Extension of the Bjorken Formula

Zi-Wei Lin
Department of Physics, East Carolina University
& Central China Normal University

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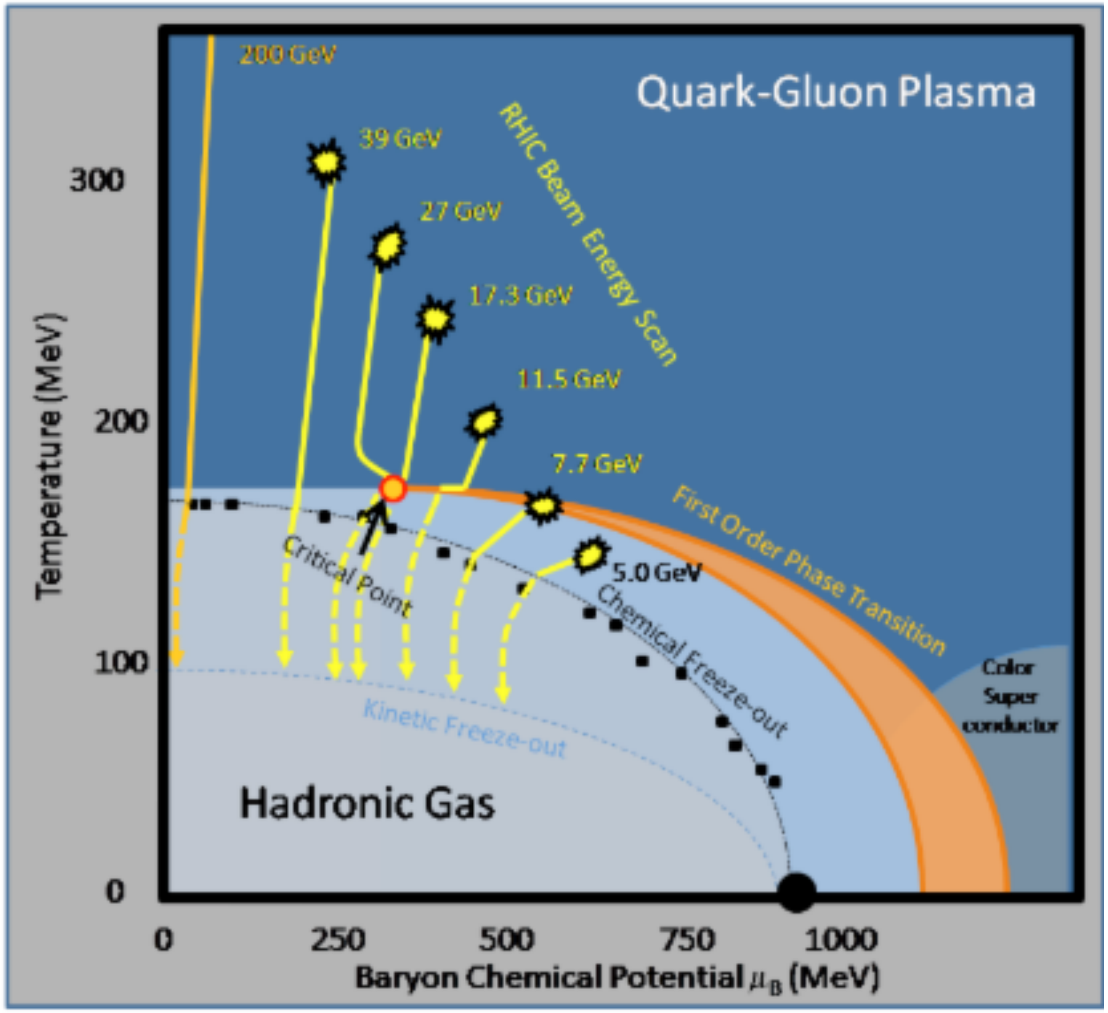
Outline

- The Bjorken formula and a major problem
- Extension of the Bjorken formula
- Applying extended formula to central Au+Au collisions
- Comparisons with results from transport model AMPT
- Summary

Based on ZWL, arXiv:1704.08418

The Bjorken formula

- Trajectory of a collision depends on the time evolution of energy density & net-baryon density.
- For Beam Energy Scan energies, trajectory is important for effects from critical point.
- We need to estimate/calculate the initial energy density, including its peak value and time dependence: ε^{max}, $\varepsilon(t)$



from STAR arXiv:1007.2613

The Bjorken formula

A common model is the Bjorken formula:

$$\epsilon(\tau) = \frac{1}{\tau\, A_T}\frac{dE_T(\tau)}{dy}$$

PHYSICAL REVIEW D VOLUME 27, NUMBER 1 1 JANUARY 1983

Highly relativistic nucleus-nucleus collisions: The central rapidity region

J. D. Bjorken
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510*
(Received 13 August 1982)

The space-time evolution of the hadronic matter produced in the central rapidity region in extreme relativistic nucleus-nucleus collisions is described. We find, in agreement with

At high energies, initial massless particles are produced from a pancake (at z=0) at t=0.

For partons in a thin slab of thickness **-d<z<d** in the middle (y~0) at time **t** :

$$|\tanh(y)| \approx |y| < \frac{d}{t}$$

Energy within the slab is

$$E = N\frac{d\langle E\rangle}{dy}\Delta y = N\frac{d\langle E\rangle}{dy}\frac{1}{\cancel{2}}\left[\frac{2d}{t}\right] . \qquad (3)$$

It follows that the central energy density ϵ is

$$\epsilon \approx \frac{N}{\mathscr{A}}\frac{d\langle E\rangle}{dy}\frac{1}{\cancel{2}t} . \qquad (4)$$

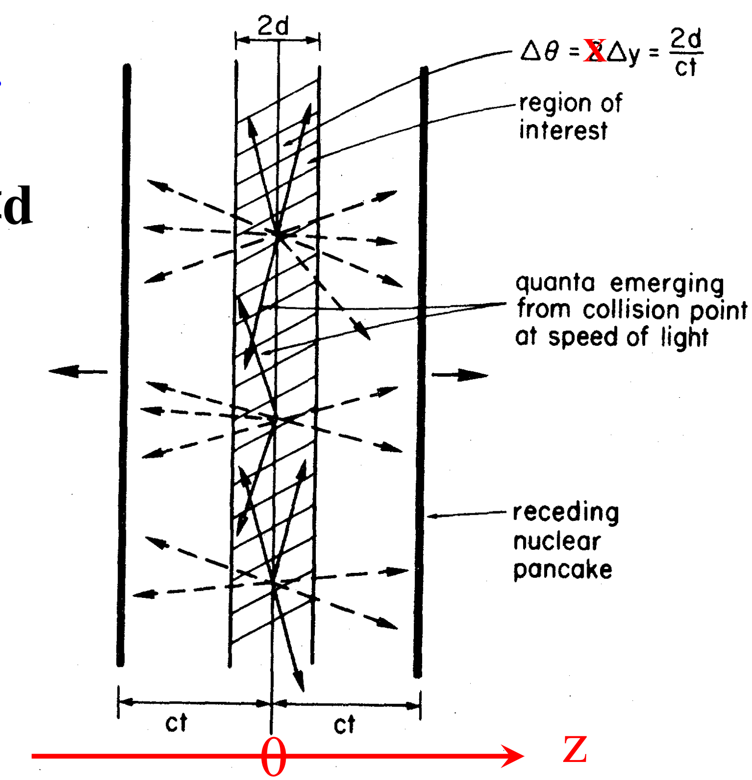


FIG. 2. Geometry for the initial state of centrally produced plasma in nucleus-nucleus collisions.

The Bjorken formula

A common model is
the Bjorken formula:

$$\epsilon(\tau) = \frac{1}{\tau\, A_T} \frac{dE_T(\tau)}{dy}$$

- Although $\frac{dE_T(\tau)}{dy}$ evolves with time,

one often uses the known final experimental value as an estimate.

