Ultrarelativistic Fluid spintronics in Hadron collisions

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Based on Phys.Rev.C76:044901,2007 ongoing work with Leonardo Tinti and Master student David Montenegro , and lots of questions I cant answer!
And thanks to Mike for the invite and the title buzzword!

Some phenomenology and back to the envelope reasoning

But no theory why this phenomenology is fundamentally incomplete

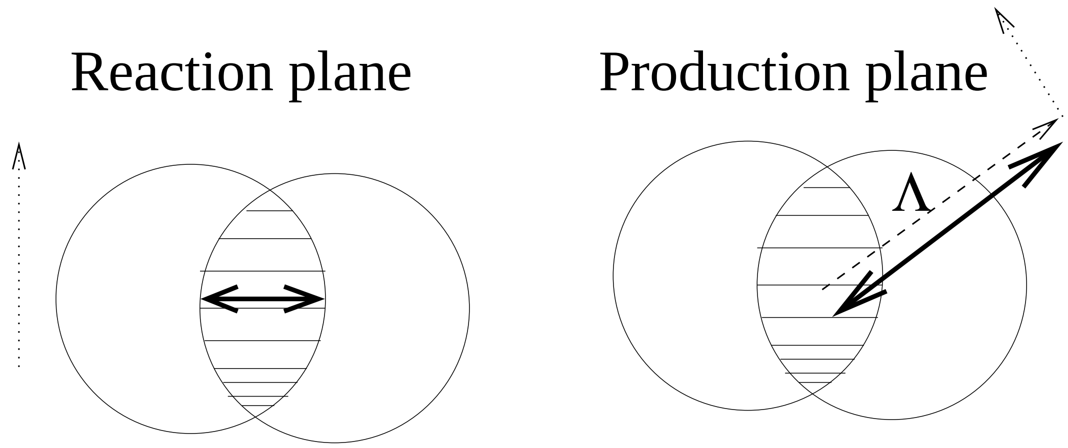
Another way to see hydro EFT could help

And how it could help to elucidate things

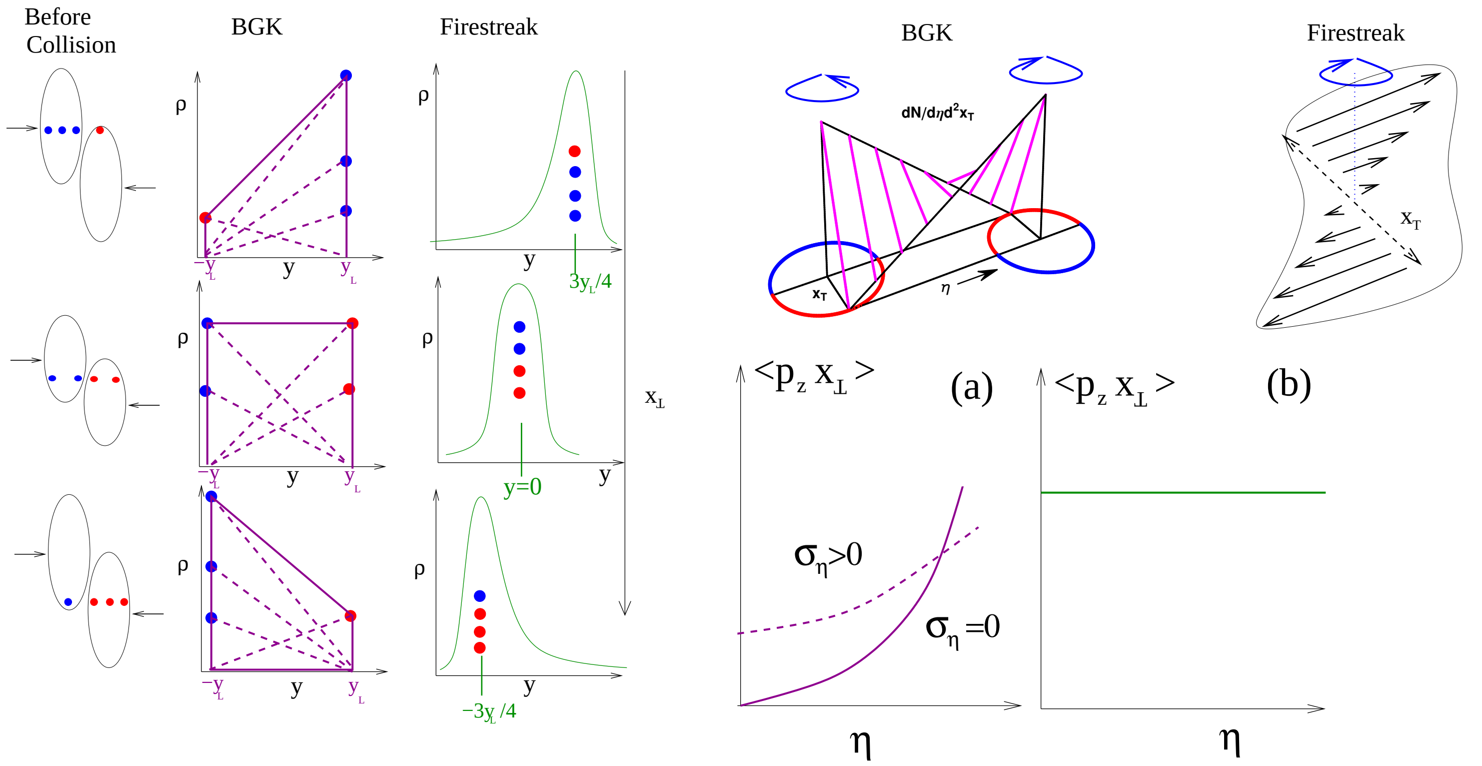
Questions and prospects experimentalists will be there first!

