Double Beta Decay and Effective Operators

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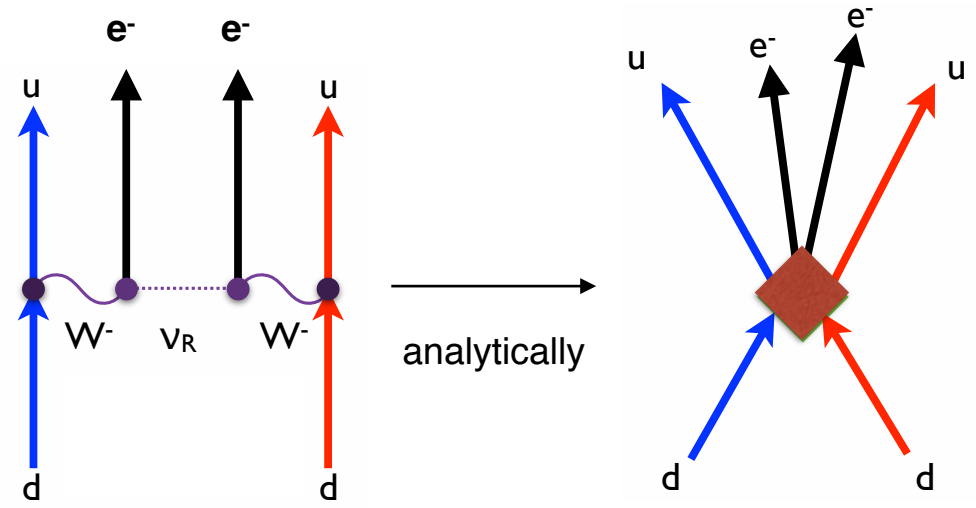
In standard effective operator treatments the problem is factored into two pieces

- the development of nucleon-level operators
- corrections to account for the errors associated with use of nucleon-level operators in truncated many-body SM spaces

The nucleon-level operators are most frequently determined by one of two methods

- by matching to experiment: this is how we determine g_A
- by direct calculation, e.g., lattice QCD evaluation of short-range mechanisms

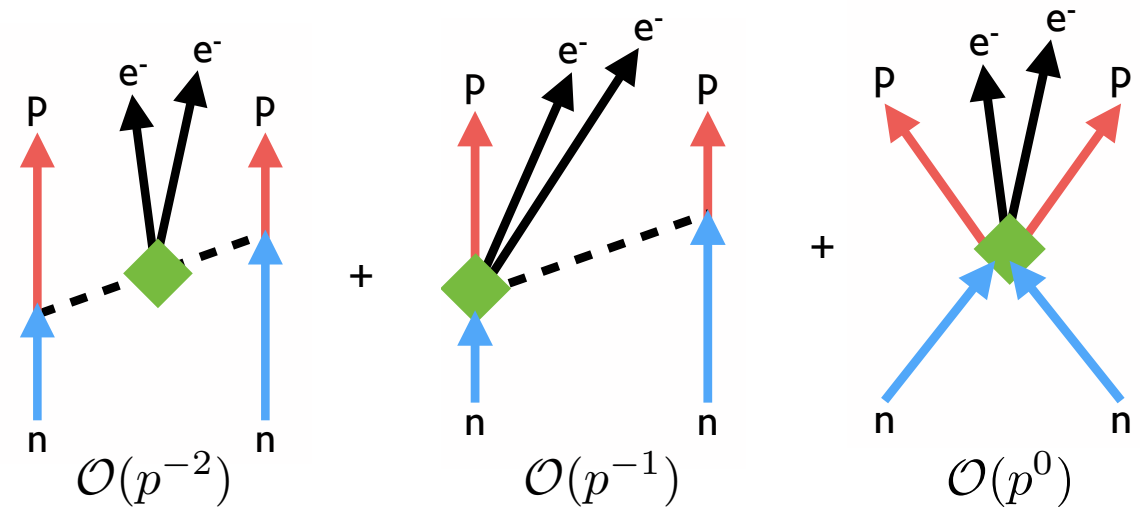
An example of a LQCD-generated nucleon-level operator we will hear about is



elementary amplitude

see Prezeau, Ramsey-Musolf, Vogel,
PRD 68 (2003) 034016
Amy Nicholson's workshop talk

chiral perturbation
power counting organizes
contributions dimensionally,
in terms of nucleon-level
low-energy constants
of unknown size



$\mathcal{O}(p^{-2})$ $\mathcal{O}(p^{-1})$ $\mathcal{O}(p^{0})$

Whatever the origin on the nucleon-level operator, work must be done to evolve that operator to the nuclear level

Feasibility of this is, in my view, dependent on the nuclear structure tools one employ

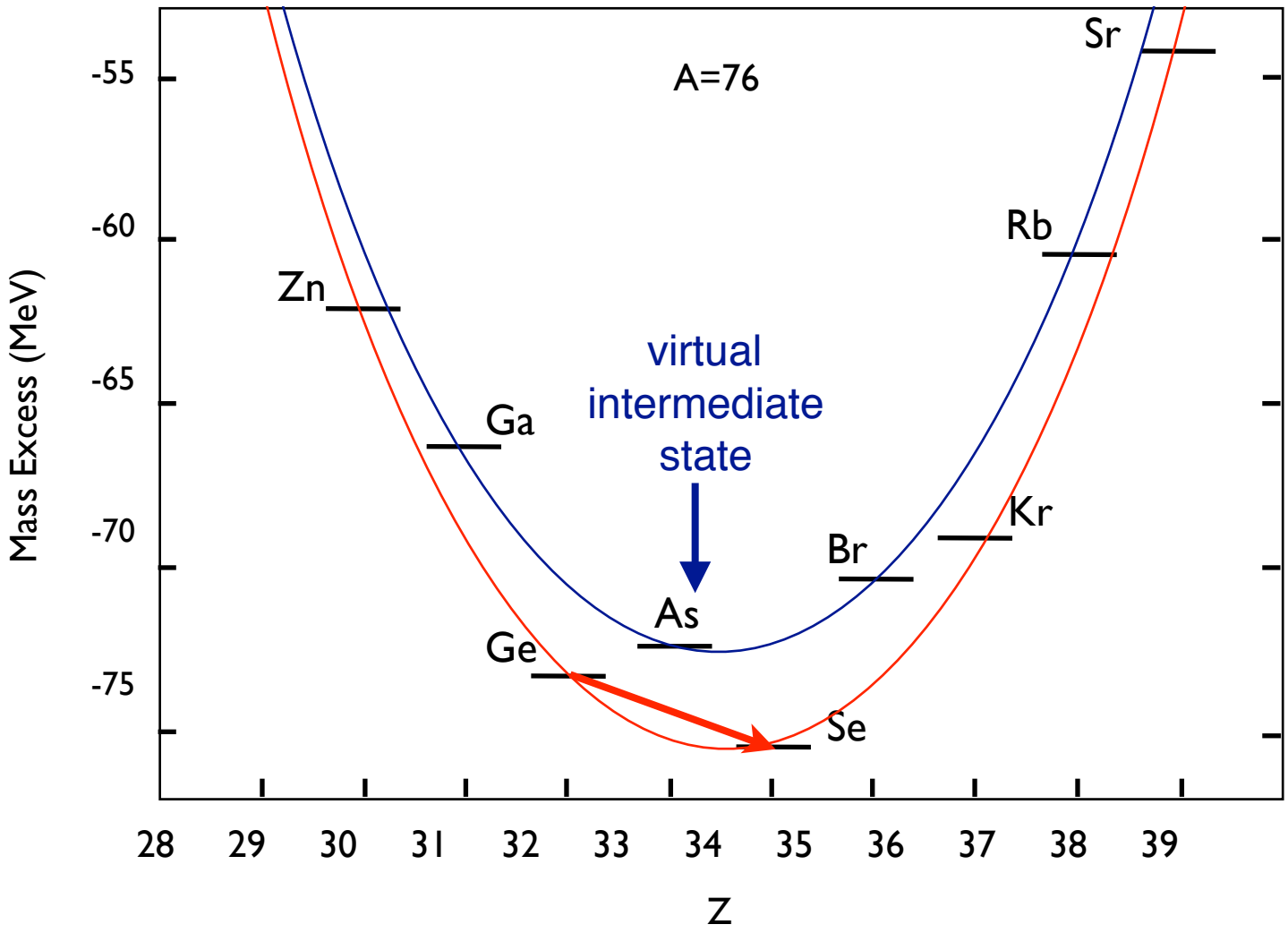
In the SM, there is usually a clean division of the Hilbert space, P+Q; modern technology generally allows us to use a complete P (e.g., full $0\hbar\omega$ space) that has attractive properties