- The Bjorken formula then diverges as $\tau \to 0$,

so we can assume a finite formation time τ_F for the initial particles, the Bjorken formula then becomes

$$\epsilon_{Bj}(\tau_F) = \frac{1}{\tau_F\, A_T} \frac{dE_T}{dy}$$

A major problem with the Bjorken formula

Bjorken, PRD 27 (1983)

In spite of Fig.1,
the Bjorken formula neglects
finite thickness of (boosted) nuclei
→ it is only valid at high energies
where crossing time << τ_F

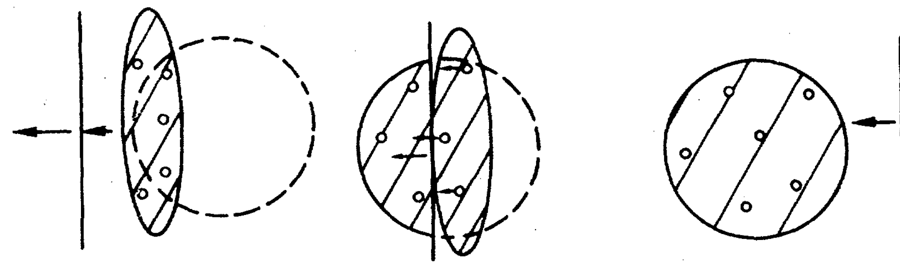


FIG. 1. Schematic of the evolution of a compressed "baryon fireball" in nucleus-nucleus collisions, according to the mechanism of Anishetty, Koehler, and McLerran (Ref. 8).

From PHENIX NPA757 (2005):

Eq. (5) here is essentially identical[5] to Eq. (4) of Bjorken's result [74], and so is usually referred to as the *Bjorken energy density* $\varepsilon_{\mathrm{Bj}}$. It should be valid as a measure of peak energy density in created particles, on very general grounds and in all frames, as long as two conditions are satisfied: (1) A finite formation time τ_{Form} can meaningfully be defined for the created secondaries; and (2) The thickness/"crossing time" of the source disk is small compared to τ_{Form}, that is, $\tau_{\mathrm{Form}} \gg 2R/\gamma$. In particular, the validity of Eq. (5) is completely independent of the shape of the $dE_T(\tau_{\mathrm{Form}})/dy$ distribution to the extent that

[5] A (well-known) factor of 2 error appears in the original.

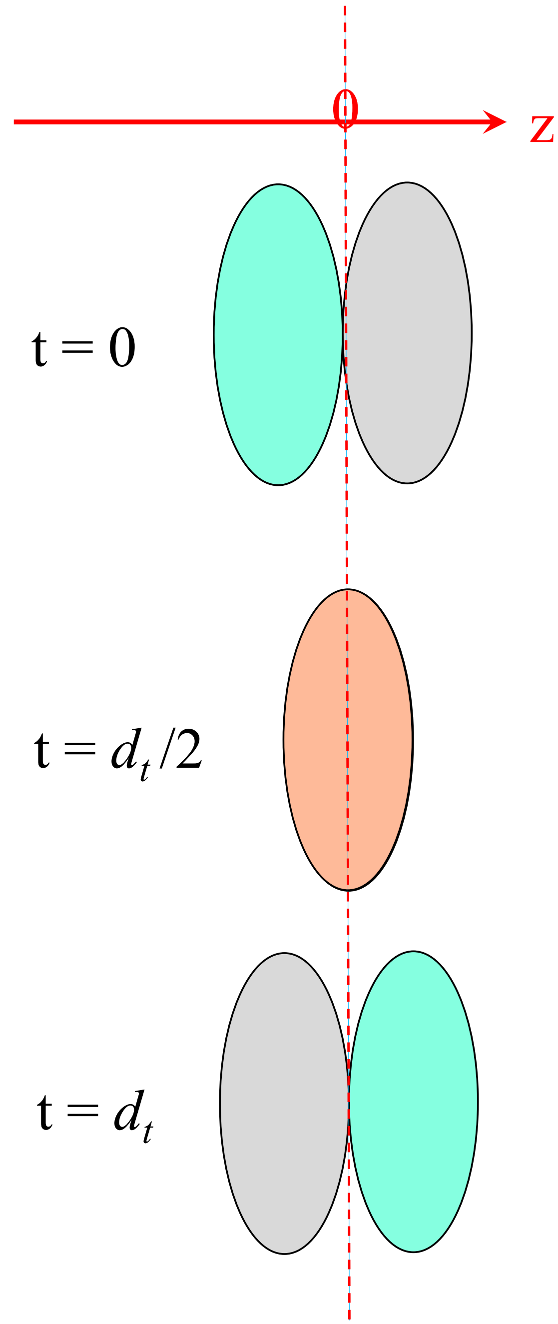
A major problem with the Bjorken formula

Considering central A+A collisions
in the center-of-mass frame
& using the hard sphere model for nucleus:

crossing time
(or duration time of the initial energy
production) is

$$d_t = \frac{2R_A}{\sinh y_{CM}} = \frac{2R_A}{\gamma\,\beta}$$

So the initial energy production
goes on throughout time $[0, d_t]$
with a certain time profile



A major problem with the Bjorken formula

Using the hard sphere model for nucleus:

$$R_A = 1.12A^{1/3}\ fm$$

crossing time ***d_t*** is

$$d_t = \frac{2R_A}{\sinh y_{CM}} = \frac{2R_A}{\gamma\,\beta}$$

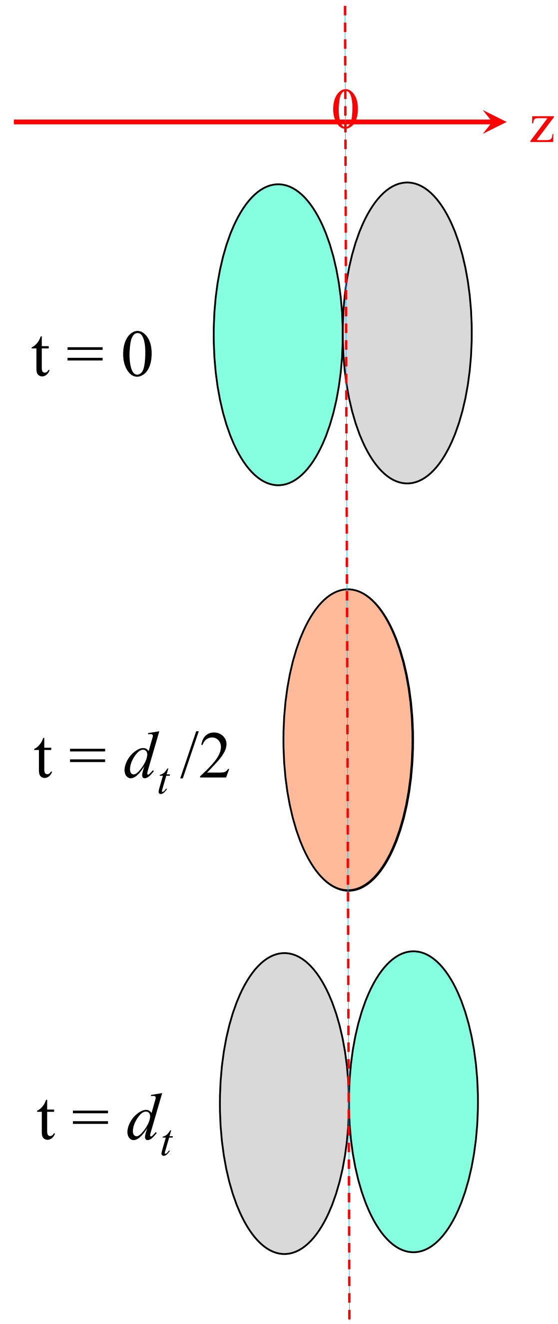
For central Au+Au collisions:

$\sqrt{s_{NN}}$ (GeV)	5	11.5	27	50	200
d_t (fm/c)	5.3	2.2	0.91	0.49	0.12

crossing time << τ_F

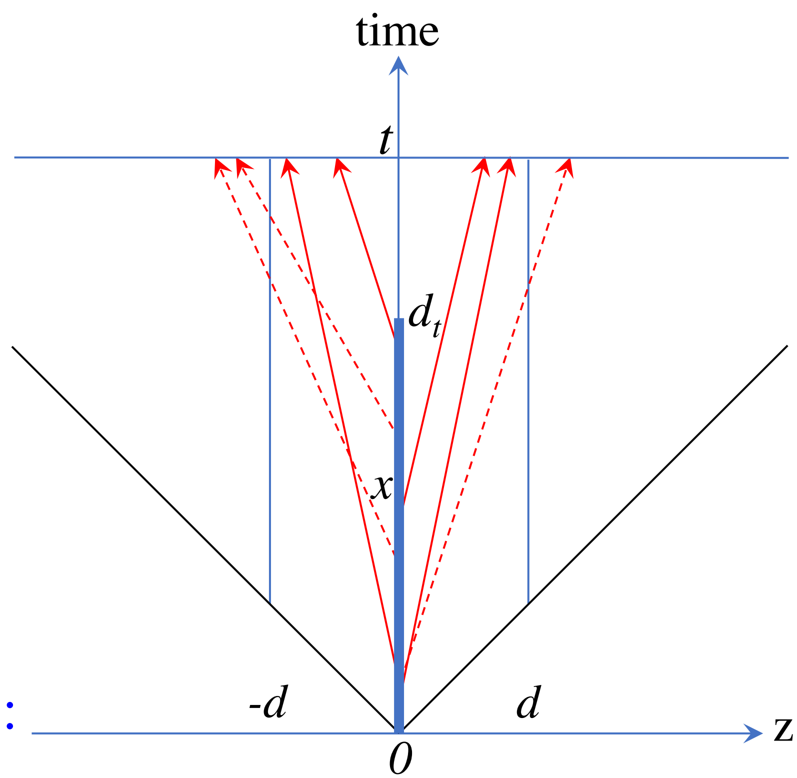
→ the Bjorken formula is only valid for

$\sqrt{s_{NN}} \gg 50GeV$ *for* $\tau_F = 0.5\ fm/c$



Extension of the Bjorken formula

My goal here is fix this problem
& have a Bjorken-type formula that's
also valid at low energies.



Particles around 0 rapidity could be produced at any time **x** within **[0,d_t]** and propagate to observation time **t**.

Initial massless particles
are produced from a pancake (at z=0)
during production time $x \in [0, d_t]$.

For partons in a thin slab of thickness **-d<z<d**
in the middle (y~0), at observation time **t** >d_t:

$$|\tanh(y)| \approx |y| < \frac{d}{t - x}$$

Write the production rate of initial $dE_T/dy(y\sim0)$
at production time **x** as $\frac{d^2E_T}{dy\,dx}$

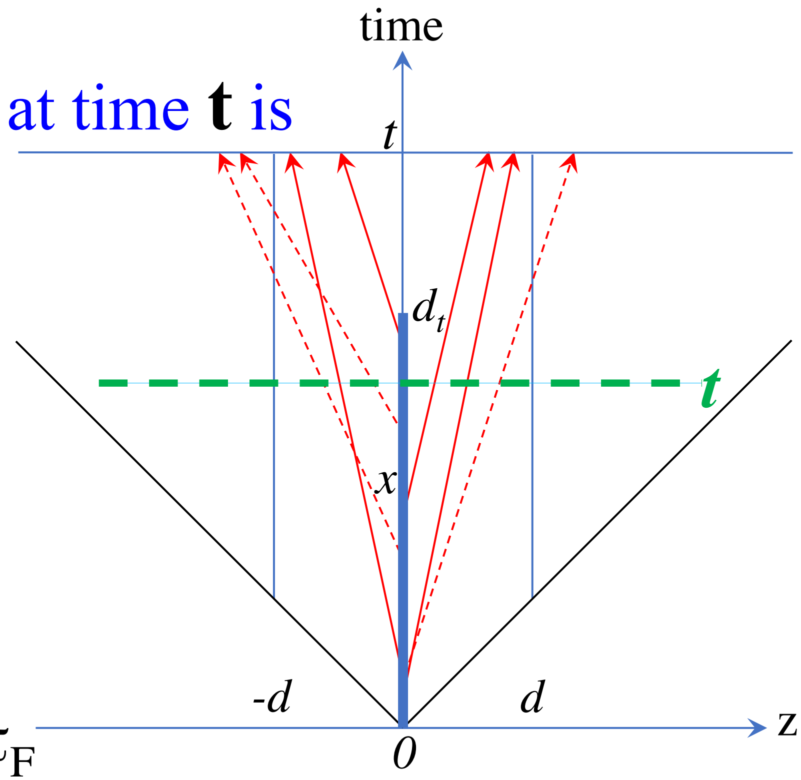
→ average energy density within the slab at time **t** is

$$\frac{E}{2d\,A_T} = \frac{1}{A_T}\int_0^{d_t} \frac{d^2E_T}{dy\,dx}\frac{dx}{(t-x)}$$

Extension of the Bjorken formula

Average energy density within the slab at time **t** is

$$\frac{E}{2d\,A_T} = \frac{1}{A_T}\int_0^{d_t} \frac{d^2E_T}{dy\,dx}\,\frac{dx}{(t-x)}$$



This usually diverges as $t \to 0$,
like the Bjorken formula.

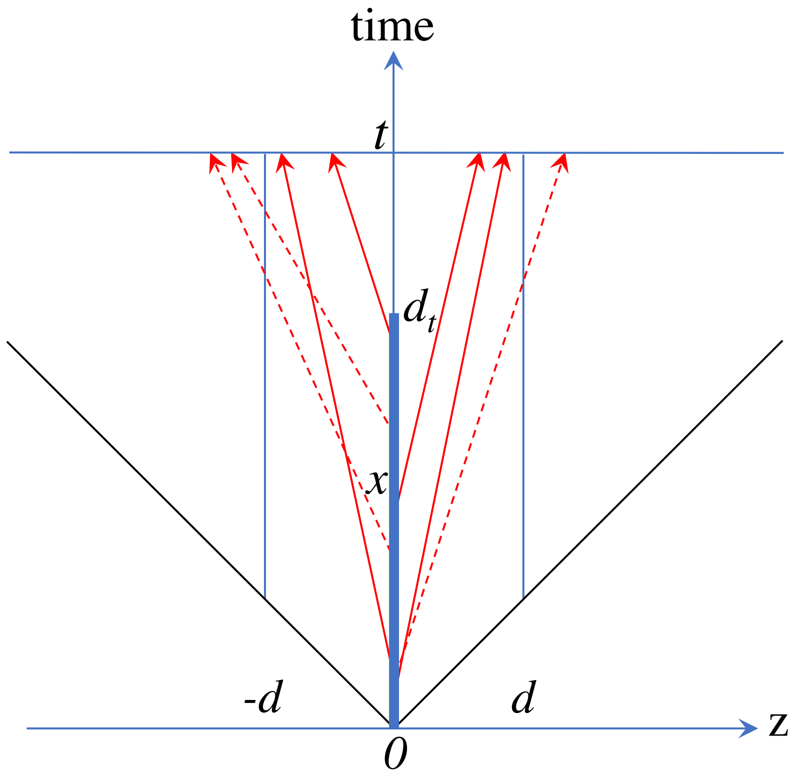
So we assume a finite formation time τ_F
for initial particles, then at any time $t \geq \tau_F$

$$\varepsilon(t) = \frac{1}{A_T}\int_0^{t-\tau_F} \frac{d^2E_T}{dy\,dx}\,\frac{dx}{(t-x)}$$

this applies to any time (even during the crossing time).