Phenomenology of polarization in a medium

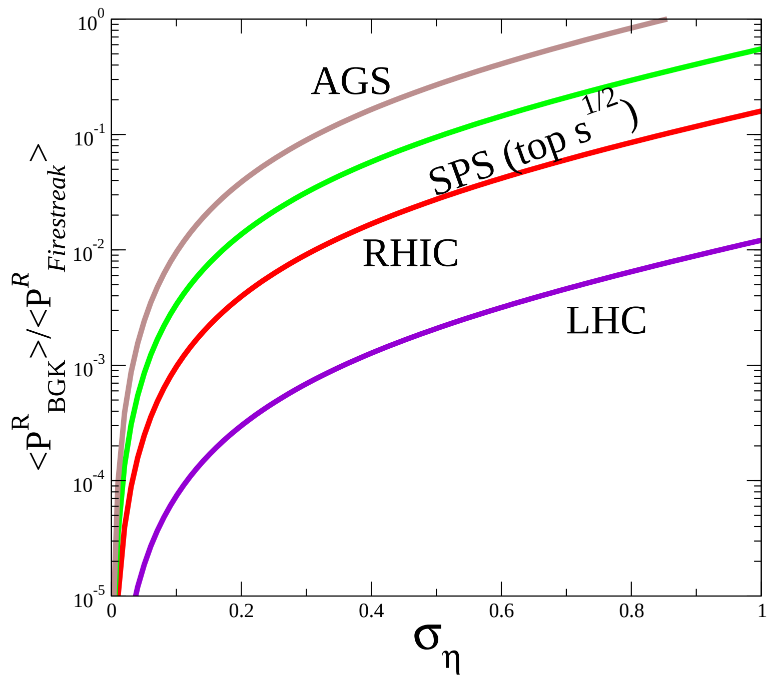
Historically polarization measurements focused on production plane, since it is known hadronic interactions generate it via a spin-orbit process. Global dynamics makes a reaction plane search interging in the context of hydro.



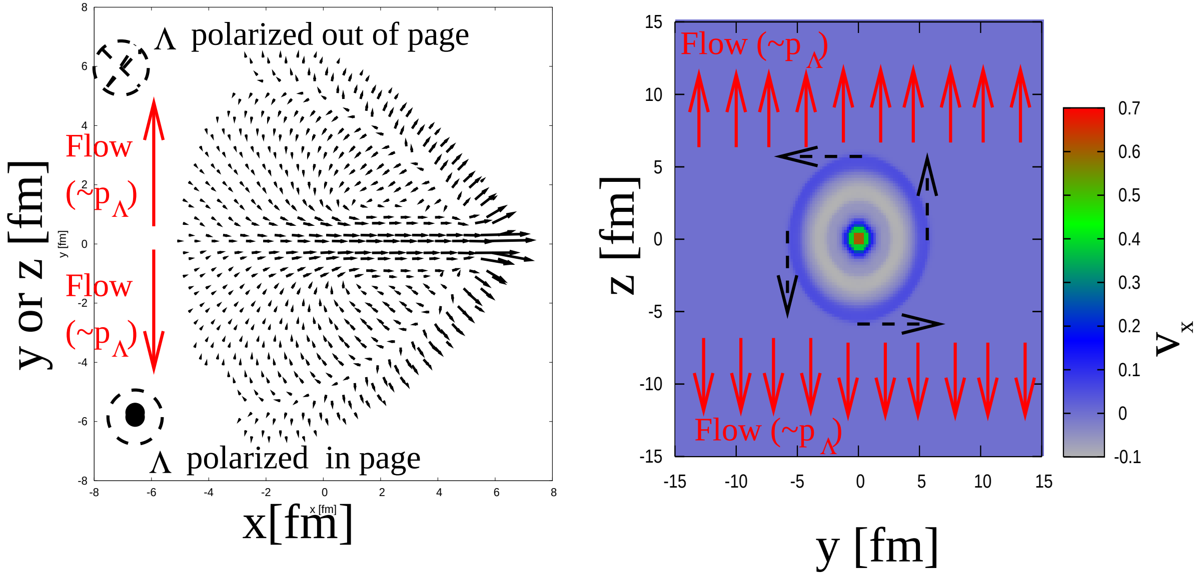
Definition of production and reaction plane. The beam line (traditionally the z axis) is perpendicular to the sheet. The dotted line, with arrow, indicates the direction of polarization of the produced Λ.



Polarization critically sensitive to initial transparency. The variation reported in the earlier talk naturally interpreted as approach to transparency



Ratio of BGK to firestreak predictions as a function of $\sqrt{s}$ and σ_η, the correlation length between spacetime and flow rapidity. Correlation between position and momentum rapidity also helps



If hydrodynamic interpretation is correct, might also be worth looking for jet-plane polarization. Vorticity generated by a fast "jet" traversing the system in the positive x direction. The arrows in the left panel show the momentum density of fluid elements in the x-y plane, while the contour in the right panel shows the x-component of the velocity in the y-z plane. After the Mach cone, the vortex?

There is a "small" problem: these are <u>all</u> back of the envelope calculations essentially ignoring all dynamics

- How does a hydrodynamic system evolve when polarized particles (eg quarks and gluons) are present and when vorticity is non-zero?

- How is polarization transferred to vorticity within a thermalized medium?

And we didnt do this because we still dont really know how to do this, on a fundamental level!