- nonspurious, and usually complete in the sense that the GT operator remains in P (or can be patched to do so)

One can enumerate all states in Q, and thus write down the exact effective operator

Contrasts with approaches like QRPA or IBA that involve much less complete Ps, and Qs hard to enumerate explicitly

Excellent phenomenological H^{eff} s exist. The missing ingredient is the complementing O^{eff}: the SM may be our only opportunity to generate O^{eff} in a controlled way



The treatment of the intermediate states differs between the 0ν and 2ν processes

Non relativistic effective theory

Full Theory:

$$H|\psi\rangle = E|\psi\rangle$$

Effective Theory:

$$PH^{\rm eff}P|\psi\rangle = EP|\psi\rangle$$

$$P + Q = 1$$

$$|\psi\rangle = P|\psi\rangle + Q|\psi\rangle$$

Wave functions used in $\beta\beta$ calculations for heavy nuclei are taken from models. Modern SM potentials are typically tuned phenomenologically: impressively predictive, with e.g. GCN5082 being state of there art.

To employ these wave functions in an ET context, they must be given an interpretation. The natural interpretation

$$|\Psi_{SM}\rangle \;\leftrightarrow\; P|\Psi\rangle$$

This makes sense: the H.O. SM omits both the short-range physics that creates the hole in the two-nucleon correlation function, and long-range corrections repairing SM H.O. over-confinement

Wave functions carry a nontrivial normalization: in calculations where an exact effective theory is executed in a SM basis, the $0\hbar\omega$ SM includes only about 30% of the wave function — the rest spreads

The normalization is

$$\langle \Psi_{SM} | \Psi_{SM} \rangle \leftrightarrow \langle \Psi | P | \Psi \rangle = 1 - \langle \Psi | Q | \Psi \rangle$$

with the normalization constant so defined the effective operator is

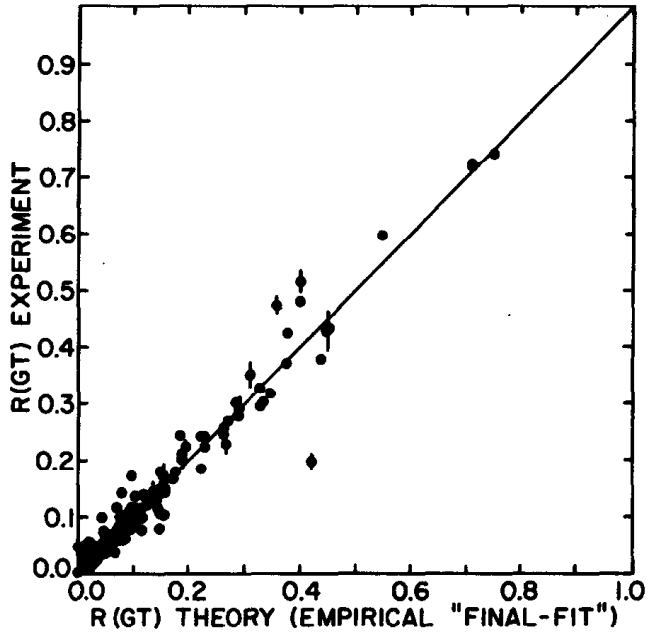
$$\langle \Psi^f_{SM} | \frac{E_f}{E_f - HQ} \, \hat{O} \, \frac{E_i}{E_i - QH} | \Psi^i_{SM} \rangle$$

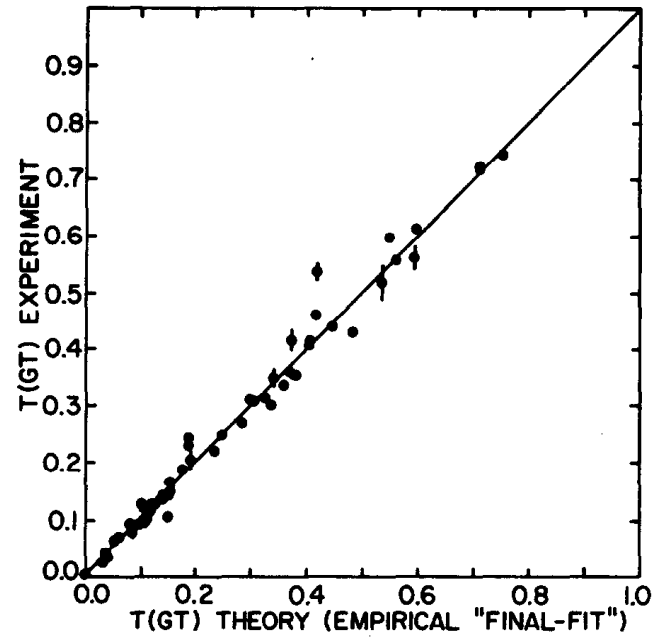
Often, at low q, the normalization is most of the physics