To proceed, we will take specific form for the time profile $\dfrac{d^2E_T}{dy\,dx}$

Extension of the Bjorken formula



For analytical results of
the initial energy production,
we make minimal extensions
to the Bjorken formula framework:

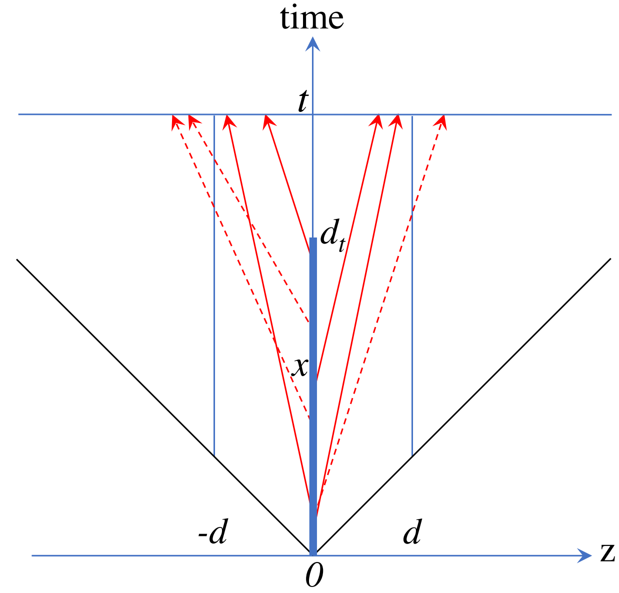
- assume massless particles
- neglect secondary interactions
- neglect transverse expansion
- neglect finite width in **z** for productions of initial particles at mid-rapidity

except for numerical results from AMPT.

These dynamics can be numerically modeled by transport models like string melting AMPT after including finite thickness ZWL, in progress
or by hydrodynamical models Shen & Heinz, PRC 85 (2012); 86 (2012) (E); Oliinychenko et al., PRC 91 (2015).

Extension of the Bjorken formula: 1) the uniform profile

$$\varepsilon(t) = \frac{1}{A_T}\int_0^{t-\tau_F} \frac{d^2E_T}{dy\,dx}\frac{dx}{(t-x)}$$

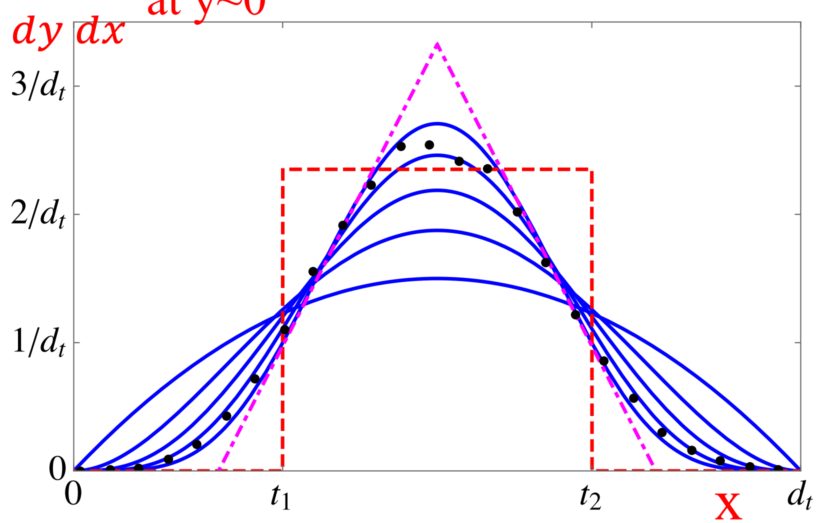


Simplest profile:
initial energy (at y~0) is produced uniformly from time t_1 to t_2 (with $t_{21} \equiv t_2 - t_1$):

$$\frac{d^2E_T}{dy\,dx} = \frac{1}{t_{21}}\frac{dE_T}{dy} \qquad \text{for } x \in [t_1, t_2]$$

→ solution:

$$\epsilon_{\rm uni}(t) = \frac{1}{A_{\rm T} t_{21}}\frac{dE_{\rm T}}{dy}\ln\left(\frac{t-t_1}{\tau_{\rm F}}\right), \text{if } t \in [t_1+\tau_{\rm F}, t_2+\tau_{\rm F}];$$
$$= \frac{1}{A_{\rm T} t_{21}}\frac{dE_{\rm T}}{dy}\ln\left(\frac{t-t_1}{t-t_2}\right), \text{if } t \geq t_2+\tau_{\rm F}.$$



Extension of the Bjorken formula: 1) the uniform profile

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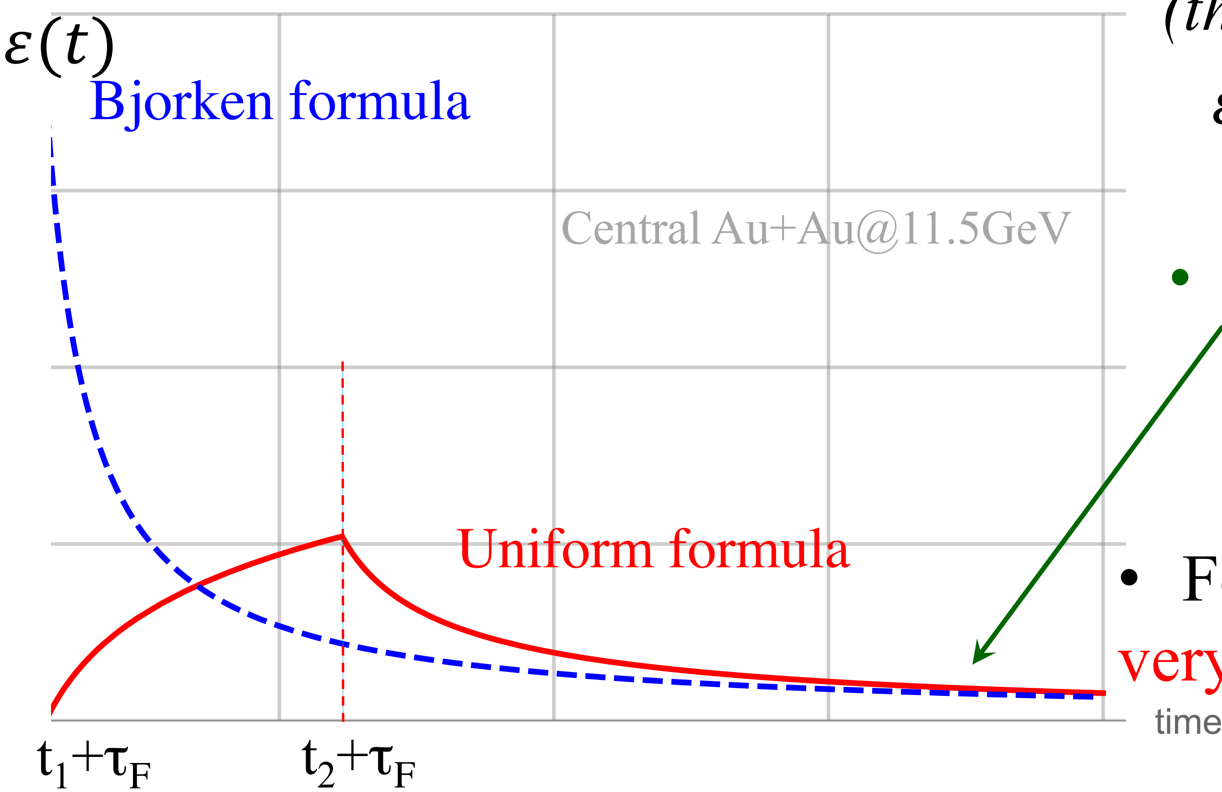
$$= \frac{1}{A_{\rm T} t_{21}} \frac{dE_{\rm T}}{dy} \ln\left(\frac{t-t_1}{t-t_2}\right), \text{if } t \geq t_2+\tau_{\rm F}.$$

- For $t_1 = 0$ & $t_2/\tau_F \to 0$ *(thin nuclei/high energy)*: $\varepsilon_{uni}(t) \to \varepsilon_{Bj}(t)$