GT,Betz,Gyulassy

An ideal hydrodynamic medium is locally isotropic, while polarization is not. Hence polarization has to go as the breakdown of isotropy, which in hydrodynamics is controlled by the mean free path. Some dimensional analysis later it turns out...

$$\left\langle P_q^i \right\rangle \sim \tanh\left[\vec{\zeta}_i\right] \sim \vec{\zeta}_i$$

$$\vec{\zeta}_i = \frac{l_{mfp}}{T}\left(\epsilon_{ijk}\frac{d\left\langle \vec{p}_k \right\rangle}{d\vec{x}_j}\right)$$

But is this really true that in ideal hydrodynamics there is no polarization? After all, in a co-moving frame...

Becattini,Chandra,Del Zanna,Grossi, 1303.3431

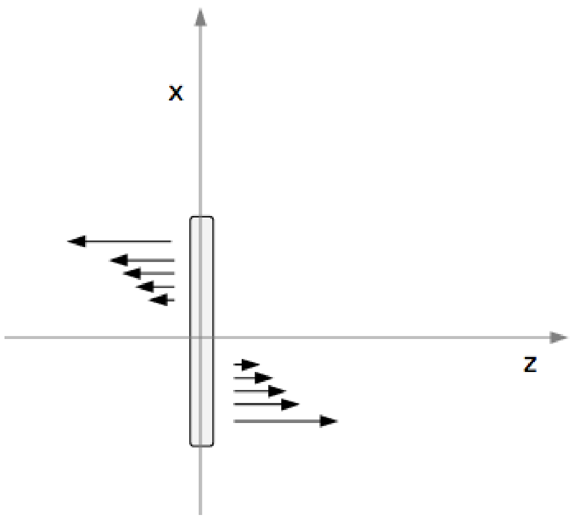
GC ensemble with angular momentum as a conserved quantity, fermions (1 species)

$$\exp\left(-\frac{p_\mu u^\mu}{T}\right) \to \exp\left(-\frac{p_\mu u^\mu}{T}\right)(\bar{u},\bar{v})\exp\left[\frac{\Sigma_{\mu\nu}\omega^{\mu\nu}}{T}\right]\left(\begin{array}{c} u \\ v \end{array}\right)$$

And Fermi-Dirac statistics. Here

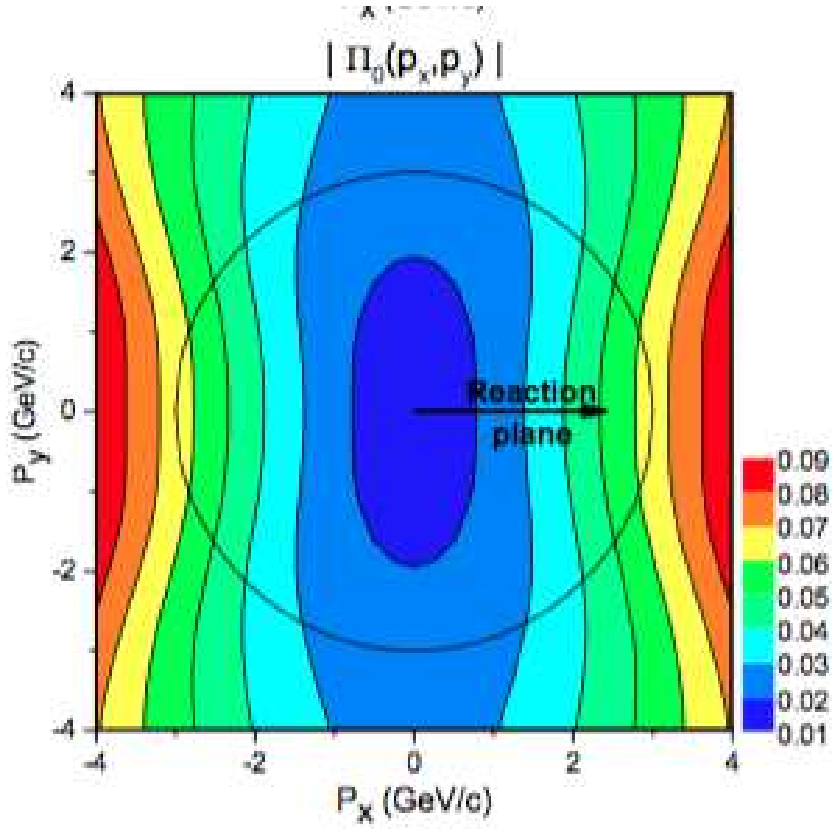
- $\omega^{\mu\nu}$ vorticity tensor
- $\Sigma^{\mu\nu}$ spin projection tensor $\sim \left(\begin{array}{cc} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{array}\right)$

A rotating statistical model (Becattini, Piccinini,Rizzo, 0711.5253



Treat event as equilibrated and spinning from initial angular momentum (firestreak model), assign anuglar momentum accordingly.
But this is a globally equilibratd system, not a locally equilibrated one. Most likely a rough estimate!