Single β Decay

Brown and Wildenthal empirical renormalization behavior for both exclusive and inclusive GT responses

$$g_A^{eff} \sim 1.0$$





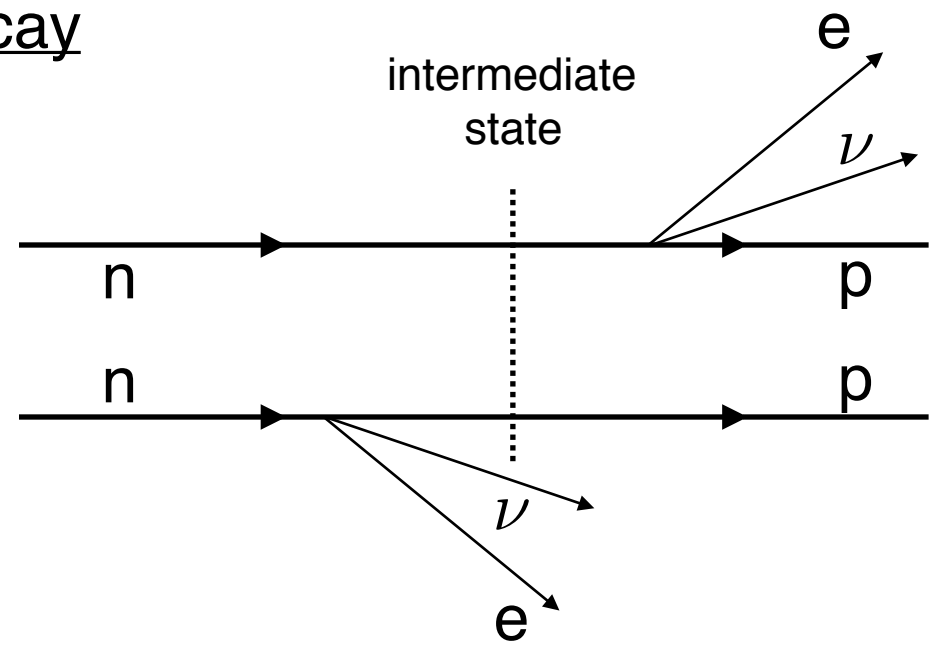
These results are interesting because of the particularly simple nature of the GT operator

Bare operators used between SM wave functions with unit normalization overestimate GT strengths

$$(g_A^{eff})^2 \text{ single } \beta \text{ decay } \leftrightarrow (g_A^{eff})^4 \ 2\nu \ \beta\beta \text{ decay}$$

$2\nu\ \beta\beta$ Decay

There was a time when 2ν decay was calculated using the closure approximation, introducing a free parameter, the average excitation energy



time-dependent perturbation theory

Replaced in modern SM calculations by a Lanczos Green's function method

Initial state $|I\rangle$ calculated, then $|v\rangle = \sum_i \sigma_0\ \tau_+\ |I\rangle$

Then

$$
\begin{aligned}
H|v_1\rangle &= \alpha_1|v\rangle + \beta_1|v_2\rangle \\
H|v_2\rangle &= \beta_1|v_1\rangle + \alpha_2|v_2\rangle + \beta_2|v_3\rangle \\
H|v_3\rangle &= \beta_2|v_2\rangle + \alpha_3|v_3\rangle + \beta_3|v_4\rangle \\
&\vdots
\end{aligned}
$$

$$\frac{1}{E_I - H}|v\rangle = g_1(E_I)|v_1\rangle + g_2(E_I)|v_2\rangle + \cdots$$

$$g_1(E_I) = \cfrac{1}{E_I - \alpha_1 - \cfrac{\beta_1^2}{E_I - \alpha_2 - \cfrac{\beta_2^2}{E_I - \alpha_3 - \cfrac{\beta_3^2}{\ddots}}}}$$

This algorithm reconstructs the Green's function by extracting from H the exact energy moments of the intermediate-state GT operator, recursively

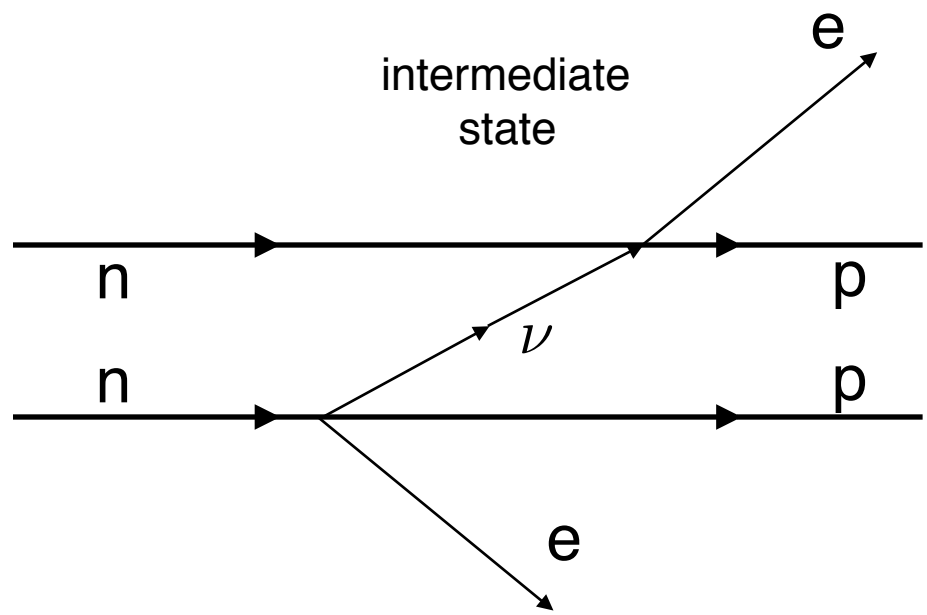
Both SM and Green's function calculations can now be executed in Hilbert spaces of 10^{11} (Bigstick, MFDn) — allowing nearly all targets of interest to be treated without truncation

The approach is a numerically sophisticated way of evaluating

$$\sum_J \sigma_0(i)\tau_+(i)\ |J\rangle\ \frac{1}{E_I - E_J}\ \langle J|\ \sum_j \sigma_0(j)\tau_+(j)$$

which explicitly involves a sum over a product of GT matrix elements, making an obvious connection to allowed beta decay, and supporting the use of $(g_A^{\rm eff})^4$

$0\nu\ \beta\beta$ Decay



The case of 0ν decay is somewhat different: prior to any model truncation the summation over intermediate states can be performed

The argument is based on more rapid growth of the relativistic phase space for the neutrino, which allows it to dominate the energy denominator

So within this controlled approximation, the summation over intermediate states can be done before the SM truncation

The norm of $|\Psi^{eff}\rangle$ enters quadratically, not to the forth power

Not at all clear that fitting $(g_A^{eff})^4$ to 2ν rates, then using it in 0ν calculations is justified

$$M_F^{0\nu} \sim \langle 0_f^+ || \frac{1}{2} \sum_{i,j=1}^{A} \tau_+(i)\tau_+(j) F(r_{ij}) || 0_i^+ \rangle$$

