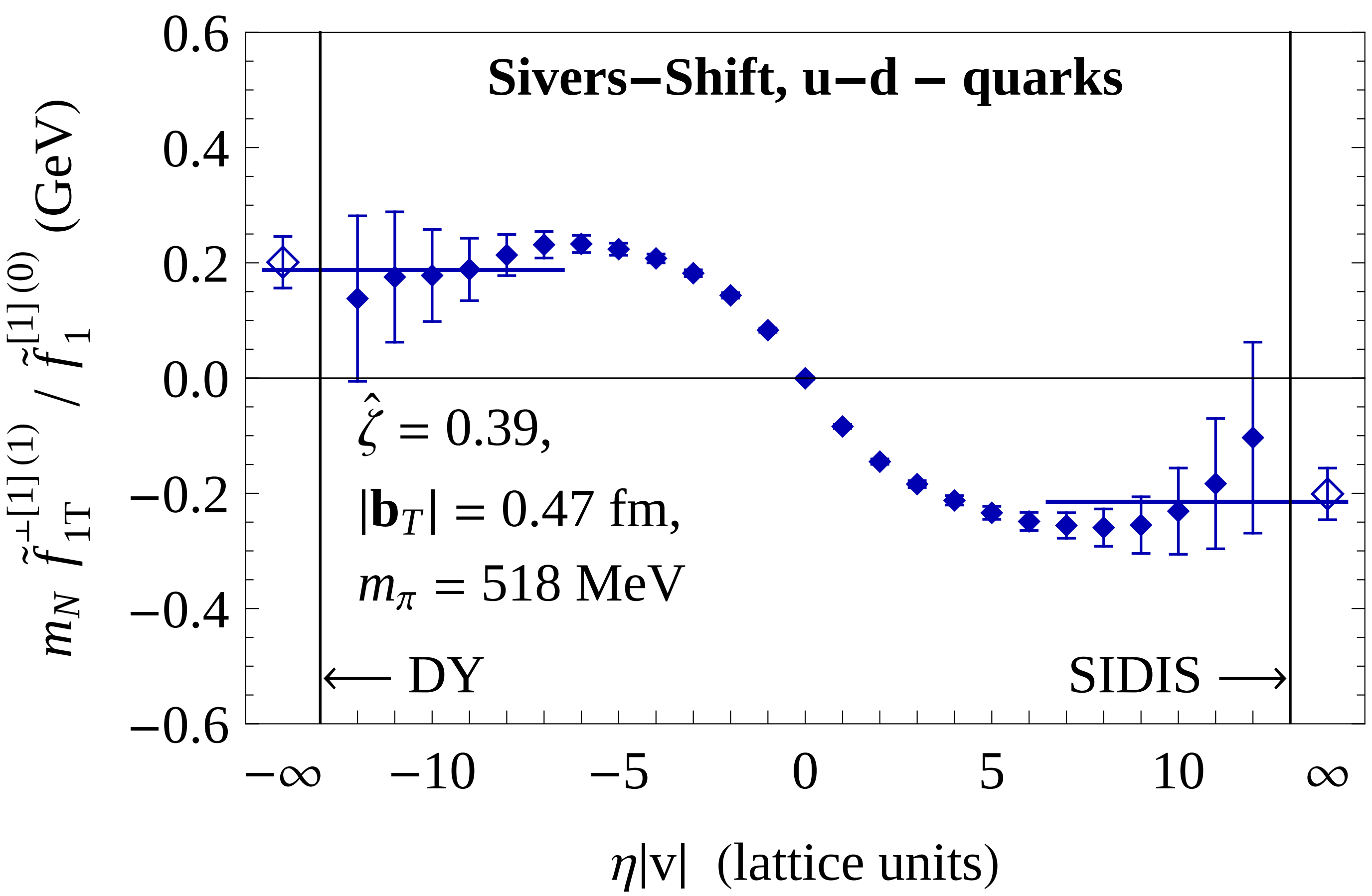
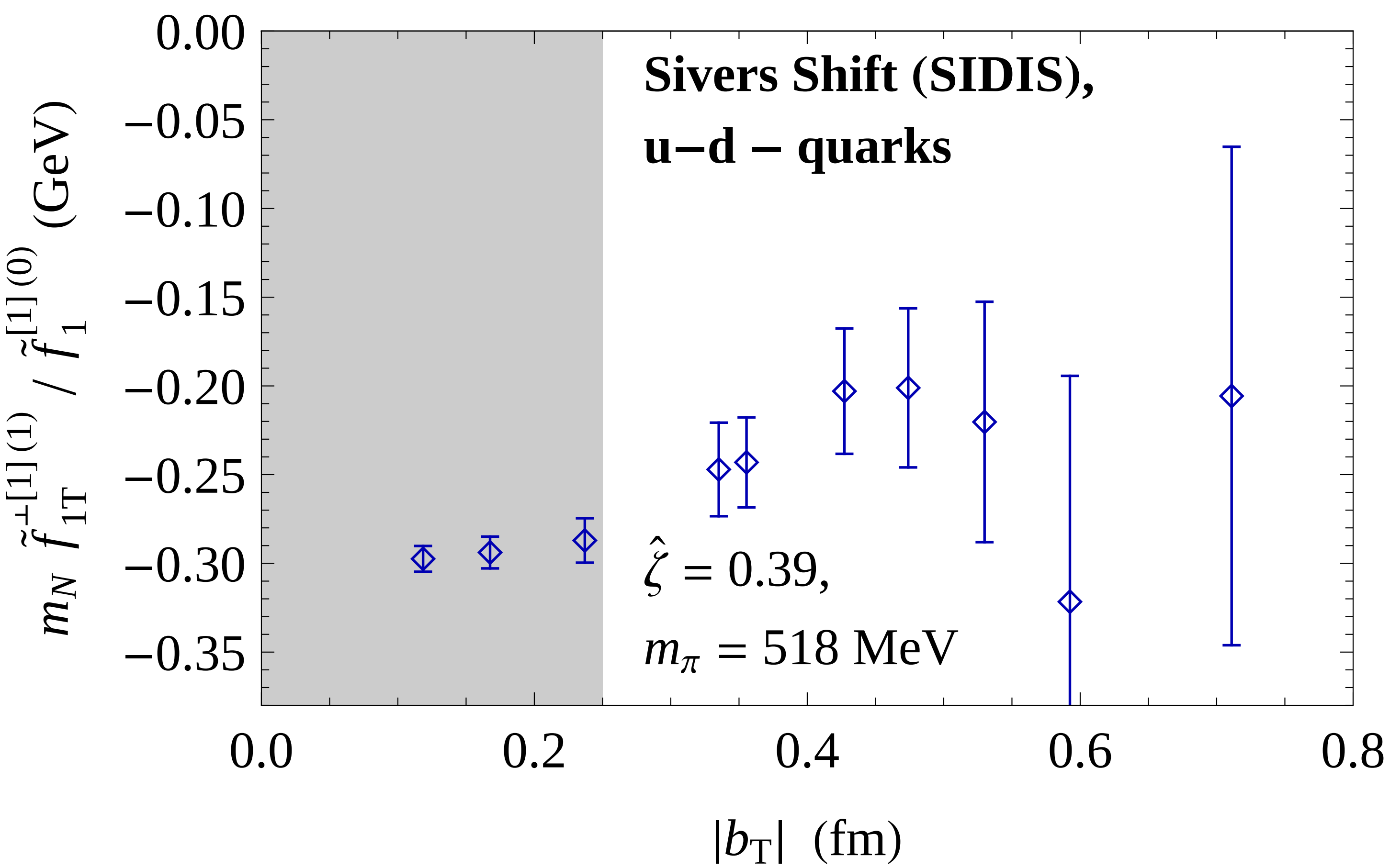


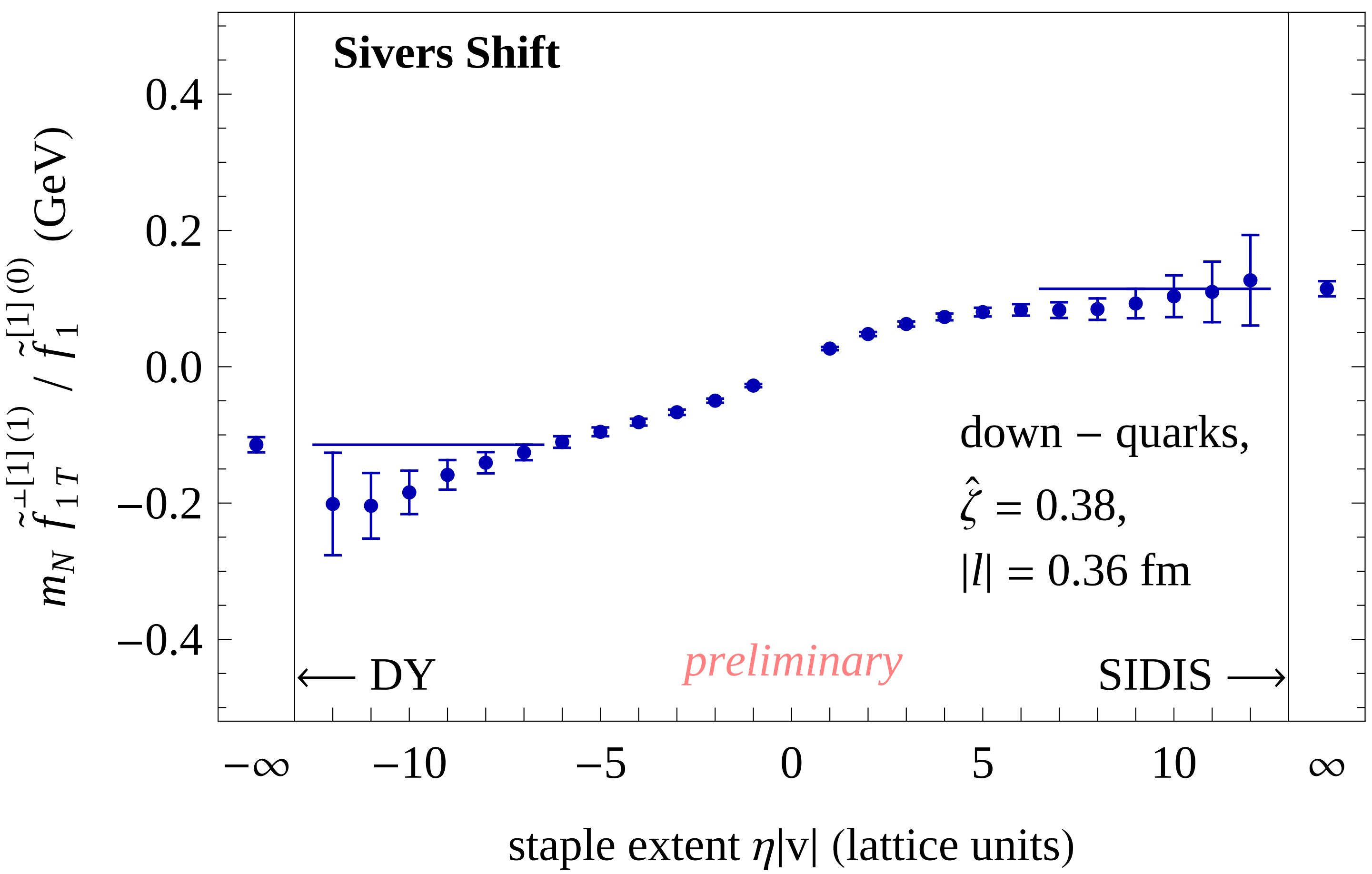
Results: Sivers shift

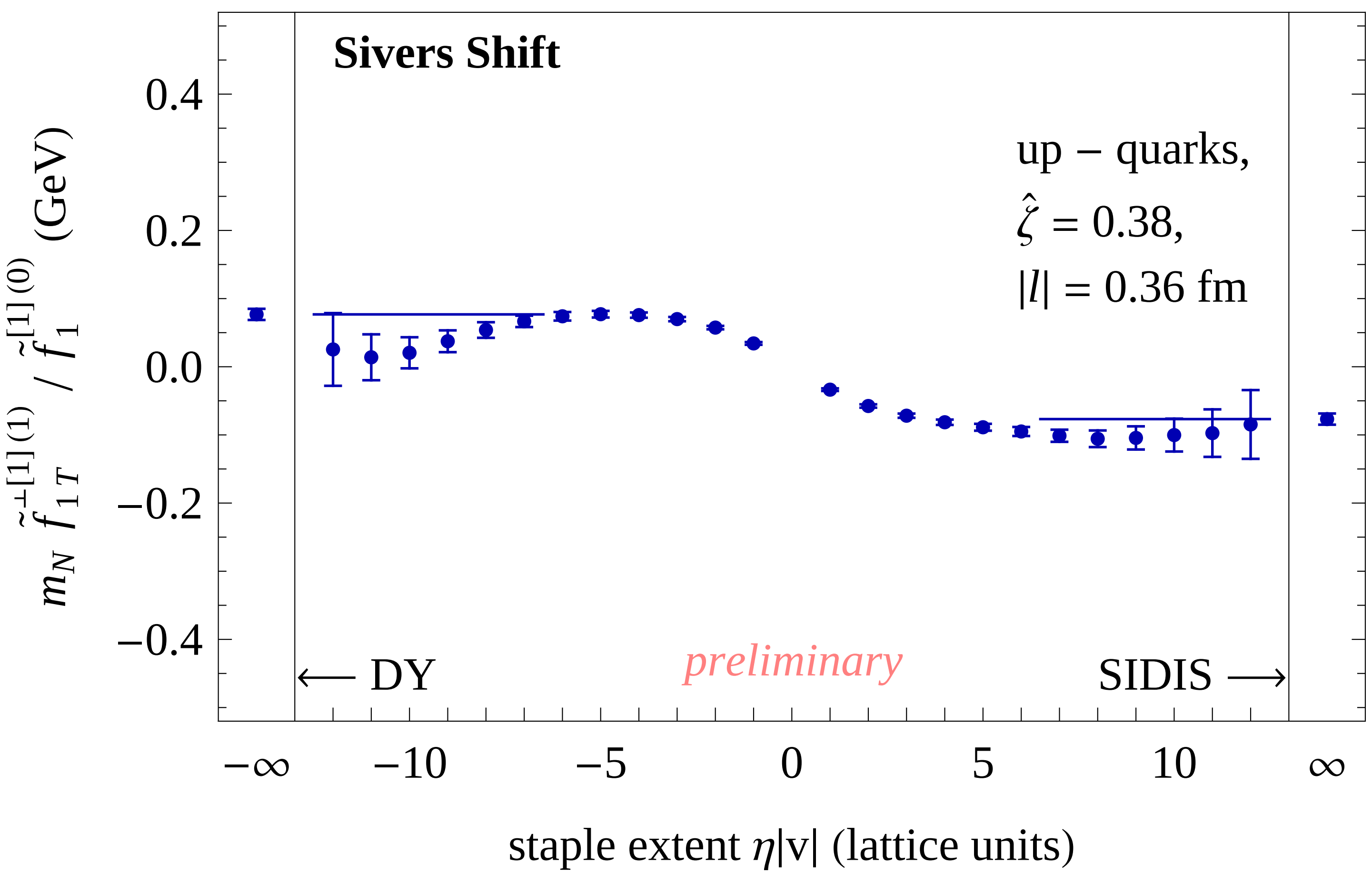
Dependence of SIDIS limit on $|b_T|$



Results: Sivers shift

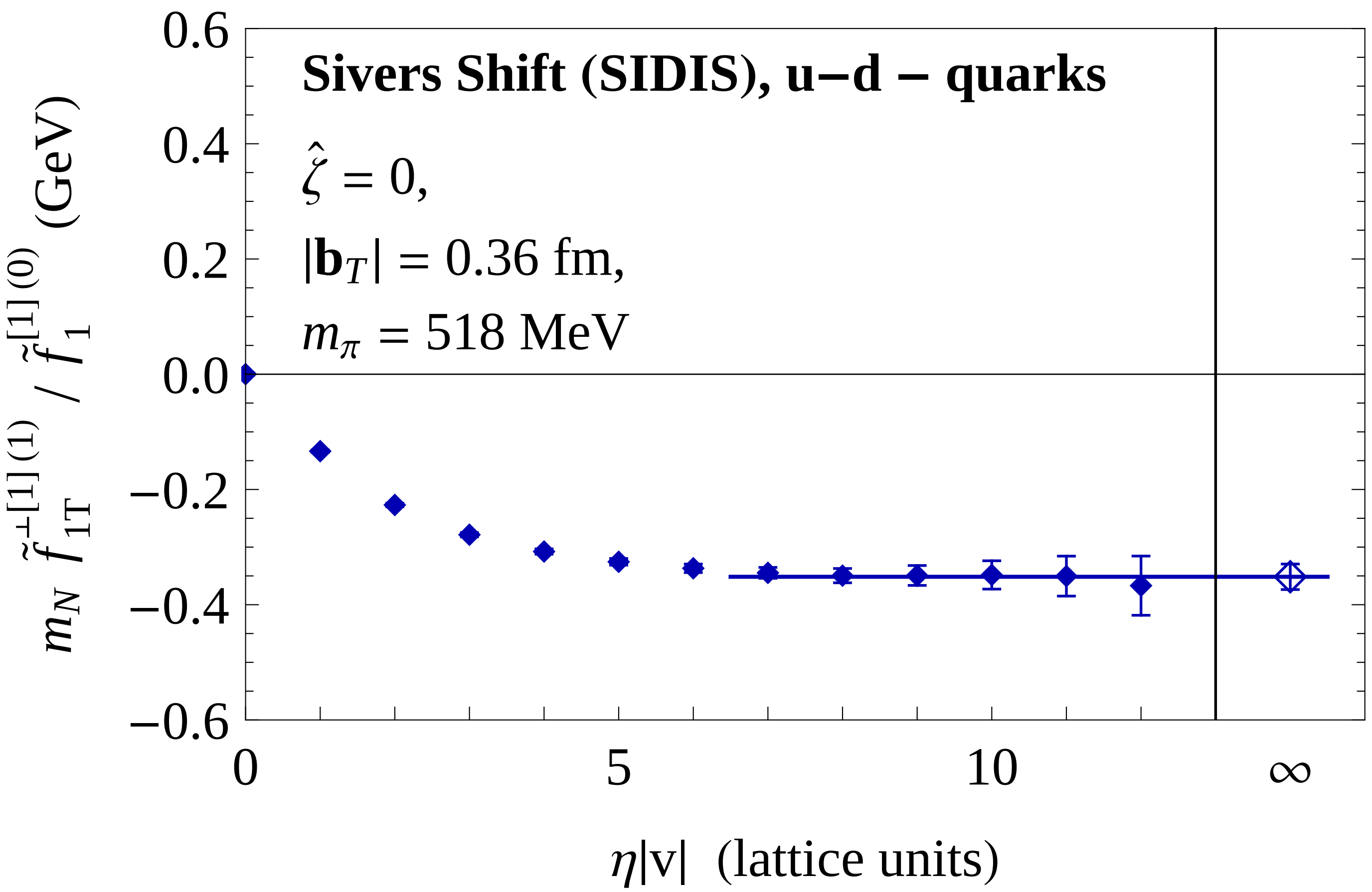
Dependence on staple extent; flavor separated





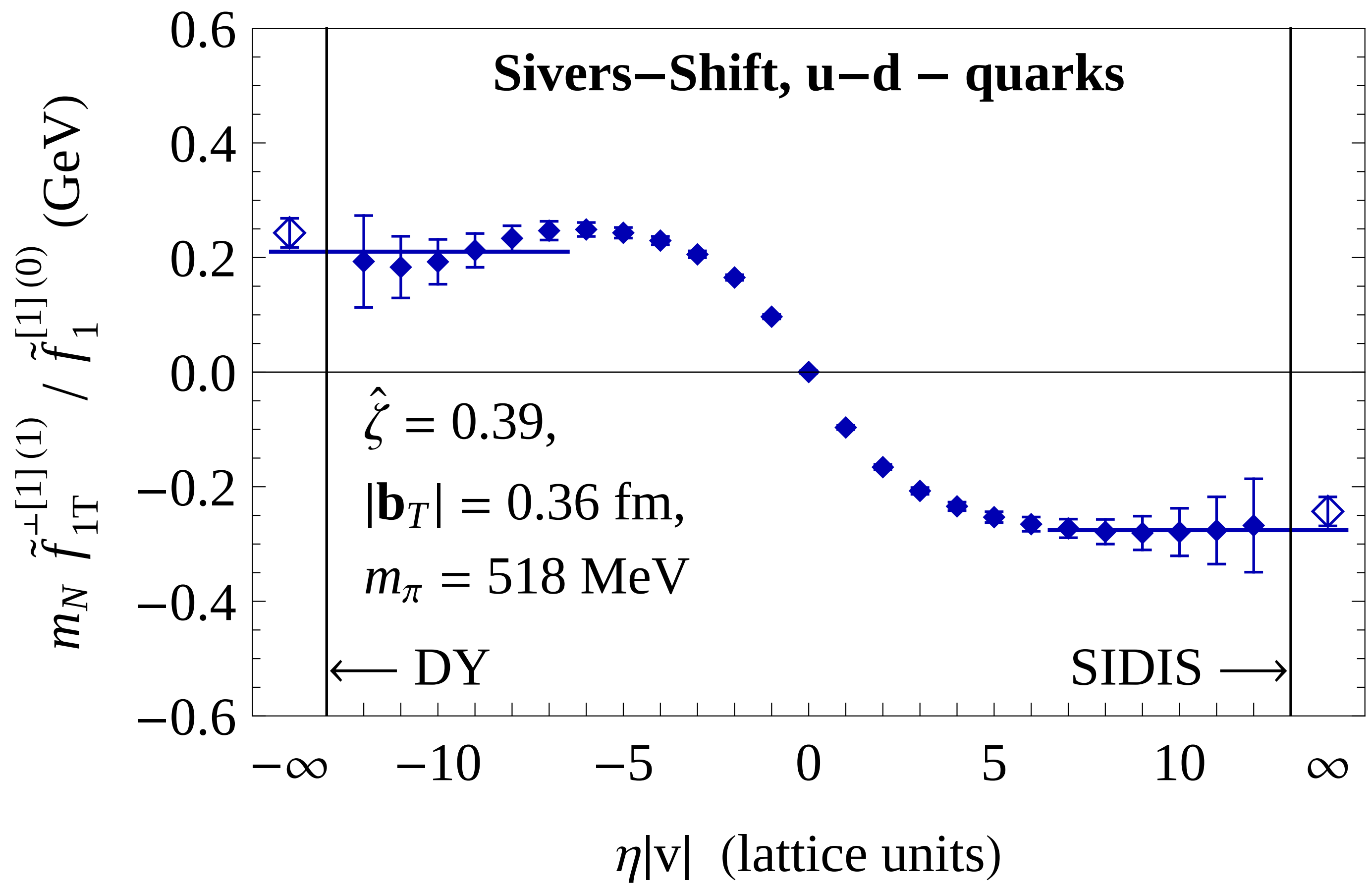
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$



Results: Sivers shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$



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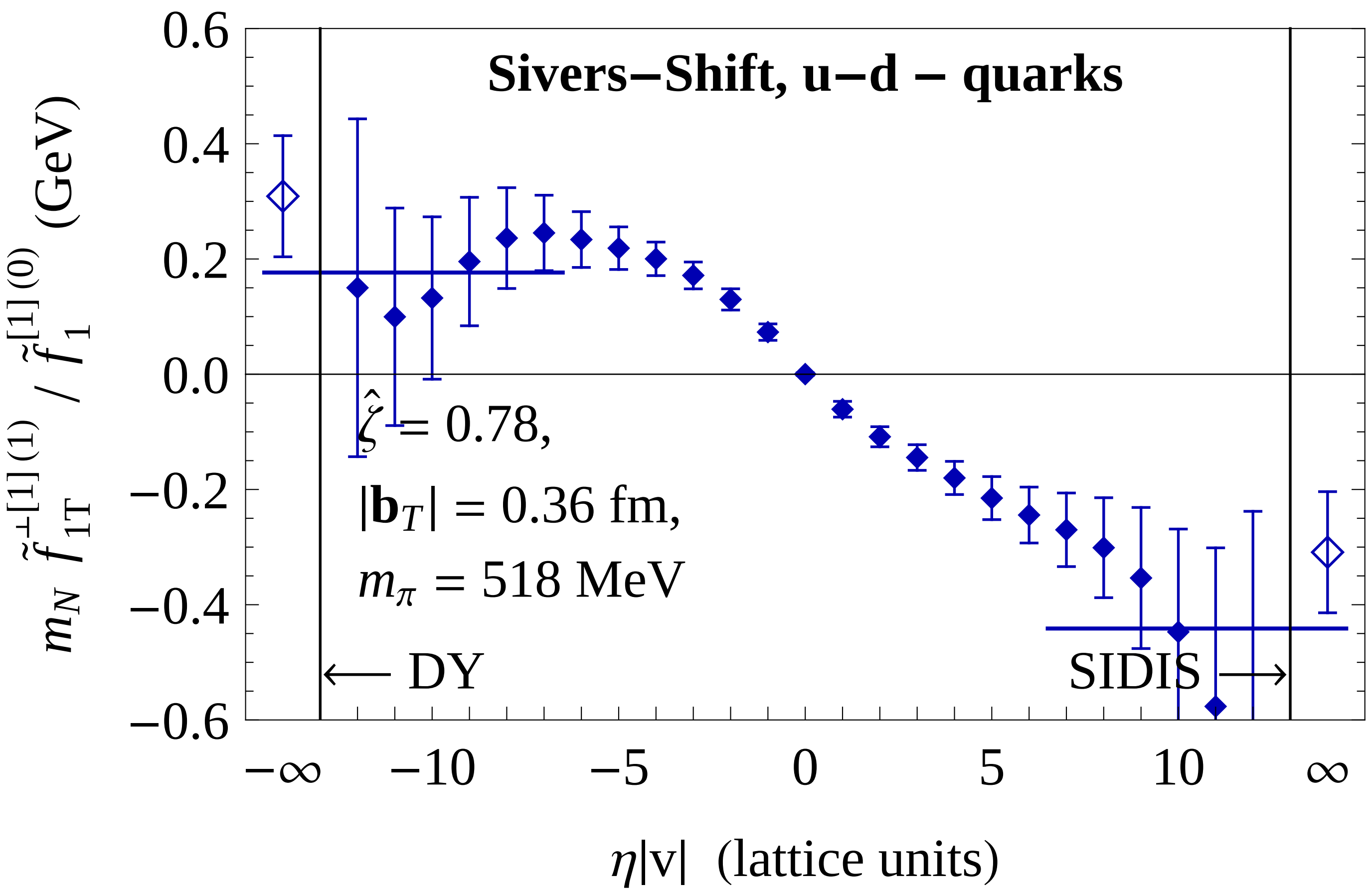

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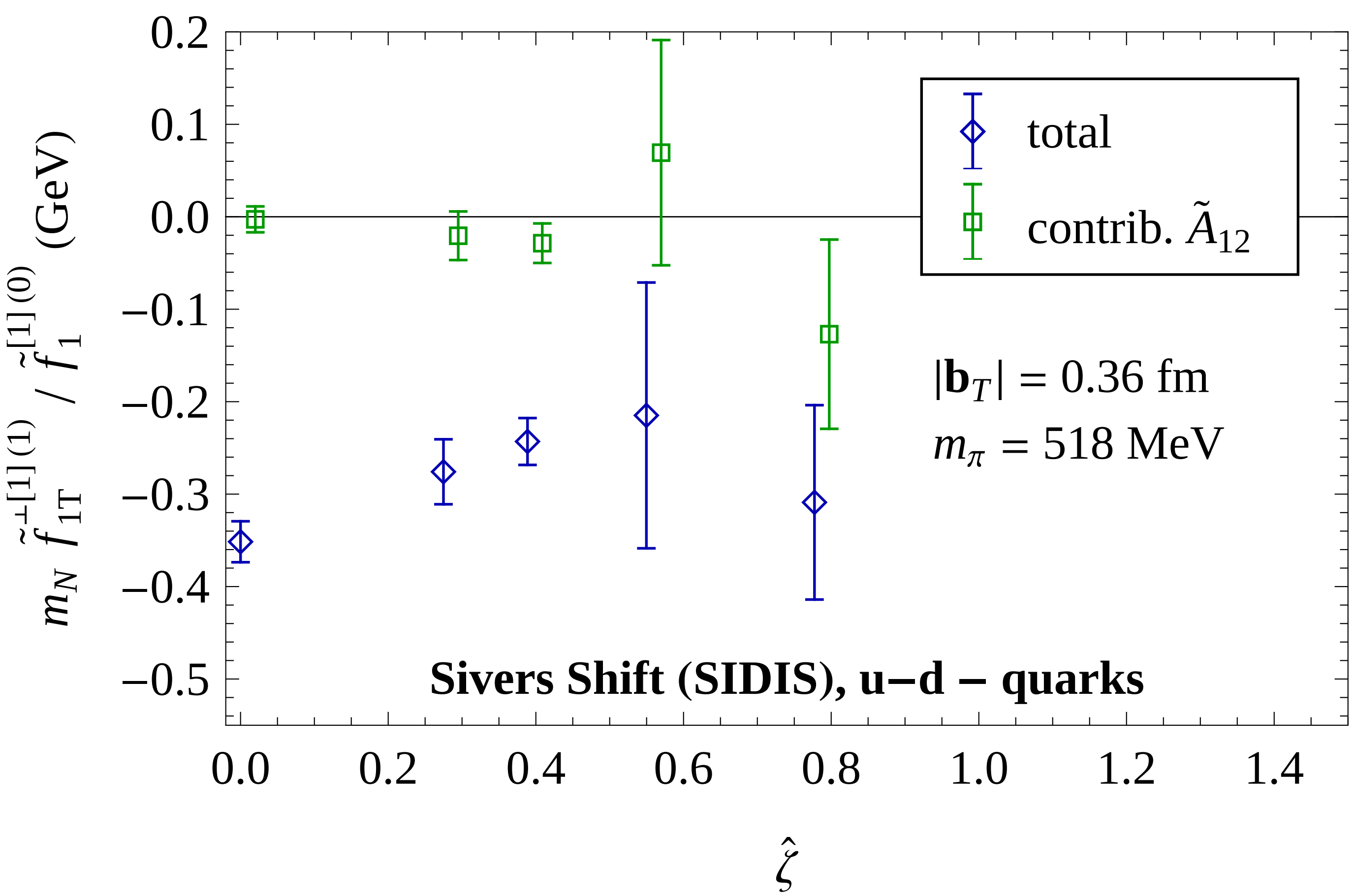
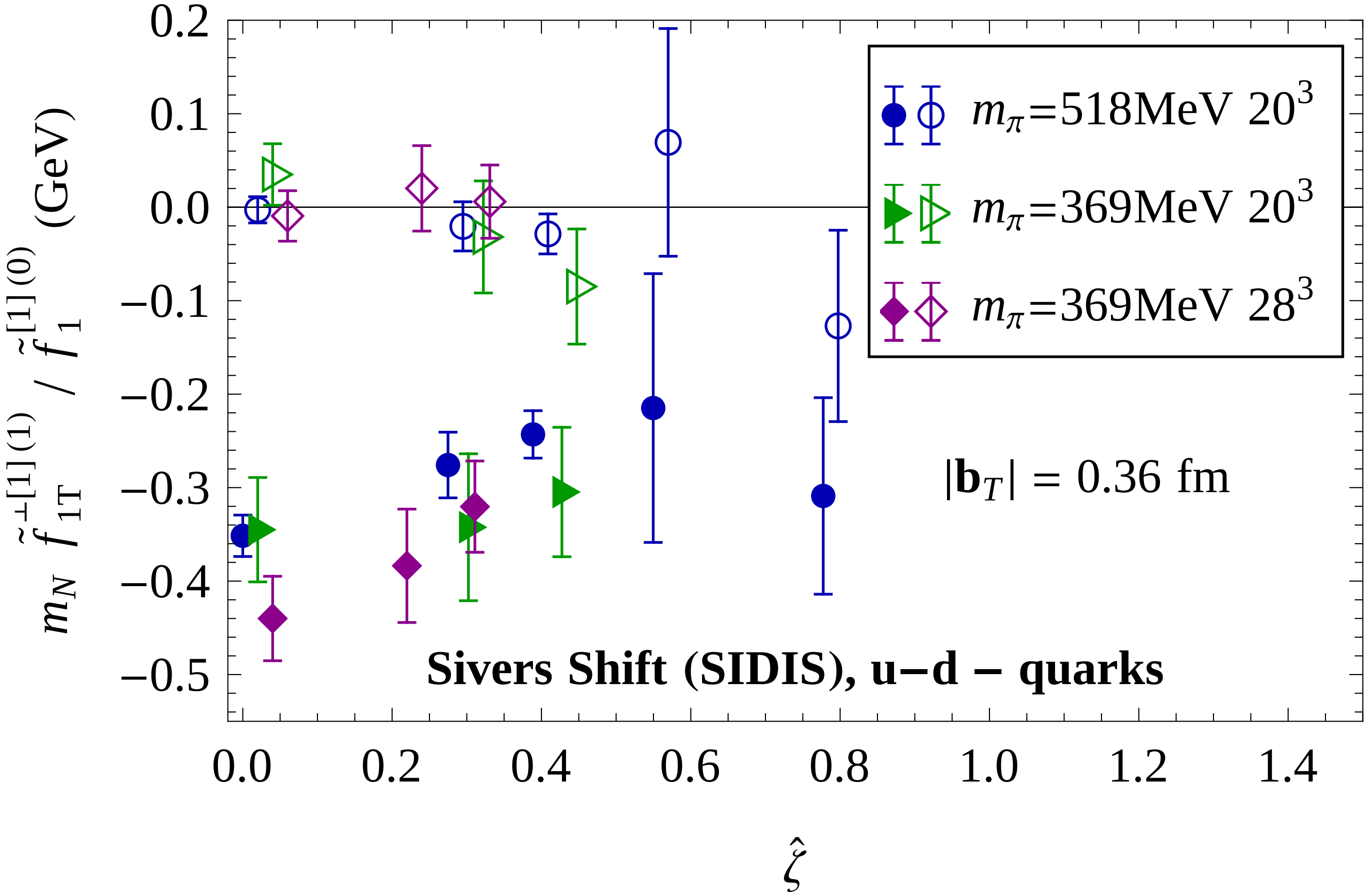
Dependence of SIDIS limit on $\hat{\zeta}$



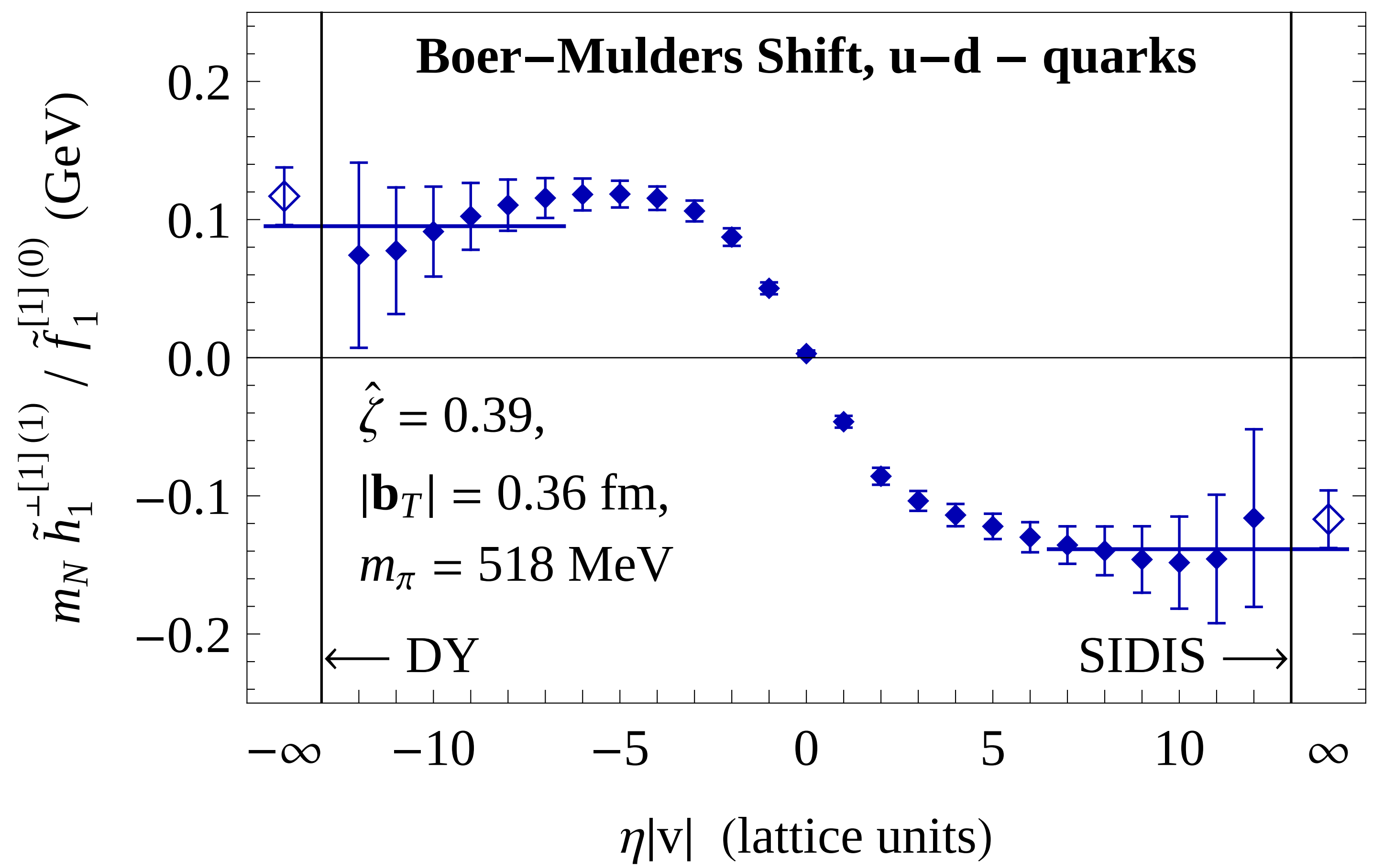
Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$, all three ensembles