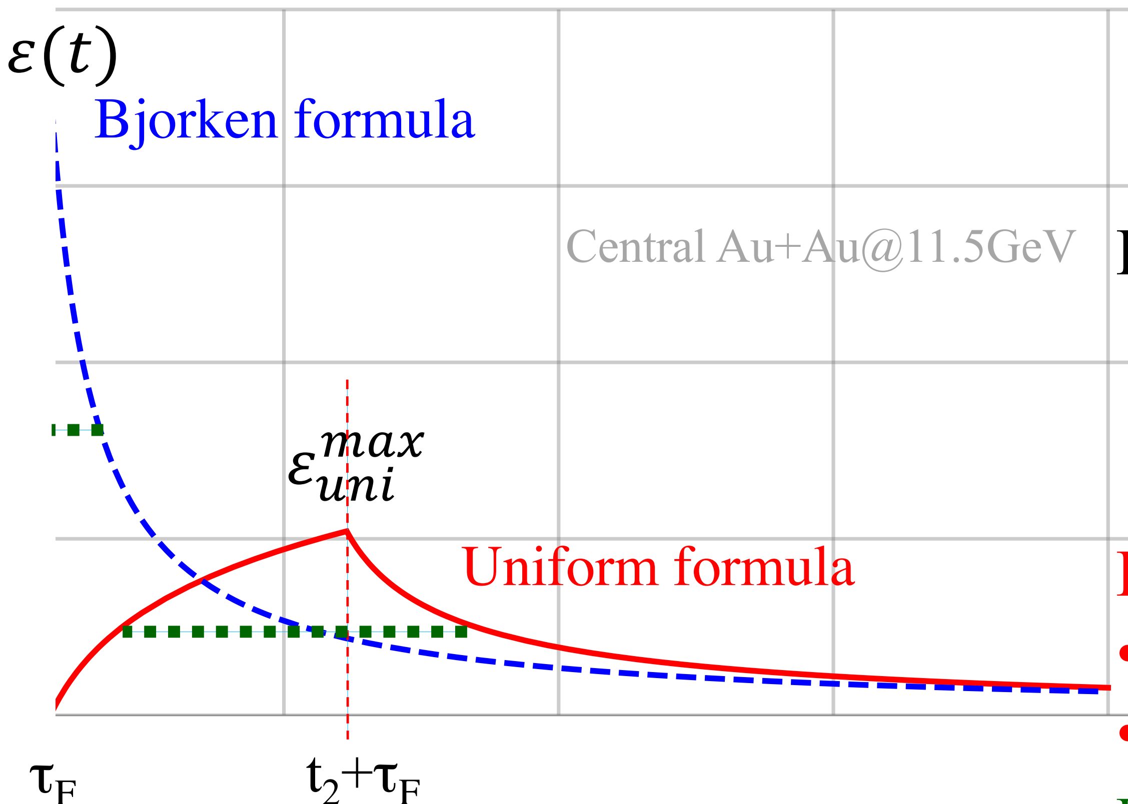
- When $t \gg (t_2+\tau_F)$: $\varepsilon_{uni}(t) \to \varepsilon_{Bj}(t)$
- For t !>> $(t_2+\tau_F)$: very different from Bjorken

Extension of the Bjorken formula: 1) the uniform profile

Peak energy density $\epsilon_{\rm uni}^{max} = \epsilon_{\rm uni}(t_2+\tau_{\rm F}) = \dfrac{1}{A_{\rm T} t_{21}} \dfrac{dE_{\rm T}}{dy} \ln\left(1+\dfrac{t_{21}}{\tau_{\rm F}}\right)$

Let $t_1 = 0$ for simplicity:

ratio over Bjorken: $\dfrac{\epsilon_{\rm uni}^{max}}{\epsilon_{\rm Bj}(\tau_{\rm F})} = \dfrac{\tau_{\rm F}}{t_{21}} \ln\left(1+\dfrac{t_{21}}{\tau_{\rm F}}\right). \le 1$



For $t_{21}/\tau_F \to 0$ (high energy):
ratio $\to 1$ *($\to$ Bjorken)*

For t_{21}/τ_F >>1 (low energy):
ratio $\to 0$;

$\varepsilon_{uni}^{max} \propto \ln\left(\frac{1}{\tau_F}\right)$, not $\frac{1}{\tau_F}$

Peak energy density:

- << Bjorken value
- much less sensitive to τ_F

FWHM width in t >> Bjorken

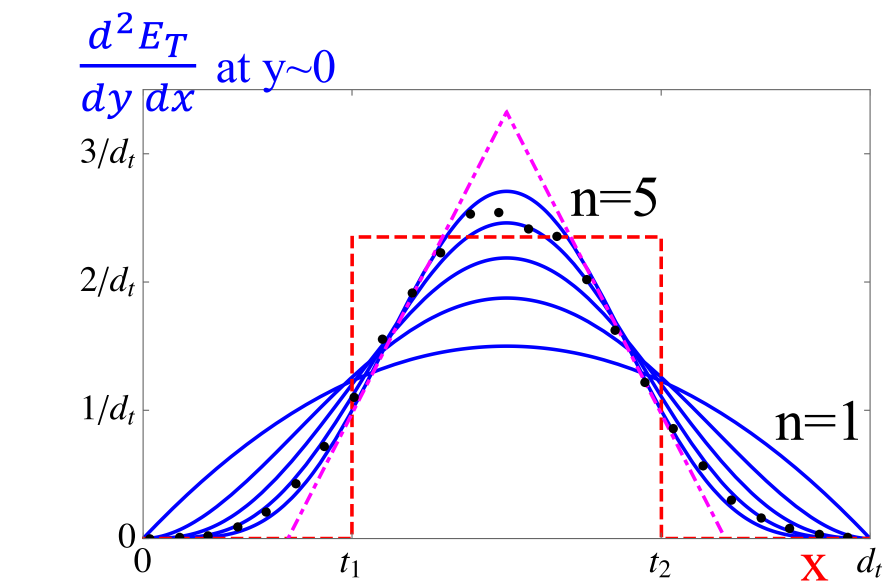
Extension of the Bjorken formula: 2) the beta profile

More realistic profile:

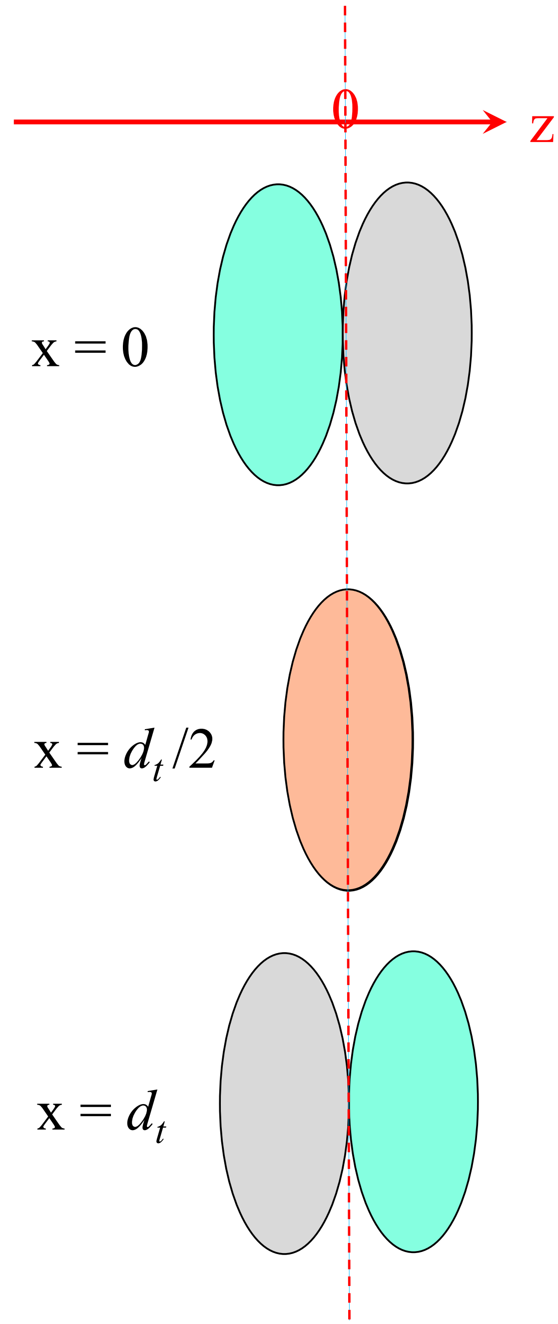
~0 energy is produced at x = 0 & d_t,
most energy is produced at x = $d_t/2$:

$$\frac{d^2E_T}{dy\,dx} = a_n[x(d_t - x)]^n \frac{dE_T}{dy} \quad \text{for } x \in [0, d_t]$$

n=0: reduces to a uniform profile



Circles: *time profile of initial partons within mid-η_s from string melting AMPT for central Au+Au @11.5 GeV.*