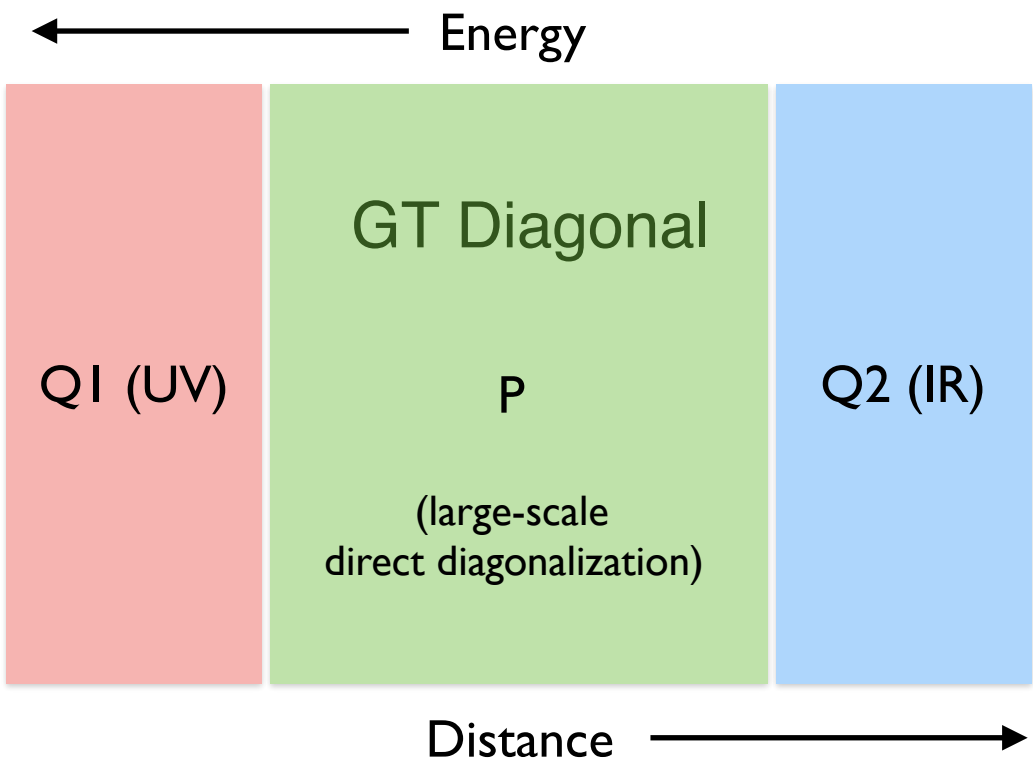
$$F(r_{ij}),\ G(r_{ij}) \ \sim \frac{1}{r_{ij}}$$

$$M_{GT}^{0\nu} \sim \langle 0_f^+ || \frac{1}{2} \sum_{i,j=1}^{A} \vec{\sigma}(i) \cdot \vec{\sigma}(j) \tau_+(i)\tau_+(j) G(r_{ij}) || 0_i^+ \rangle$$

This should give one some pause: it does not naively support a rescaling of the operators by $(g_A^{eff})^4$

One of the goals of the $\beta\beta$ Topical Collaboration is to test such assumptions, while developing a sounder approach to effective operators in weak decays

The GT operator does not change $\hbar\omega$: it does not link P, Q
It does not carry momentum, and thus should have
negligible probability of linking Q1 and Q2



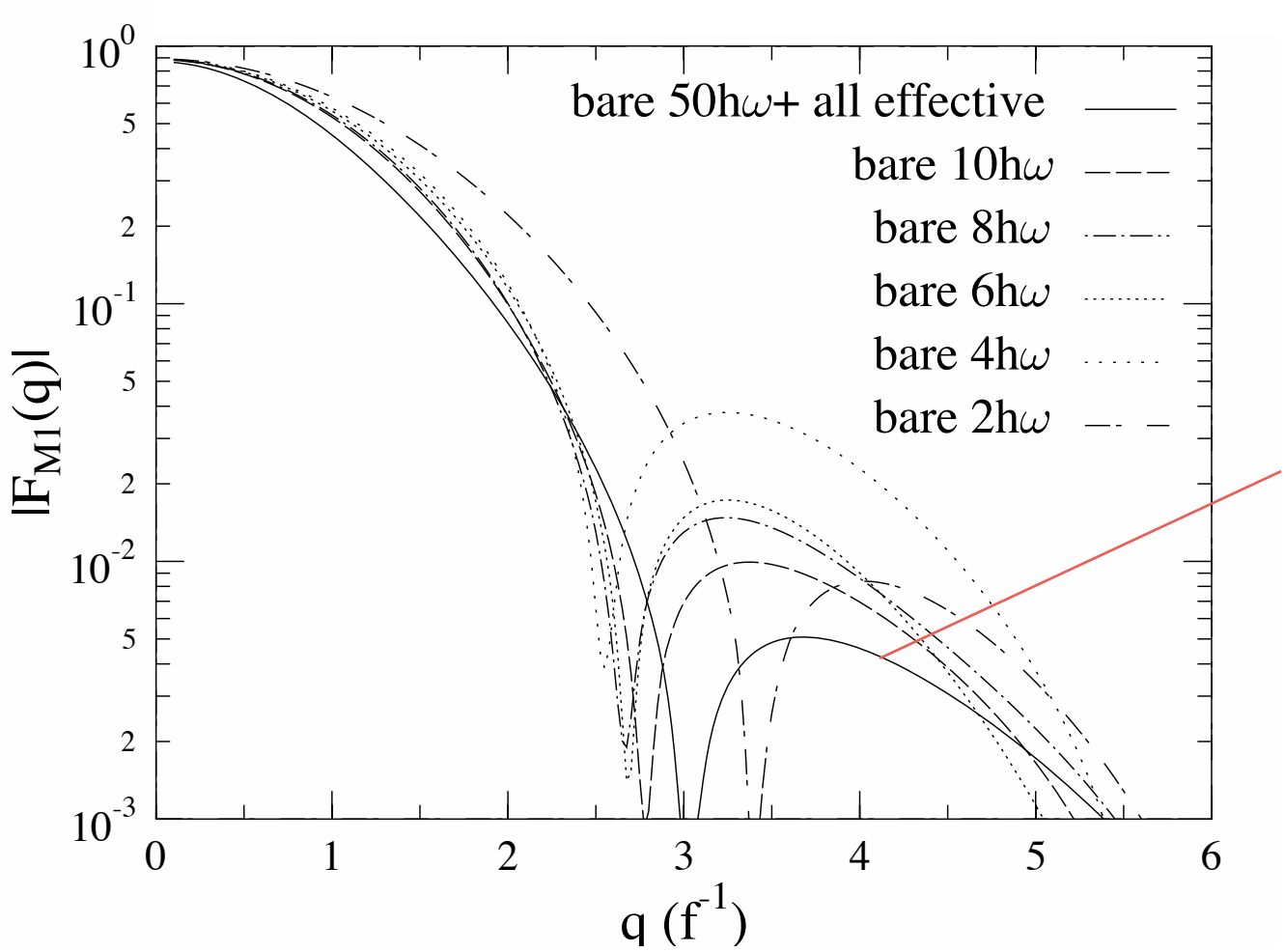
The sum rule holds in both in a unit-normalized P and P+Q:
consequently one concludes that if the sum is overestimated
with bare operators in P, typical matrix elements in Q are
smaller than those in P - qualitatively makes sense

However, what can we do quantitatively?