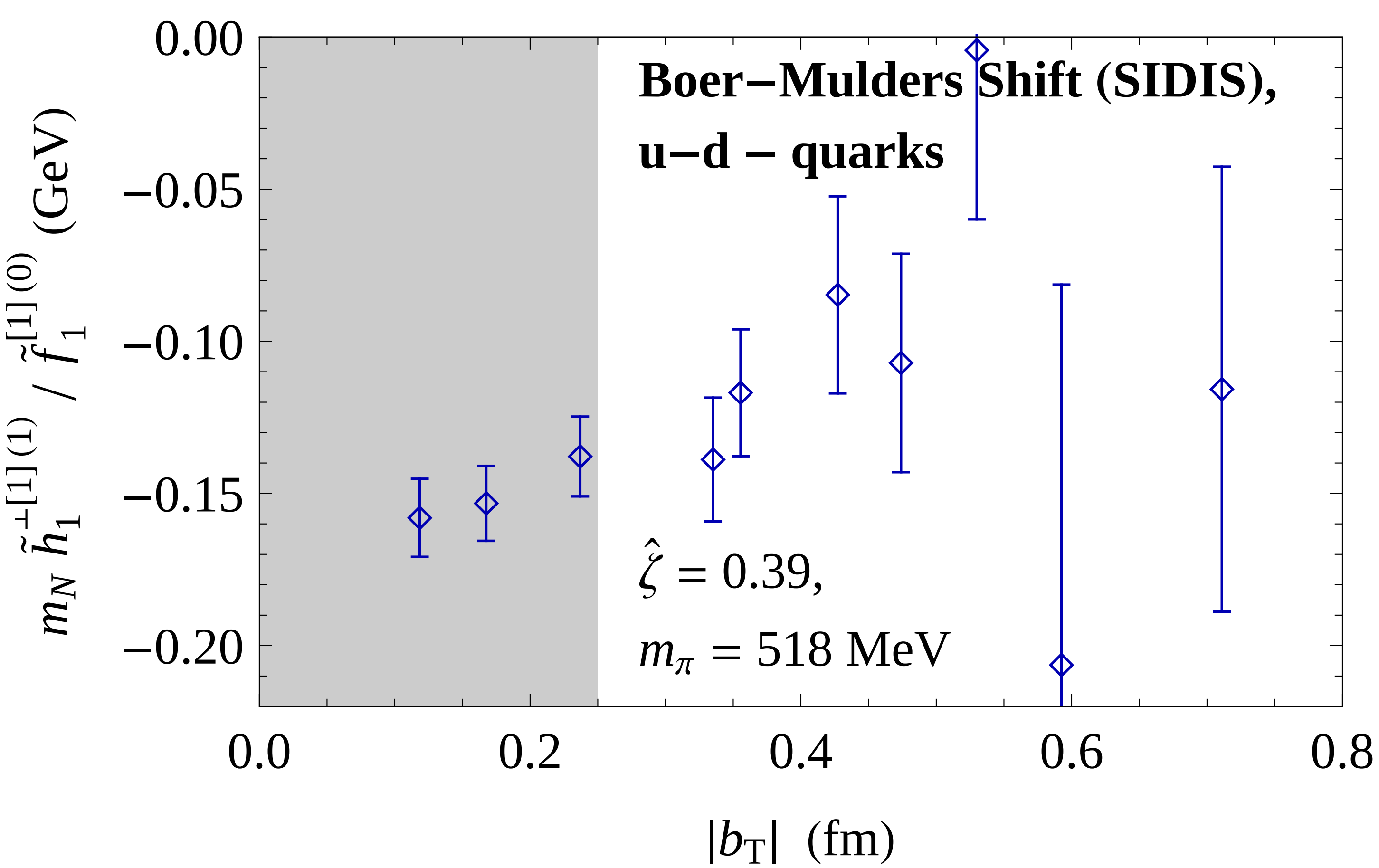
Results: Boer-Mulders shift

Dependence on staple extent



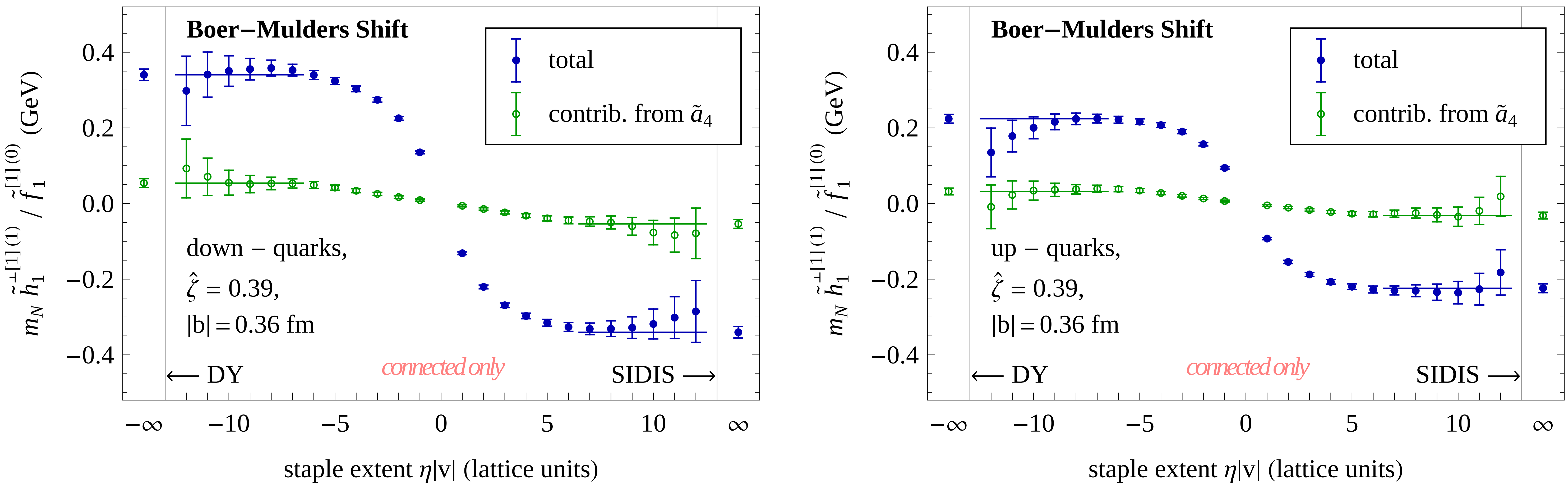
Results: Boer-Mulders shift

Dependence of SIDIS limit on $|b_T|$



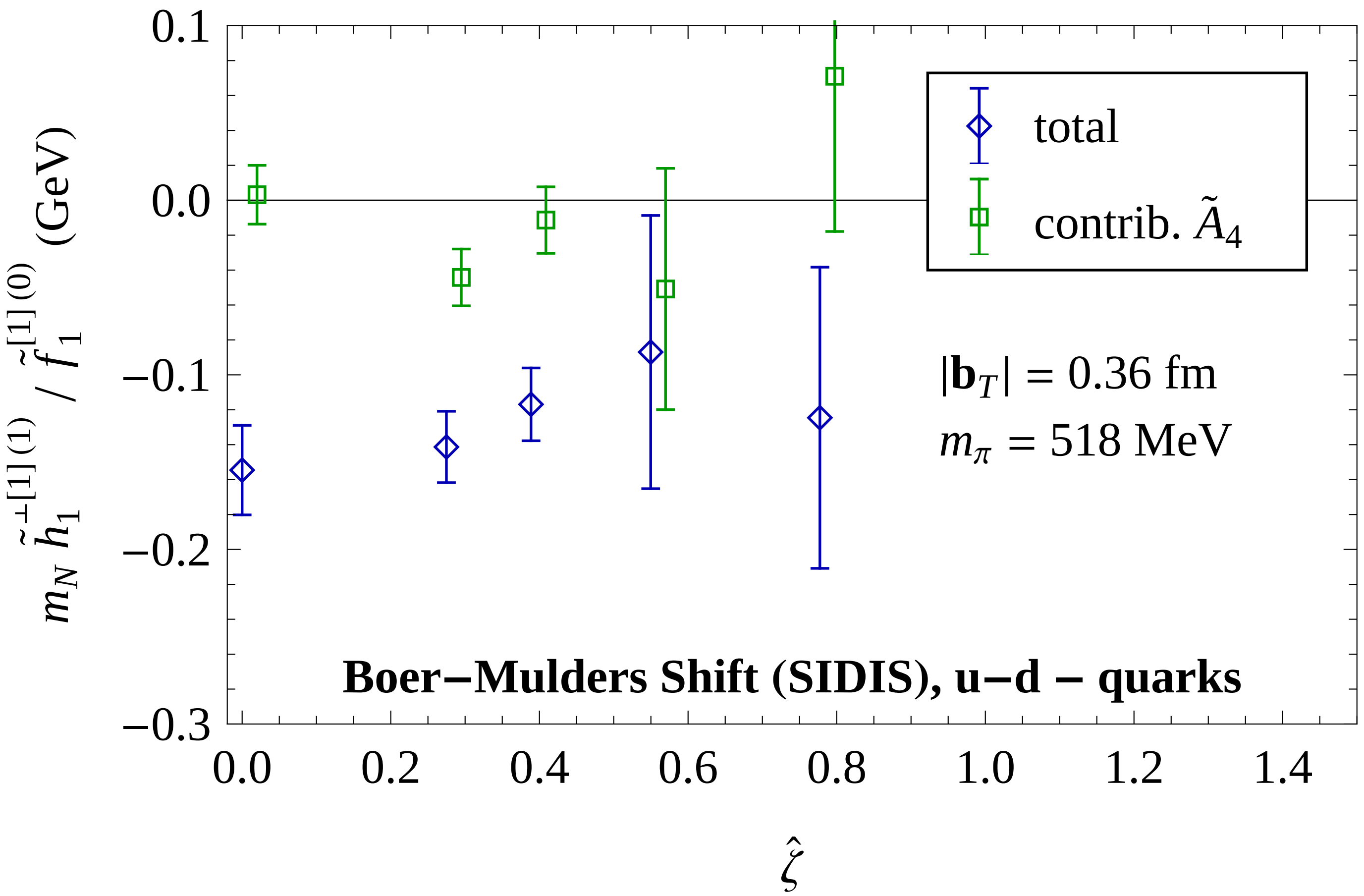
Results: Boer-Mulders shift

Dependence on staple extent; flavor separated



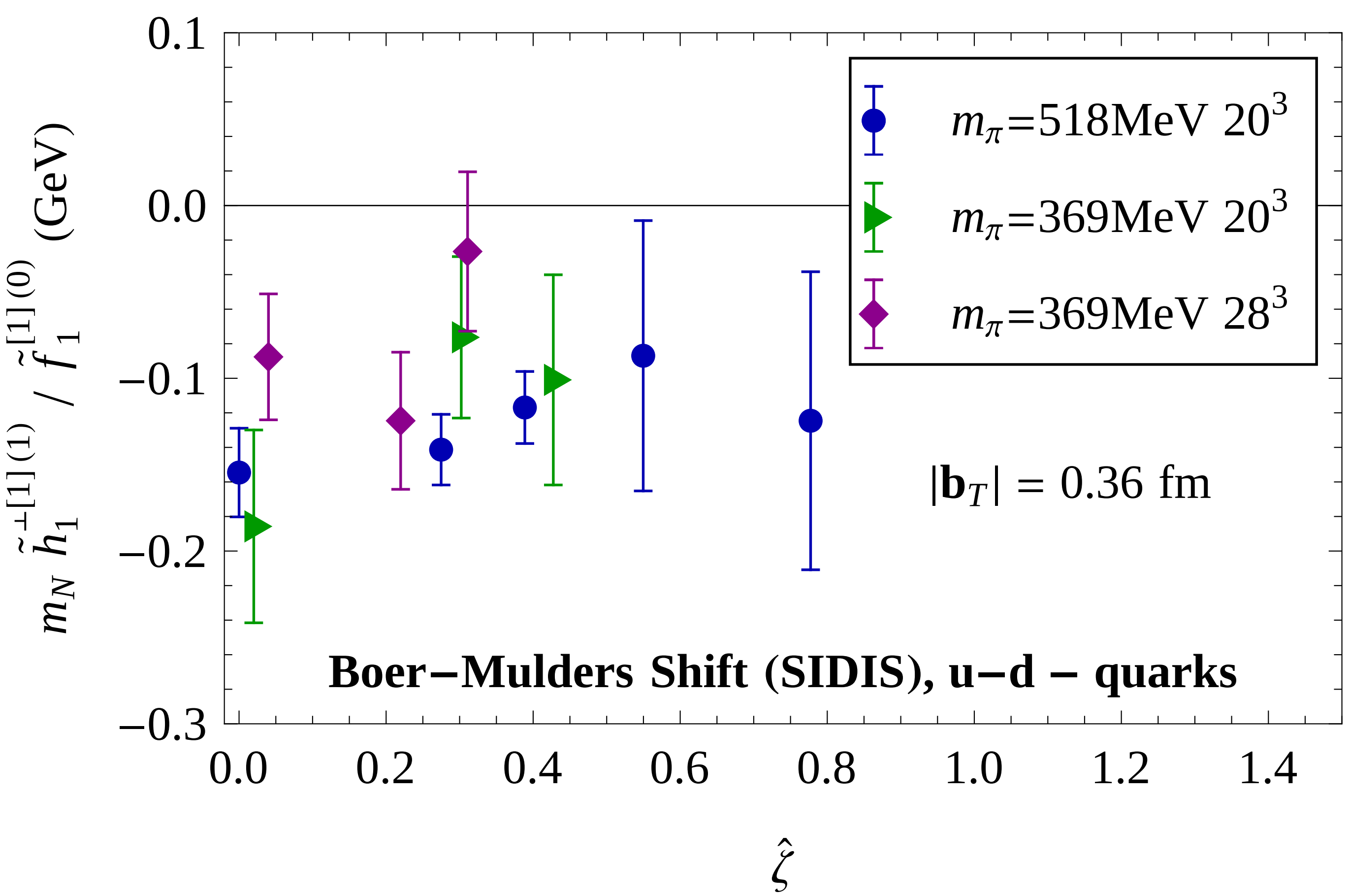
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$



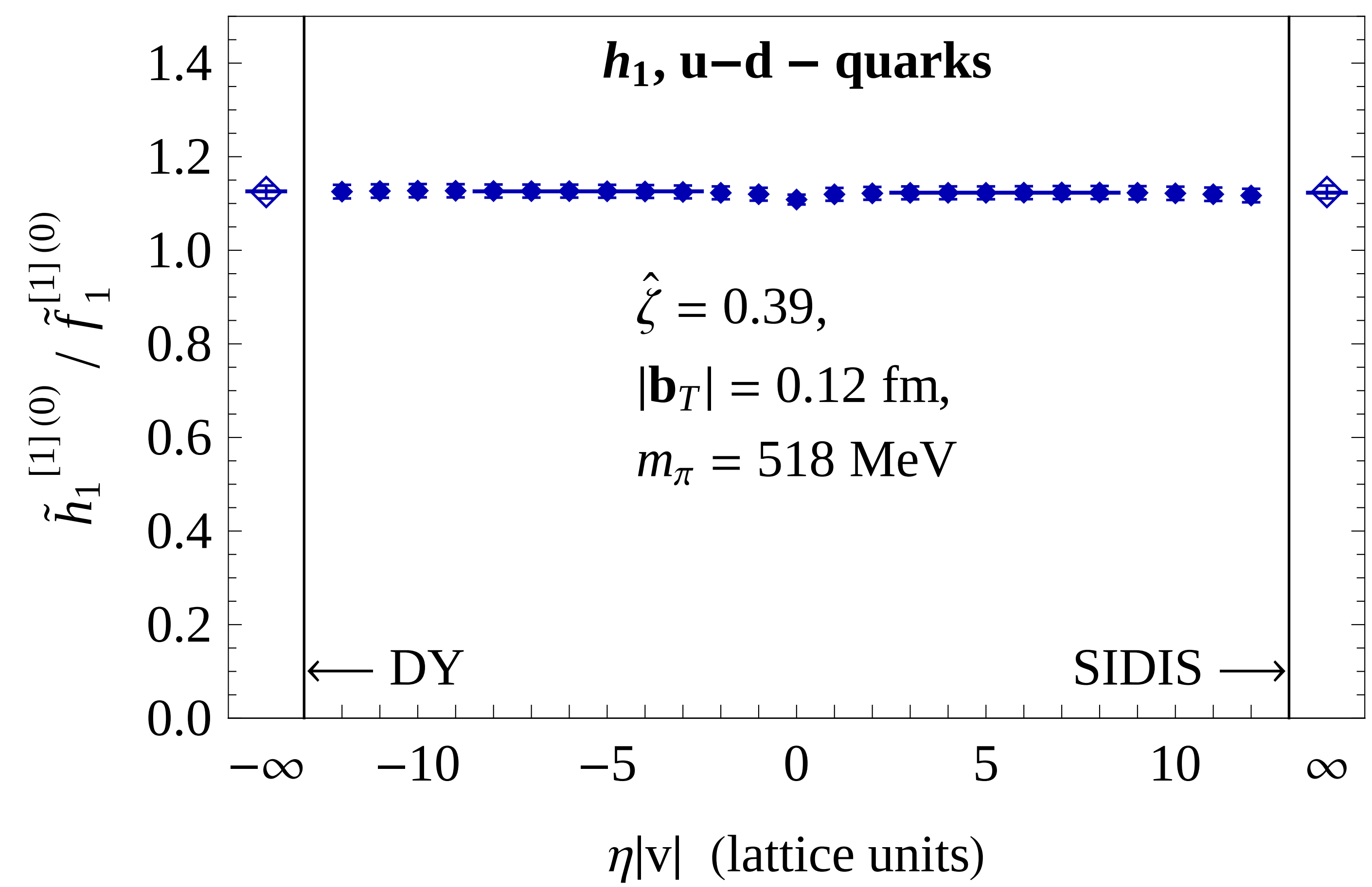
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$, all three ensembles



Results: Transversity

Dependence on staple extent; sequence of panels at different $|b_T|$



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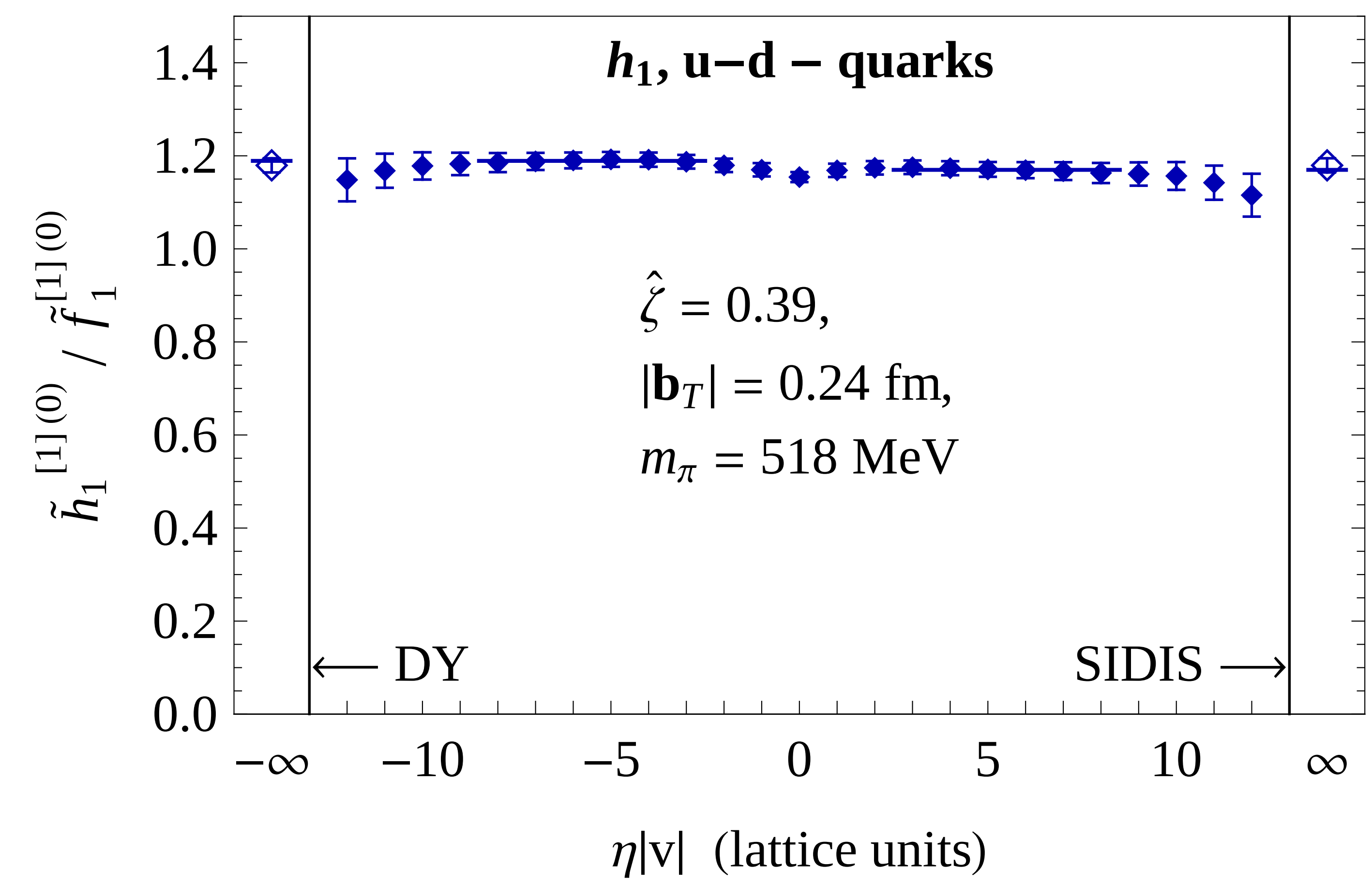

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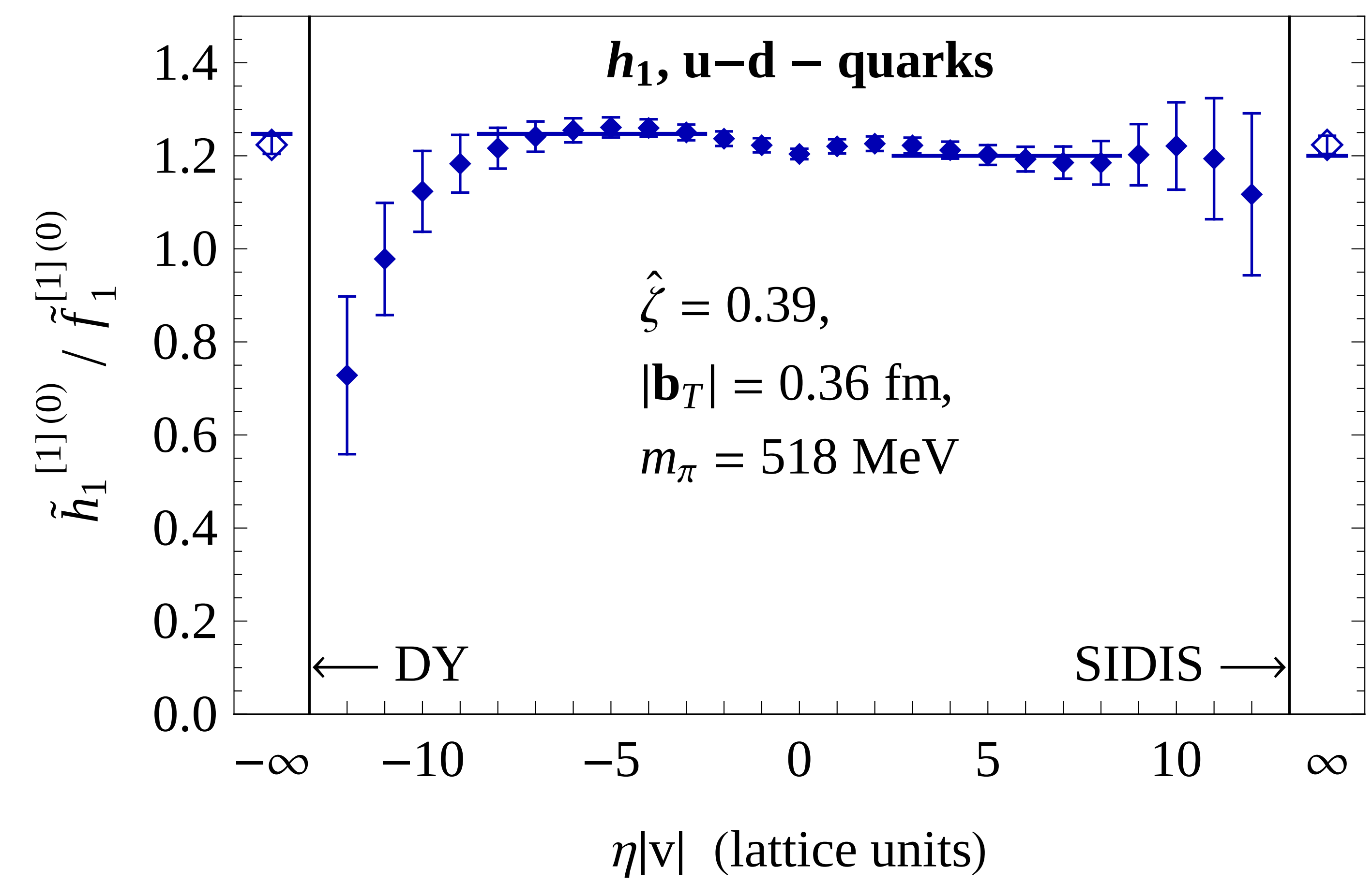
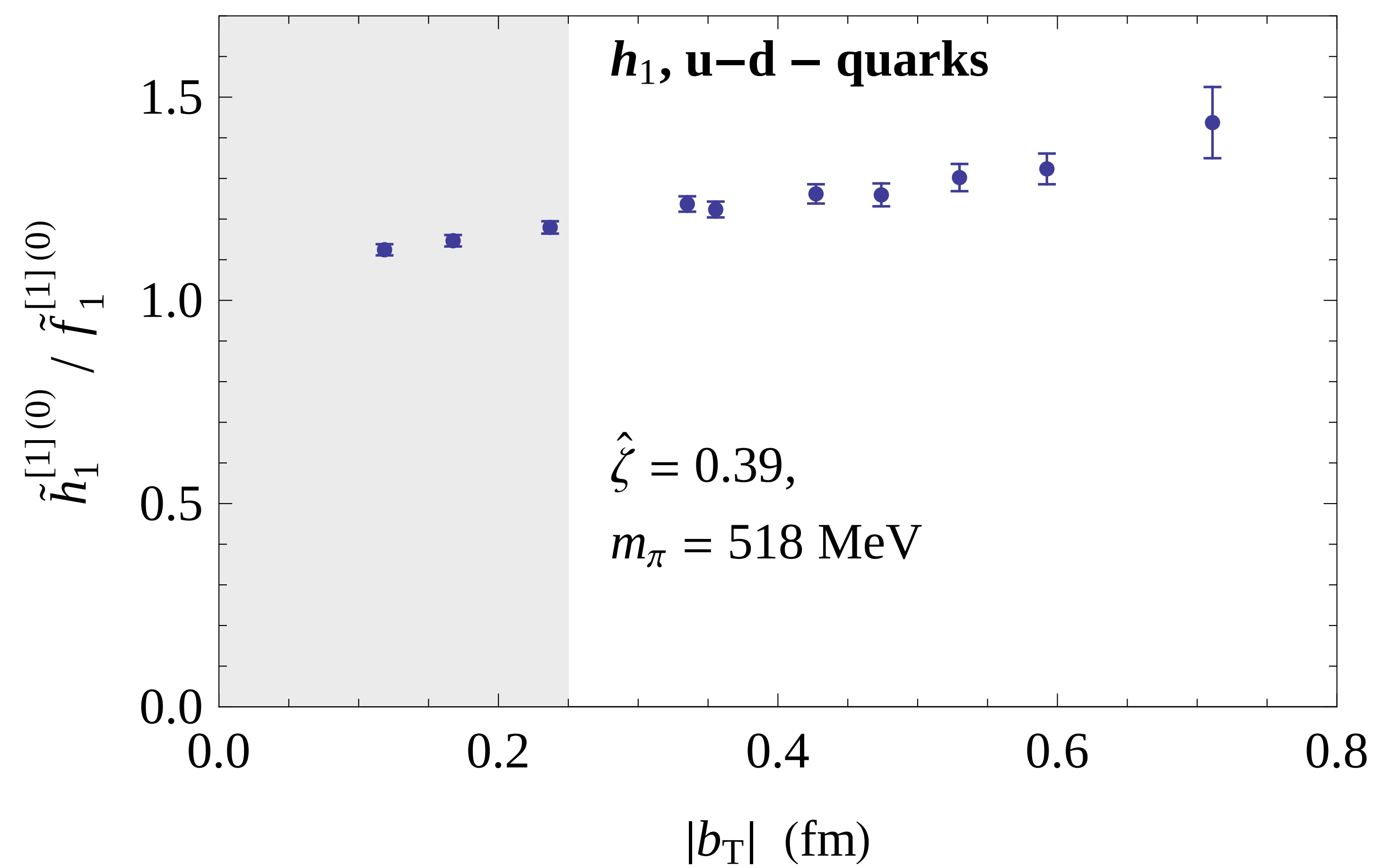