Extension of the Bjorken formula: 2) the beta profile

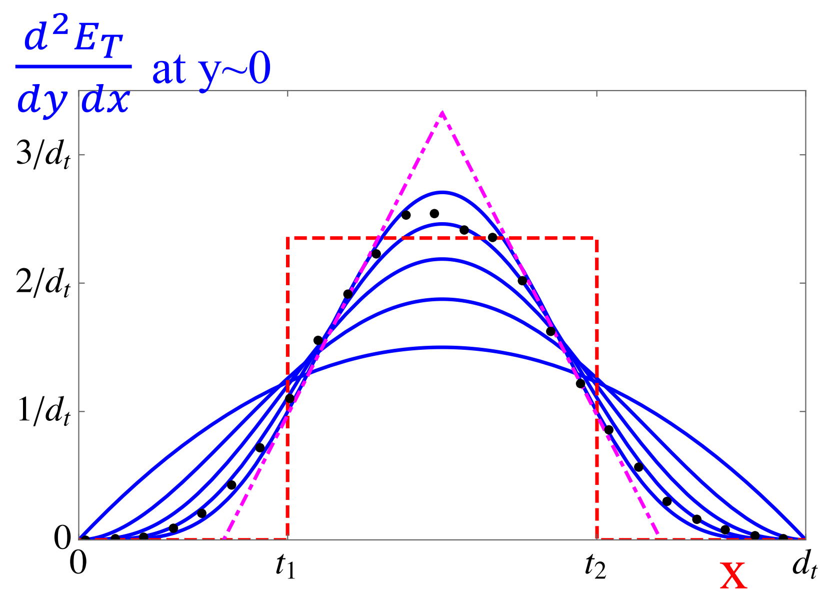
→ solution:

$$\epsilon_{\rm beta}(t) = \frac{1}{A_{\rm T}} \frac{dE_{\rm T}}{dy} \frac{[(t-\tau_{\rm F})/d_t]^{n+1}}{(n+1)B(n+1,n+1)\,t} * F_1\left[n+1,-n,1,n+2,\frac{t-\tau_{\rm F}}{d_t},\frac{t-\tau_{\rm F}}{t}\right], \quad \text{if } t \in [\tau_{\rm F}, d_t+\tau_{\rm F}];$$

$$= \frac{1}{A_{\rm T}} \frac{dE_{\rm T}}{dy} \frac{1}{t} *{}_2F_1\left[1,n+1,2n+2,\frac{d_t}{t}\right], \quad \text{if } t \geq d_t+\tau_{\rm F}.$$

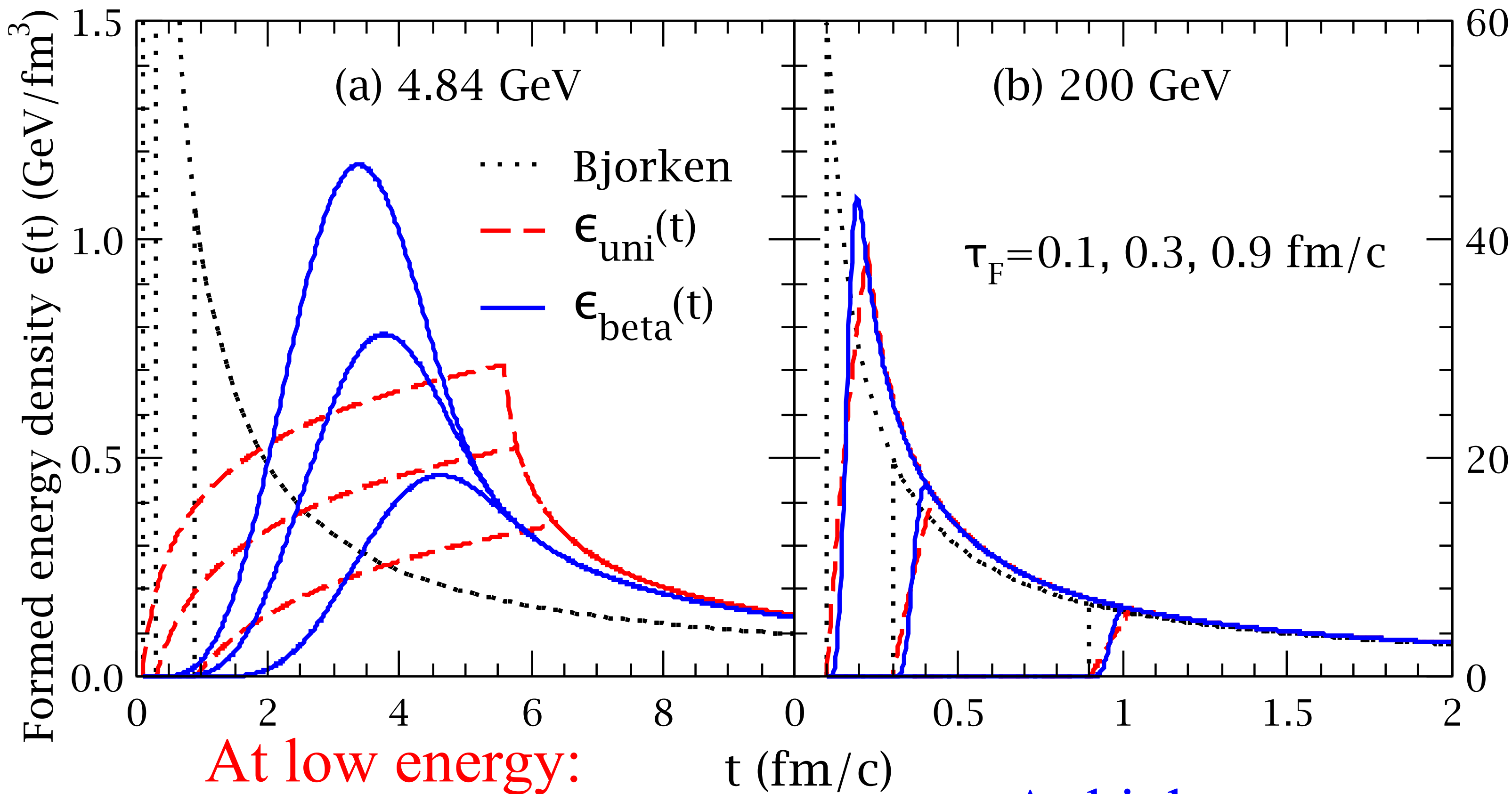
B: the Beta function,
F₁: Appell hypergeometric function of 2 variables,
₂F₁: the Gaussian hypergeometric function.

Next we take **n=4**, since it well describes the AMPT time profile.



Applying extended formula to central Au+Au collisions

The uniform time profile (with $t_1 = 0$ & $t_2 = d_t$),
the beta time profile for n = 4 & the Bjorken formula:



At low energy:

ε^{max}

- << Bjorken value;
- is much less sensitive to τ_F :

factor of 2.1 or 2.5 change (not factor of 9) when τ_F changes from 0.1 to 0.9 fm/c