Example of shell-by-shell renormalization for the M1 operator — 3He, integrate out exactly

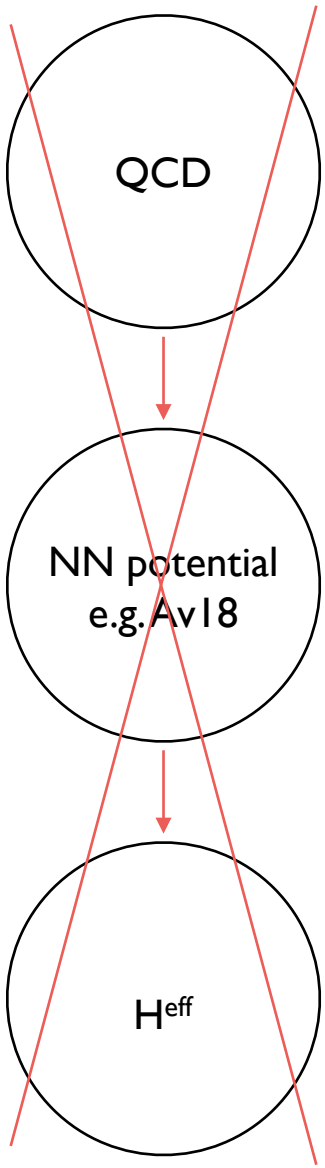
state	amplitude					
	$0\hbar\omega$	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\ \hbar\omega$	exact
	(31.1%)	(57.4%)	(70.0%)	(79.8%)	(85.5%)	(100%)
$\|0,1\rangle$	0.5579	0.5579	0.5579	0.5579	0.5579	0.5579
$\|2,1\rangle$	0.0000	0.0463	0.0461	0.0462	0.0462	0.0463
$\|2,2\rangle$	0.0000	-0.4825	-0.4824	-0.4824	-0.4824	-0.4826
$\|2,3\rangle$	0.0000	0.0073	0.0073	0.0073	0.0073	0.0073
$\|4,1\rangle$	0.0000	0.0000	-0.0204	-0.0204	-0.0204	-0.0205
$\|4,2\rangle$	0.0000	0.0000	0.1127	0.1127	0.1127	0.1129
$\|4,3\rangle$	0.0000	0.0000	-0.0419	-0.0420	-0.0421	-0.0423

While effective operator corrections are systematic at small q, they are effectively random at large q

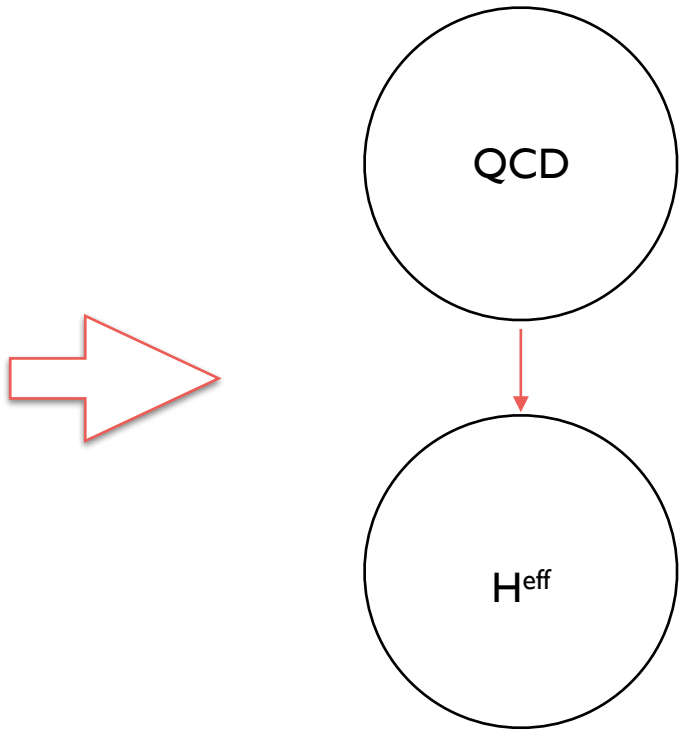


the effective operators correct all results

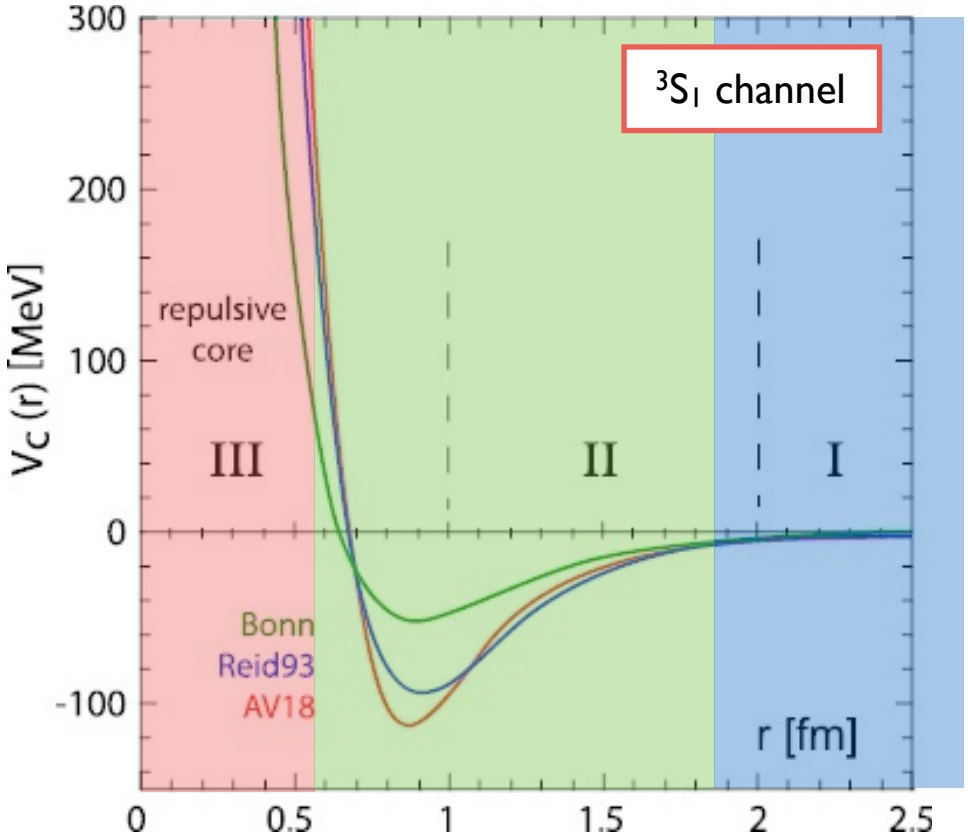
So at least this is encouraging for our allowed MEs of interest