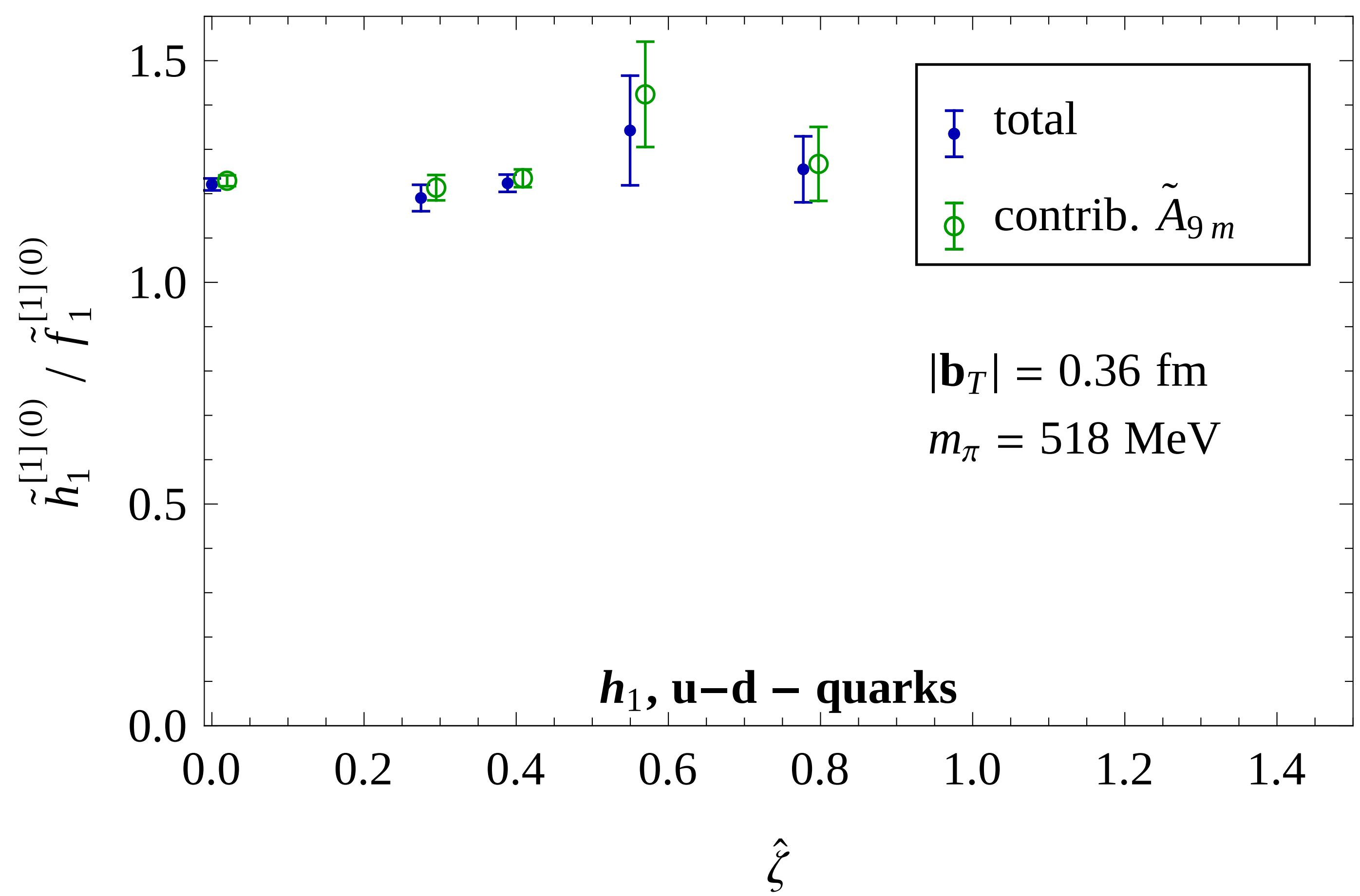
Results: Transversity

Dependence of SIDIS/DY limit on $|b_T|$



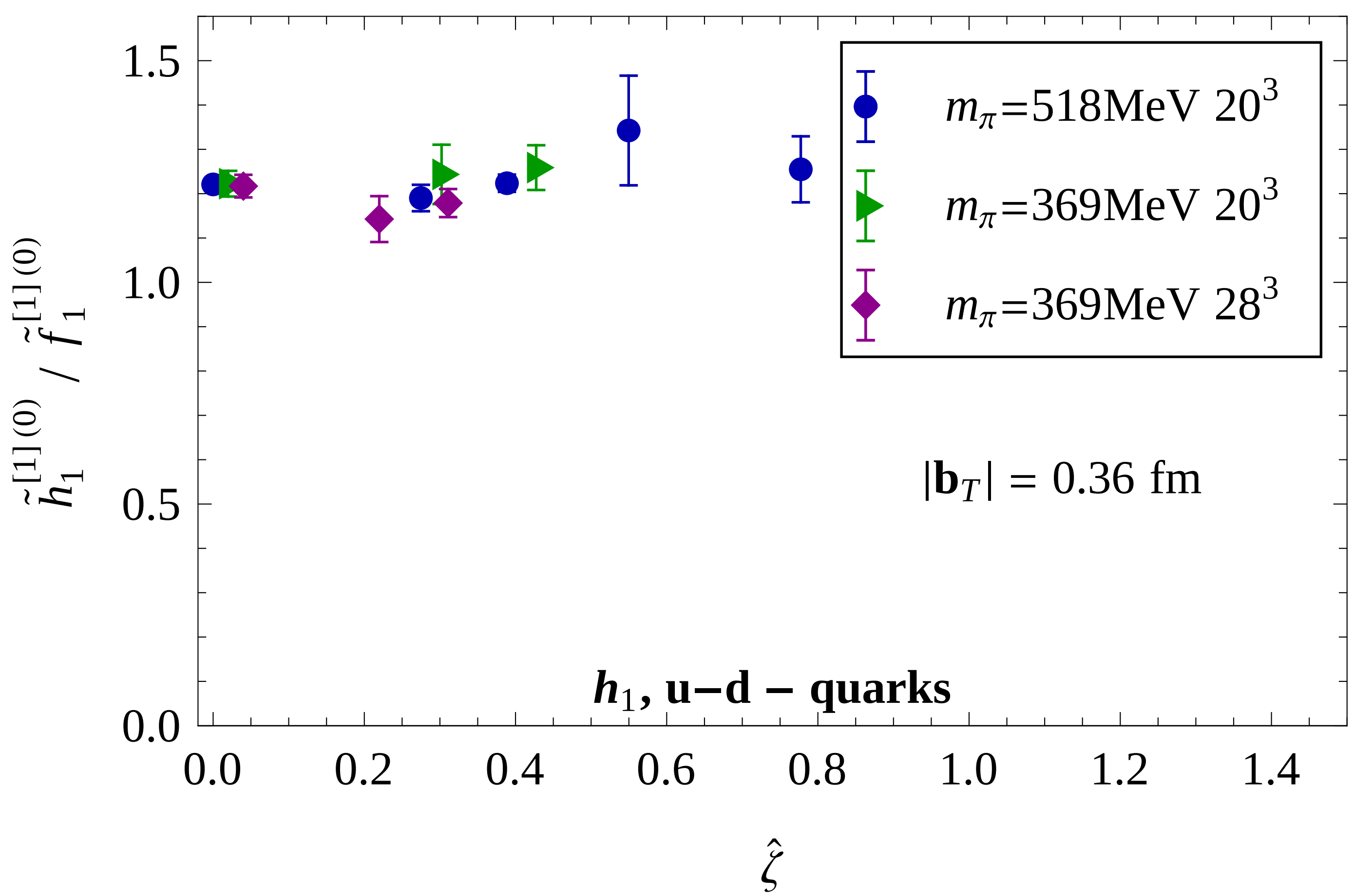
Results: Transversity

Dependence of SIDIS/DY limit on $\hat{\zeta}$



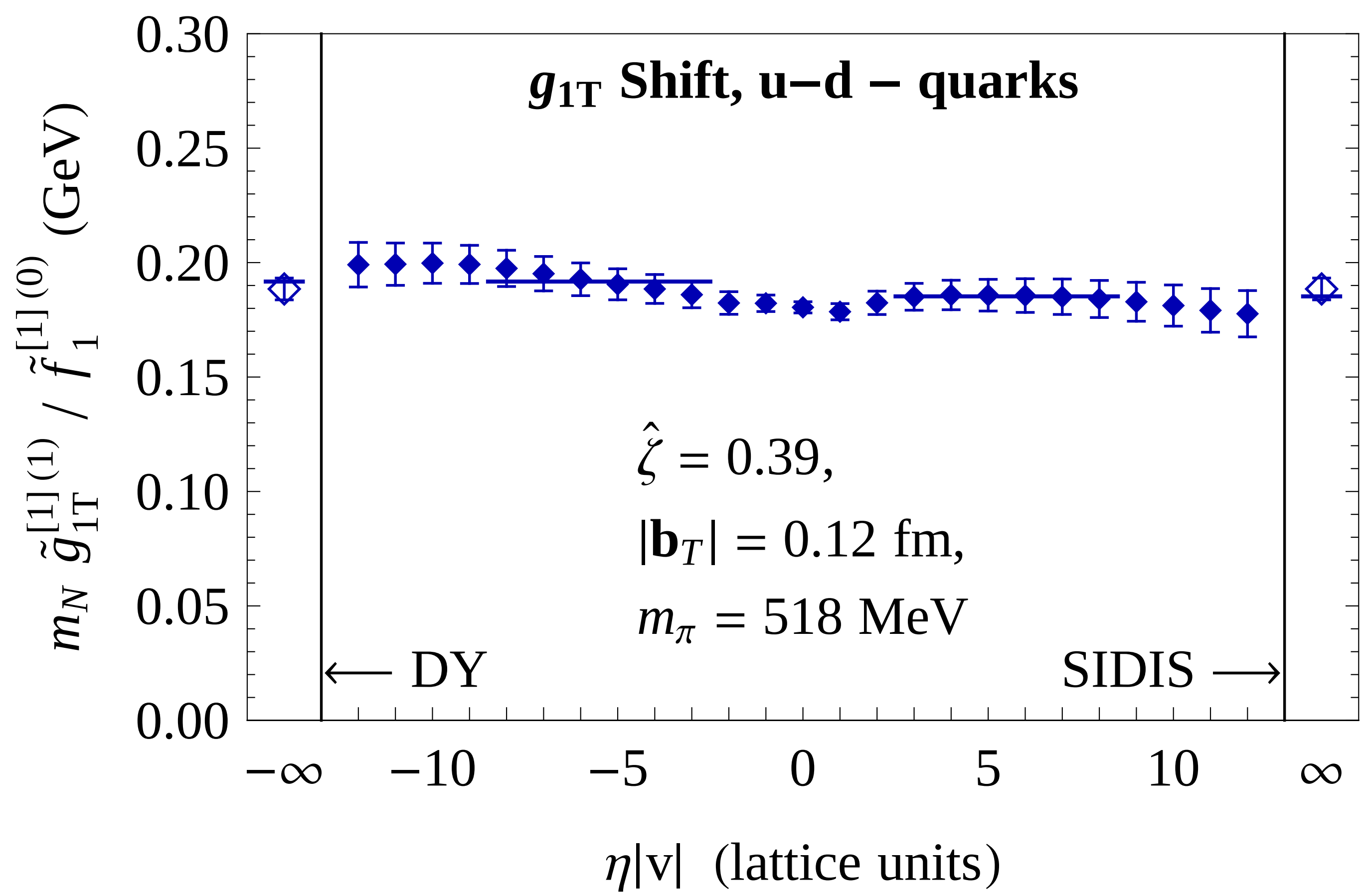
Results: Transversity

Dependence of SIDIS/DY limit on $\hat{\zeta}$, all three ensembles



Results: g_{1T} worm gear shift

Dependence on staple extent; sequence of panels at different $|b_T|$



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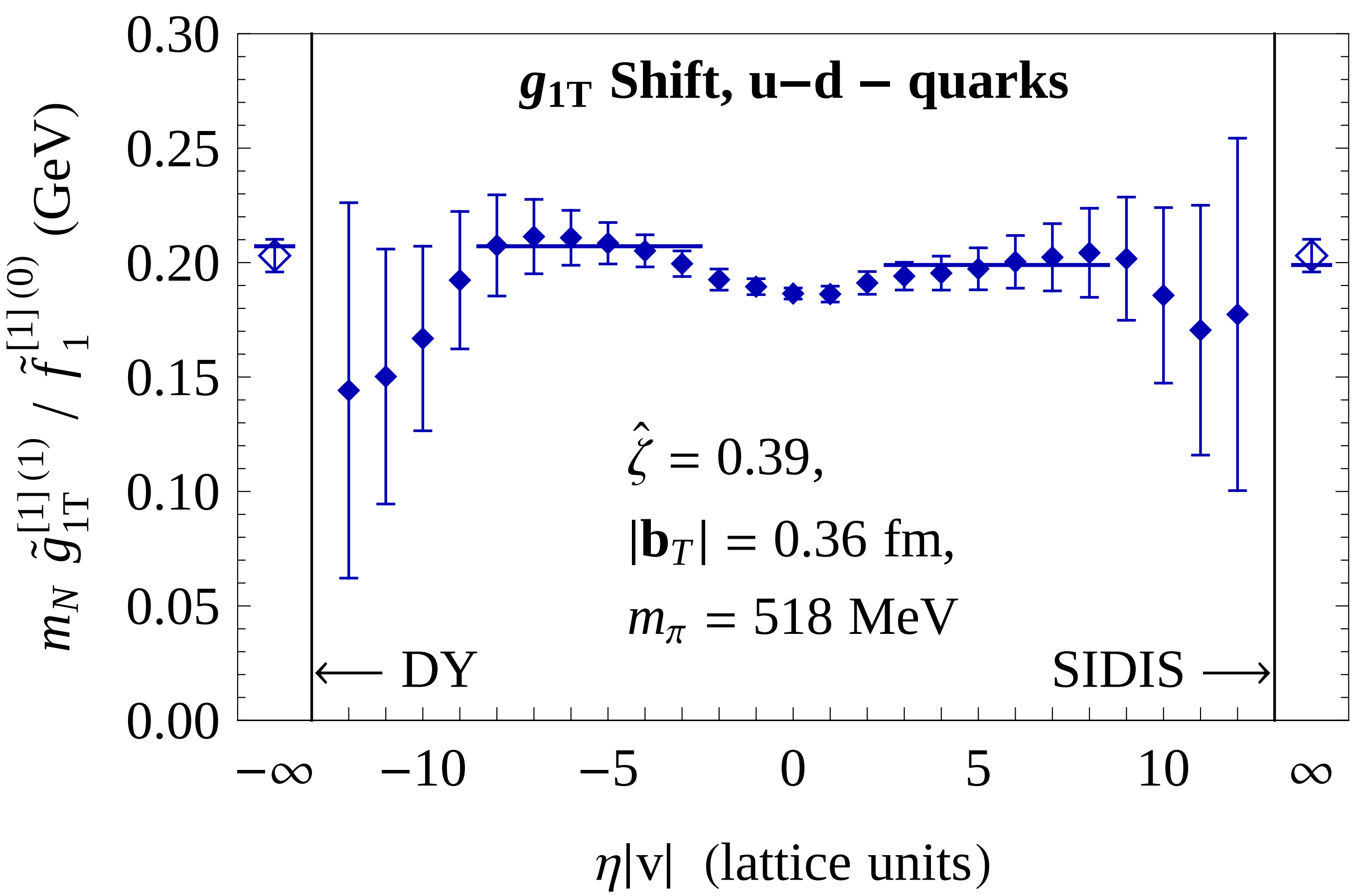

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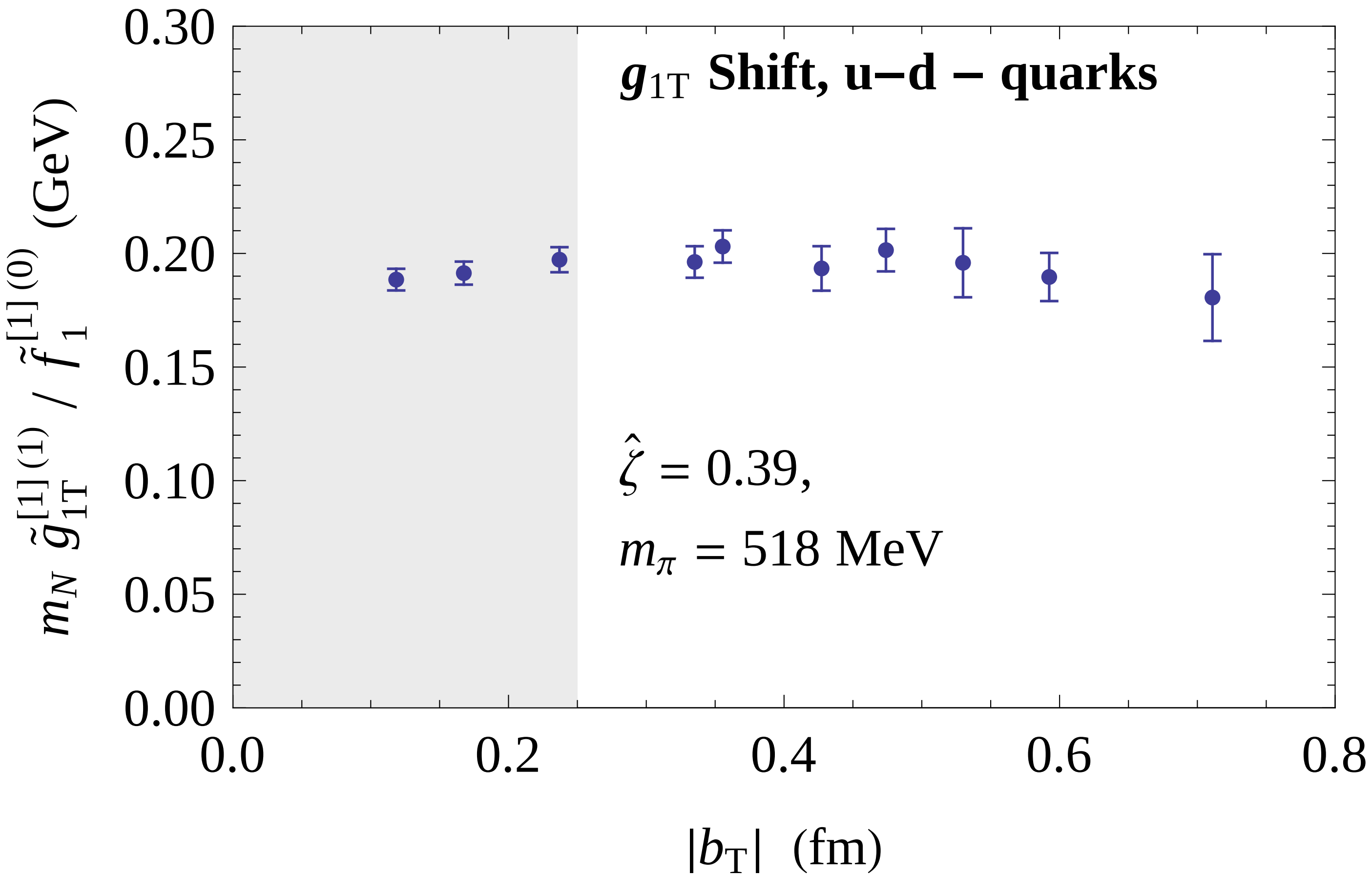

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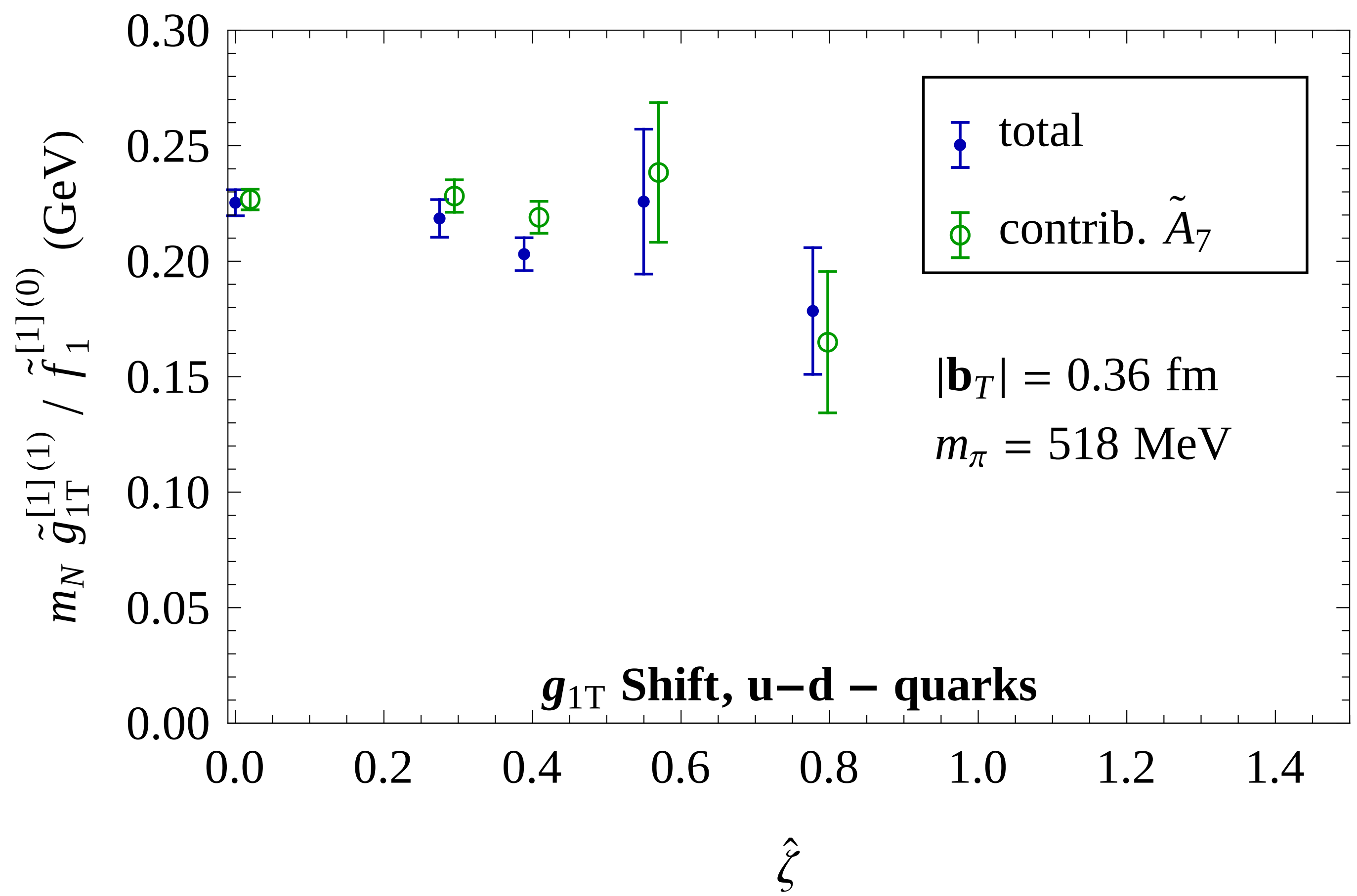

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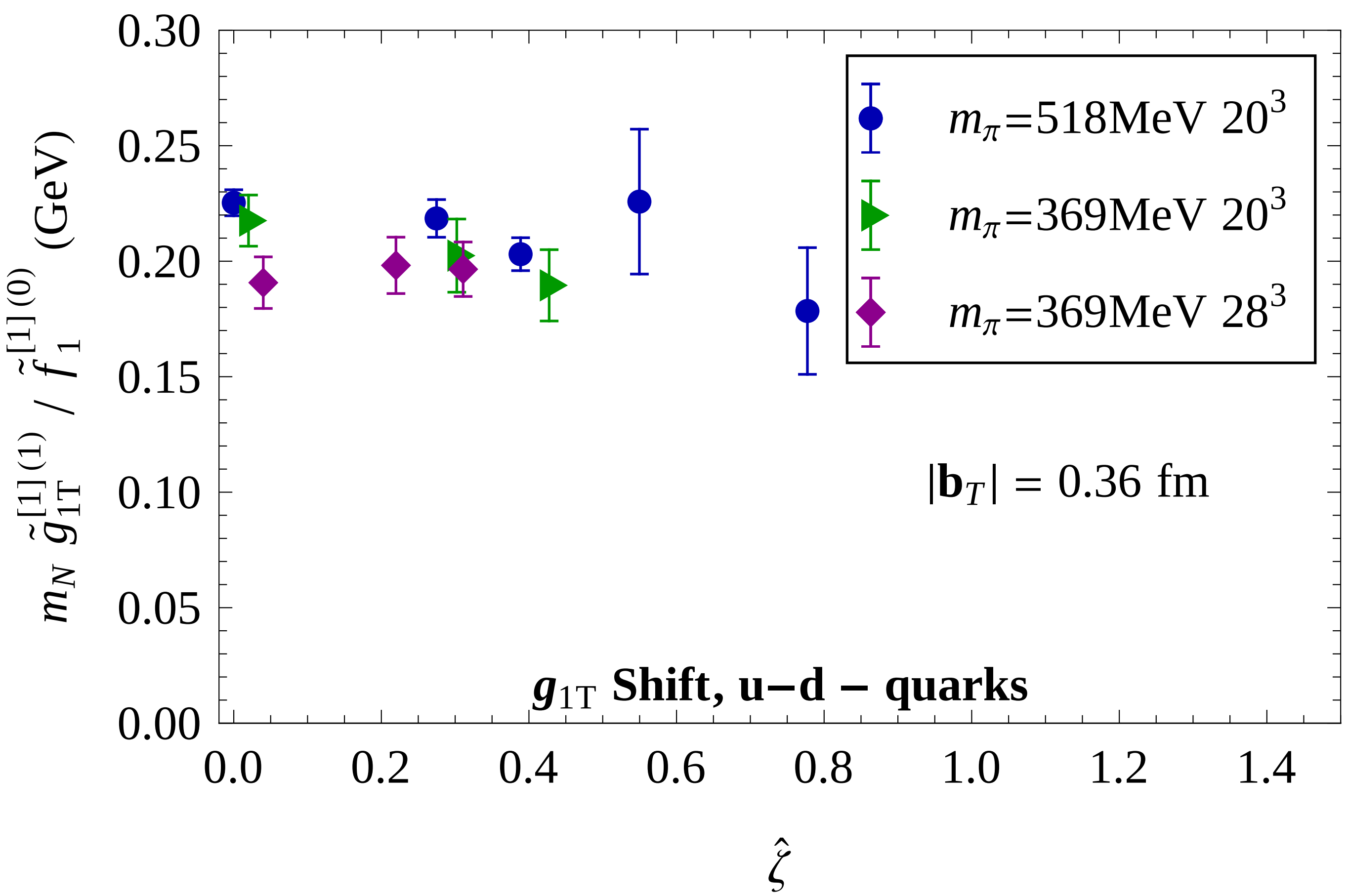
Dependence of SIDIS/DY limit on $\hat{\zeta}$, all three ensembles



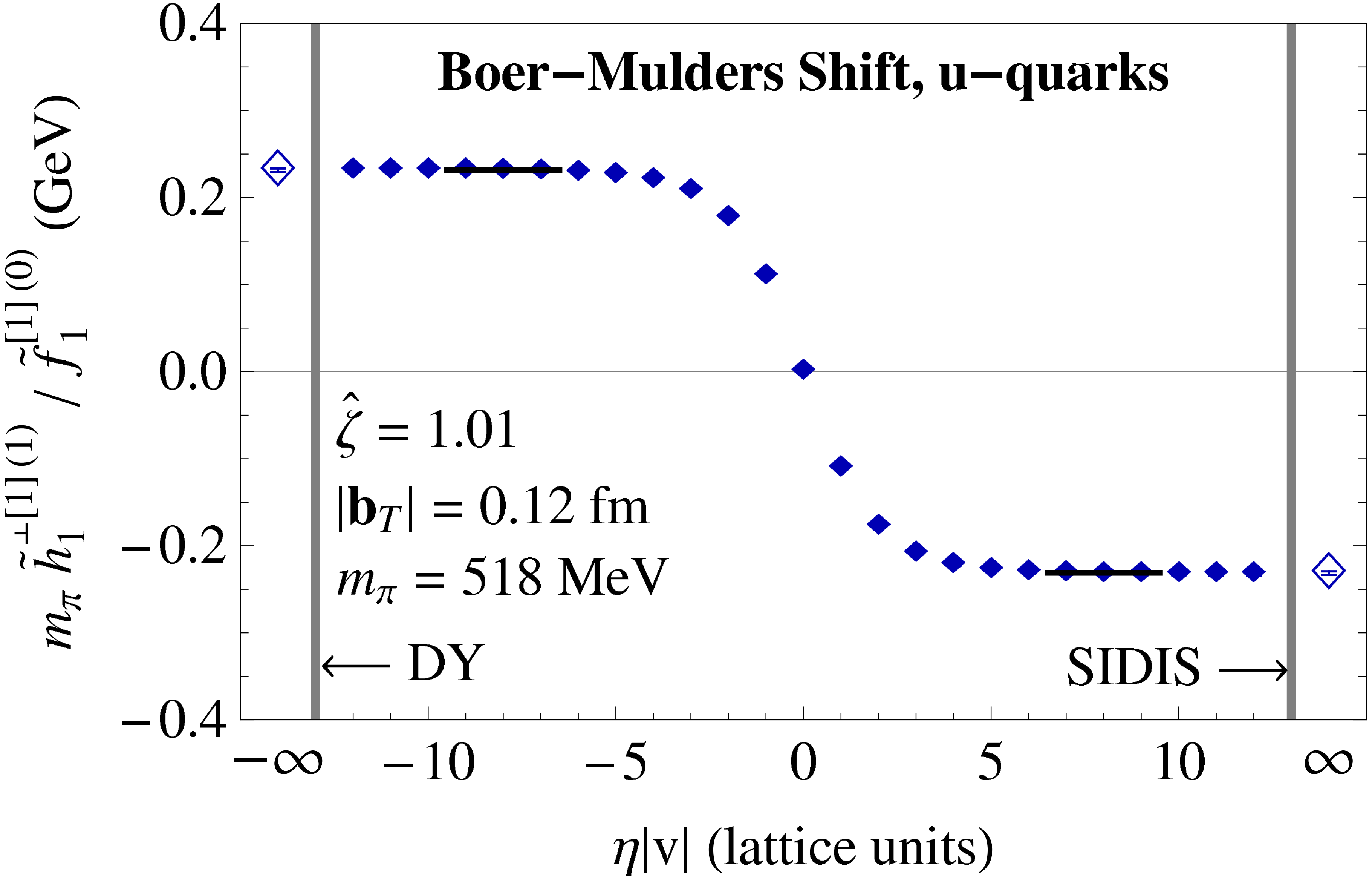
Challenges

- The limit $\hat{\zeta} \to \infty$: Approaching the light cone
- Discretization effects, soft factor cancellation on the lattice in TMD ratios
- Progress toward the physical pion mass

Approaching the light cone (with a pion)

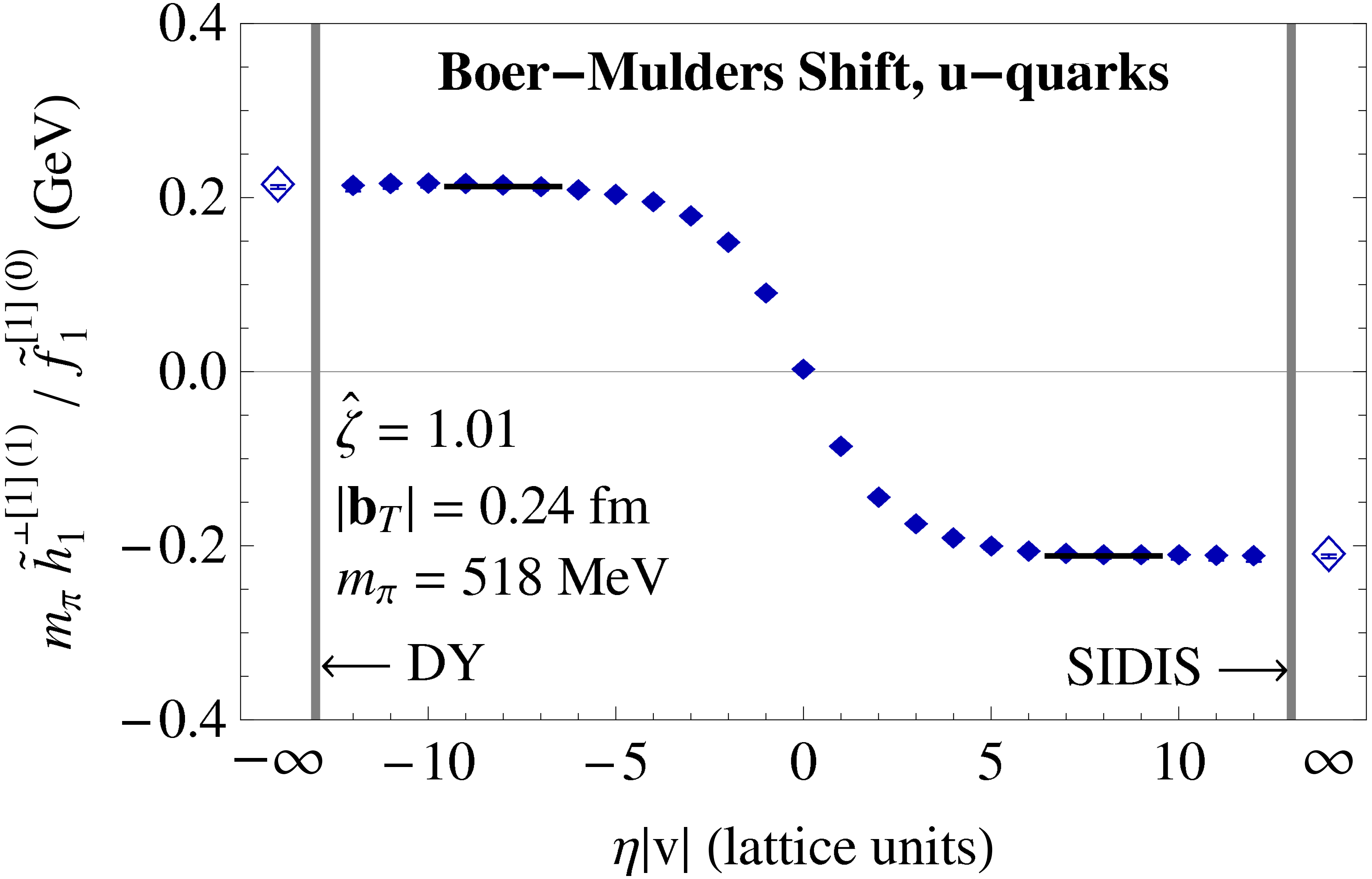
Results: Boer-Mulders shift (pion)

Dependence on staple extent; sequence of panels at different $|b_T|$



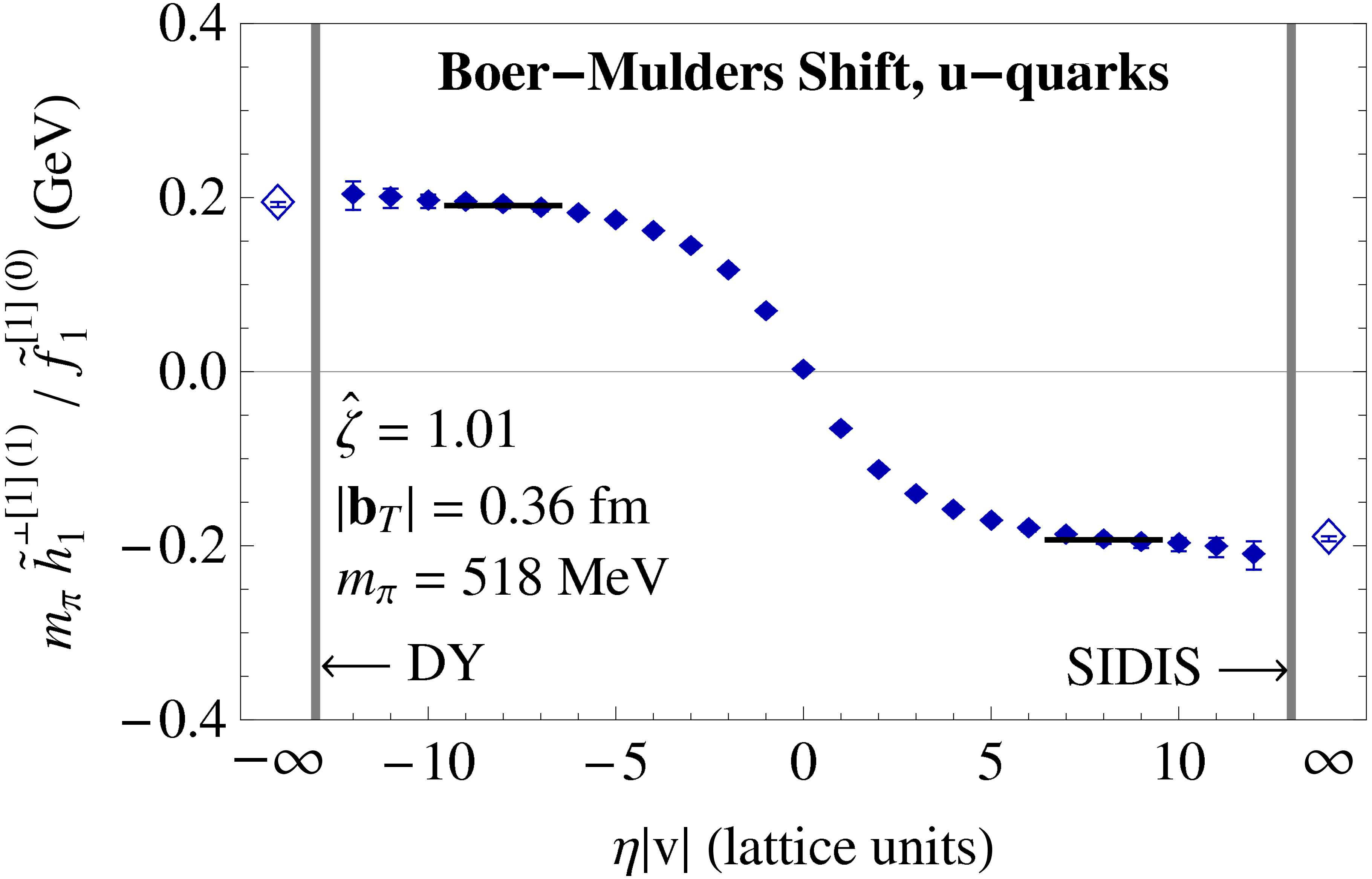
Results: Boer-Mulders shift (pion)

Dependence on staple extent; sequence of panels at different $|b_T|$



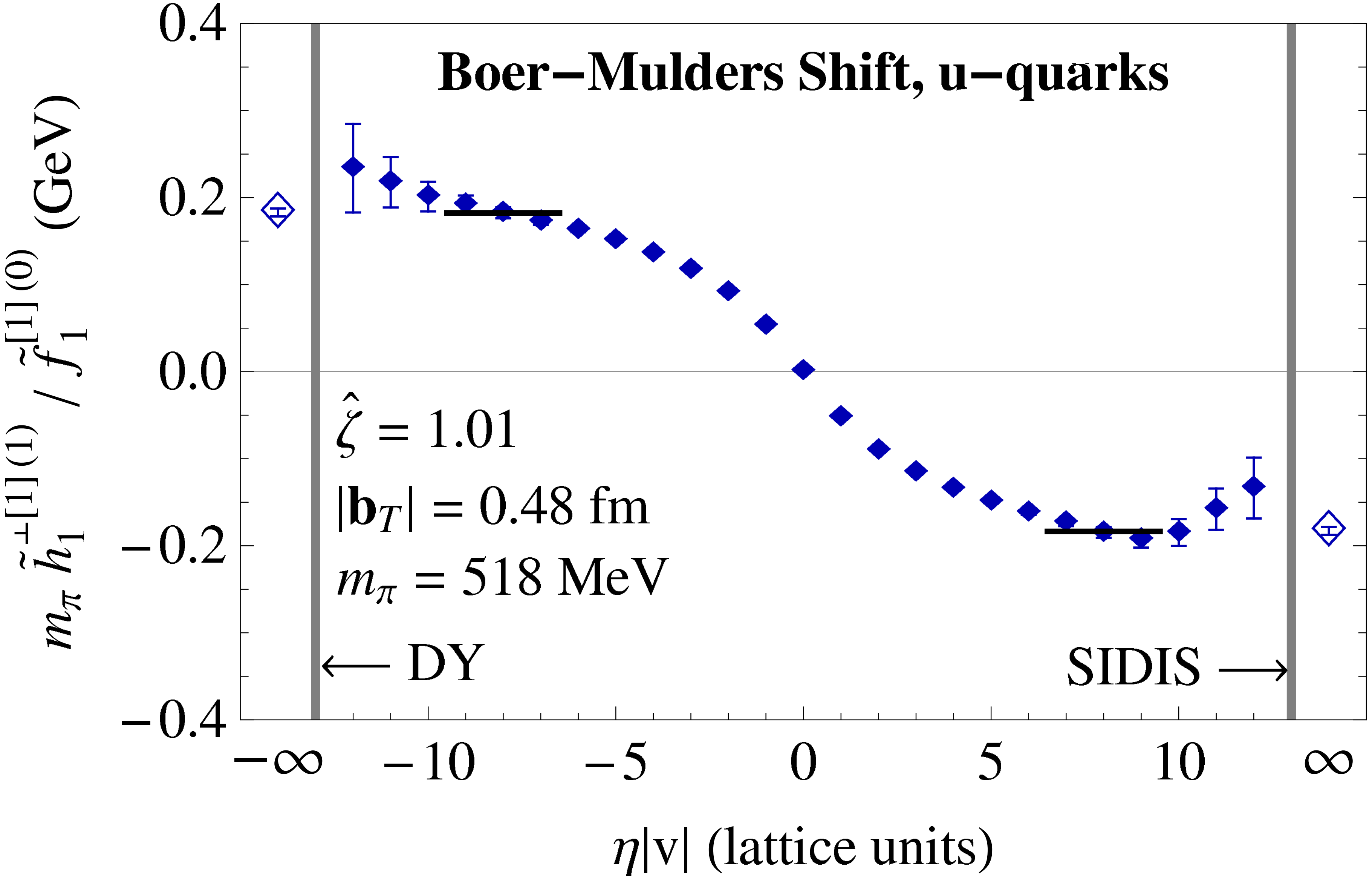
Results: Boer-Mulders shift (pion)