Becattini and Csernai, 1304.2247 and upcoming Put formulae above within a Cooper-Frye formula, couple to hydro code, get a polarization density



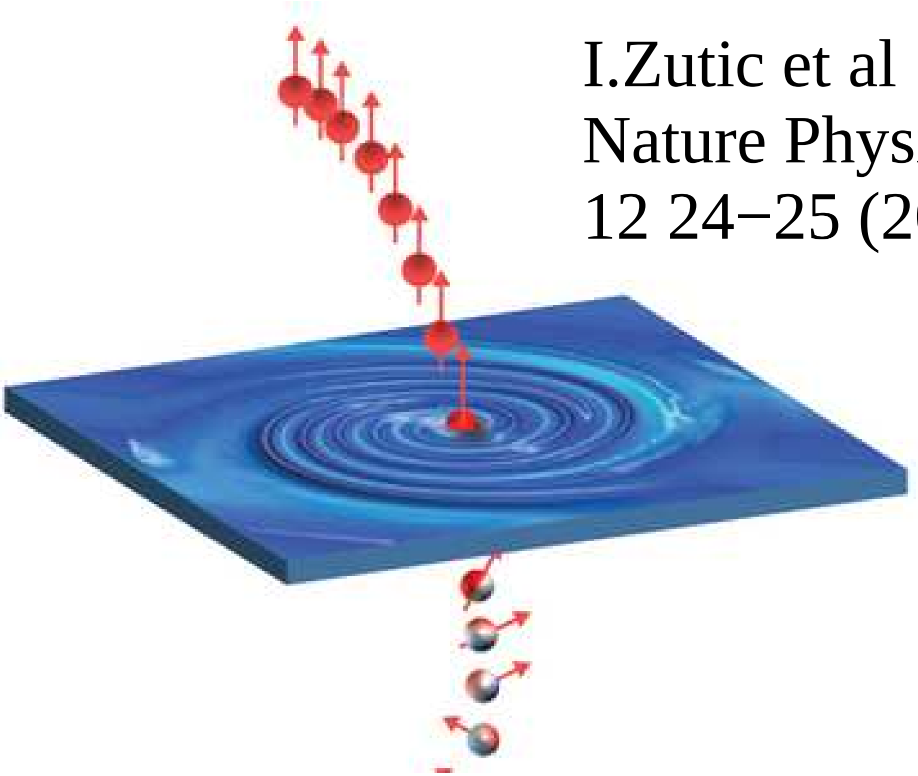
But I think this has fundamental issues

Cooper-Frye formula based on ideal isotropic hydro.

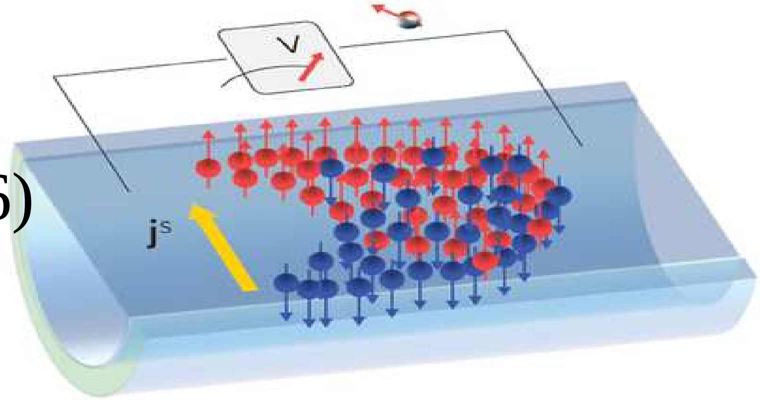
$$d\Sigma^{\mu}(T_{\mu\nu}^{hydro} - T_{\mu\nu}^{particles}) = d\Sigma^{\mu}(s_{\mu}^{hydro} - s_{\mu}^{particles}) = 0$$

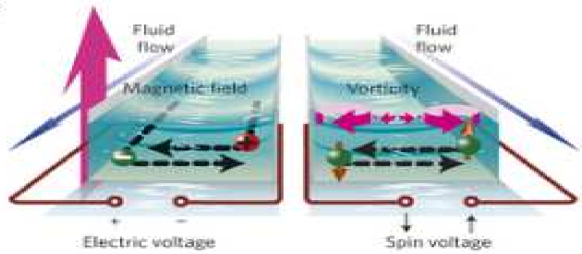
- Non-trivial $d\Sigma_{\mu}$ affects spin
- (More generally) polarization/vorticity coexist and interact in medium, not just freezeout. CF is detailed balance

Need relativistic version of theory incorporating vorticity and spin



I.Zutic et al
Nature Physics
12 24−25 (2016)





Zutic, Matos-Abiague, "Spin Hydrodynamics", Nature Physics **12** 24-25
Takahashi et al", Nature Physics **12** 52-56 (2016)

What is ideal hydro?

Entropy conserved always at maximum at each point in spacetime

Local isotropy in the comoving frame

Vorticity is conserved (Kelvins theorem)

Continuum limit when you break up cells, intensive results stay the same

With polarization, only the first has a chance of being realized even in the ideal limit. Which means no ideal hydro limit is defined for mediums with polarization. "viscous, transport etc. should be on top of this undefined limit for strong coupling! Related to nonlocality of vorticity. (Weyssenhoff,Halbach, Becattini, Tinti have partial definitions, but cant resolve contradiction above)

A note to AdS/CFT fans... this stuff (probably) doesent concern you

Fermion polarization always suppressed by factors of N_c, boson polarization unobservable (gauge dependent). Landau and Lifshitz (also D.Rishke,B Betz et al): Hydrodynamics has <u>three</u> length scales

$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

Weakly coupled: Ensemble averaging in Boltzmann equation good up to $\mathcal{O}\left((1/\rho)^{1/3}\partial_\mu f(...)\right)$
Strongly coupled: classical supergravity requires $\lambda \gg 1$ but $\lambda N_c^{-1} = g_{YM} \ll 1$ so

$$\frac{1}{TN_c^{2/3}} \ll \frac{\eta}{sT} \quad \left(\quad or \quad \frac{1}{\sqrt{\lambda}T}\right) \ll L_{macro}$$

Why is $l_{micro} \ll l_{mfp}$ necessary? Without it, microscopic fluctuations (which come from the finite number of DoFs and have nothing to do with viscosity) will drive fluid evolution.