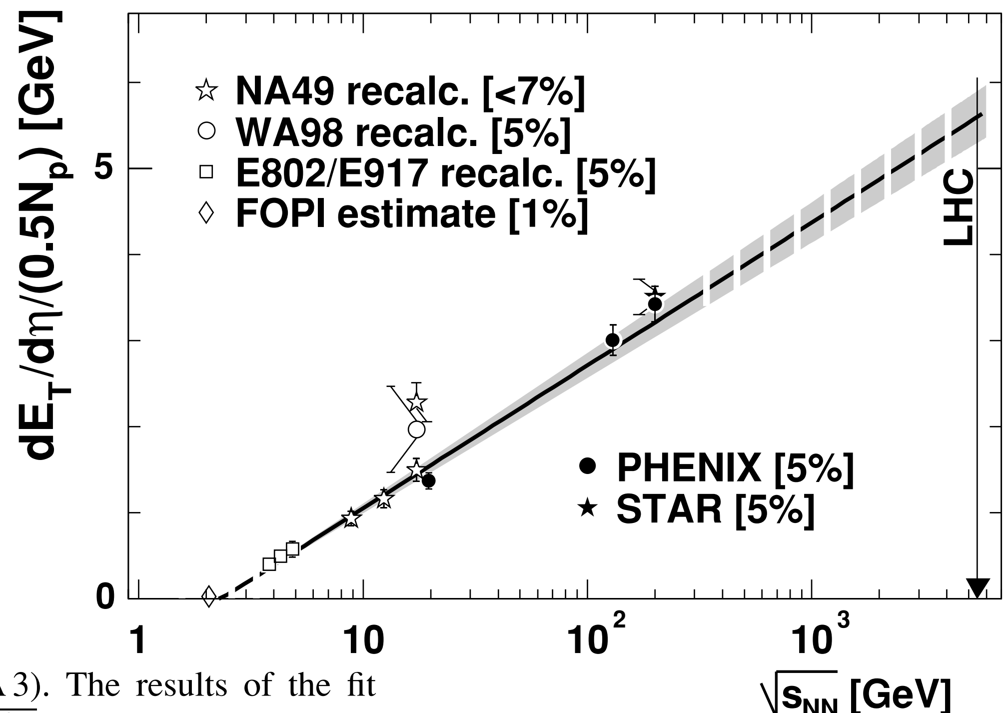
At high energy, solution ~ Bjorken

Applying extended formula to central Au+Au collisions

We use the hard sphere model for nucleus: $\pi R_A^2, R_A = 1.12 A^{1/3}\ fm$

$$\frac{dE_T}{dy} = 1.25\frac{dE_T}{d\eta} = 0.456 N_{part}\ ln\left(\frac{\sqrt{s_{NN}}}{2.35GeV}\right)$$

$\& \ N_{part} = 2A$ for central collisions.



from the E_T fit (see Appendix A 3). The results of the fit $dX/d\eta = (0.5N_p \cdot A)\ln(\sqrt{s_{NN}}/\sqrt{s_{NN}^0})$ are

for E_T, $\sqrt{s_{NN}^0} = 2.35 \pm 0.2$ GeV and $A = 0.73 \pm 0.03$ GeV,

PHENIX PRC 71 (2005)

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$\& \ N_{part} = 2A$ for central collisions.

NOTE: in ZWL, arXiv:1704.08418, I wrote:

$$\frac{dE_T}{dy} = 1.25\frac{dE_T}{d\eta} = 0.913N_{part}\ ln\left(\frac{\sqrt{s_{NN}}}{2.35GeV}\right)$$

$\& \ N_{part} = A$ for central collisions.

Both *Npart* should be ***0.5Npart*** *(or Npart from each nucleus);*

this typo does not affect $\frac{dE_T}{dy}$ or the results.

from the E_T fit (see Appendix A 3). The results of the fit $dX/d\eta = (0.5N_p \cdot A)\ln(\sqrt{s_{NN}}/\sqrt{s_{NN}^0})$ are

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PHENIX PRC 71 (2005)

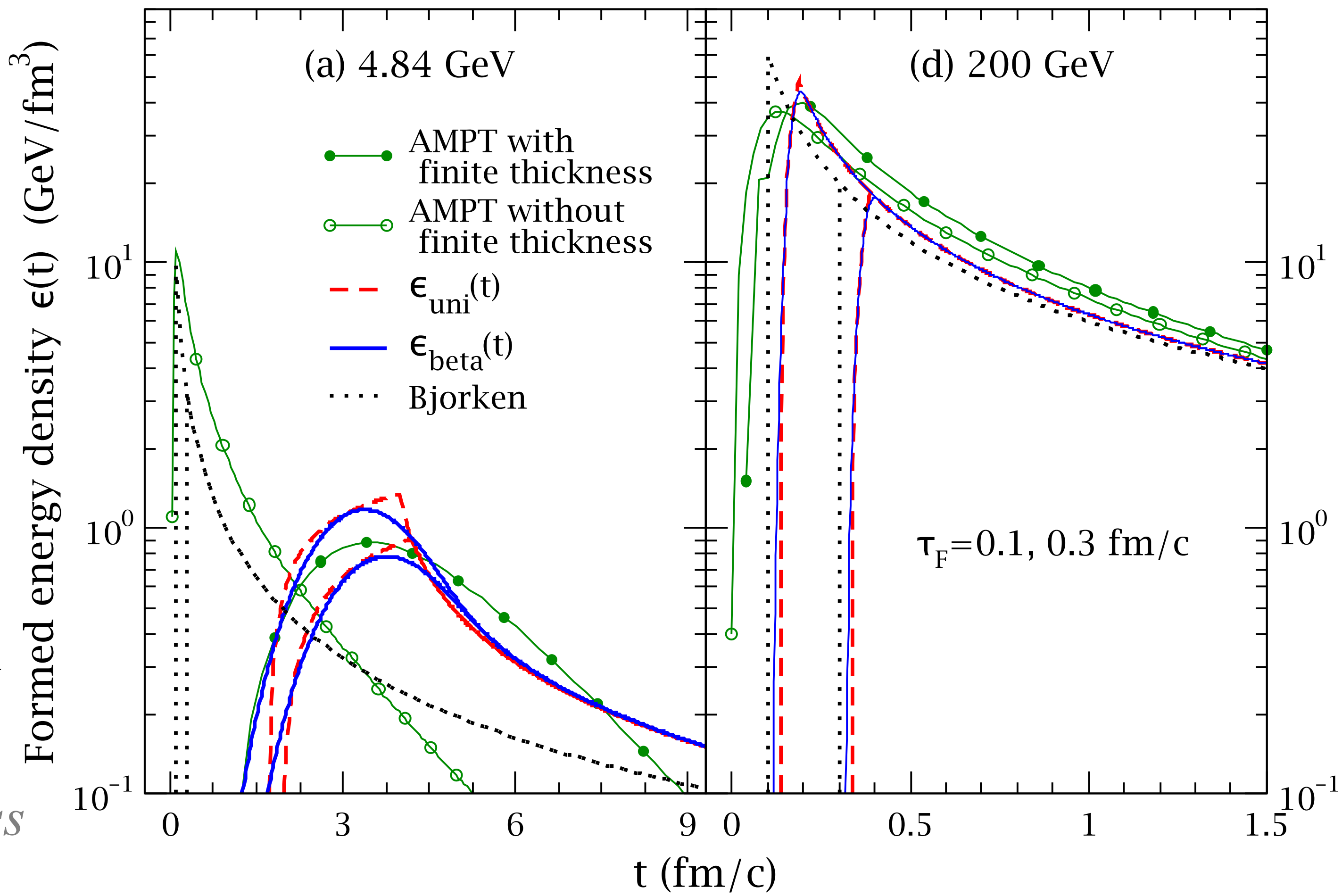
Comparisons with results from transport model AMPT

String melting AMPT is improved by including finite thickness. ZWL, in progress

Overall:

- AMPT with F.T. ~ our model;
- AMPT w/o F.T. ~ Bjorken formula.
- *Small effect of F.T. at 200 GeV*

F.T.=finite thickness



$\varepsilon_{uni}(t) \sim \varepsilon_{beta}(t)$,

since we set t_1 & t_2 of the uniform profile so that it has the same mean & standard deviation as the beta profile.