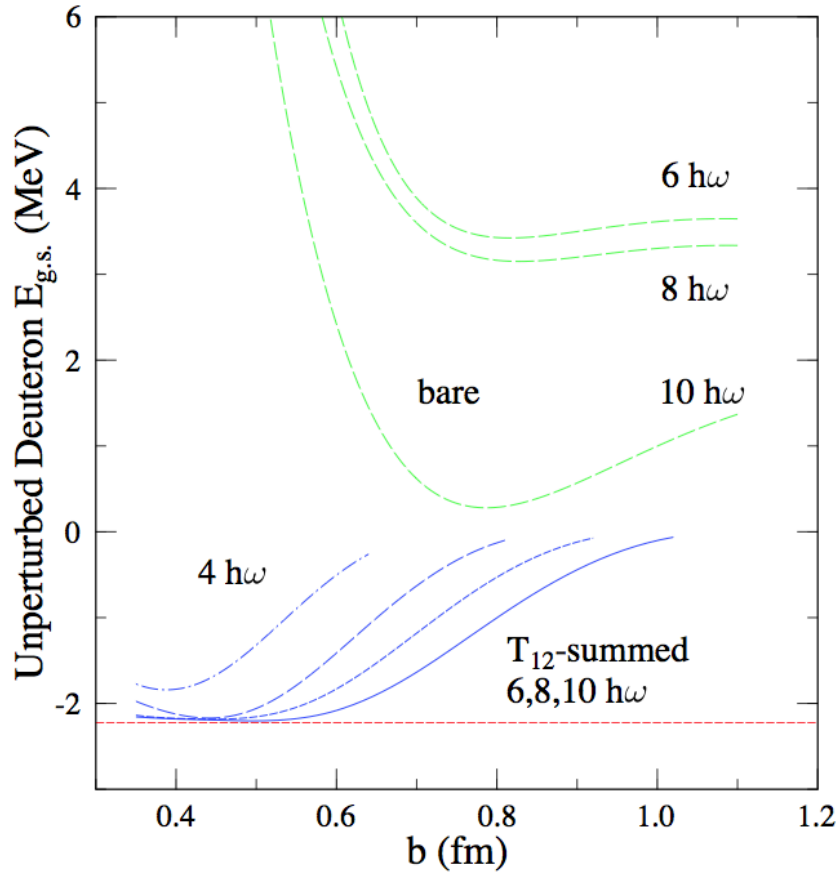


Our method for H^{eff} / O^{eff}



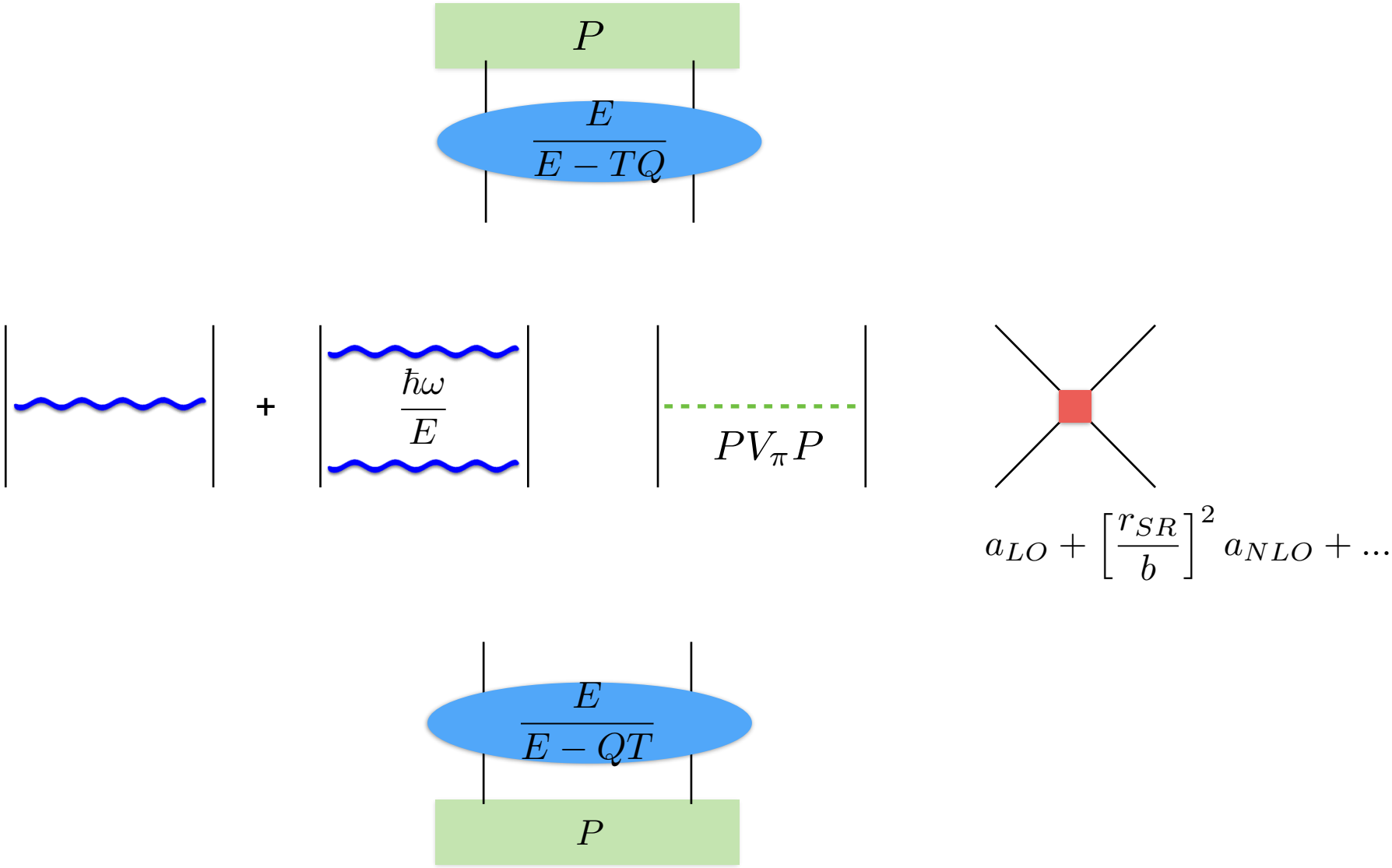
A true nonrelativistic ET



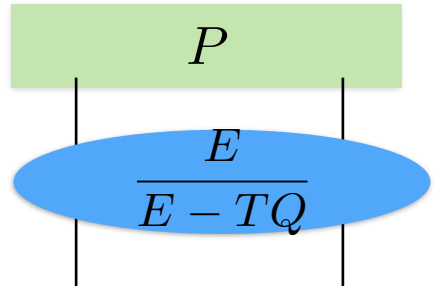


$$H^{\text{eff}} = PH\frac{1}{E-QH}QHP \rightarrow P\frac{E}{E-TQ}\left[T - T\frac{Q}{E}T + V + V\frac{1}{E-QH}QV\right]\frac{E}{E-QT}P$$

SOLVE SELF-CONSISTENTLY IN E: WH/Tom Luu Form of the BH Equation

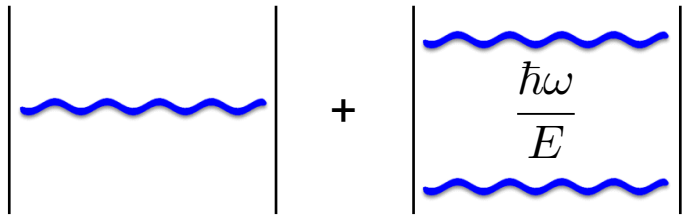


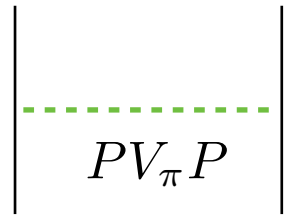
the correct chiral interaction for the HO "SM" - not the form folks use. Rapidly convergent

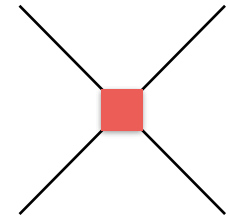


FAR INFRA-RED (asymptotic LS operator)