Dependence on staple extent; sequence of panels at different $|b_T|$



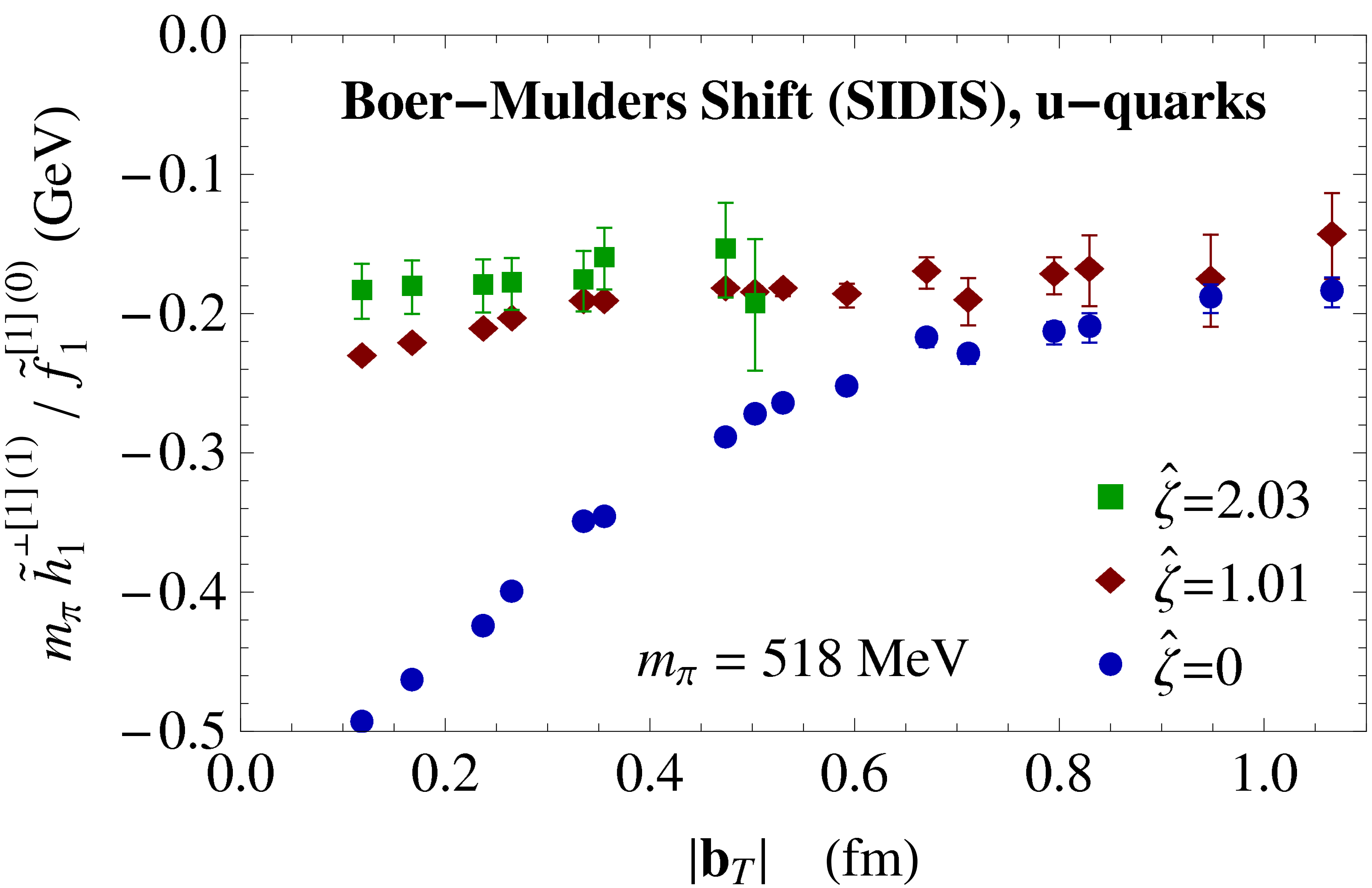
Results: Boer-Mulders shift (pion)

Dependence on staple extent; sequence of panels at different $|b_T|$



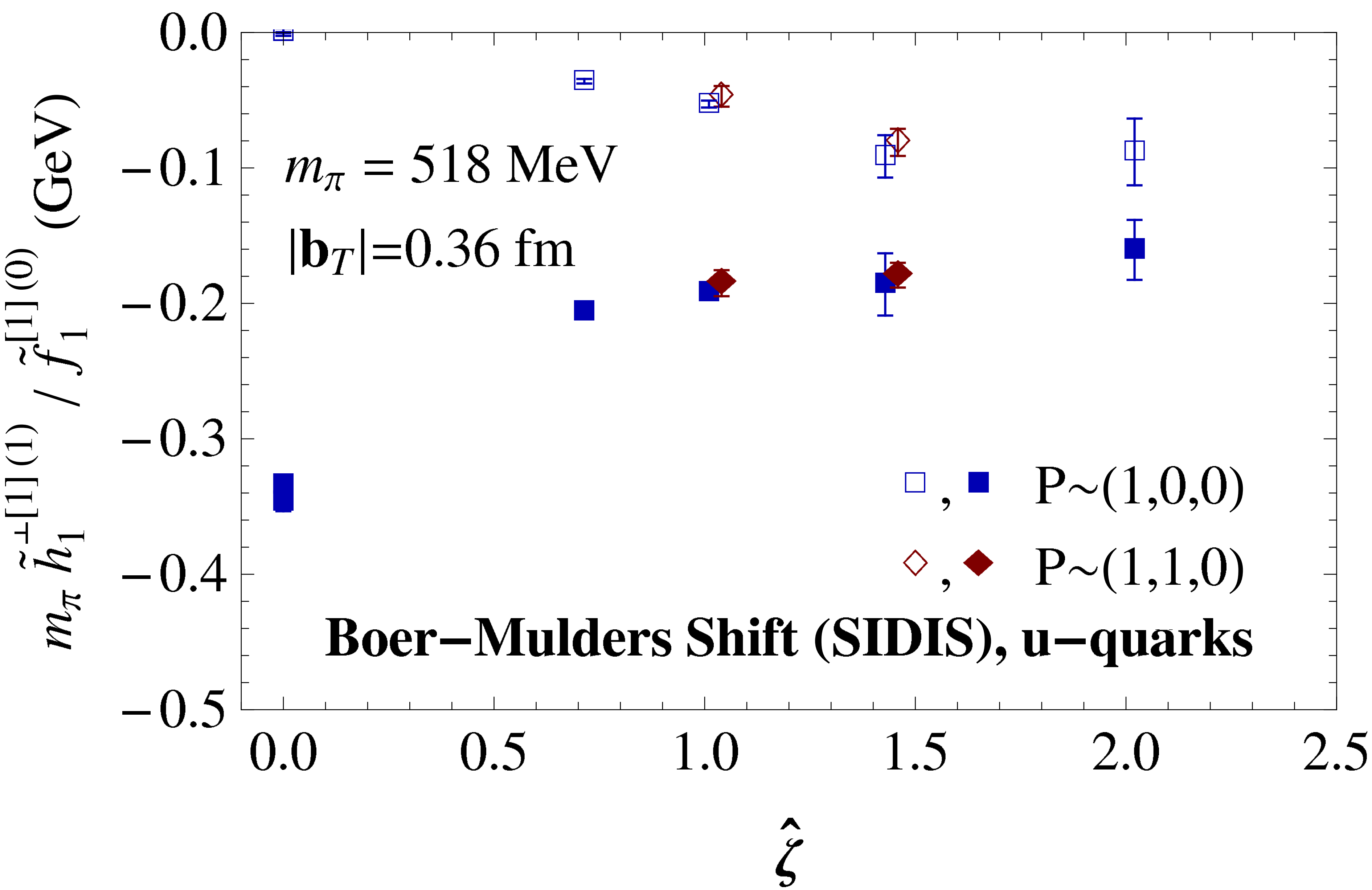
Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $|b_T|$



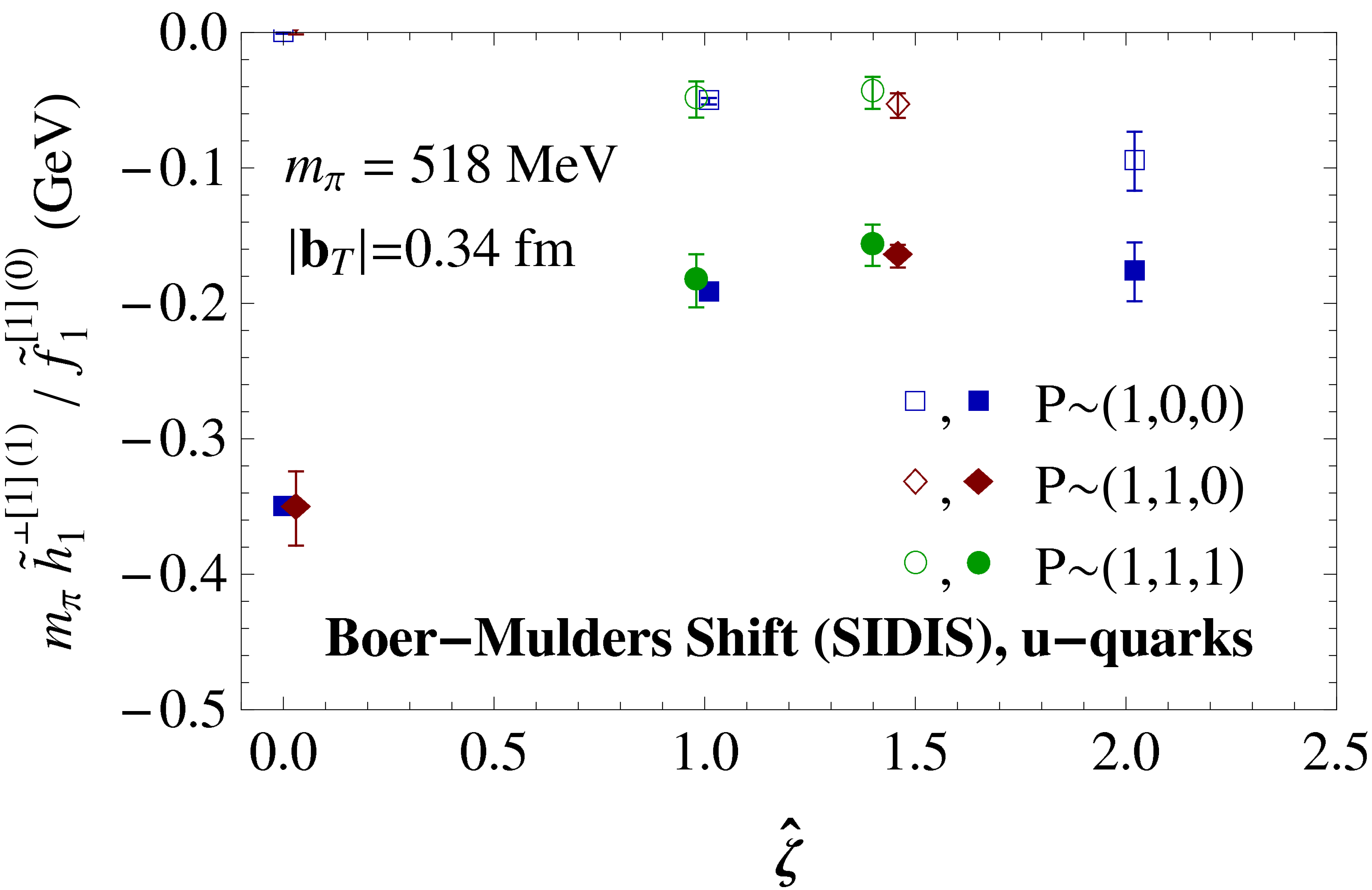
Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $\hat{\zeta}$; open symbols: contribution $\widetilde{A}_4$ only



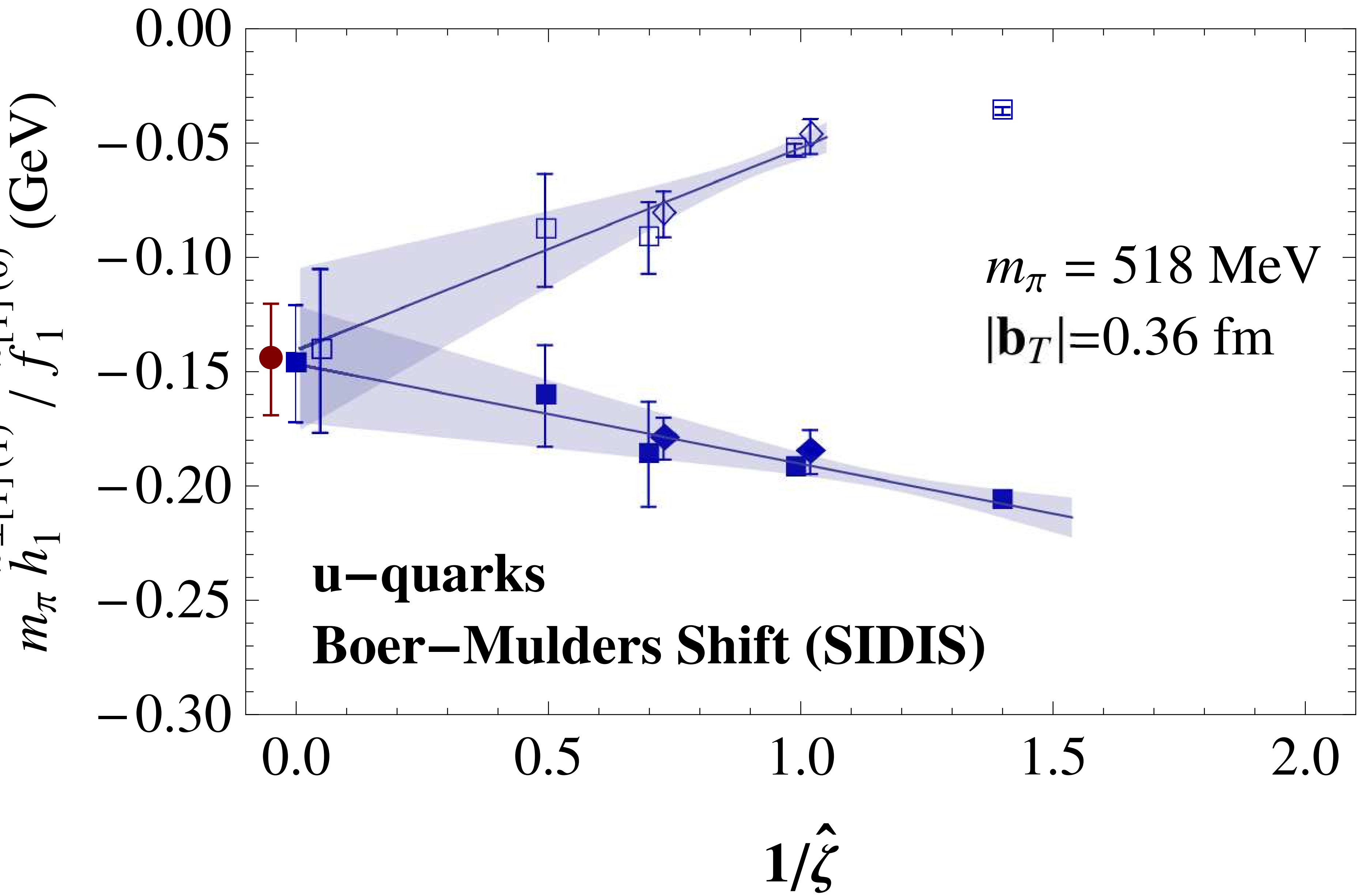
Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $\hat{\zeta}$; open symbols: contribution $\widetilde{A}_4$ only



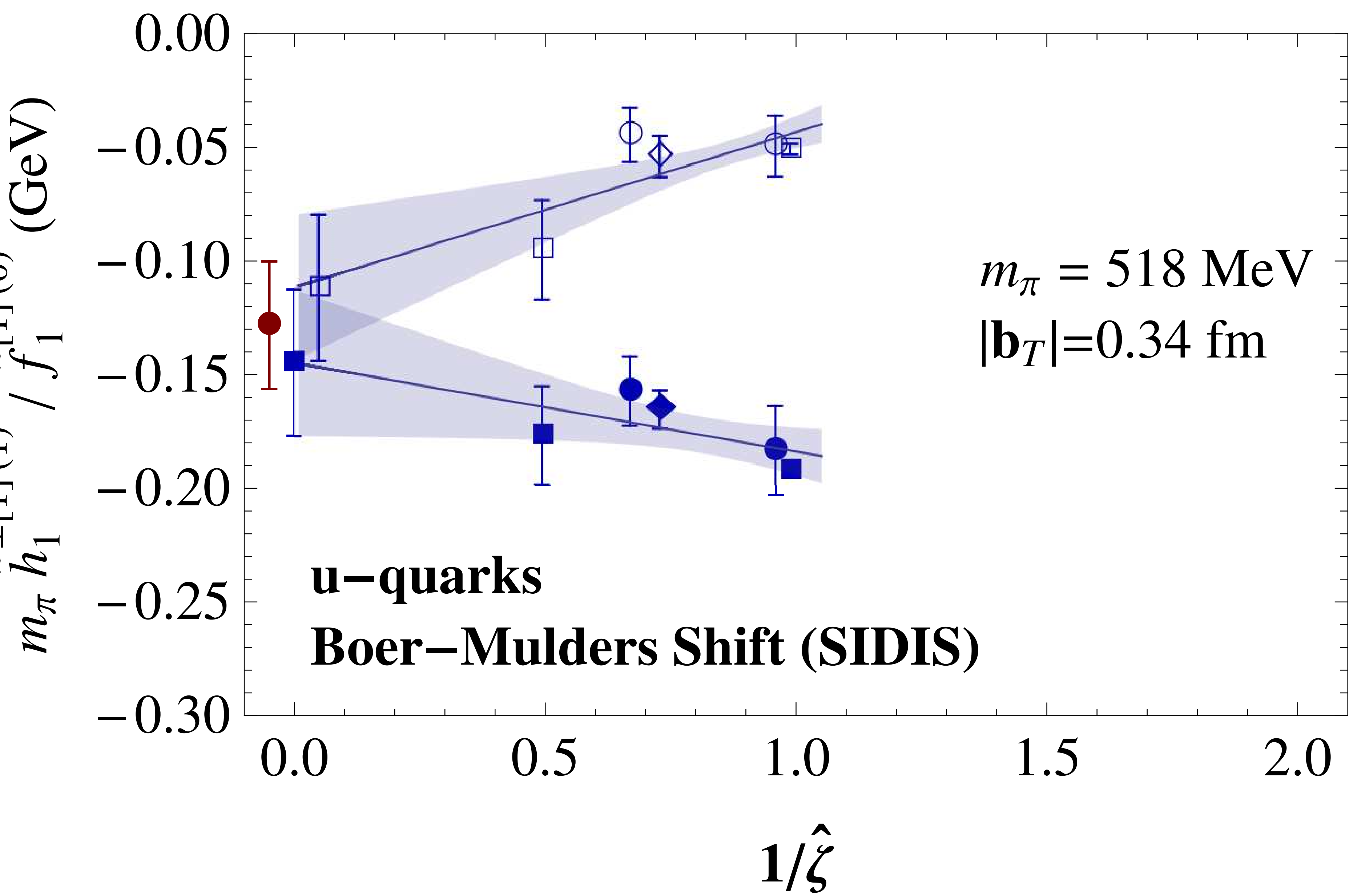
Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $\hat{\zeta}$; fit function $a + b/\hat{\zeta}$



Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $\hat{\zeta}$; fit function $a + b/\hat{\zeta}$



Discretization effects:

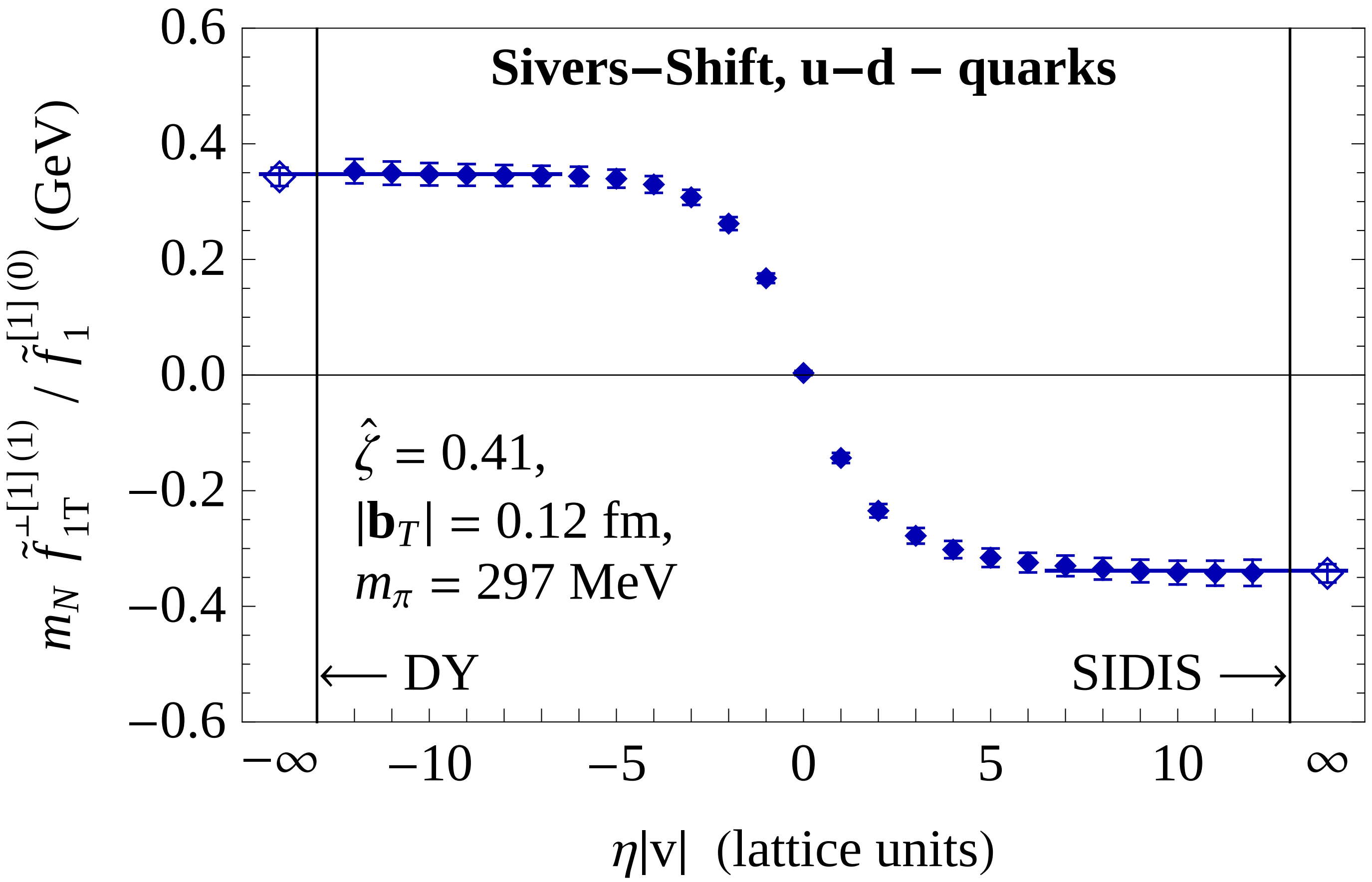
Comparison of

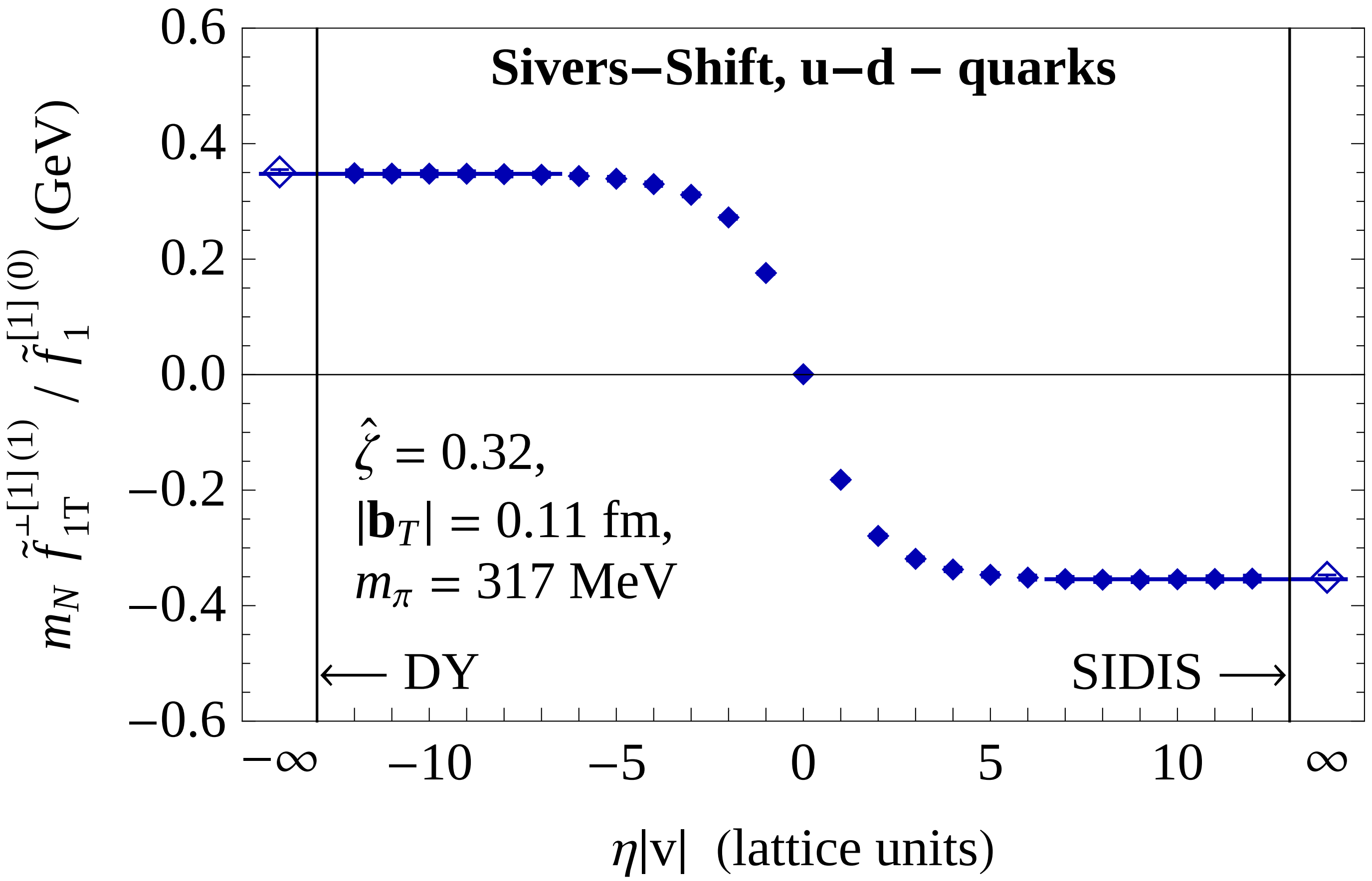
RBC/UKQCD DWF ensemble ($m_\pi = 297\,$MeV, $a = 0.084\,$fm)

with clover ensemble ($m_\pi = 317\,$MeV, $a = 0.114\,$fm)
produced by K. Orginos and JLab collaborators

Results: Sivers shift

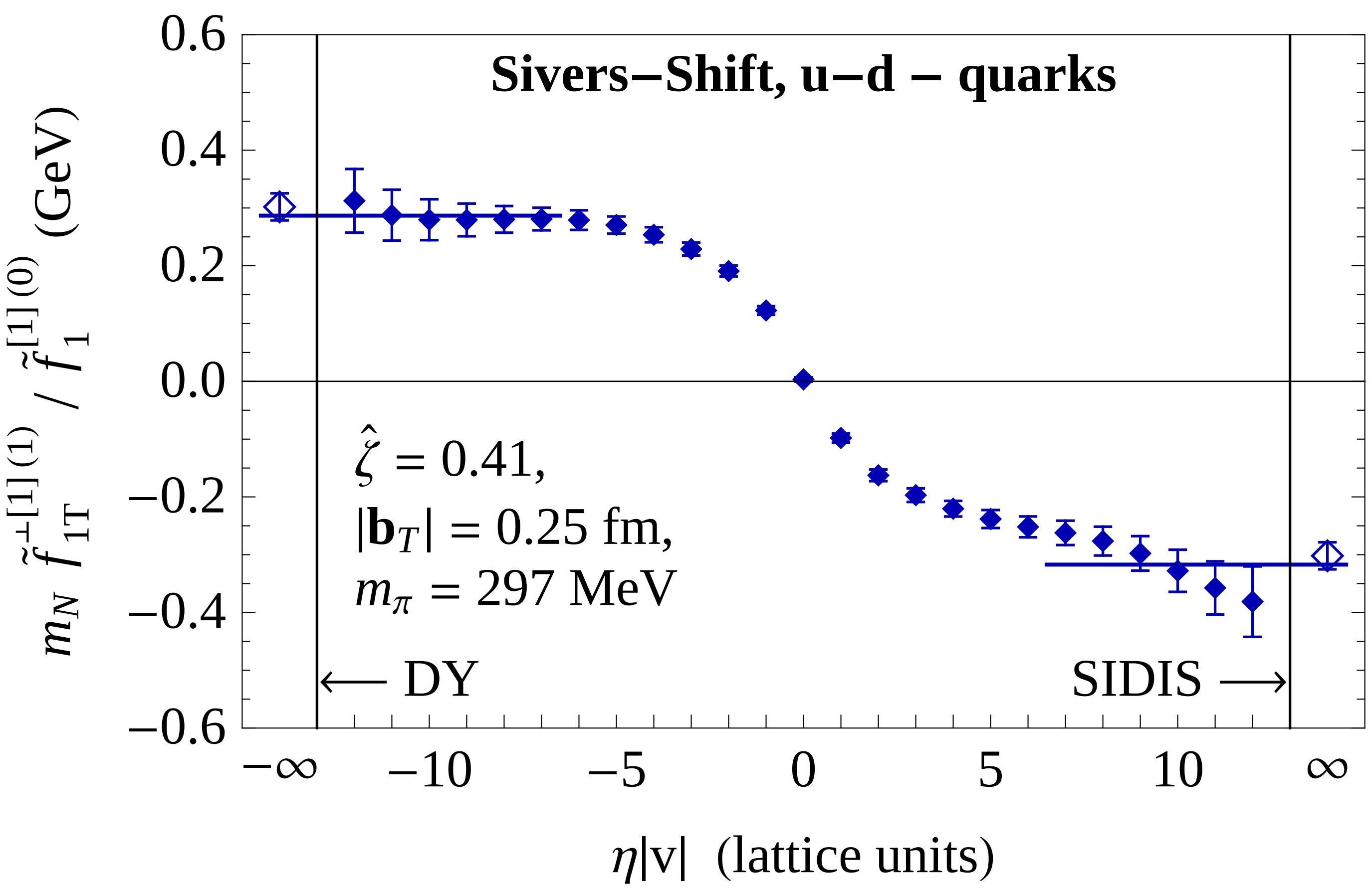
Dependence on staple extent; sequence of panels at different $|b_T|$

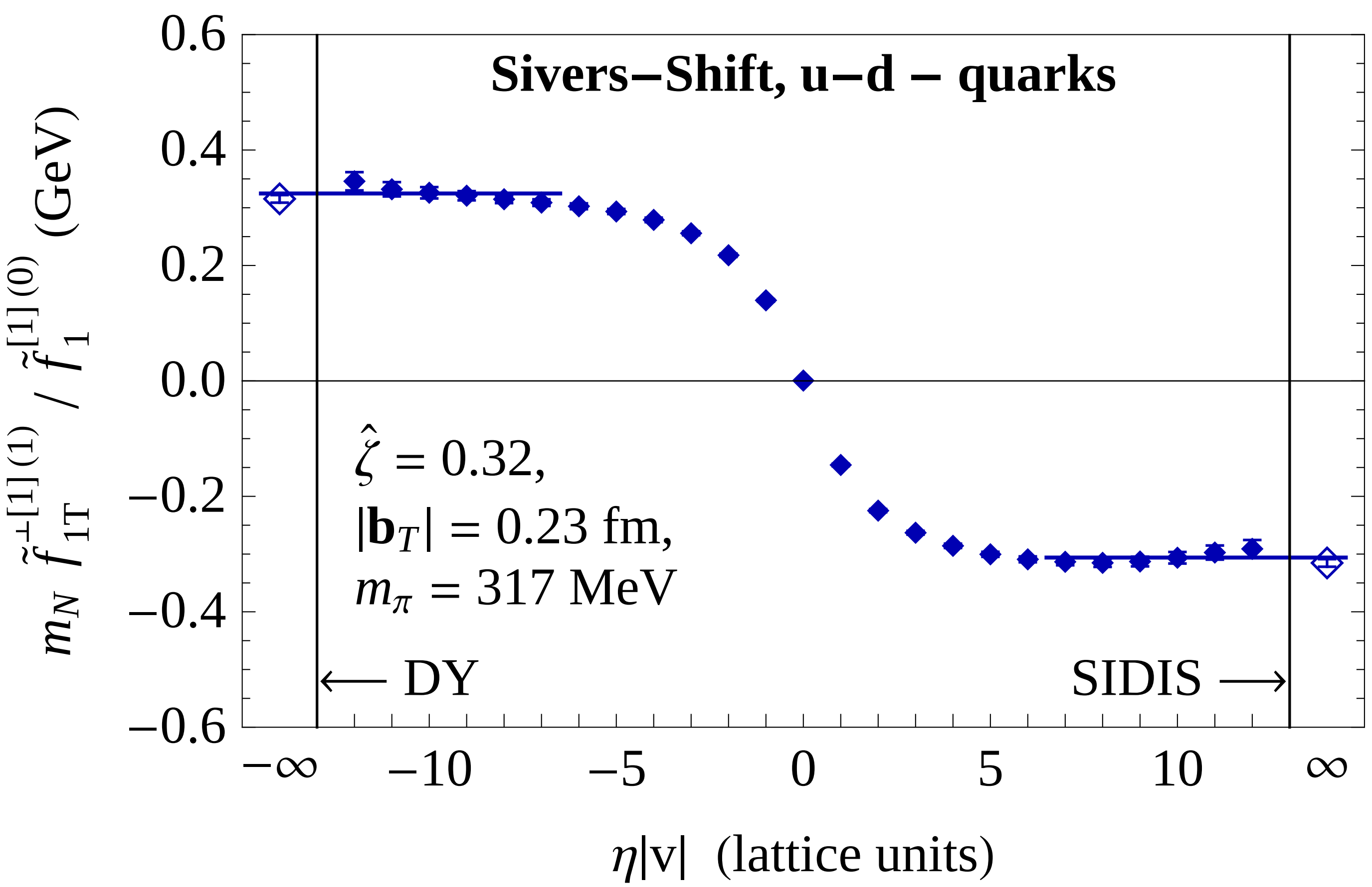




Results: Sivers shift

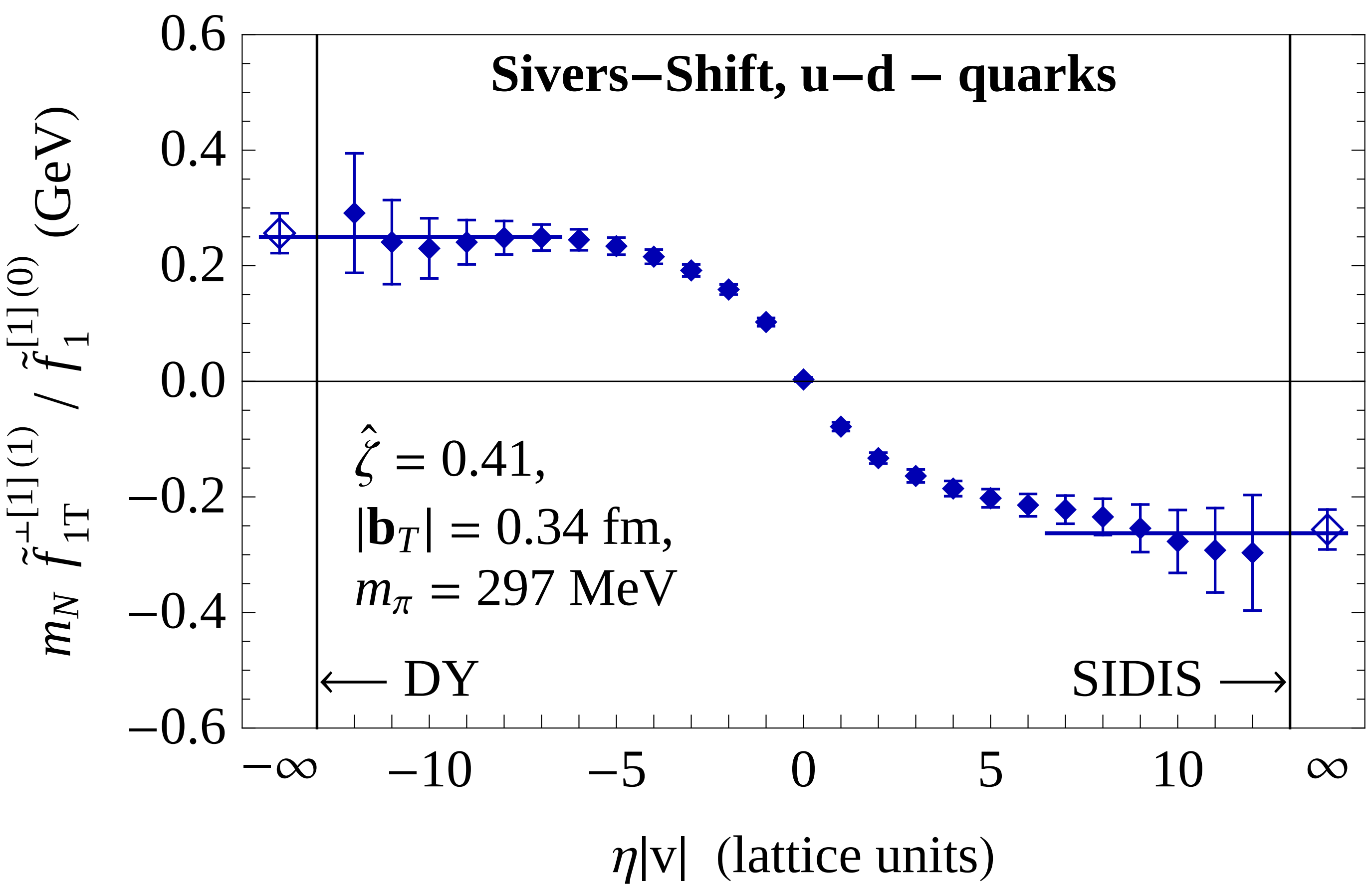
Dependence on staple extent; sequence of panels at different $|b_T|$