Extension of the Bjorken formula: 3) the triangular time profile

We can also use
a symmetric triangular profile
for $x \in [t_1, t_2]$. $\rightarrow$ solution:

$$\epsilon_{\rm tri}(t) = \frac{4}{A_{\rm T} t_{21}^2}\frac{dE_{\rm T}}{dy}\left[-t + t_1 + \tau_{\rm F} + (t-t_1)\ln\left(\frac{t-t_1}{\tau_{\rm F}}\right)\right], \quad \text{if } t \in [t_1+\tau_{\rm F}, t_{mid}+\tau_{\rm F}];$$

$$= \frac{4}{A_{\rm T} t_{21}^2}\frac{dE_{\rm T}}{dy}\left[t - t_2 - \tau_{\rm F} + (t-t_1)\ln\left(\frac{t-t_1}{t-t_{mid}}\right) + (t_2 - t)\ln\left(\frac{t-t_{mid}}{\tau_{\rm F}}\right)\right], \text{if } t \in [t_{mid}+\tau_{\rm F}, t_2+\tau_{\rm F}];$$

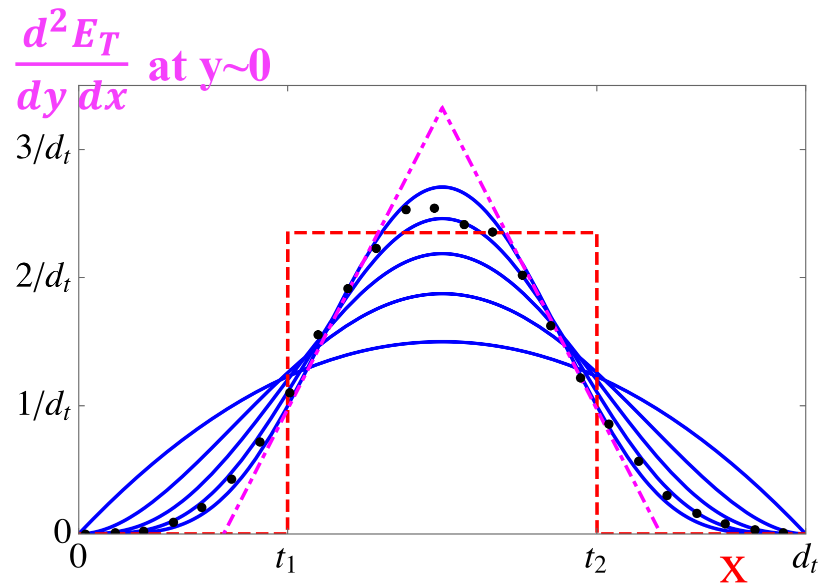
$$= \frac{4}{A_{\rm T} t_{21}^2}\frac{dE_{\rm T}}{dy}\left[(t-t_1)\ln\left(\frac{t-t_1}{t-t_{mid}}\right) + (t_2 - t)\ln\left(\frac{t-t_{mid}}{t-t_2}\right)\right], \text{ if } t \ge t_2 + \tau_{\rm F}.$$

Advantage:

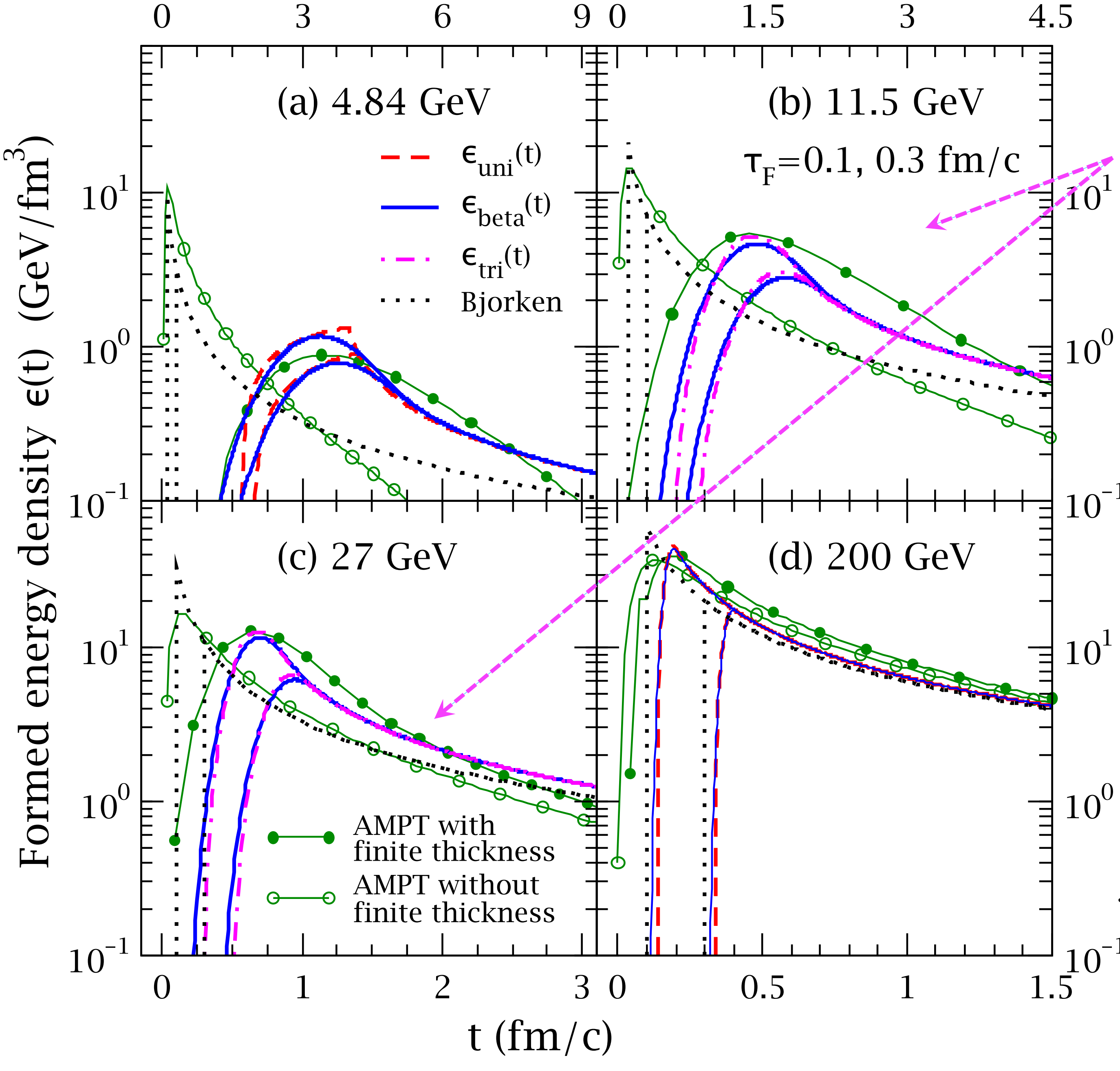
convenient analytical solution of ε^{max} and the corresponding time:

$$\epsilon_{\rm tri}^{max} = \epsilon_{\rm tri}\left((t_1 + t_2 + \tau_{\rm F} + \sqrt{\tau_{\rm F}}\sqrt{2\,t_{21} + \tau_{\rm F}}\,)/2\right)$$

$$= \frac{2}{A_{\rm T} t_{21}}\frac{dE_{\rm T}}{dy}\left[-1 - \frac{\tau_{\rm F}}{t_{21}} + \sqrt{\frac{\tau_{\rm F}}{t_{21}}}\sqrt{2 + \frac{\tau_{\rm F}}{t_{21}}} + 2\ln\left(\frac{1 + \sqrt{1 + 2\,t_{21}/\tau_{\rm F}}}{2}\right)\right].$$



Comparisons with results from transport model AMPT



$\varepsilon_{tri}(t) \sim \varepsilon_{beta}(t)$ after setting t_1 & t_2 for the same mean & standard deviation.

Note: AMPT has *variable τ_F, Woods-Saxon, secondary scatterings, transverse expansion, finite width in z.*

Summary

The Bjorken formula is only valid
when τ_F is much bigger than the finite crossing time:
$\sqrt{s_{NN}} \gg \sim 50 GeV$ for central Au+Au collisions.

We have analytically extended the Bjorken formula:

- also valid at low energies
- approaches the Bjorken formula at high energies or late times.
- comparisons with AMPT confirm key features of the solutions.

At low energies (compared to the Bjorken formula):

- solution is much less sensitive to the formation time τ_F ;
- the maximum energy density is much lower;
- the width of the energy density time evolution is much bigger.

This provides a general model for the initial energy production of relativistic heavy ion collisions, especially at low energies.

Thank you!

Also thanks to Miklos Gyulassy
for careful reading of the manuscript
and helpful comments.