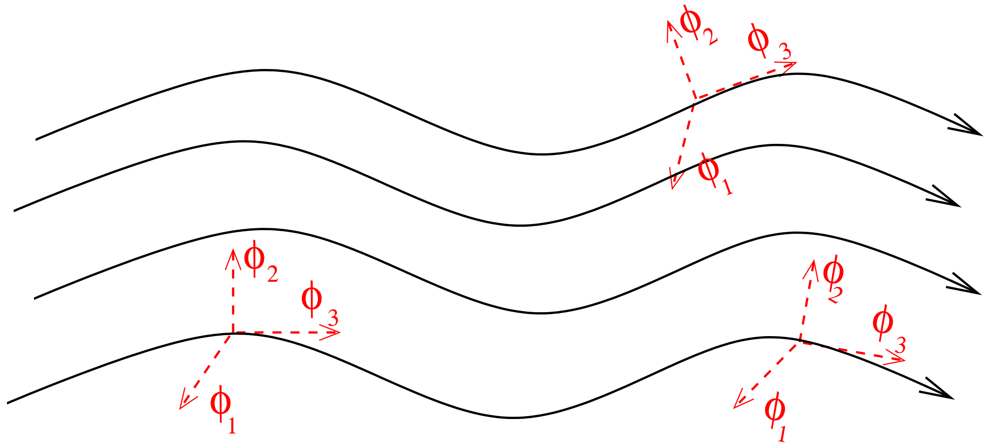
$\Delta\rho/\rho \sim C_V^{-1} \sim N_c^{-2}$, thermal fluctuations "too small" to be important! (Lifshitz+Landau has hydrodynamical fluctuation both from thermal $\sim C_V$ and dissipative $\sim Kn$ sources)

But we know this approximation is far from perfect, $N_c = 3 \ll \infty$ and $dN/dy \sim 10^{1-3} \ll \infty$

So first scale is always non-negligible. It also controls polarization distribution. Understanding role of polarization is "similar" to understanding role of fluctuations: Lagrangian hydrodynamics and functional integrals
Let us try to define hydro without reference to microscopic DoFs :No quasi particles, AdS/CFT, just hydro! This is a bottom-up EFT

Hydro as EFT fields: (Nicolis et al,1011.6396 (JHEP))

Continuus mechanics (fluids, solids, jellies,...) is written in terms of 3-coordinates $\phi_I(x^\mu), I = 1...3$ of the position of a fluid cell originally at $\phi_I(t=0,x^i), I = 1...3$. (Lagrangian hydro . NB: no conserved charges)



The system is a Fluid if it's Lagrangian obeys some <u>symmetries</u> (Ideal hydrodynamics $\leftrightarrow$ Isotropy in comoving frame) Solutions generally <u>break these</u>, Excitations (Sound waves, vortices etc) can be thought of as "Goldstone bosons".

Translation invariance at Lagrangian level $\leftrightarrow$ Lagrangian can only be a function of $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ Now we have a "continuus material"!

Homogeneity/Isotropy means the Lagrangian can only be a function of $B = \mathrm{det} B^{IJ}, \mathrm{diag} B^{IJ}$
The comoving fluid cell must not see a "preferred" direction $\Leftarrow SO(3)$ invariance

Invariance under Volume-preserving diffeomorphisms means the Lagrangian can only be a function of B (actually $b = \sqrt{B}$)
In all fluids a cell can be infinitesimally deformed
(with this, we have a fluid. If this last requirement is not met, Nicolis et all call this a "Jelly")

A few exercises for the bored public Check that L = -F(B) leads to

$$T_{\mu\nu} = (P+\rho)u_\mu u_\nu - P g_{\mu\nu}$$

provided that

$$\rho = F(B)\ , \qquad p = F(B) - 2F'(B)B\ , \qquad u^\mu = \frac{1}{6\sqrt{B}}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{IJK}\,\partial_\alpha\phi^I\partial_\beta\phi^J\partial_\gamma\phi^K\ .$$

(A useful formula is $\frac{db}{d\partial_\mu\phi_I}\partial_\nu\phi_I = u^\mu u^\nu - g^{\mu\nu}$)
Equation of state chosen by specifying $F(b)$. "Ideal": $\Leftrightarrow F(B) \propto b^{2/3}$
b is identified with the entropy and $b\frac{dF(B)}{dB}$ with the microscopic temperature.
u^μ fixed by $u^\mu\partial_\mu\phi^{\forall I} = 0$

You can also show that

$$\partial_\mu \left(\underbrace{b}_{=s} u^\mu \right) = 0 \quad , \quad s = -\frac{dP}{dT} = \frac{p+\rho}{T}$$

Ie, b is the conserved quantity corresponding to our earlier group. Up to dimensional factor corresponds to microscopic entropy. Can also write everything in terms of $K^\mu = bu^\mu$

Chemical potentials (neglected here) would be implemented by complexifying ϕ_I and promoting them to internal space vectors

An infinite number of global conserved charges for every closed path, vorticity is conserved. Corresponding to infinite-D diffeomorphism invariance

Ideal hydrodynamics and the microscopic scale

The most general Lagrangian is

$$L = T_0^4 F\left(\frac{B}{T_0^4}\right) \quad , \quad B = T_0^4 \det B^{IJ} \quad , \quad B^{IJ} = \left|\partial_\mu \phi^I \partial^\mu \phi^J\right|$$

Where $\phi^{I=1,2,3}$ is the comoving coordinate of a volume element of fluid.

NB: $T_0 \sim \Lambda g$ microscopic scale, includes thermal wavelength <u>and</u> $g \sim N_c^2$ (or μ/Λ for dense systems). $T_0 \to \infty \Rightarrow$ classical limit
It is therefore natural to identify T_0 with the microscopic scale!