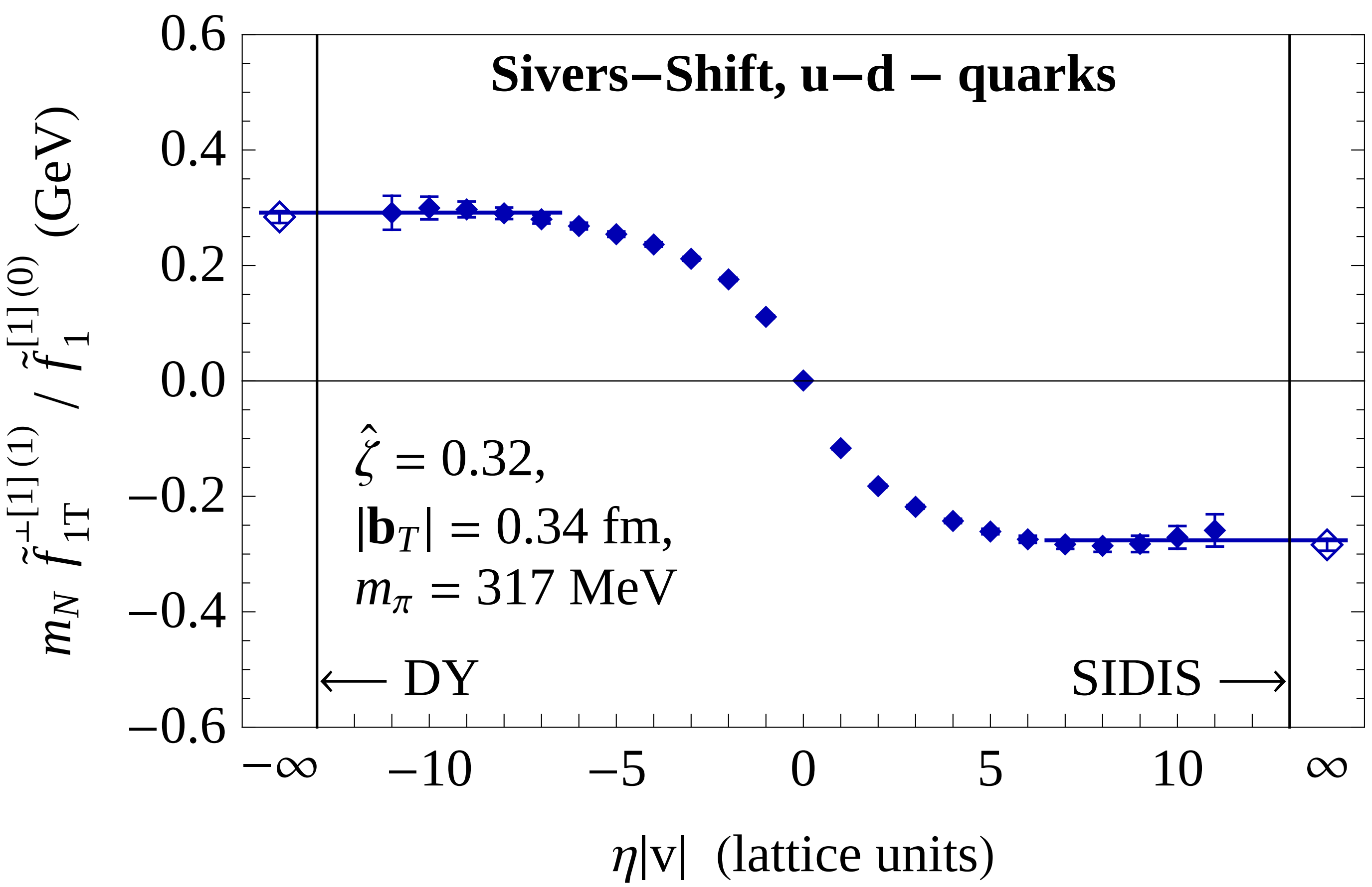




Results: Sivers shift

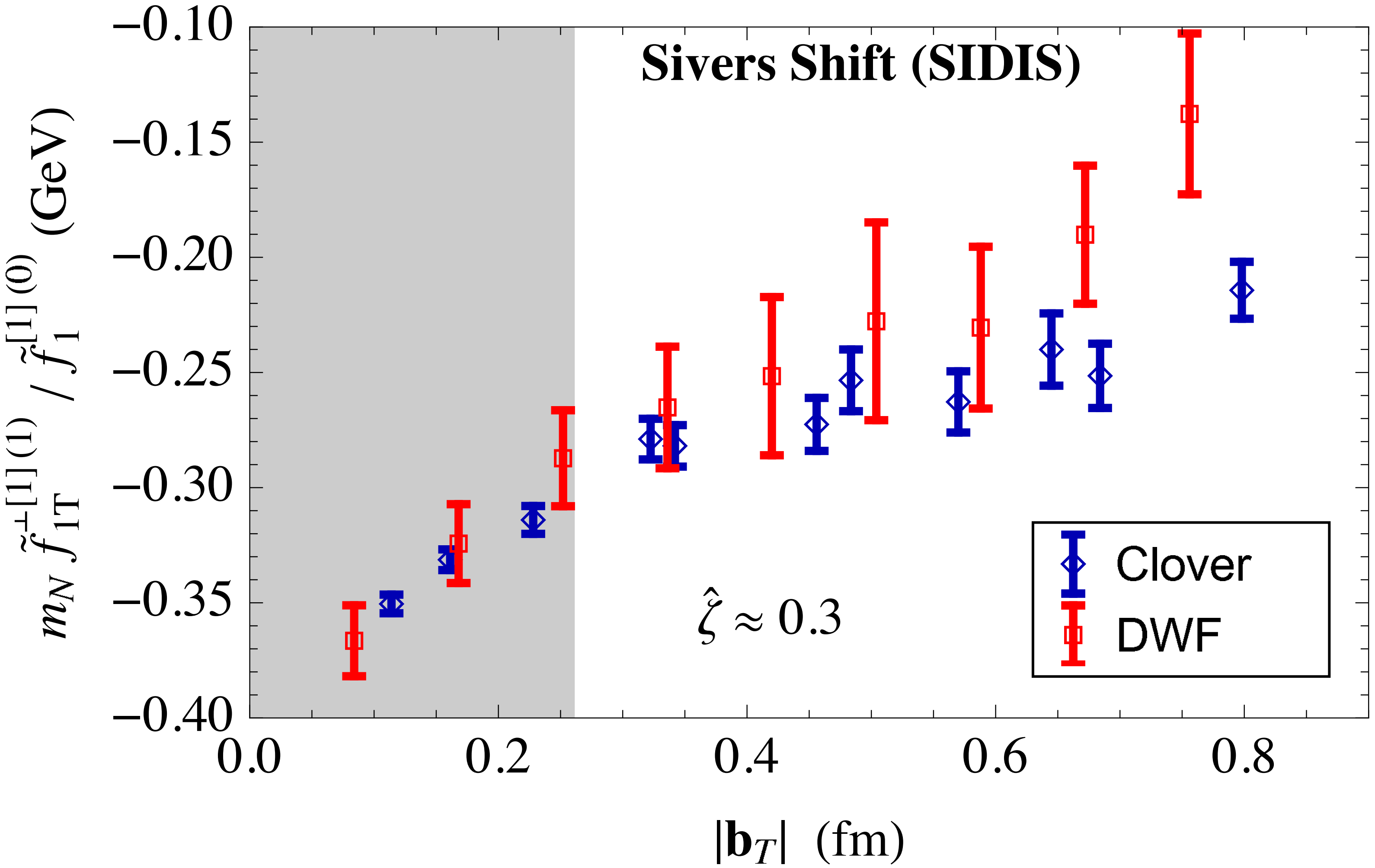
Dependence on staple extent; sequence of panels at different $|b_T|$





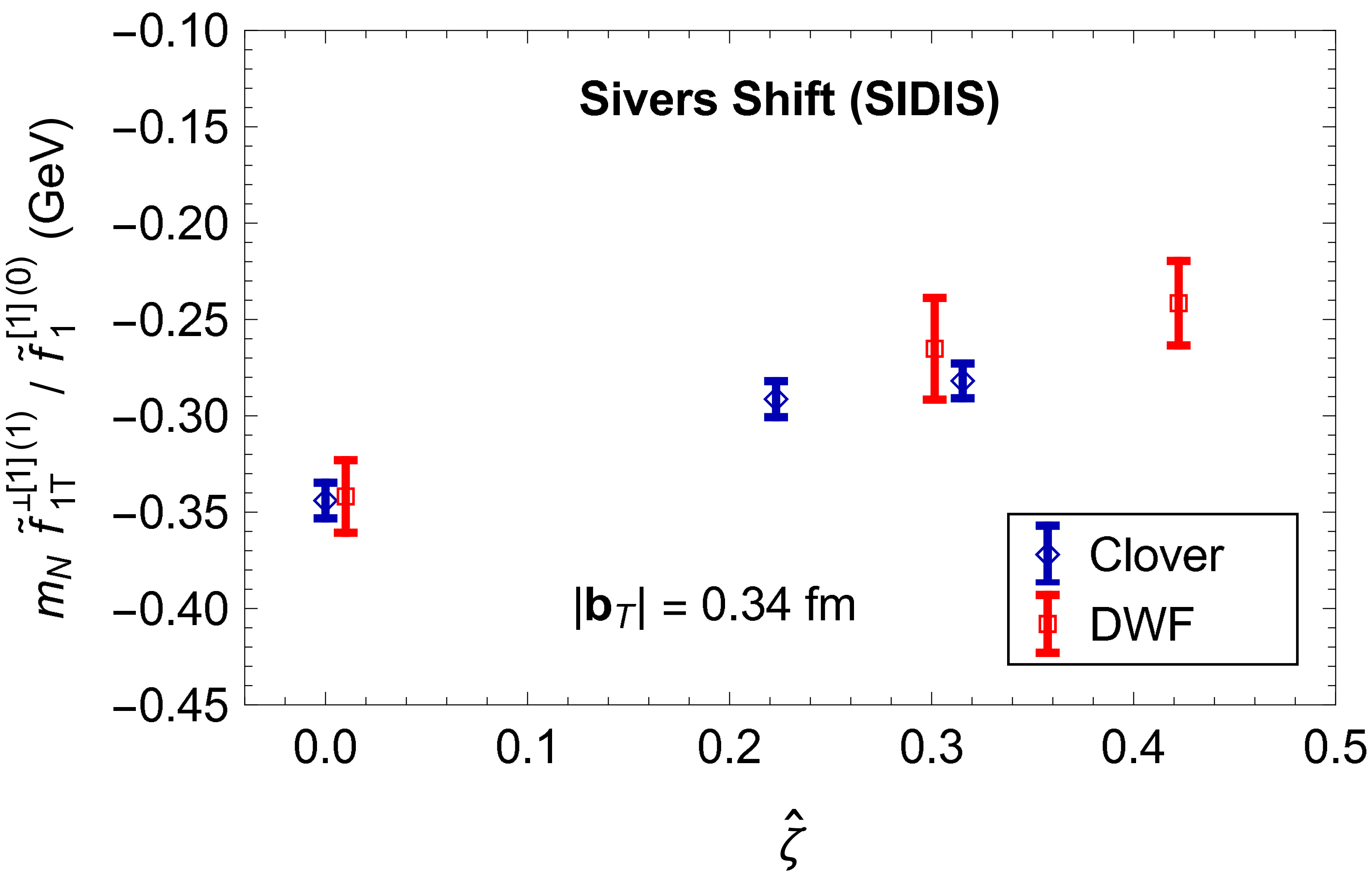
Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$



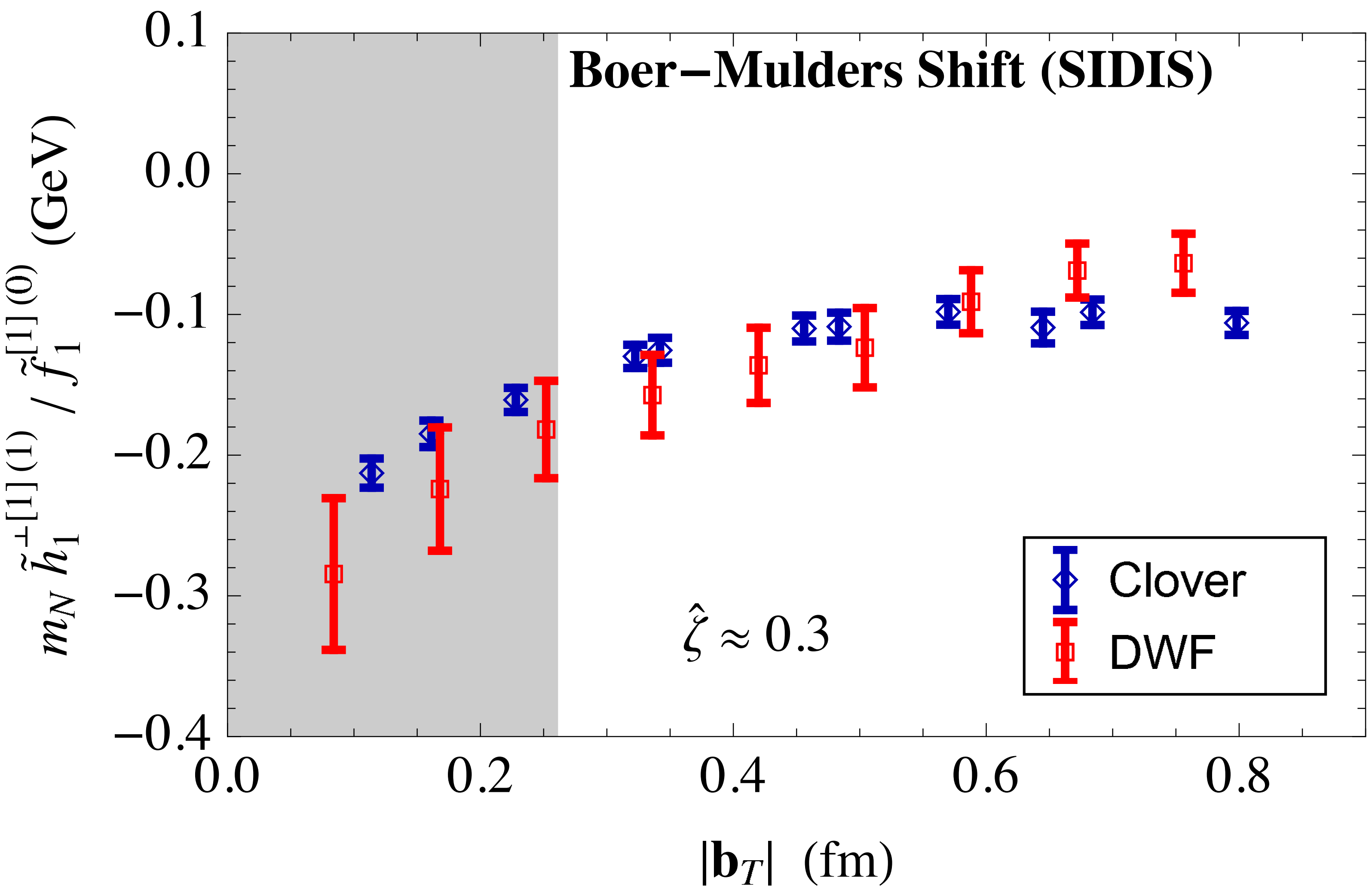
Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$