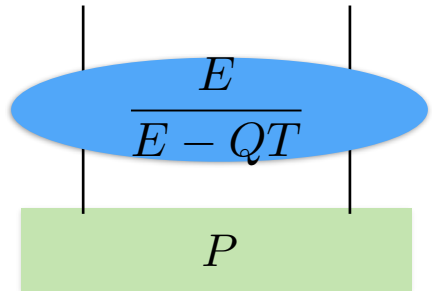
FAR INFRA-RED REGULATED, NEAR IR UV







$$a_{LO} + \left[\frac{r_{SR}}{b}\right]^2 a_{NLO} + \ldots$$



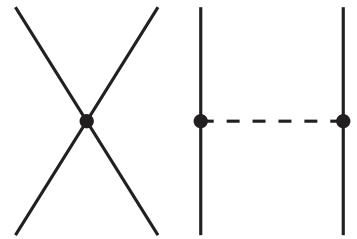
FAR INFRA-RED

the cutoff IS P

CONTRAST WITH WHAT IS USED IN THE BEST MODERN TREATMENTS OF STRUCTURE

LO

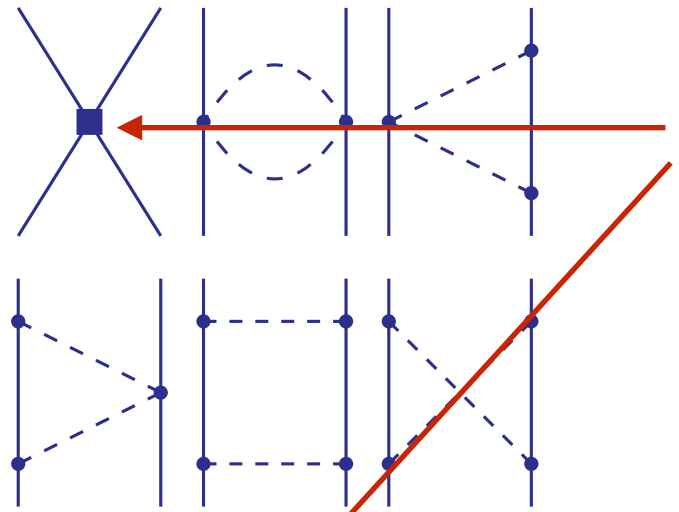
$(Q/\Lambda_\chi)^0$



wrong tensor-force regulator

NLO

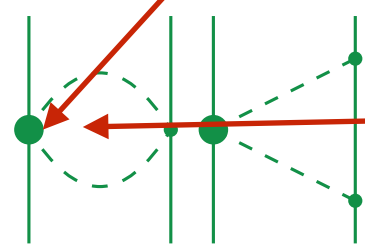
$(Q/\Lambda_\chi)^2$



momentum cutoffs inconsistent with the energy cutoff of the HO SM

NNLO

$(Q/\Lambda_\chi)^3$



cutoffs inconsistent with the translational invariance of the HO

+...

$$H^{\rm eff} = PH\frac{1}{E-QH}QHP \to P\frac{E}{E-TQ}\left[T - T\frac{Q}{E}T + V + V\frac{1}{E-QH}QV\right]\frac{E}{E-QT}P$$

$$a_{LO}^{3S1}\delta(\vec{r}) + a_{NLO}^{3S1}(\overleftarrow{\nabla}^2_{HO}\delta(\vec{r}) + \delta(\vec{r})\overrightarrow{\nabla}^2_{HO}) + \cdots$$

second quantized form:
HO raising/lowering
operators

$$\sim a_{LO}^{3S1}|n'\ell=0\rangle\langle n\,\ell=0| +$$

$$a^{3S1}{}_{NLO}\left(\boldsymbol{a}^\dagger\odot\boldsymbol{a}^\dagger|n'\ell=0\rangle\langle n\ell=0| + |n'\ell=0\rangle\langle n\,\ell=0|\tilde{\boldsymbol{a}}\odot\tilde{\boldsymbol{a}}\right) \;+\; \cdots$$

It is a HO phonon expansion

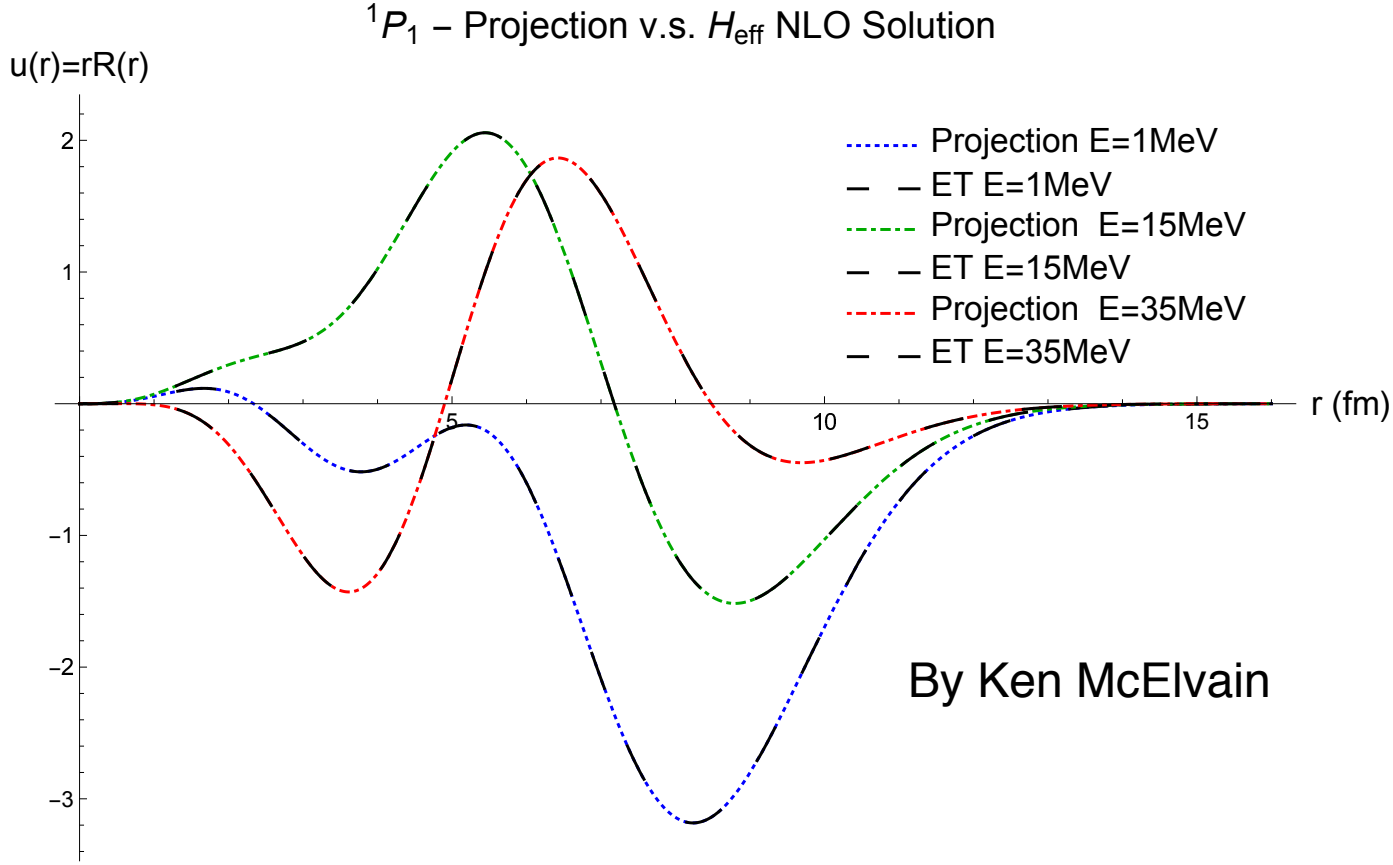
The LECs are determined directly from experiment: there is no potential outside P, there is no renormalization