Kn behaves as a gradient, T_0 as a Planck constant!!!

At $T_0 < \infty$ quantum and thermal fluctuations can produce sound waves and vortices, "weighted" by the usual path integral prescription!

$$\mathcal{Z} = \int \mathcal{D}\phi_i \exp\left[-T_0^4 \int F(B) d^4 x \right], \langle \mathcal{O} \rangle \sim \frac{\partial \ln \mathcal{Z}}{\partial \ldots} \left(eg. \quad \left\langle T_{\mu\nu}^{x} T_{\mu\nu}^{x'} \right\rangle = \frac{\partial^2 \ln \mathcal{Z}}{\partial g_{\mu\nu}(x) \partial g_{\mu\nu}(x')} \right)$$

$T_0 \sim n^{-1/3}$, unlike Knudsen number, behaves as a "Planck constant"

For analytical calculations fluid can be perturbed around a <u>hydrostatic</u> ($\phi_I = \vec{x}$) background

$$\phi_I = \vec{x} + \underbrace{(\vec{\pi}_L)}_{sound} + \underbrace{(\vec{\pi}_T)}_{vortex}$$

Polarization likely to dramatically change things here

And we discover a fundamental problem: Vortices carry arbitray small energies but stay put! No S-matrix in hydrostatic solution!

$$L_{linear} = \underbrace{\dot{\vec{\pi}}_L{}^2 - c_s^2(\nabla.\vec{\pi}_L)^2}_{sound\quad wave} + \underbrace{\dot{\pi_T}^2}_{vortex} + Interactions(\mathcal{O}\left(\pi^3, \partial\pi^3, ...\right))$$

Unlike sound waves , Vortices can not give you a theory of free particles, since they do not propagate: They carry energy and momentum but stay in the same place! Can not expand such a quantum theory in terms of free particles.

Physically: "quantum vortices" can live for an arbitrary long time, and dominate any vacuum solution with their interactions. This does not mean the theory is ill-defined, just that its strongly non-perturbative!

The big problem with Lagrangians... usually only non-dissipative terms
A first order term in the Lagrangian can always be reabsorbed as a field redefinition, i.e. is topological

But there are a few ways to fix it. We focus on coordinate doubling (Galley,but before Morse+Feschbach)

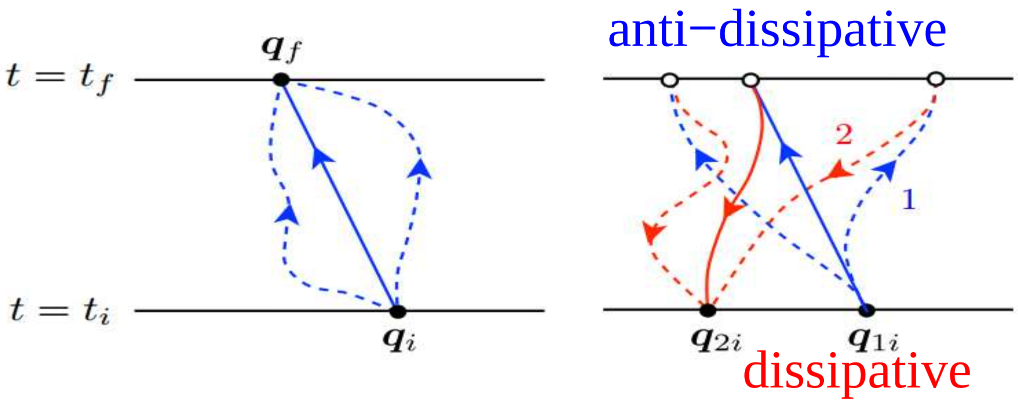
$$\phi_I \to \hat{\phi}_I = (\phi_I^+, \phi_I^-)$$

Action given by two copies plus an interaction term

$$S_{CTP} = \int_{t_f}^{t_i} d^4x \left\{ \mathcal{L}_s[\phi^+] - \mathcal{L}_s^*[\phi^-] + \mathcal{K}[\hat{\phi_\pm}] \right\}$$

The first two terms are non-dissipative, action doubled. Third term can be used to model dissipation

Dissipative extension of Hamiltons principle



$$L = \frac{1}{2}\left(\underbrace{m\dot{x}^2 - wx^2}_{SHO}\right) \rightarrow \left[\underbrace{\frac{1}{2}\left(m\dot{x_+}^2 - wx_+^2\right)}_{\mathcal{L}_1}\right] - \left[\underbrace{\frac{1}{2}\left(m\dot{x_-}^2 - wx_-^2\right)}_{\mathcal{L}_2}\right] + \left[\underbrace{\dot{x}_+ x_- - \dot{x}_- x_+}_{\mathcal{K}}\right]$$

Standard techniques give you two sets of equations, one with a damped harmonic oscillator, the other "anti-damped"

Navier-Stokes (GT,D.Montenegro, PRD, in press)

In terms of $K^\mu = bu^\mu$ the bulk term is

$$\mathcal{L}^{(1)}_{CTP} = T_o^4 \sum_{i,j,k} z_{ijk}(K^{l\gamma}K^m_\gamma)B\partial^\mu\phi^{iI}\partial^\nu\phi^{jJ}\partial_\mu K^k_\nu.$$

and the shear term is

$$\mathcal{L}^{(1)}_{CTP} = T_o^4 \sum_{i,j,k} z_{ijk}(K^{l\gamma}K^m_\gamma)BB^{-1}_{IJ}\partial^\mu\phi^{iI}\partial^\nu\phi^{jJ}\partial_\mu K^k_\nu.$$

These are the simplest terms compatible with <u>most</u> symmetries. But shear term <u>also</u> breaks volume-preserving diffeomorphism invariance. Effect of fundamental length?