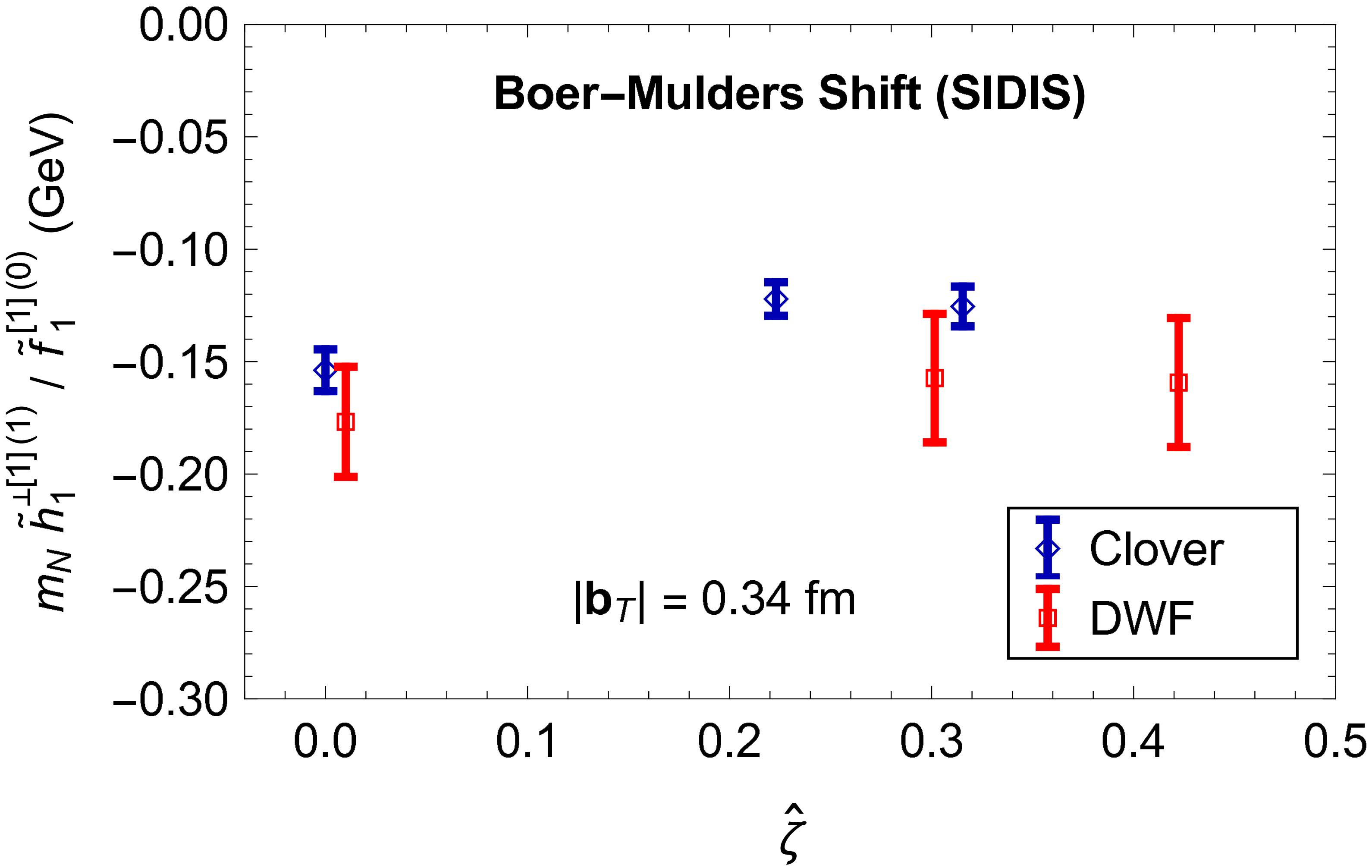
Results: Boer-Mulders shift

Dependence of SIDIS limit on $|b_T|$



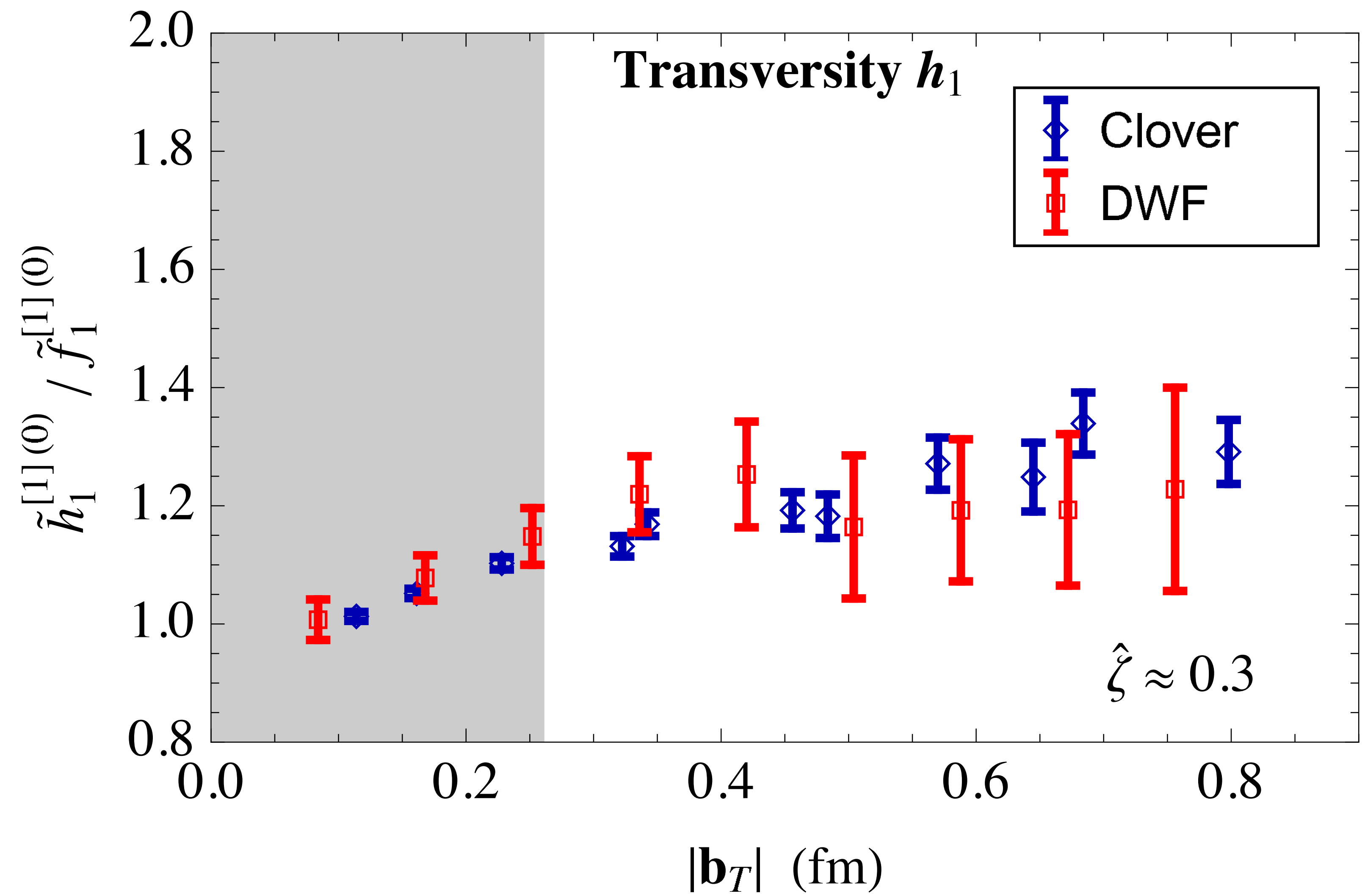
Results: Boer-Mulders shift

Dependence of SIDIS limit on $\hat{\zeta}$



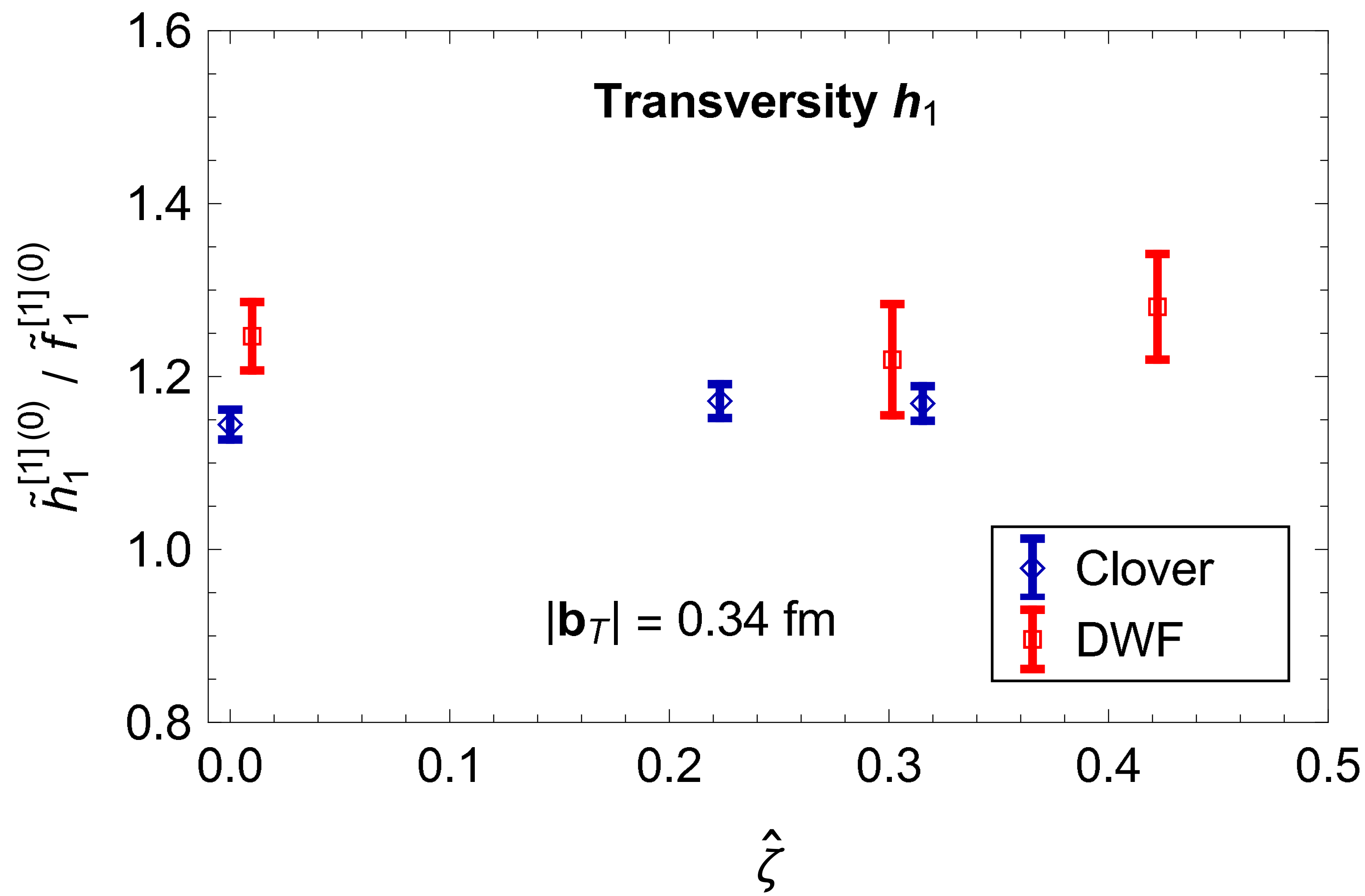
Results: Generalized Transversity

Dependence of SIDIS limit on $|b_T|$



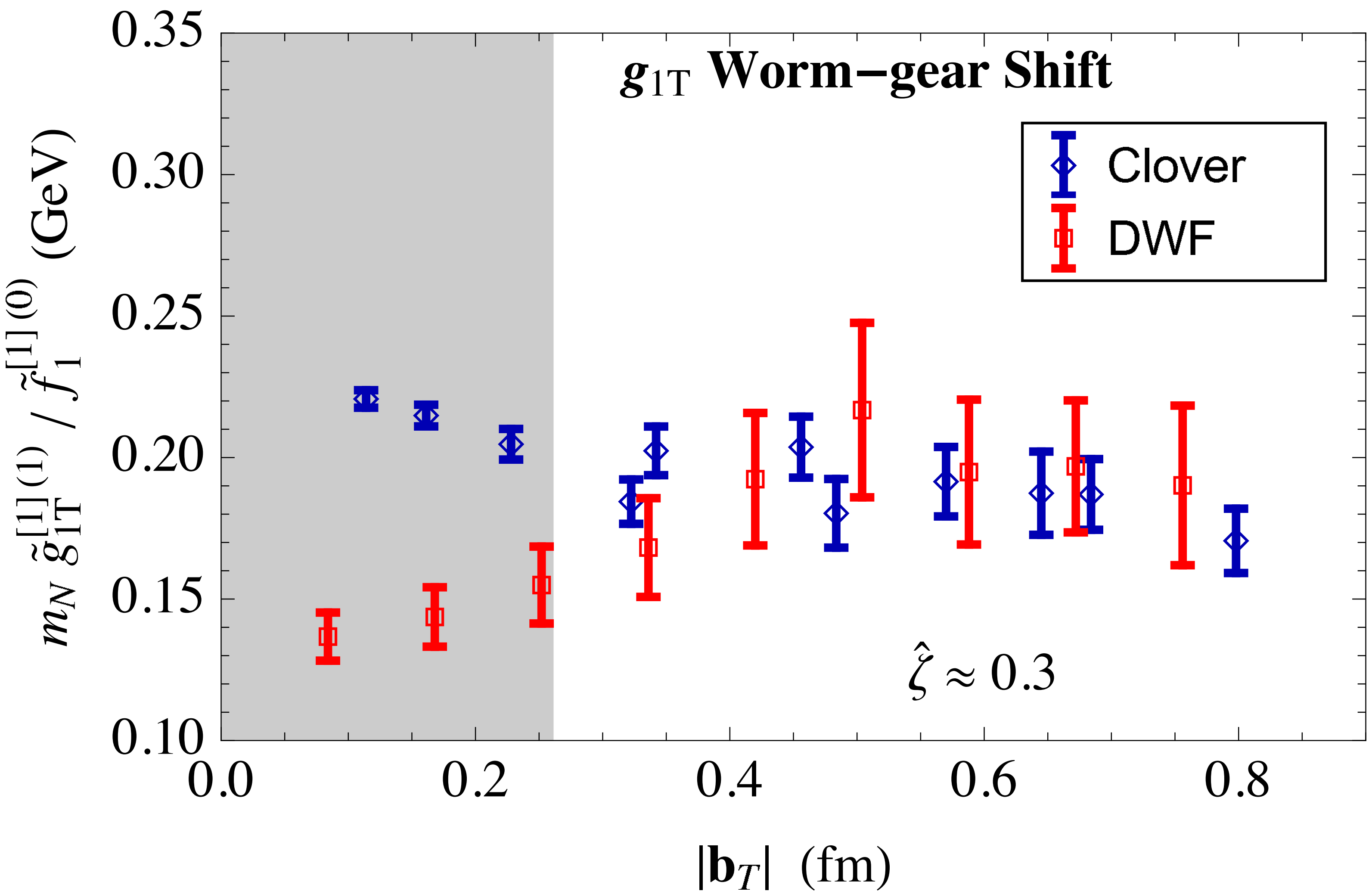
Results: Generalized Transversity

Dependence of SIDIS limit on $\hat{\zeta}$



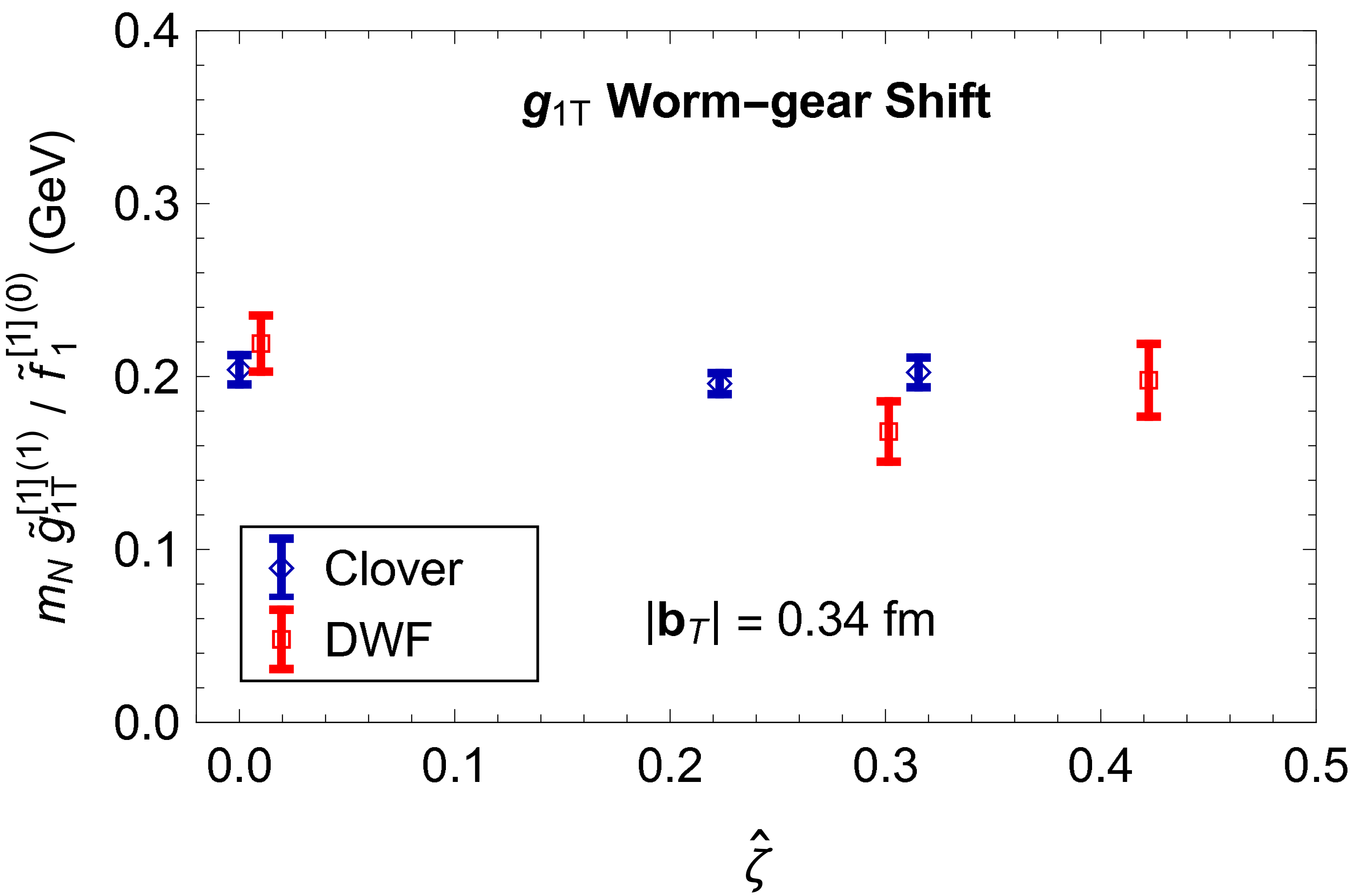
Results: g_{1T} worm gear shift

Dependence of SIDIS limit on $|b_T|$



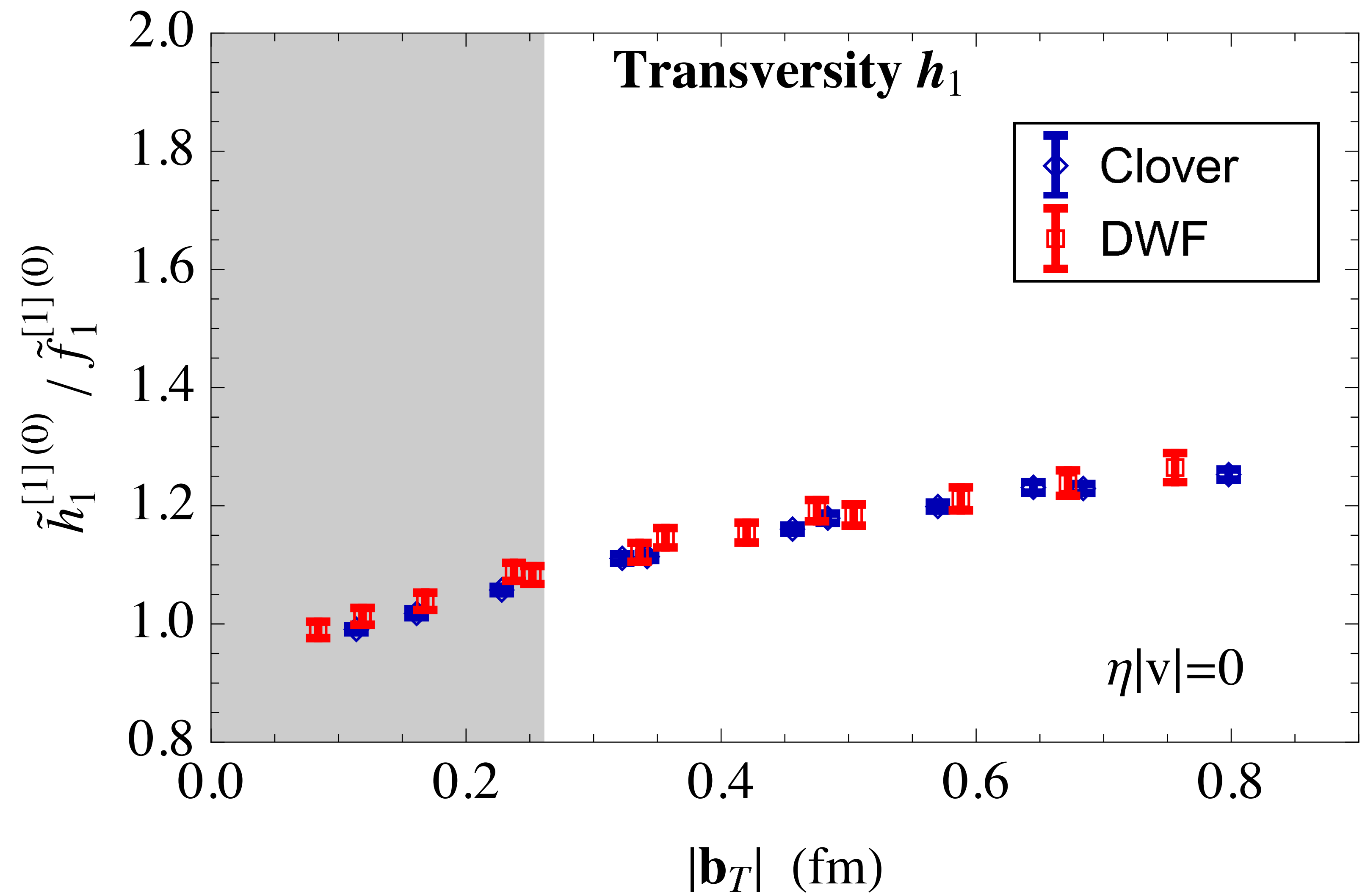
Results: g_{1T} worm gear shift

Dependence of SIDIS limit on $\hat{\zeta}$



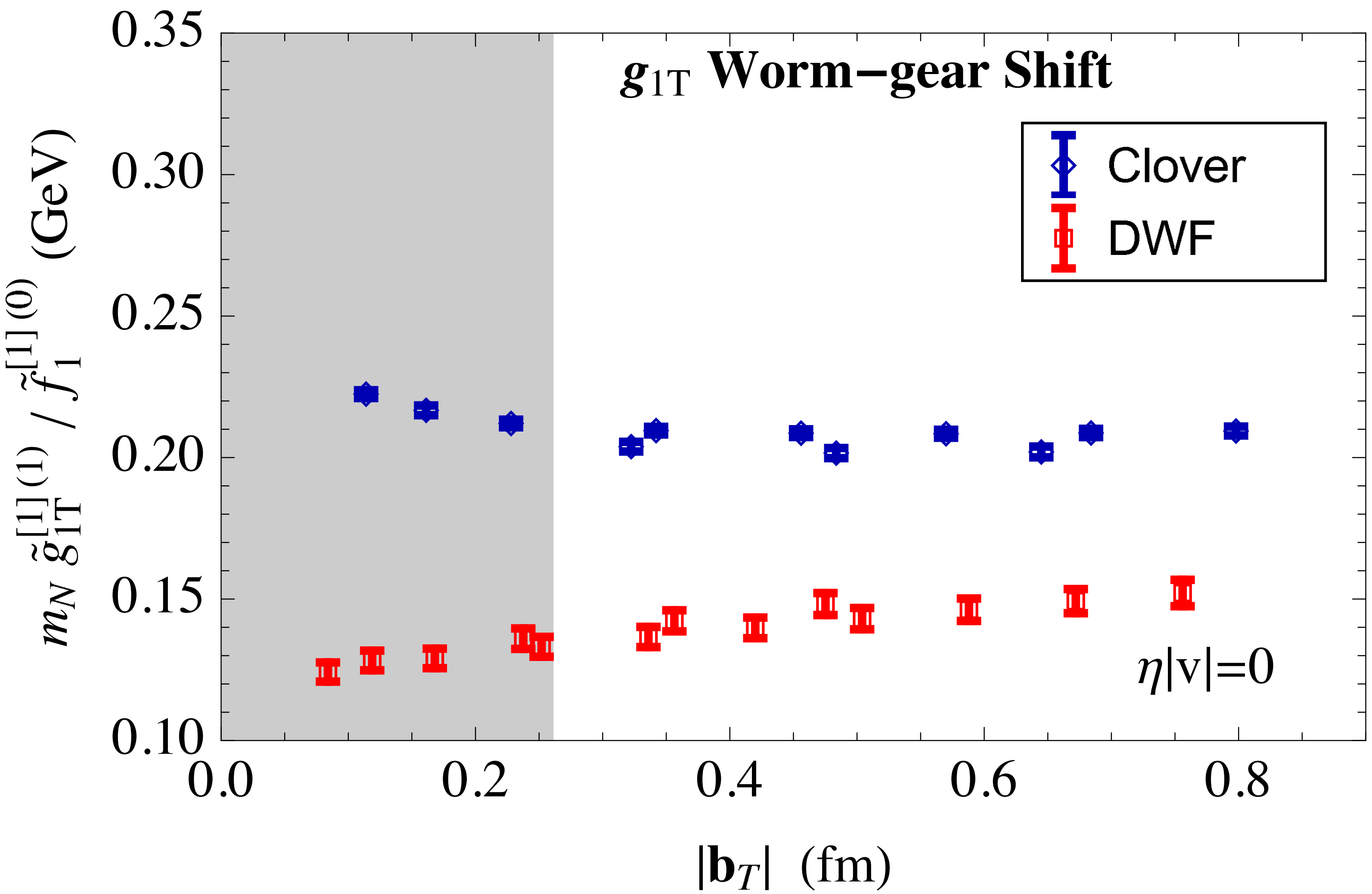
Results: Generalized Transversity, straight link

Dependence on $|b_T|$



Results: g_{1T} worm gear shift, straight link

Dependence on $|b_T|$

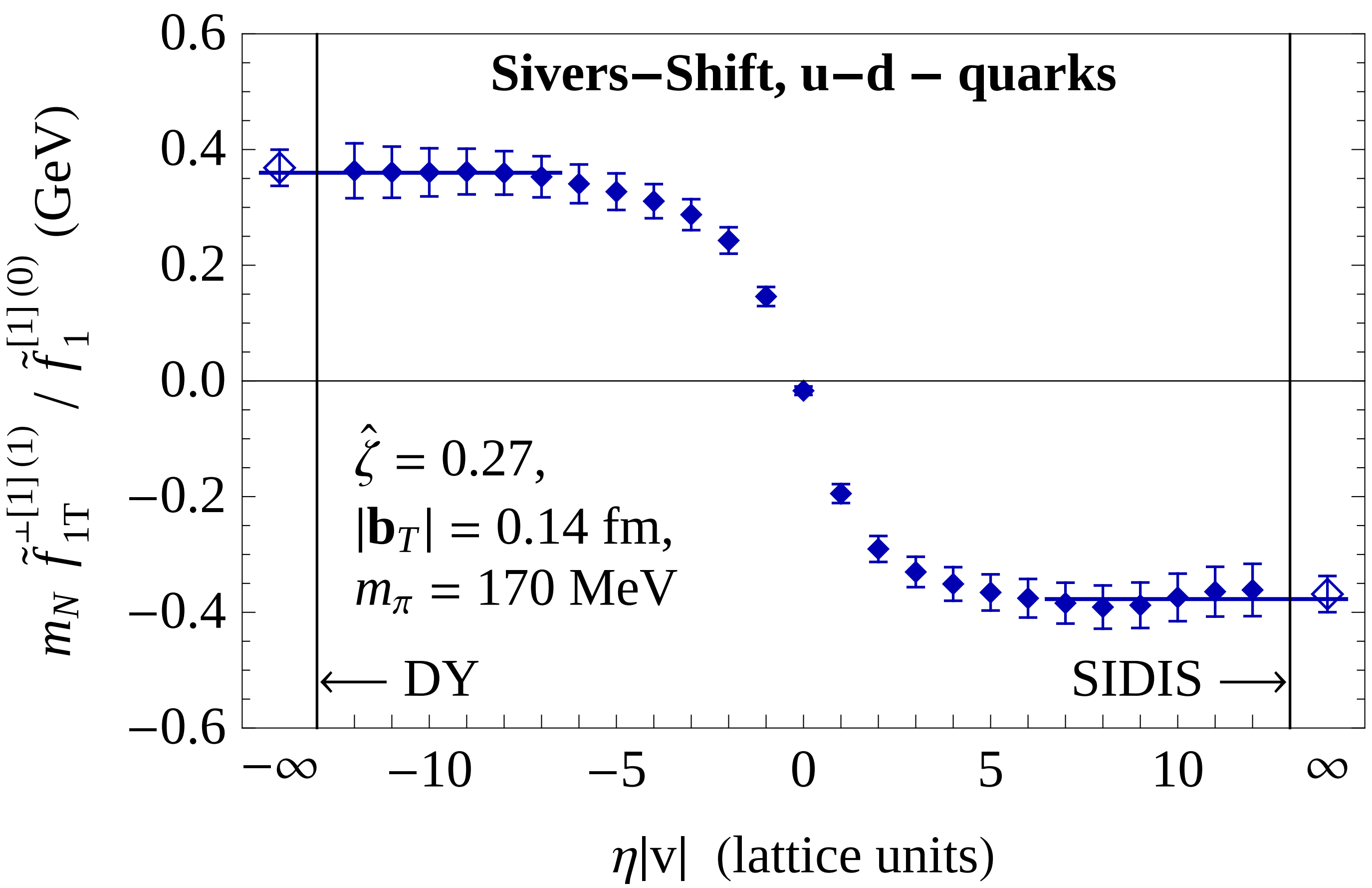


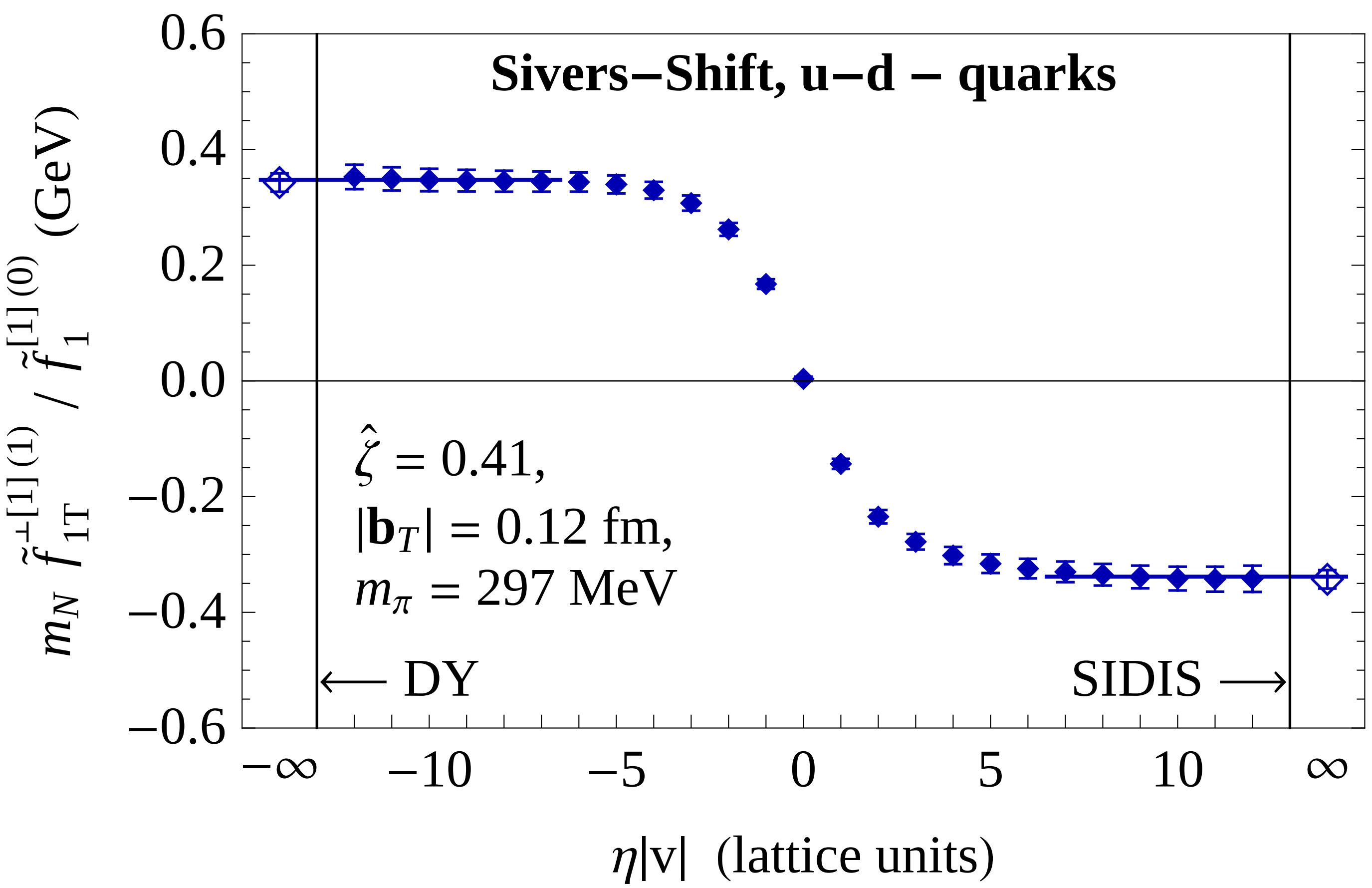
Evidence of operator mixing? ⟶ Lattice perturbation theory M. Constantinou et al.

Dependence on the pion mass

Results: Sivers shift

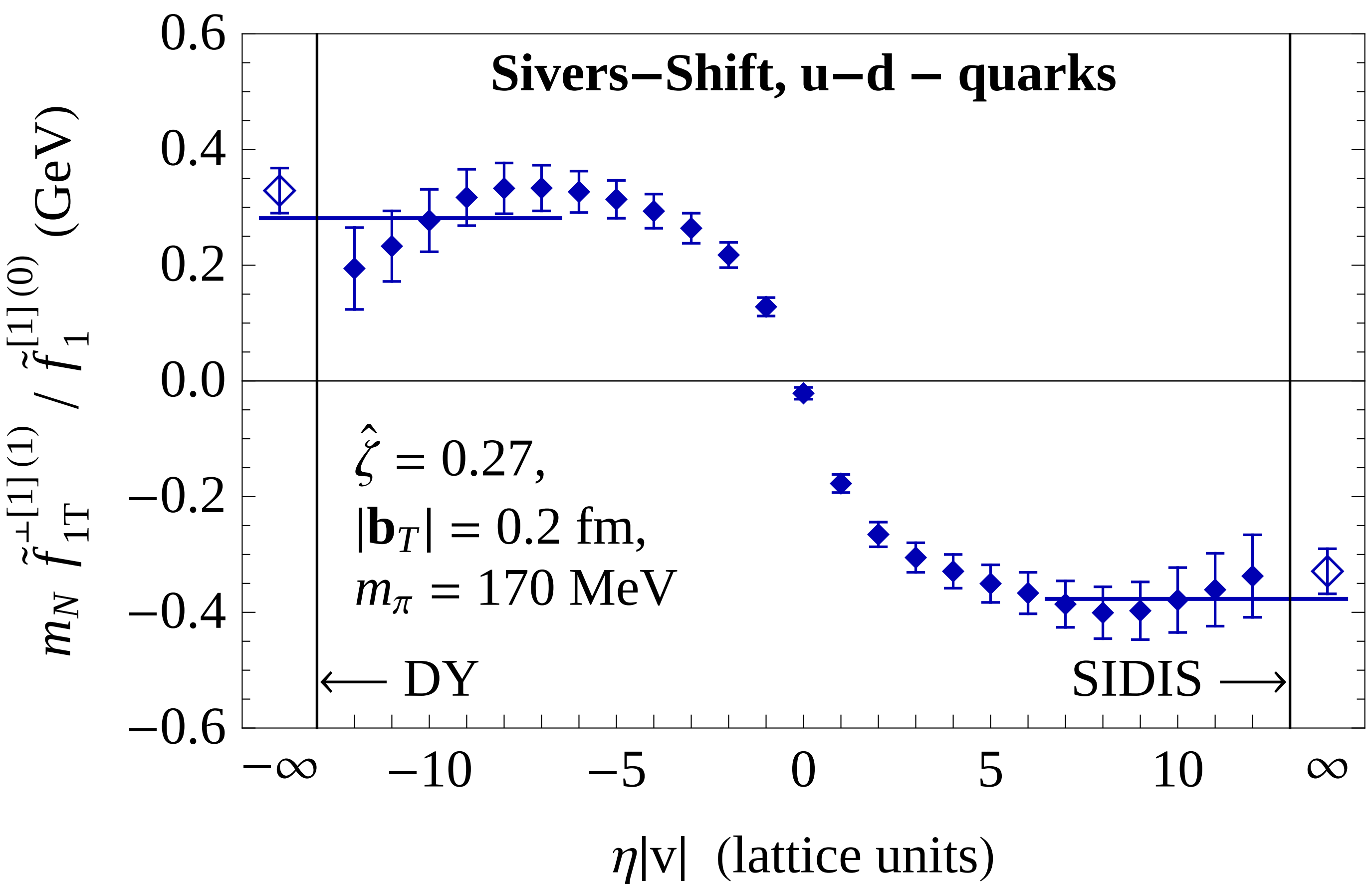
Dependence on staple extent; sequence of panels at different $|b_T|$

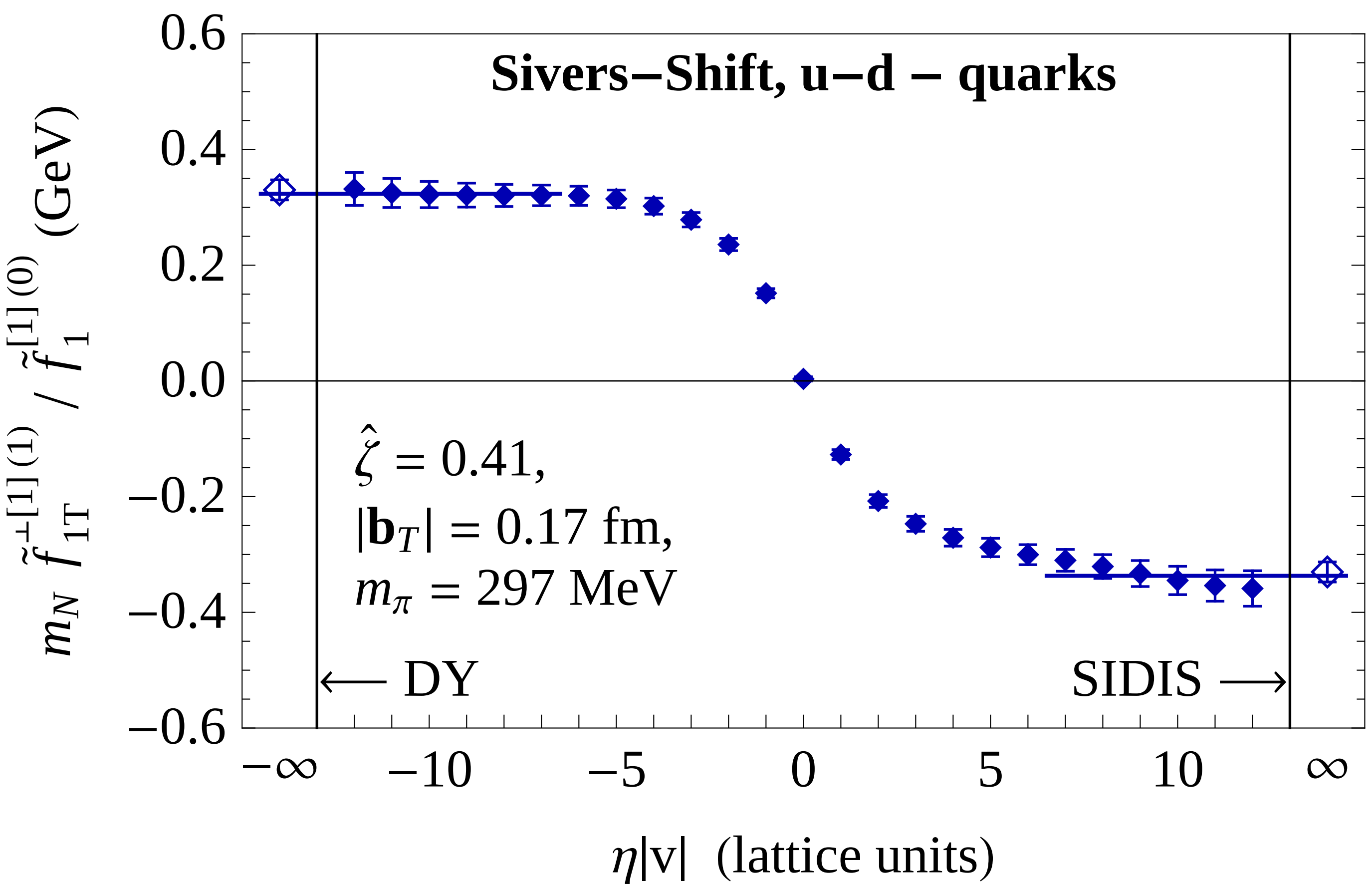




Results: Sivers shift

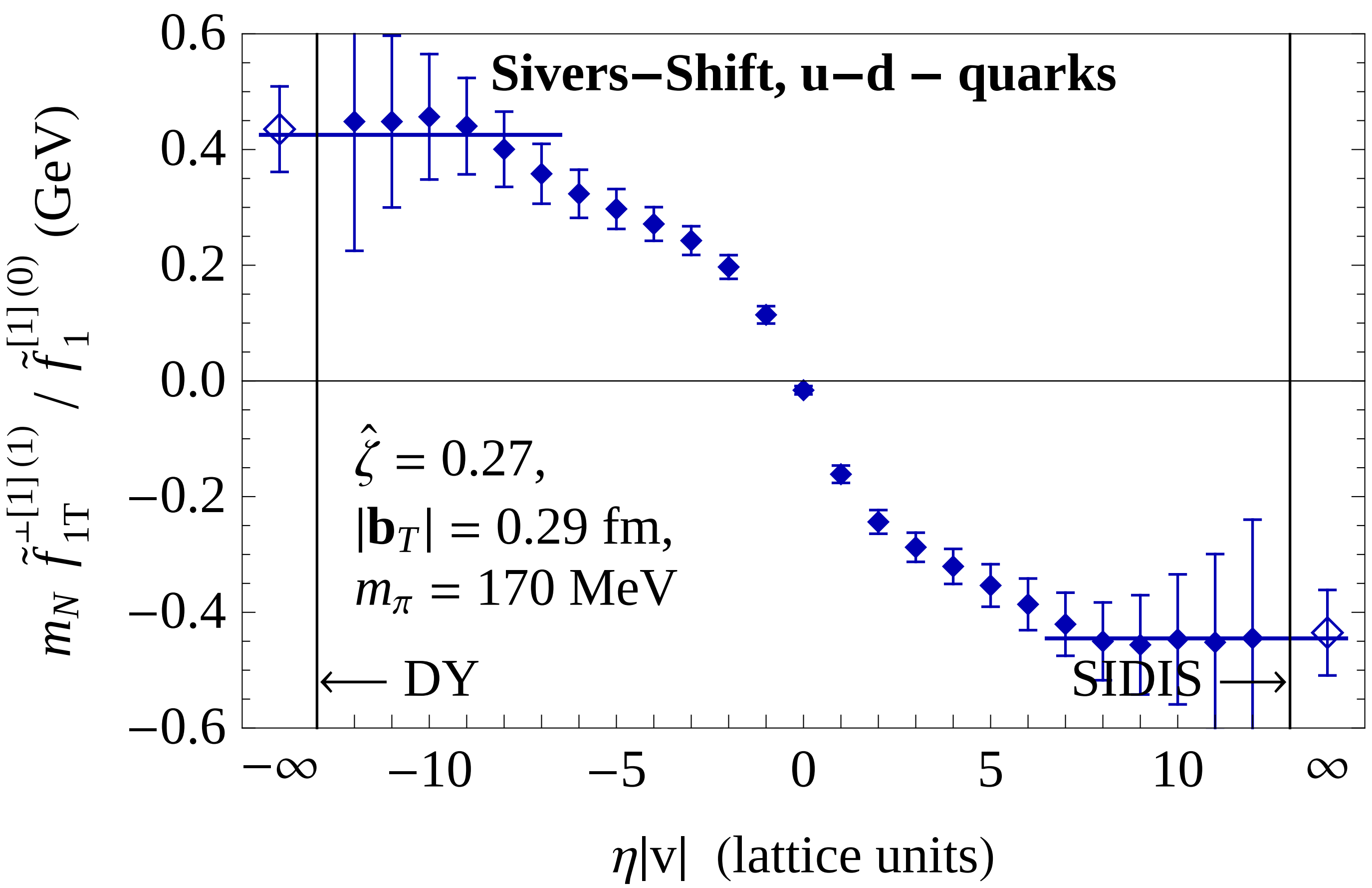
Dependence on staple extent; sequence of panels at different $|b_T|$

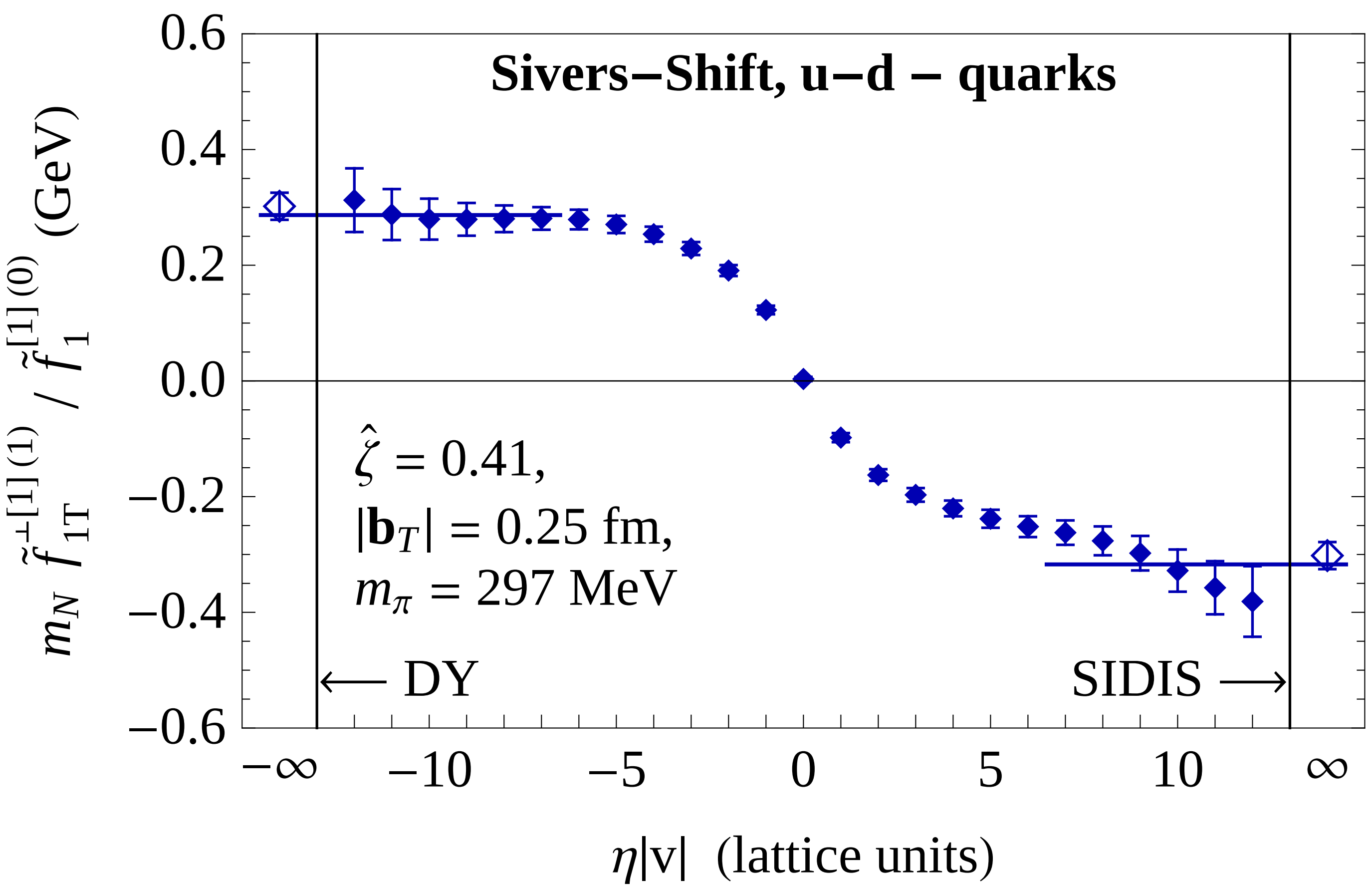




Results: Sivers shift

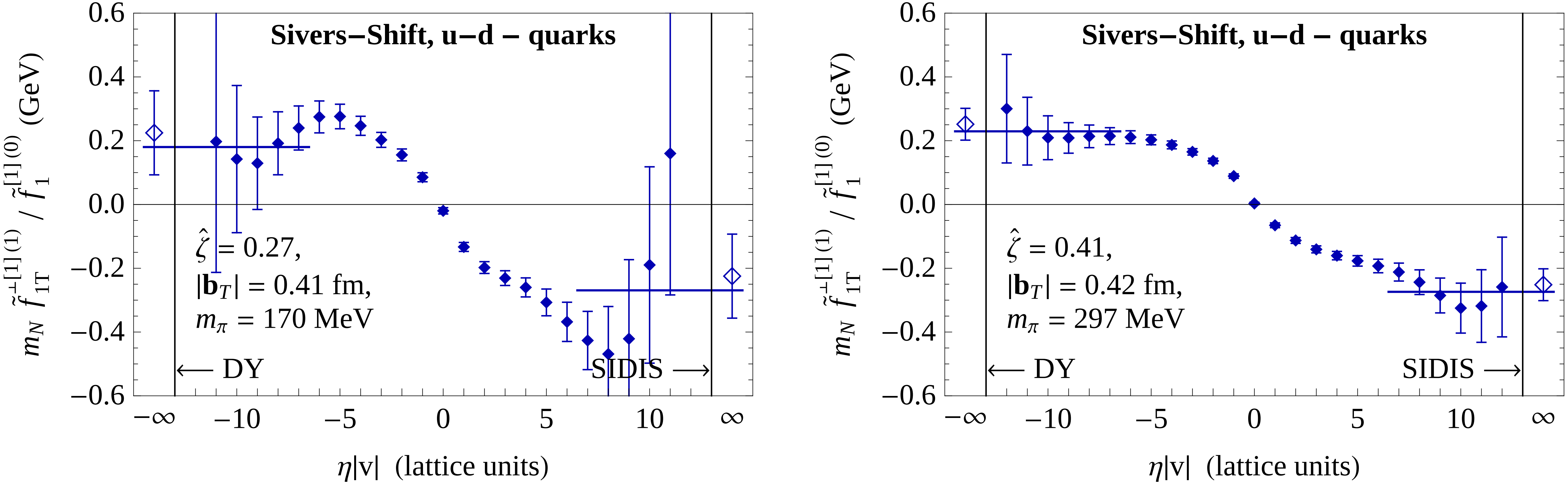
Dependence on staple extent; sequence of panels at different $|b_T|$





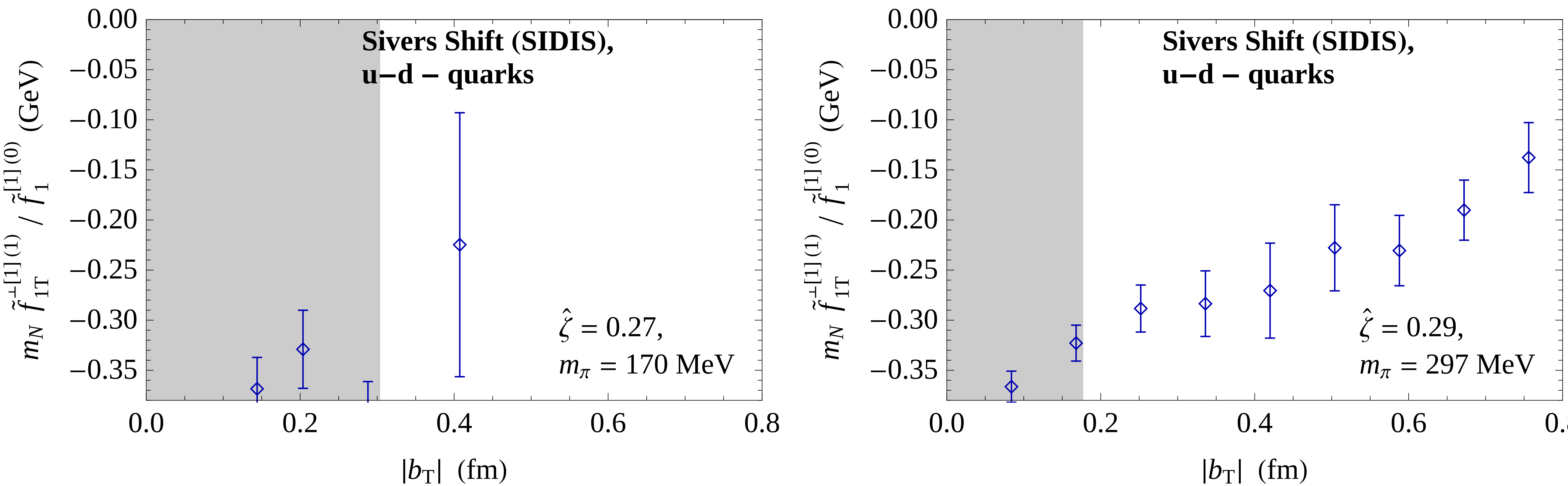
Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$



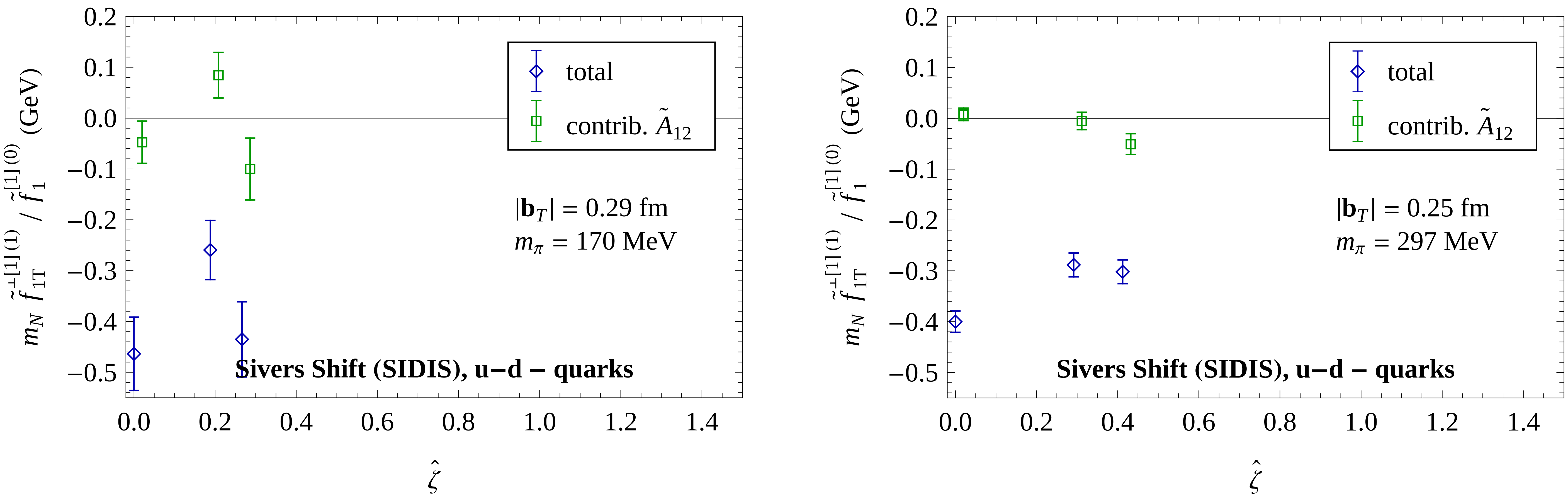
Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$