We can do this because the theory is analytically continuous in energy, treating bound states and the continuum the same

- The BH equation formulated in a finite space yields an infinite an infinite number of states

- In the case of bound states, the KE Green's function depends only on E, and the self-consistency condition is an eigenvalue equation: if you know an eigenvalue (bound-state information), one must adjust the LECs to force reproduction

- In the case of continuum states, the Green's function depends on E and δ(E): pick any E, insert the experimental δ — a solution must exist at that E, so adjust LECs to achieve this

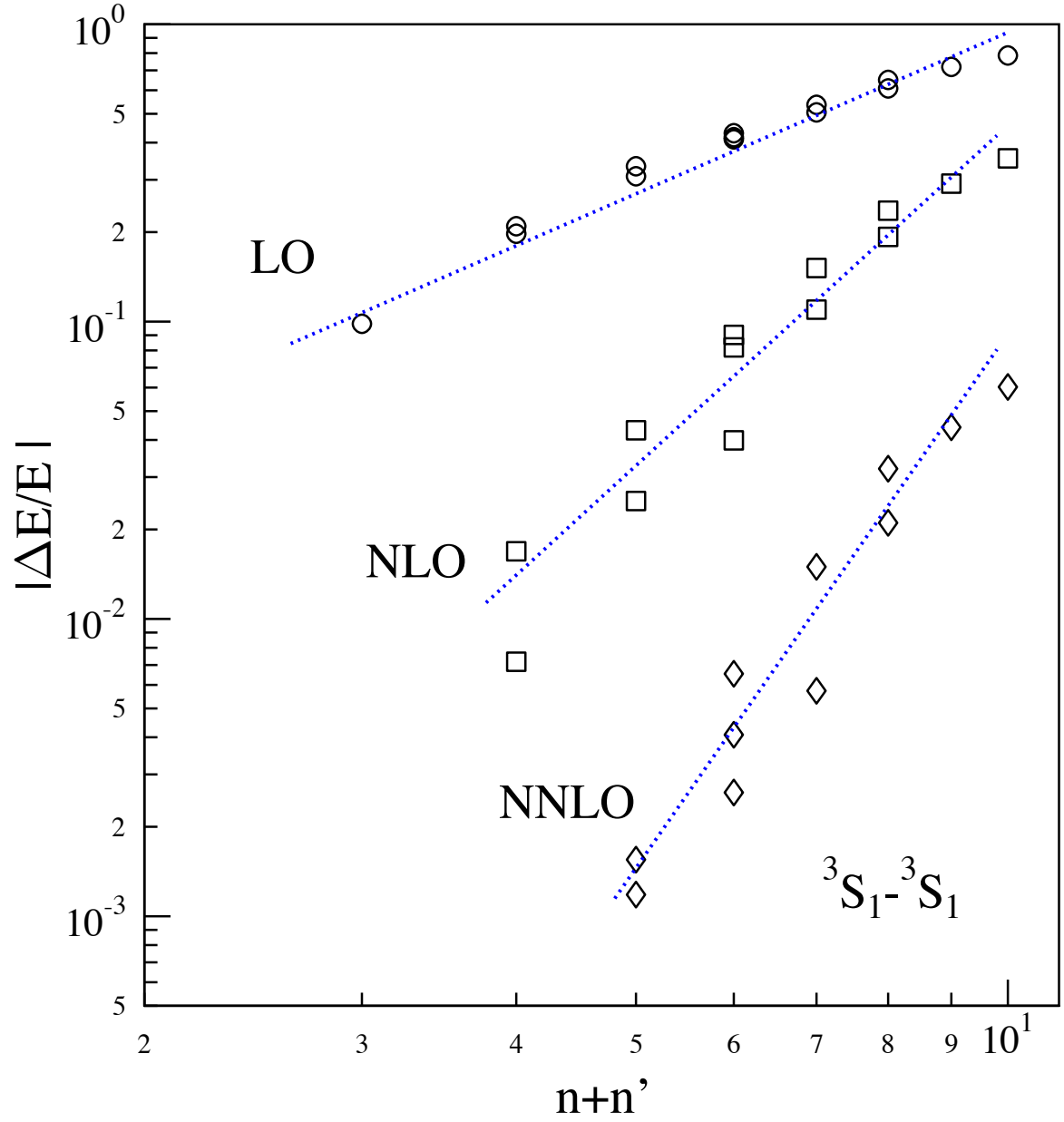


"SM" constructed directly from phase shifts: yields the exact restriction of the true wave function to P

information previously encoded in, decoded from an NN potential

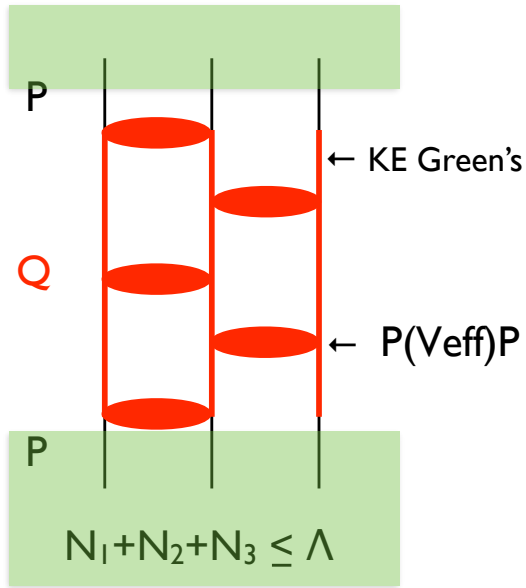
Phase shifts procedure yields deuteron binding energy of -2.2245 MeV

Results are independent of the choices made in defining P



order-by-order improvement systematic

The two-body physics so determined can then be subtracted exactly out of the N-body problem (now being done in p shell):



IR propagators

Interactions are now soft and finite in number: P by P matrices with KE propagation in between

The "ground up" double beta decay effective interactions strategy is essentially identical to that just described…

- $\beta\beta$ decay operators are scalars: the amplitude as a function of CM energy looks like a scattering process

- The formalism allows one to do the elementary process $nn \to pp + 2e^-$ as a function of energy, mapping out the amplitude in E: one amplitude imposes many constraints on LECs

- The effective theory (HOBET) yields the scattering state projected onto the SM-like P-space

- The effective operator can then be “built upward” from this amplitude, evaluating higher-body corrections systematically

- Can be compared to standard top-down methods, where a very large SM space is employed for a light nucleus, then that space is integrated downward, to evolve O^{eff}

Final Comments

- In truth, I worry a bit that so much energy is being focused on $\beta\beta$ decay: this is an highly exclusive operator, typically exhausting $\sim 0.1\%$ of the double GT sum rule

- "Walk before you run" theory suggests attacking simpler, better known (data!) operators first

- But we can certainly compute functionally exact effective operators for 2,3,4,… body systems — which would test basic assumptions about relationships between allowed β decay, $2\nu\ \beta\beta$, and $0\nu\ \beta\beta$