Going further, second order term?

Problem Causality problem for first order terms (Lagrangian unbounded), second order terms with no local equilibrium (Ostrogradski's theorem)

Solution: introduce a new degree of freedom. Keep transversality condition but drop gradient dependence

$$\Pi_{\mu\nu} = X_{IJ} \partial_\mu \phi^I \partial_\nu \phi^J$$

X_{IJ} are 6 new degrees of freedom to be fixed by initial conditions... Equivalent of Israel-Stewart off-diagonal terms

Israel-Stewart/Anisotropic hydrodynamics emerge naturally in Lagrangian approach

I-S in a lagrangian approach

$\Pi_{\mu\nu} = X_{IJ}\bar{A}^{IJ}_{\mu\nu}$ As these are not conserved quantities, the equation of motion has to be obtained from Lagrange's equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial X)} = \frac{\partial \mathcal{L}}{\partial X}$$

The Israel-Stewart equations of motion Follows easily from the Lagrangian

$$\mathcal{L} = T_0^4 F(B) + \frac{1}{2}\tau_\pi^\eta (\Pi_-^{\mu\nu} u_+^\alpha \partial_\alpha \Pi_{\mu\nu+} - \Pi_+^{\mu\nu} u_-^\alpha \partial_\alpha \Pi_{\mu\nu-})$$

$$+\frac{1}{2}\Pi_\pm^{\mu\nu}\Pi_{\mu\nu\pm} + \frac{X_{IJ\pm}}{6} \underbrace{\left[(A^\circ)^{IJ}_{\mu\nu} \partial^\mu K^\nu \right]_\pm}_{\sim\sigma_{\mu\nu}} + \mathcal{O}\left((\partial u)^2 \right)$$

Last term non-dissipative, worked out in J. Bhattacharya, S. Bhattacharyya and M. Rangamani,1211.1020

We are now ready to add polarization in the ideal hydrodynamic limit

Forget doubled Lagrangians for now (but it will be necessary when we add dissipation)

Break isotropy by introducing extra DoFs transforming as vectors

Use Lorentz and internal symmetries to construct EFT around a conserved entropy

Conserved charges (Dubovsky et al, 1107.0731(PRD))

Within Lagrangian field theory a scalar chemical potential is added by adding a $U(1)$ symmetry to system.

$$\phi_I \to \phi_I e^{i\alpha} \quad , \quad L(\phi_I, \alpha) = L(\phi_I, \alpha + y) \quad , \quad J^\mu = \frac{dL}{d\partial_\mu \alpha}$$

generally flow of b and of J not in same direction. Can impose a well-defined u^μ by adding chemical shift symmetry

$$L(\phi_I, \alpha) = L(\phi_I, \alpha + y(\phi_I)) \to L = L\left(b, y = u_\mu \partial^\mu \alpha\right)$$

A comparison with the usual thermodynamics gives us

$$\mu = y \quad , \quad n = dF/dy$$

obviously can generalize to more complicated groups

So how do we implement polarization?

Need local $\sim SO(3)$ charges and unambiguus definition of u^μ $(s^\mu \propto J^\mu)$
Chemical shift symmetry, $SO(3)_{\alpha_{1,2,3}} \to SO(3)_{\alpha_{1,2,3}(\phi^I)}$

- Polarization of many particles is a vector $\to \Psi^{\mu\nu}$ Polarization in isotropic materials of this form, spinors etc average in many-particle limit. Plus polarization and vorticity indistinguishable at coarse-graining scale

- Chemical shift: unique definition of u^μ, everything conserved flows the same way

$$\Psi_{\mu\nu} = -\Psi_{\nu\mu} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_0 \exp\left[-\sum_{i=1,2,3} \alpha_i(\phi_I)\hat{T}_i\right] \end{pmatrix}$$

$\alpha_i \to \alpha_i + \Delta\alpha_i\,(\phi_I) \Rightarrow L(b, y_{\alpha\beta} = u_\mu \partial^\mu \Psi_{\alpha\beta})$
$y_{\mu\nu} \equiv \mu_i$ for polarization vector components in comoving frame

How to combine polarization with local equilibrium?

Since polarization decreases the entropy by an amount proportinal to the DoFs and independent of polarization direction

$$b \to b\left(1 - c y_{\mu\nu} y^{\mu\nu} + \mathcal{O}\left(y^4\right)\right) \qquad , \qquad F(b) \to F(b,y) = F\left(b\left((1 - c y^2\right)\right)$$

First law of thermodynamics,

$$dE = TdS - pdV - Jd\Omega \to dF(b) = db\frac{dF}{db} + dy\frac{dF}{d(yb)}$$

$T_{\mu\nu} \to \Theta^{\mu\nu}$, the Belinfante tensor!

If field has local net direction, Noether current for a translationally invariant lagrangian NOT $T_{\mu\nu}$ but tensor incorporating twist: Belinfante-Rosenfeld In terms of the fundamental fields having transformation properties ψ_i

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{i}{2}\partial_\kappa \left[\frac{\partial L}{\partial\left(\partial_\kappa \psi^l\right)} \left(J^{\mu\nu}\right)^l_m \psi^m - \frac{\partial L}{\partial\left(\partial_\mu \psi^l\right)} \left(J^{\kappa\nu}\right)^l_m \psi^m - \frac{\partial L}{\partial\left(\partial_\nu \psi^l\right)} \left(J^{\kappa\mu}\right)^l_m \psi^m \right]$$

where $T_{\mu\nu}$ is the usual definition of energy-momentum tensor and $J^{\mu\nu}$ the appropriate representation of the Lorentz group

$$T_{\mu\nu} = \frac{\partial L}{\partial(\partial^\mu \psi)} \partial^\nu \psi - g^{\mu\nu} L \quad , \quad NB : \psi = \phi^I \quad AND \quad \Psi^i$$

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(J^{\sigma\rho} g^{\mu\sigma} - J^{\sigma\nu} g^{\mu\rho} - J^{\sigma\mu} g^{\nu\rho} + J^{\mu\rho} g^{\nu\sigma}\right)$$

Avoids antisymmetric part of canonical $T^{\mu\nu}$. (S.Weinberg, QFT1)

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Not unique alternative.