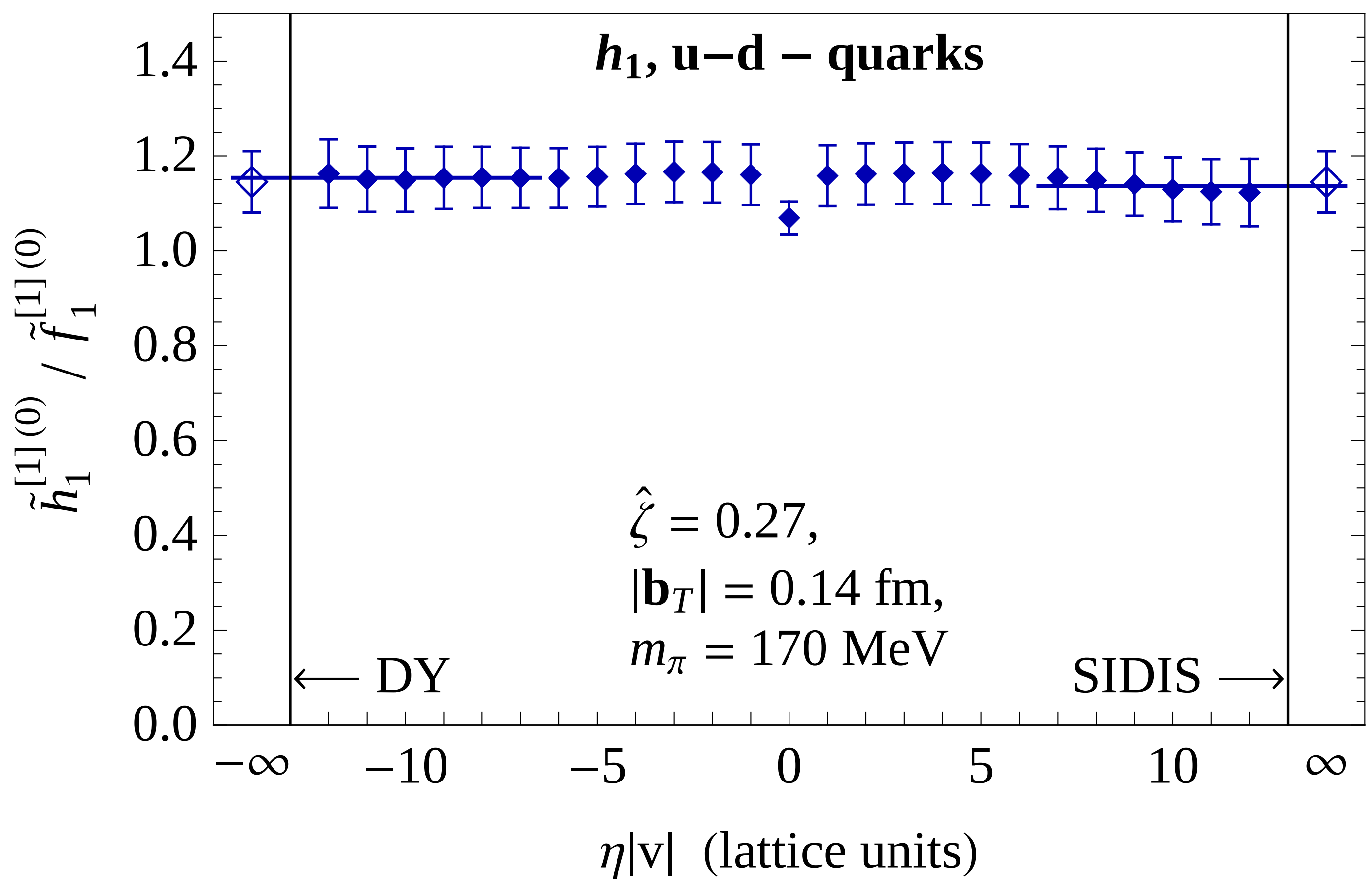
Results: Sivers shift

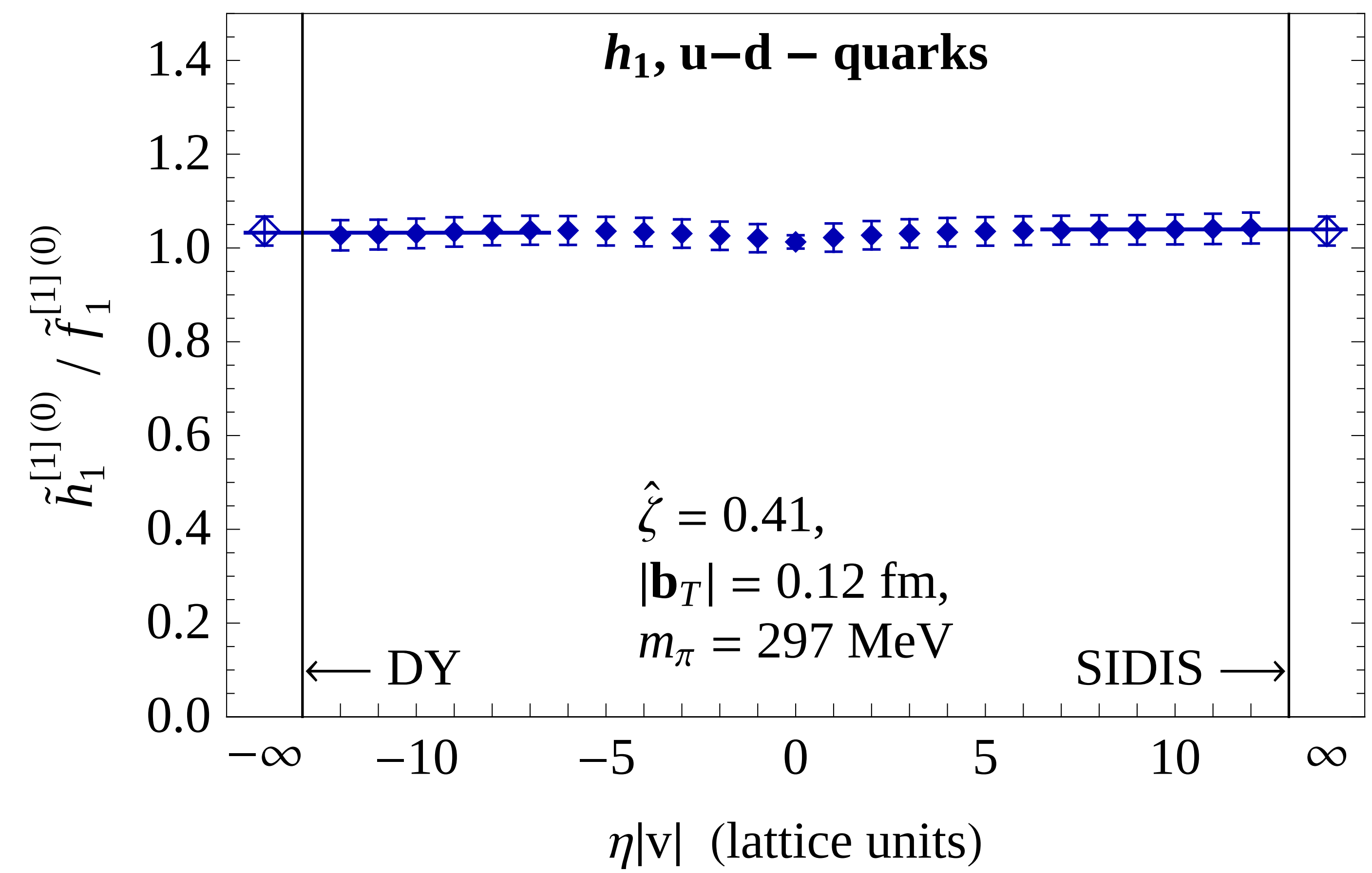
Dependence of SIDIS limit on $\hat{\zeta}$



Results: Transversity

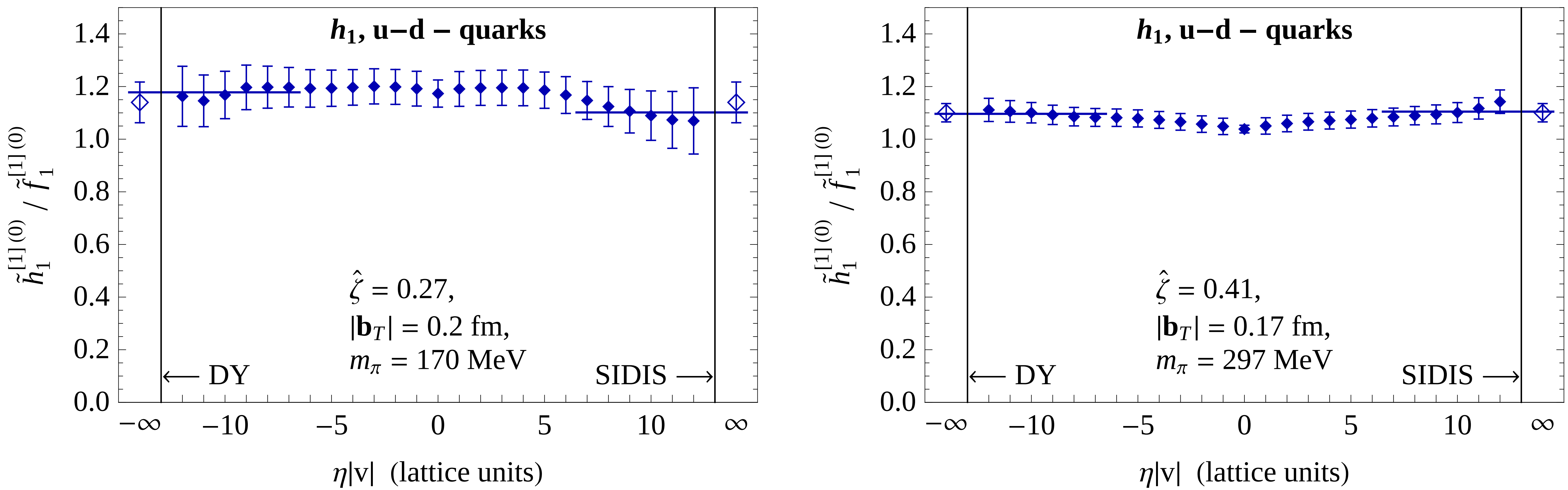
Dependence on staple extent; sequence of panels at different $|b_T|$





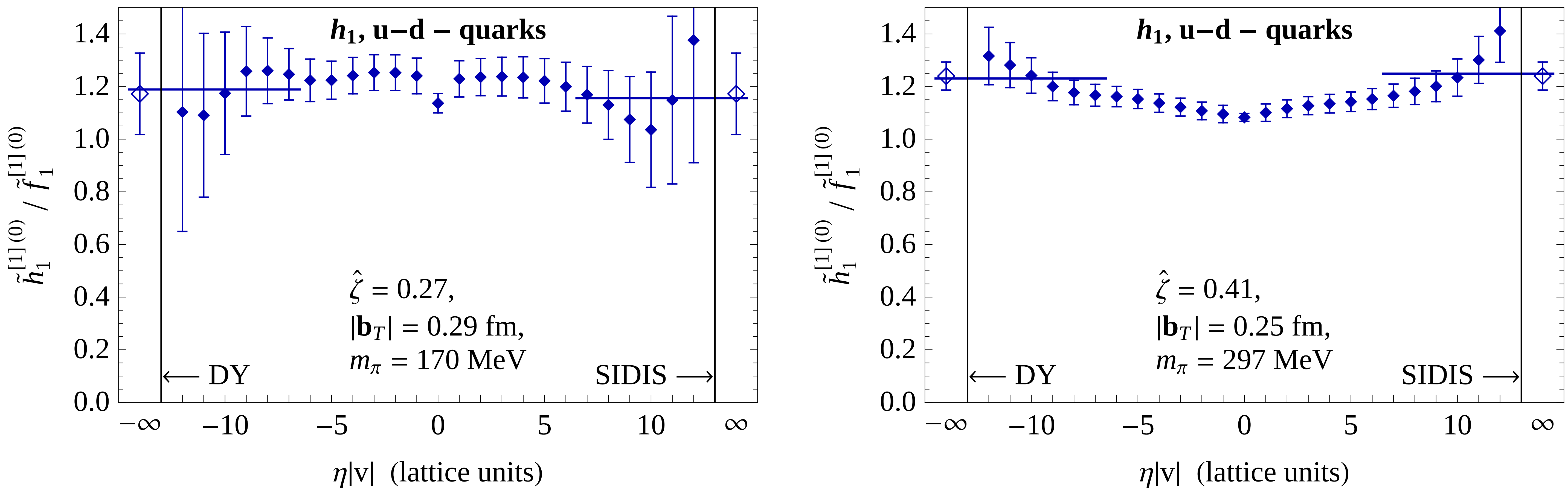
Results: Transversity

Dependence on staple extent; sequence of panels at different $|b_T|$



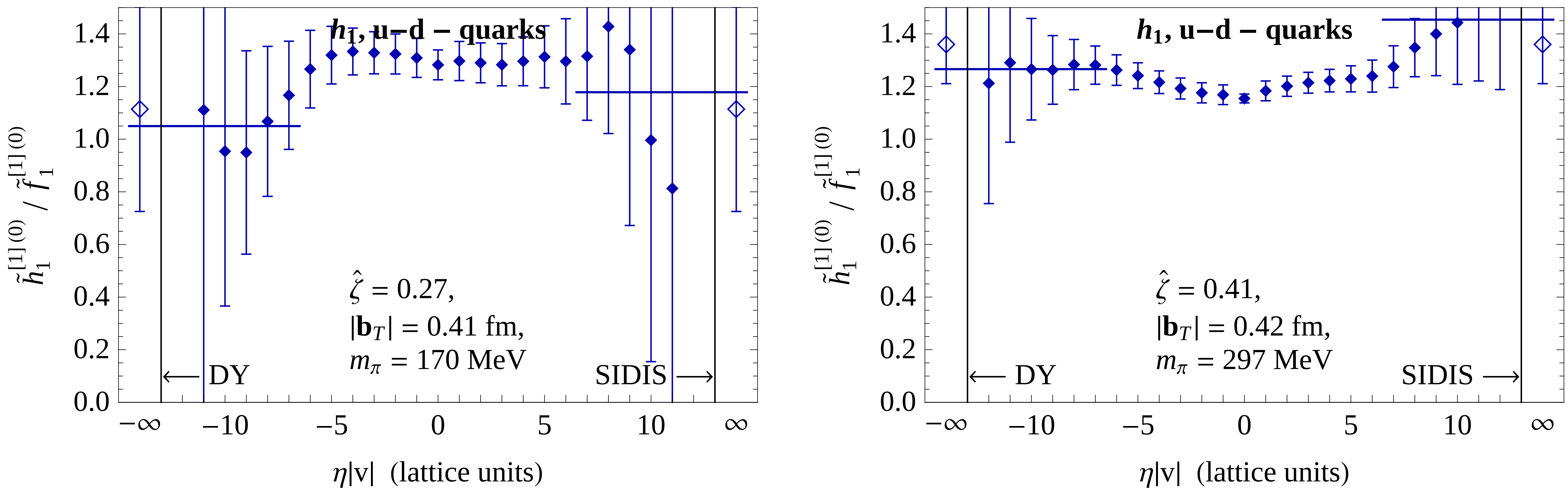
Results: Transversity

Dependence on staple extent; sequence of panels at different $|b_T|$



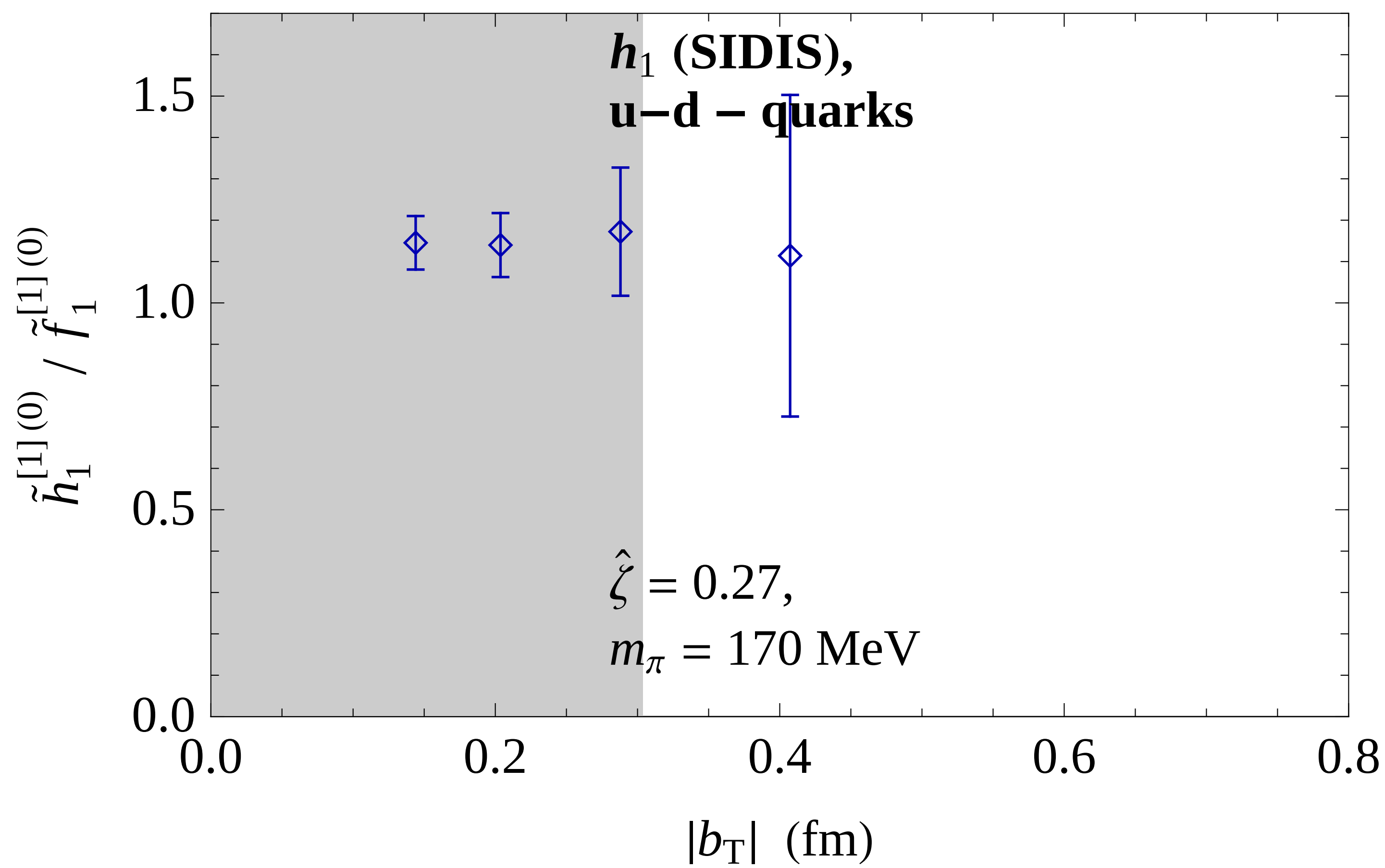
Results: Transversity

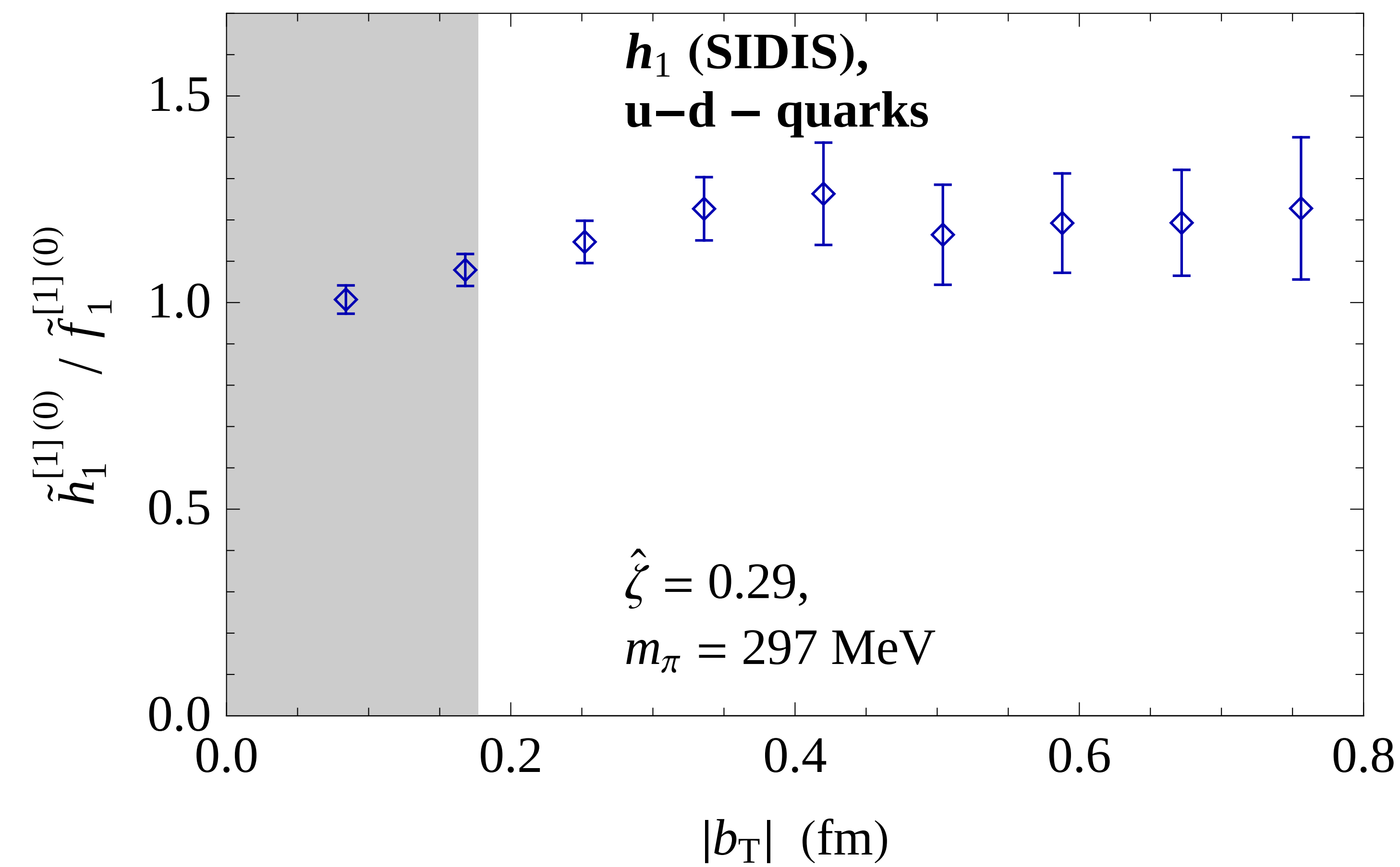
Dependence on staple extent; sequence of panels at different $|b_T|$



Results: Transversity

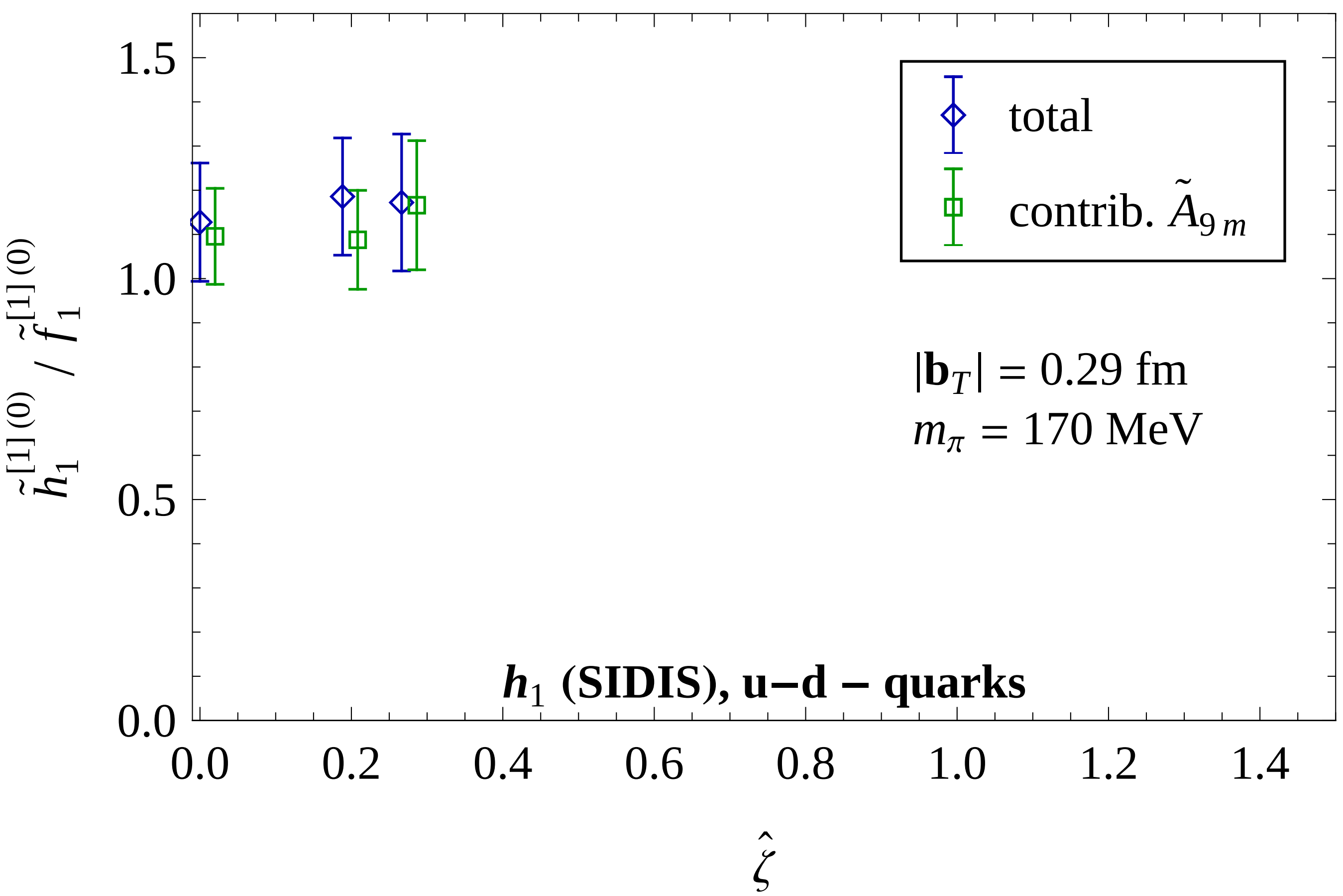
Dependence of SIDIS/DY limit on $|b_T|$

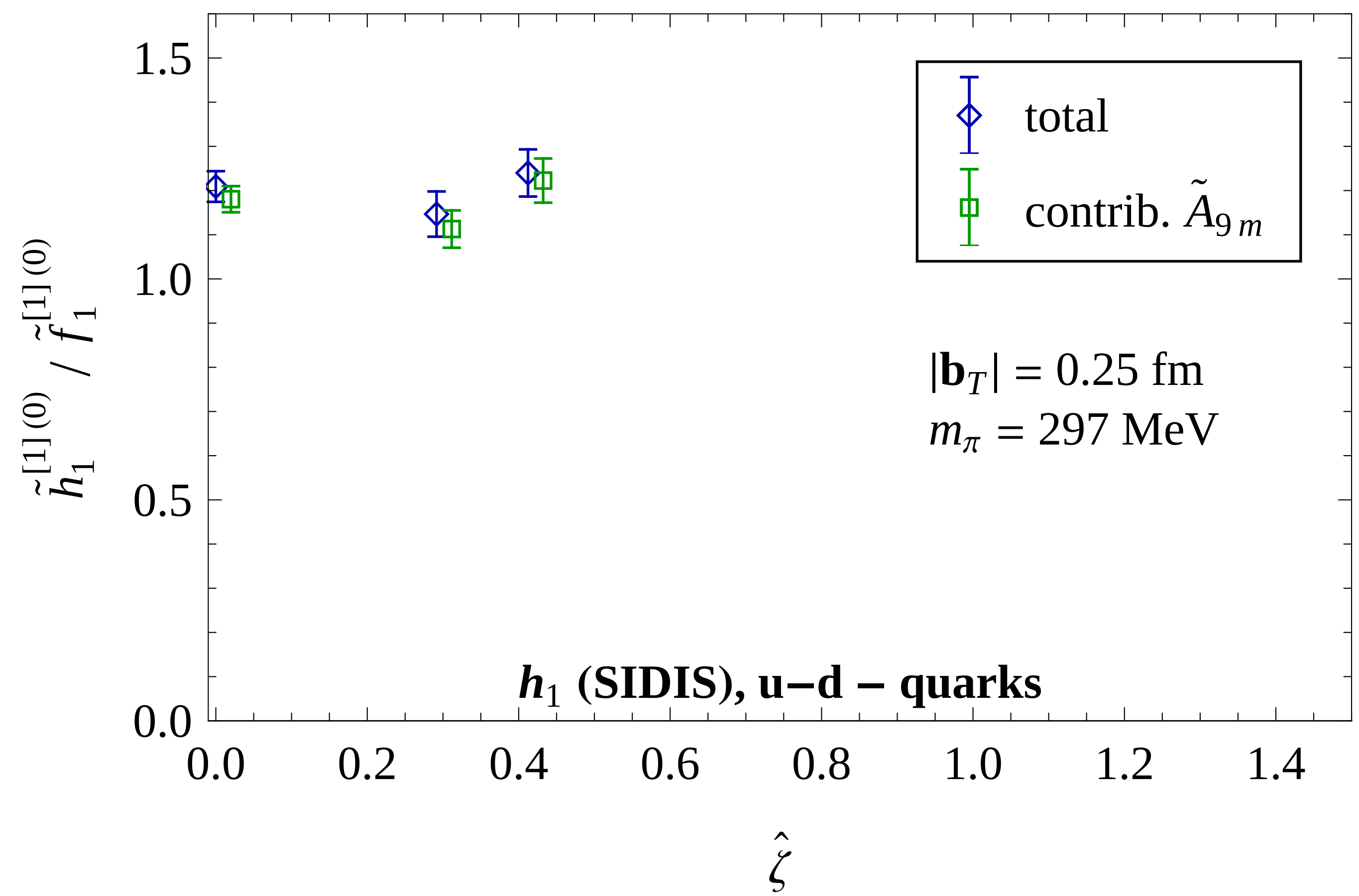




Results: Transversity

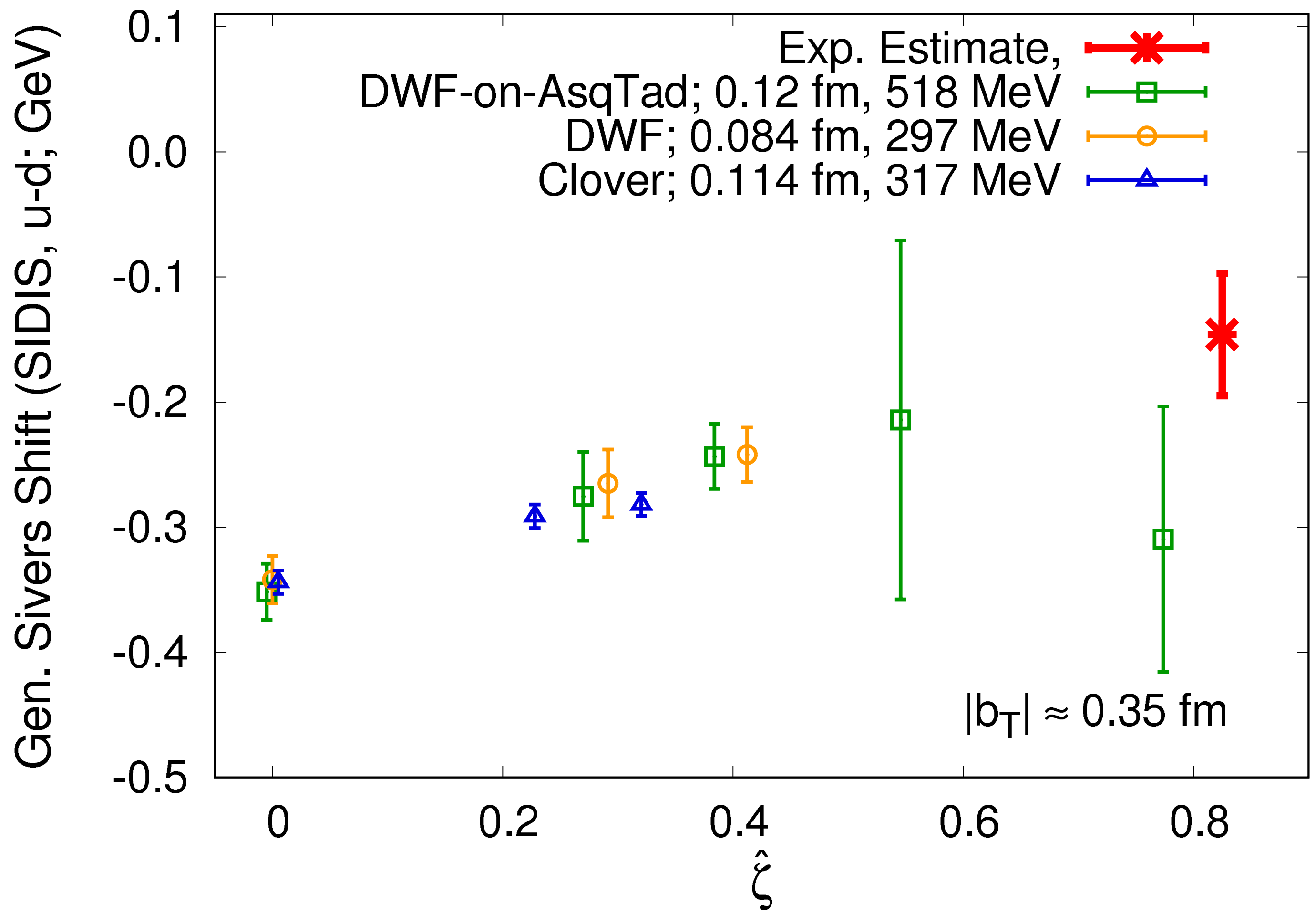
Dependence of SIDIS/DY limit on $\hat{\zeta}$





Results: Sivers shift summary

Dependence of SIDIS limit on $\hat{\zeta}$



Experimental value from global fit to HERMES, COMPASS and JLab data,
M. Echevarria, A. Idilbi, Z.-B. Kang and I. Vitev, Phys. Rev. D 89 (2014) 074013

Proton spin decompositions

$$\frac{1}{2} = \frac{1}{2}\sum_q \Delta q + \sum_q L_q + J_g \qquad \text{(Ji)}$$

$$\frac{1}{2} = \frac{1}{2}\sum_q \Delta q + \sum_q \mathcal{L}_q + \Delta g + \mathcal{L}_g \qquad \text{(Jaffe-Manohar)}$$

. . . and many more (in fact, we will see a continuous interpolation between the two . . .)

There isn't one unique way of separating quark and gluon orbital angular momentum – the different decompositions have different, legitimate meanings.

Quark orbital angular momentum

Interpreting terms in the energy-momentum tensor:

$$L_q \quad \sim \quad -i\psi^\dagger(\vec{r} \times \vec{D})_z\psi$$

Can be obtained from $L_q = J_q - S_q$, where S_q and J_q can be related to GPDs (Ji sum rule) – this has been used in Lattice QCD.

$$\mathcal{L}_q \quad \sim \quad -i\psi^\dagger(\vec{r} \times \vec{\partial})_z\psi \qquad \text{in light cone gauge}$$

Hitherto not accessed in Lattice QCD.

Quark Orbital Angular Momentum

$$L_3^{\mathcal{U}} = \int dx \int d^2k_T \int d^2r_T \, (r_T \times k_T)_3 \, \mathcal{W}^{\mathcal{U}}(x, k_T, r_T)$$

Wigner distribution

$$= -\int dx \int d^2k_T \, \frac{k_T^2}{m^2} \, F_{14}(x, k_T^2, k_T \cdot \Delta_T, \Delta_T^2) \Big|_{\Delta_T = 0}$$

Generalized transverse momentum-dependent parton distribution (GTMD)

$$= \epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2) \gamma^+ \mathcal{U}[-z/2, z/2] \psi(z/2) \mid p, S \rangle \Big|_{z^+ = z^- = 0, \, \Delta_T = 0, \, z_T \to 0}$$

Y. Hatta, X. Ji, M. Burkardt:
Staple-shaped $\mathcal{U}[-z/2, z/2] \longrightarrow$ Jaffe-Manohar OAM
Straight $\mathcal{U}[-z/2, z/2] \longrightarrow$ Ji OAM

Connection to GTMDs – A. Metz, M. Schlegel, C. Lorcé, B. Pasquini . . .

Direct evaluation of quark orbital angular momentum

$$L_3^{\mathcal{U}} = \int dx \int d^2k_T \int d^2r_T \, (r_T \times k_T)_3 \, \mathcal{W}^{\mathcal{U}}(x, k_T, r_T) \qquad \text{Wigner distribution}$$

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2] \psi(z/2) \mid p, S\rangle \big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}}{\langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2] \psi(z/2) \mid p, S\rangle \big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}}$$

n : Number of valence quarks

$p' = P + \Delta_T/2$, $p = P - \Delta_T/2$, P, S in 3-direction, $P \to \infty$

This is the same type of operator as used in TMD studies – generalization to off-forward matrix element adds transverse position information

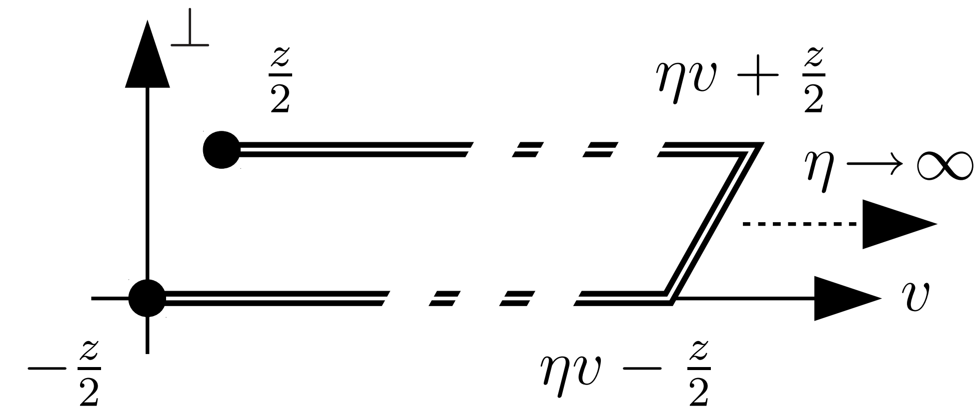
Direct evaluation of quark orbital angular momentum

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{\epsilon_{ij}\frac{\partial}{\partial z_{T,i}}\frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle\Big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}}{\langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle\Big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}}$$

Role of the gauge link $\mathcal{U}$:

Y. Hatta, M. Burkardt:

- Straight $\mathcal{U}[-z/2, z/2] \longrightarrow$ Ji OAM
- Staple-shaped $\mathcal{U}[-z/2, z/2] \longrightarrow$ Jaffe-Manohar OAM
- Difference is torque accumulated due to final state interaction



Direct evaluation of quark orbital angular momentum

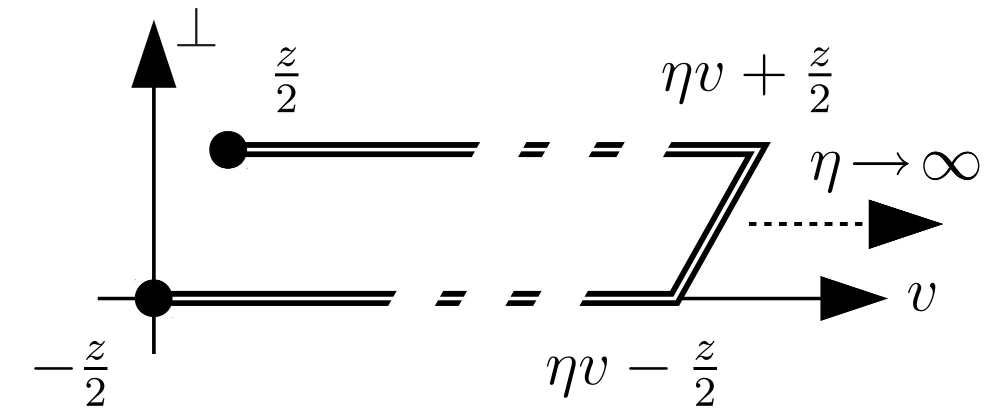
$$\frac{L_3^{\mathcal{U}}}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2] \psi(z/2) \mid p, S \rangle \big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T \to 0}}{\langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2] \psi(z/2) \mid p, S \rangle \big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T \to 0}}$$

Role of the gauge link $\mathcal{U}$:

Direction of staple taken off light cone (rapidity divergences)

Characterized by Collins-Soper parameter $\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$

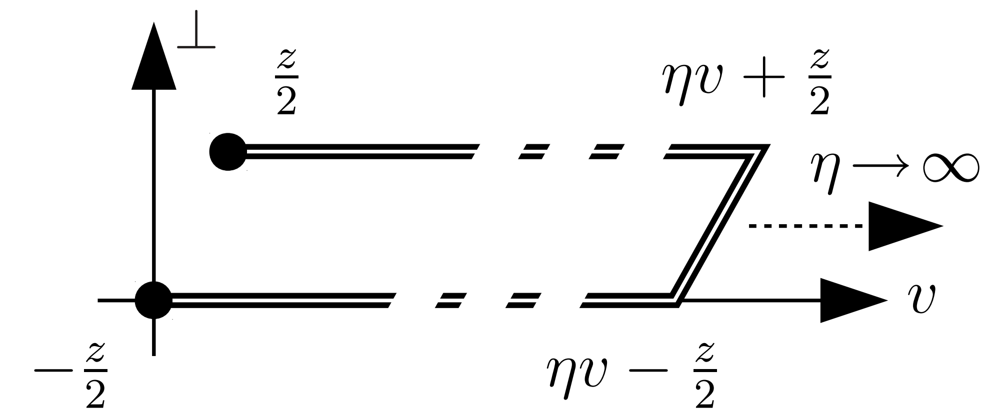
Are interested in $\hat{\zeta} \longrightarrow \infty$; synonymous with $P \longrightarrow \infty$ in the frame of the lattice calculation $(v = e_3)$



Direct evaluation of quark orbital angular momentum

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{\epsilon_{ij}\frac{\partial}{\partial z_{T,i}}\frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2)\gamma^{+}\mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle\big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}}{\langle p', S \mid \overline{\psi}(-z/2)\gamma^{+}\mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle\big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}}$$

Parameters to consider: Δ, $\hat{\zeta}$, z, η



Direct evaluation of quark orbital angular momentum

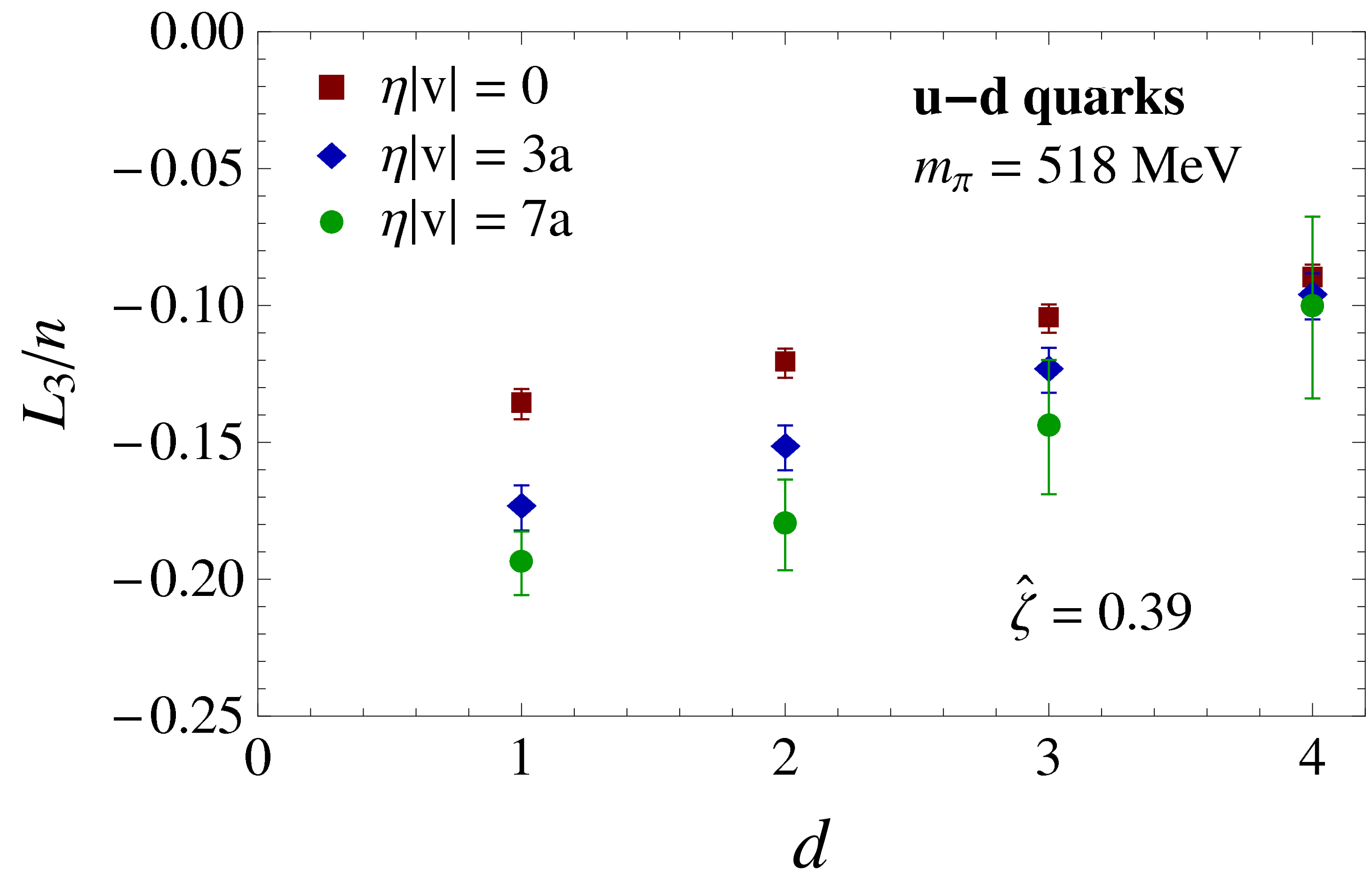
$$\frac{L_3^{\mathcal{U}}}{n} = \frac{\epsilon_{ij}\frac{\partial}{\partial z_{T,i}}\frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle \big|_{z^+=z^-=0,\ \Delta_T=0,\ z_T\to 0}}{\langle p', S \mid \overline{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle \big|_{z^+=z^-=0,\ \Delta_T=0,\ z_T\to 0}}$$

Dataset contains only one value of $|\Delta_T| = 4\pi/aL \approx 1$ GeV

Substantial underestimate of $\partial f/\partial \Delta_T$ by using

$$\left.\frac{\partial f}{\partial \Delta_{T,j}}\right|_{\Delta_{T,j}=0} = \frac{1}{2\Delta_{T,j}}(f(\Delta_{T,j}) - f(-\Delta_{T,j}))$$

Direct evaluation of quark orbital angular momentum

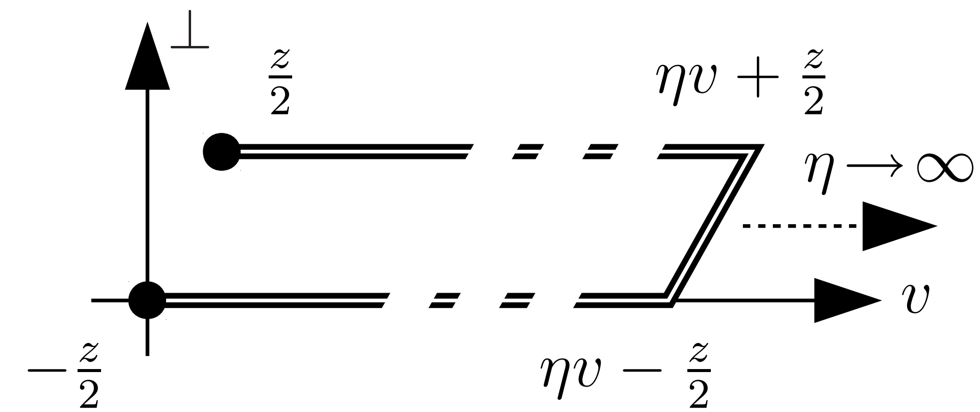


$$\left.\frac{\partial f}{\partial z_{T,i}}\right|_{z_{T,i}=0} = \frac{1}{2da}(f(dae_i) - f(-dae_i))$$

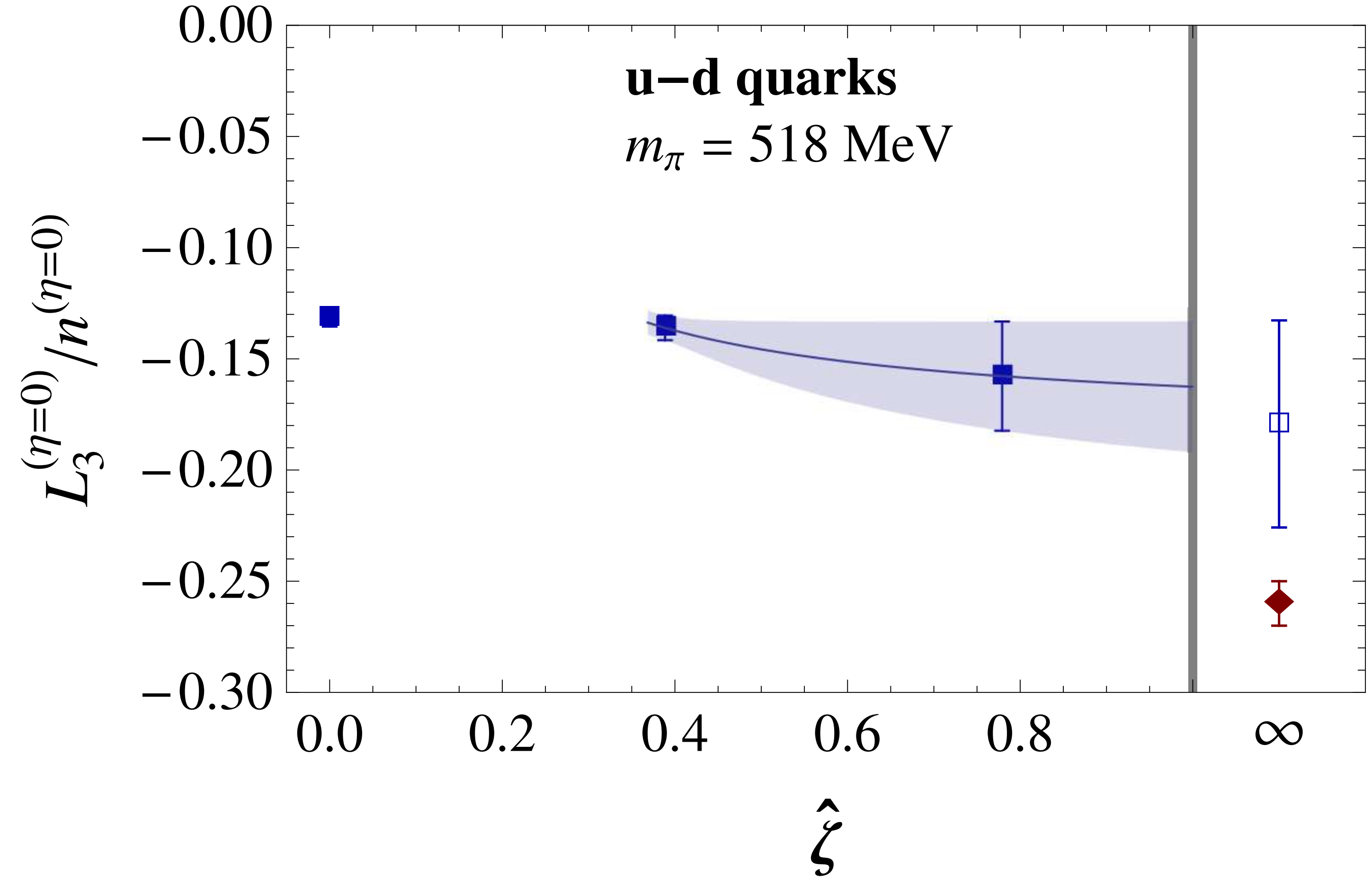
Direct evaluation of quark orbital angular momentum

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{\epsilon_{ij}\frac{\partial}{\partial z_{T,i}}\frac{\partial}{\partial \Delta_{T,j}} \langle p', S \mid \overline{\psi}(-z/2)\gamma^{+}\mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle\big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\rightarrow 0}}{\langle p', S \mid \overline{\psi}(-z/2)\gamma^{+}\mathcal{U}[-z/2, z/2]\psi(z/2) \mid p, S\rangle\big|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\rightarrow 0}}$$

Remaining parameters to consider: $\hat{\zeta}$, η

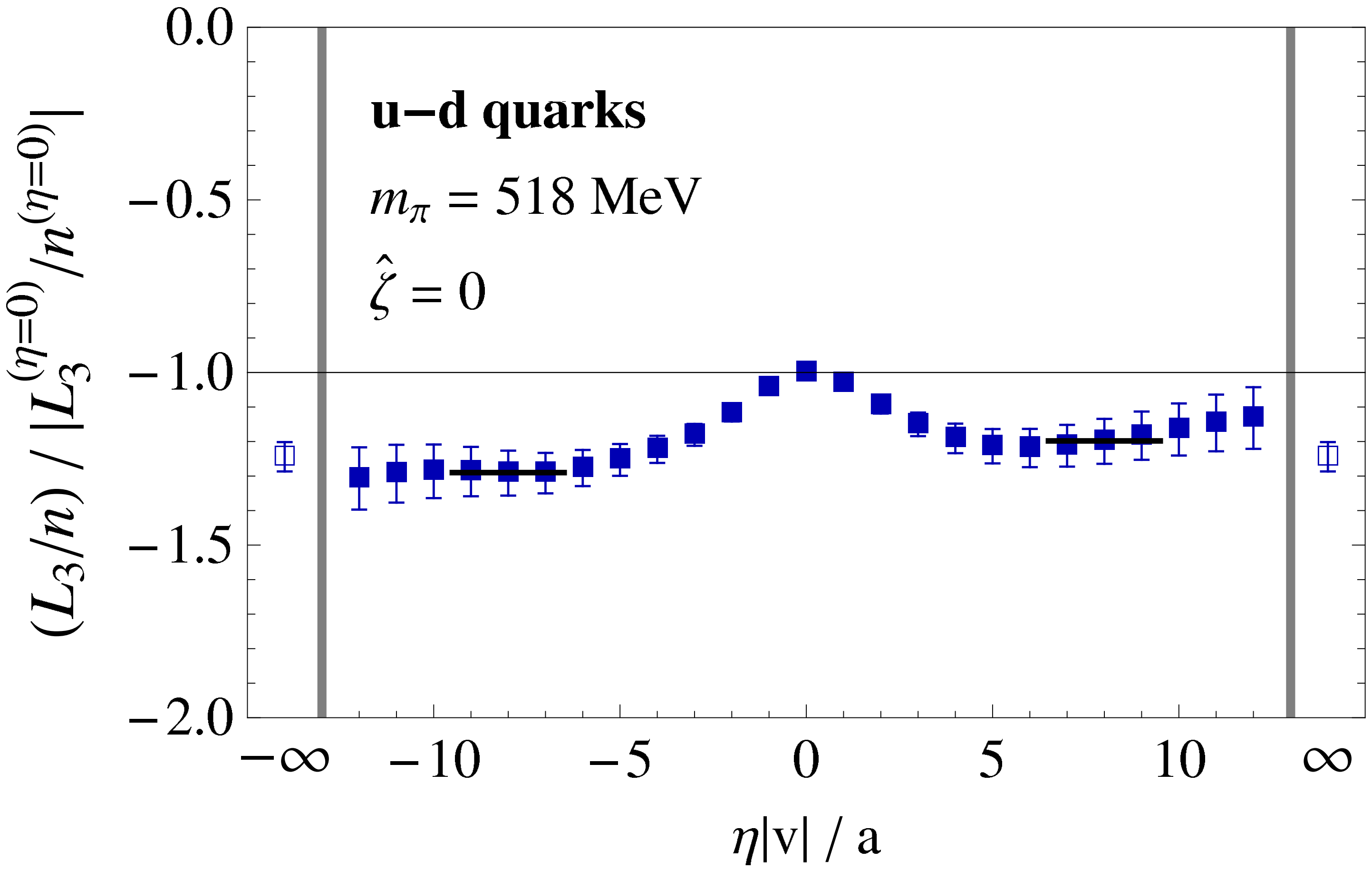


Ji quark orbital angular momentum: $\eta = 0$

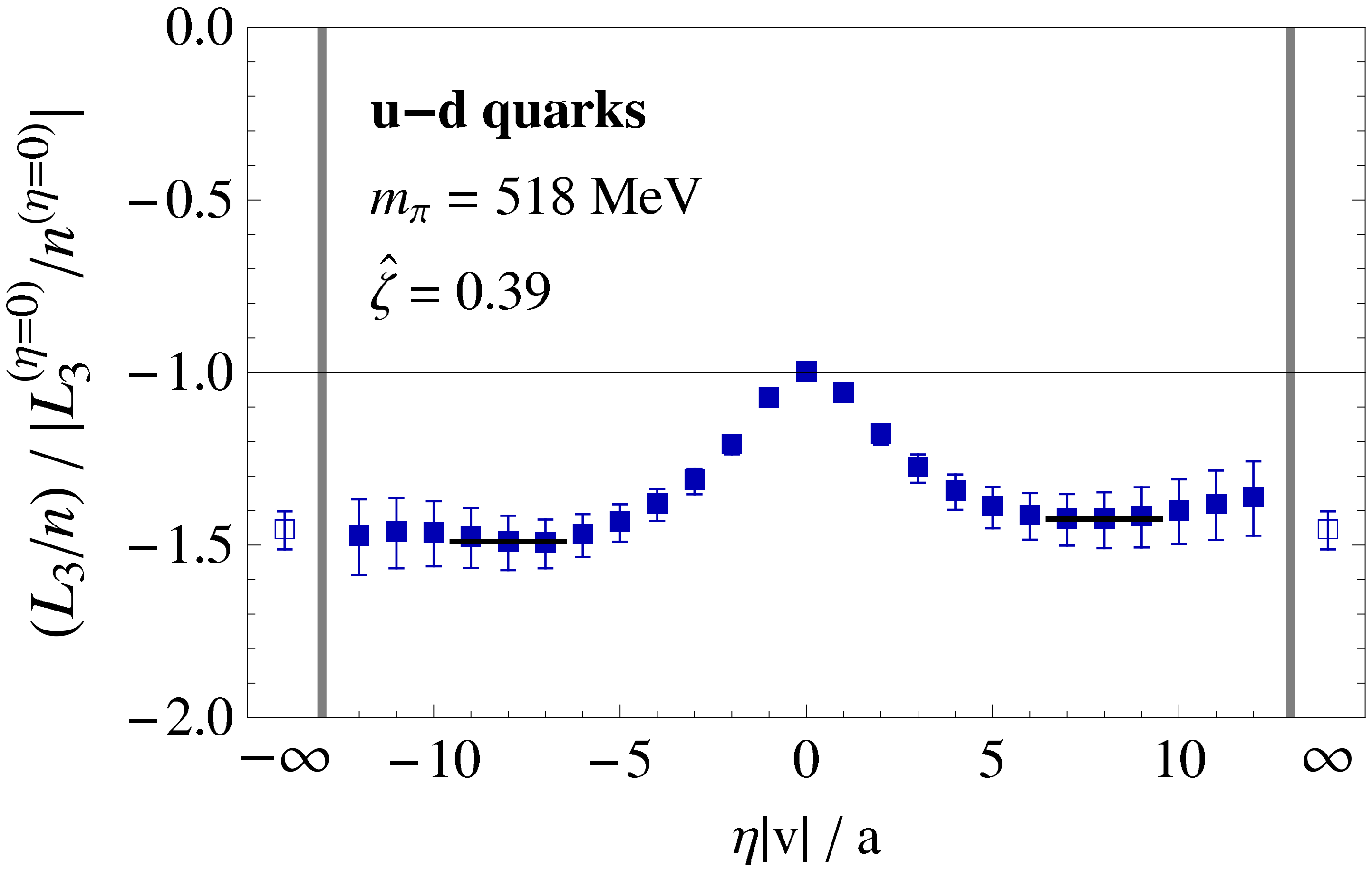


$\longrightarrow$ Signature of underestimate of $\partial f / \partial \Delta_T$

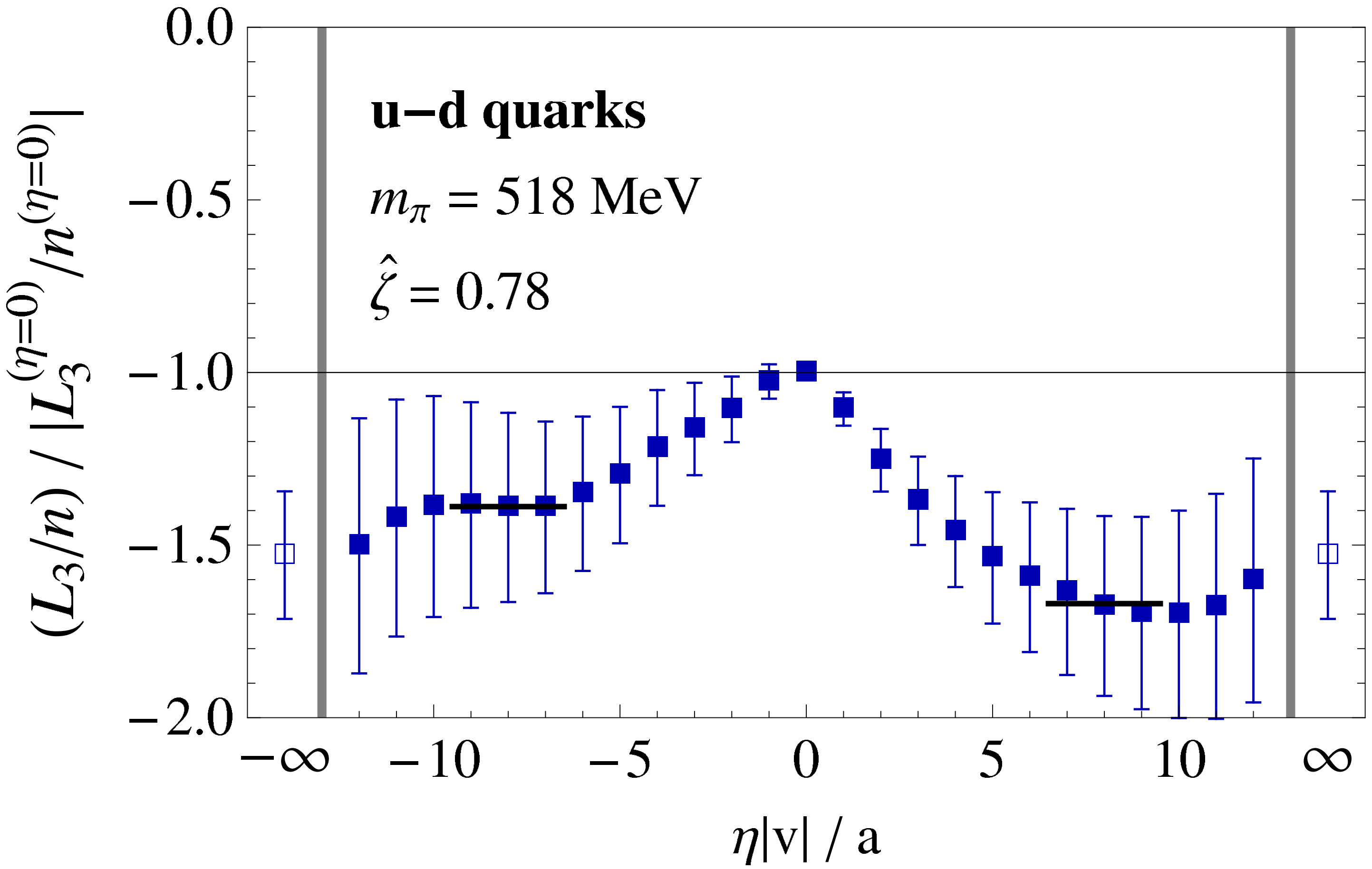
From Ji to Jaffe-Manohar quark orbital angular momentum



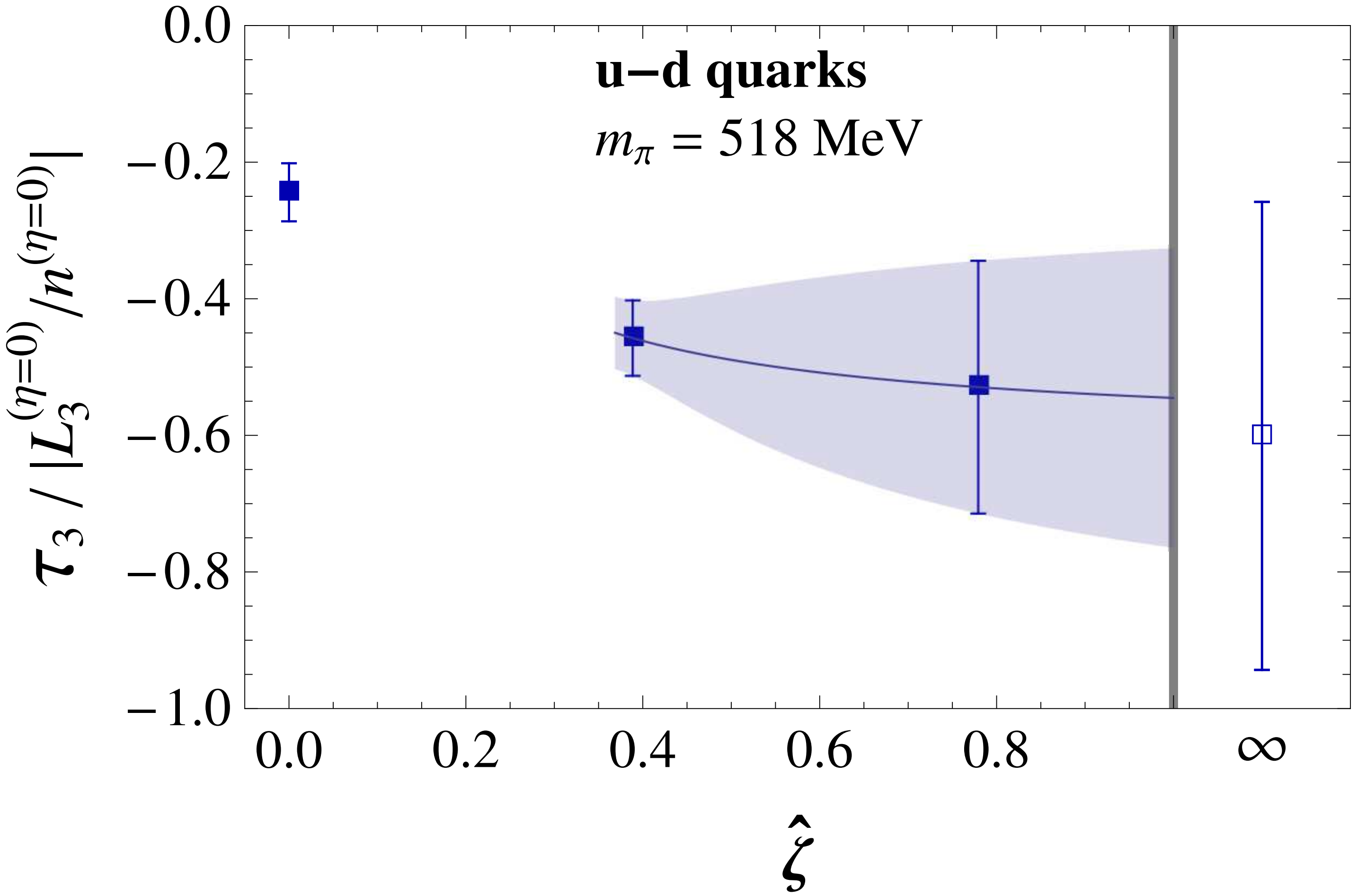
From Ji to Jaffe-Manohar quark orbital angular momentum



From Ji to Jaffe-Manohar quark orbital angular momentum



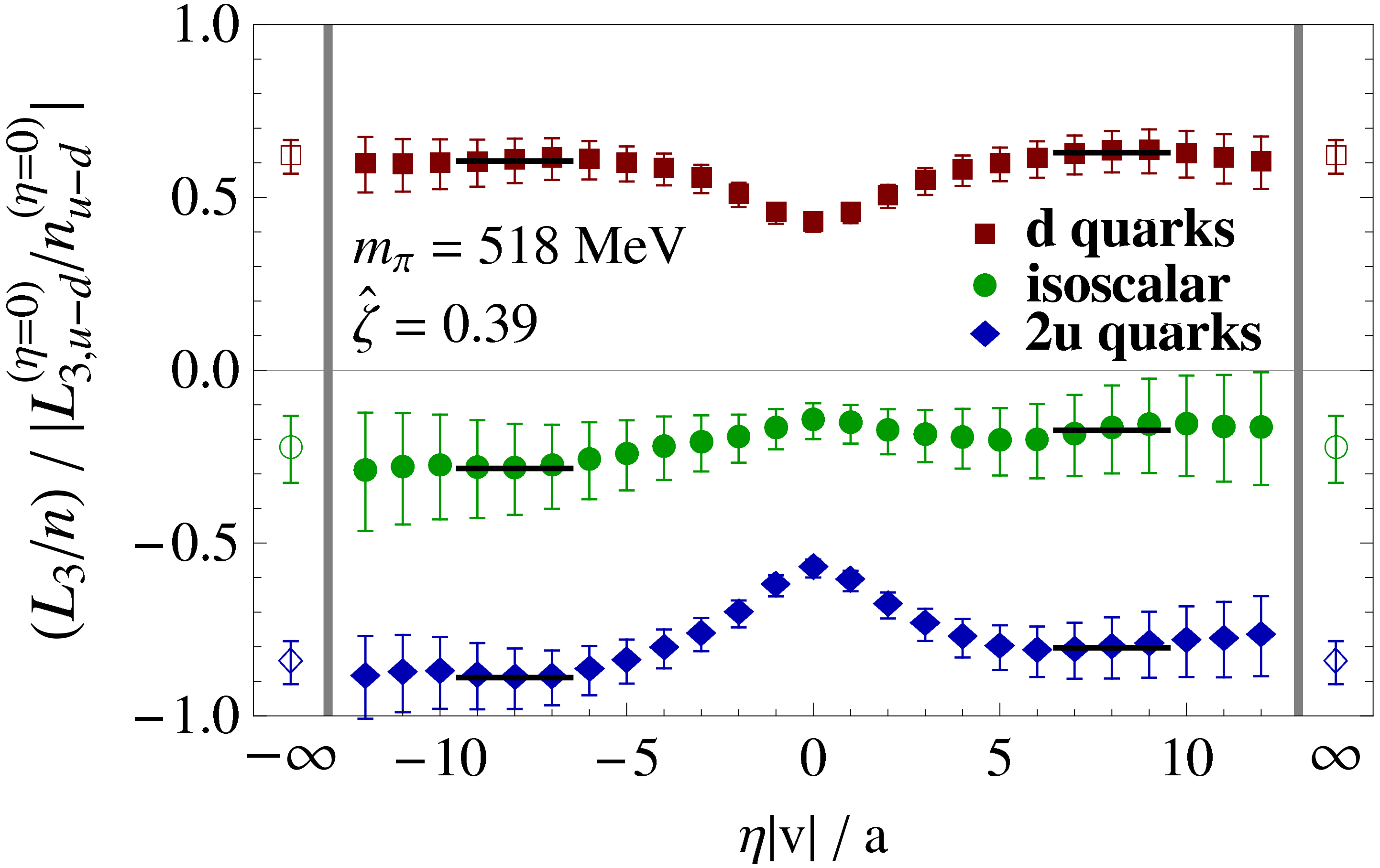
Burkardt's torque – extrapolation in $\hat{\zeta}$



$$\tau_3 = (L_3^{(\eta=\infty)}/n^{(\eta=\infty)}) - (L_3^{(\eta=0)}/n^{(\eta=0)})$$

Integrated torque accumulated by struck quark leaving proton

Flavor separation – from Ji to Jaffe-Manohar quark orbital angular momentum



Conclusions and Outlook

- Continued exploration of TMDs using bilocal quark operators with staple-shaped gauge link structures. Soft factors, multiplicative renormalizations are canceled by constructing appropriate ratios of Fourier-transformed TMDs / GTMDs.
- Exploration of challenges posed by $\hat{\zeta} \to \infty$ limit, discretization effects, physical pion mass limit.
- Generalization to mixed transverse momentum / transverse position observables (Wigner functions / GTMDs) gives direct access to quark orbital angular momentum and related observables such as quark spin-orbit coupling.
- A first comparison with experiment (Sivers shift) is encouraging.
- Current efforts concentrate on approaching the physical pion mass, improving the treatment of momentum transfer in GTMDs, and exploring further new TMD/GTMD observables (longitudinal polarization, twist-3 GTMDs).