No non-relativistic limit for same reason Pauli matrices and $g-2$ arise, so I dont think its a big disadvantage

Symmetric and Gauge-invariant lack of the former is a much bigger problem for hydrodynamics. Recovers $T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}$ Is <u>not</u> Noether current for translations (spin twisting)

The Belinfante-Rosenfeld tensor for ideal hydrodynamics

$y_{\mu\nu}$ is 4-vector of chemical potentials represented as an antisymmetric tensor By rotation symmetry, dynamics only depends on <u>gradients</u> of chemical potentials. Hence, a good representation is

$$y^{\mu\nu} = u^{\alpha}\partial_{\alpha}\Psi_{\mu\nu} = \partial_{\mu}Z_{\nu} - \partial_{\nu}Z_{\mu}$$

What this definition means is that the local polarization can be obtained by integrating the potential along the path defined by u^{μ}

$$\Psi^{\mu\nu} = \int_{\tau_0}^{\tau} d\tau \left(\partial_{\mu}Z_{\nu} - \partial_{\nu}Z_{\mu}\right)$$

where τ_0 is the "starting point" of the evolution, and

$$d\tau = \frac{\partial^4 x}{\partial^3 \phi} = u^{\mu}dx_{\mu} = \frac{1}{6b} * \left(\epsilon_{IJK}d\phi^{I} \wedge d\phi^{J} \wedge d\phi^{K}\right)$$

The Belinfante tensor for a polarized fluid

$$\psi_l \equiv Z_\sigma \qquad , \qquad \frac{\partial L}{\partial\left(\partial_\kappa \psi^l\right)} \left(J^{\mu\nu}\right)^l_m \psi^m \equiv \frac{\partial L}{\partial\left(\partial_\kappa Z^\eta\right)} \left(J^{\mu\nu}\right)^\eta_\zeta Z^\zeta$$

Using our proposal for how to modify $F(b) \to F(b(1-cy^2))$ we get

$$\frac{\partial L}{\partial\left(\partial_\kappa Z^\eta\right)} = y^\eta_\kappa g(b,y) \qquad , \qquad g(b,y) = -b \left.\frac{\partial F(X)}{\partial X}\right|_{X=b\left(1-c y_{\mu\nu} y^{\mu\nu}\right)}$$

giving

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{i}{2}\partial_\kappa \Omega^\kappa_{\mu\nu}$$

were

$$\Omega^{\kappa\nu}_\mu = g(b,y)\left(y^\kappa_\sigma \left(J_{\mu\nu}\right)^\sigma_\rho Z^\rho - y_{\mu\sigma}\left(J^\kappa_\nu\right)^\sigma_\rho Z^\rho - y_{\nu\sigma}\left(J^\kappa_\mu\right)^\sigma_\rho Z^\rho\right)$$

However, just like with Israel-Stewart hydrodynamics, the conservation equations $\partial_\mu \Theta^{\mu\nu} = 0$ of this tensor will have to be augmented with 4 explicit equations for $\Psi_{\mu\nu}$, of the form

$$-\partial_\alpha \frac{\partial L}{\partial \left(\partial_\alpha Z_\beta\right)} + \frac{\partial L}{\partial Z_\beta} = 0$$

which describe the relationship between vorticity and polarization. It is easy to thee that these reduce to

$$\partial_\beta \left(y^{\alpha\beta} g \left(b, y\right)\right) = 0$$

which can be used to simplify the conservation equations for the Belinfante tensor, since

$$\partial_\kappa \left(g(b,y) y^\kappa_\sigma \left(J^{\mu\nu}\right)^\sigma_\rho Z^\rho \right) = g(b,y) y^\kappa_\sigma \left(J^{\mu\nu}\right)^\sigma_\rho \partial_\kappa Z^\rho$$

we get that

$$\partial^\mu \partial_\kappa \Omega^\kappa_{\mu\nu} = -\partial_\mu \partial_\kappa \left(g(b,y) y^\nu_\sigma \left(J^{\mu\kappa}\right)^\sigma_\rho Z^\rho \right)$$

Since the first two terms of $\Omega^\kappa_{\mu\nu}$

$$\partial_\mu \left(g(b,y) y^\kappa_\sigma \left(J^{\mu\nu}\right)^\sigma_\rho \partial_\kappa Z^\rho \right) - \partial_\kappa \left(g(b,y) y^\mu_\sigma \left(J^{\kappa\nu}\right)^\sigma_\rho \partial_\mu Z^\rho \right)$$

will add to zero (exchange $\kappa \leftrightarrow \mu$ in one of them)

Hence, the hydrodynamic equations will be

$$\partial_\mu T^{\mu\nu}_\phi + \partial_\mu T^{\mu\nu}_Z + \partial_\mu \partial_\kappa \left(g(b,y) y^\nu_\sigma \left(J^{\mu\kappa}\right)^\sigma_\rho Z^\rho \right) = 0$$

where

$$T^{\mu\nu}_\phi = (\rho' + p') u^\mu u^\nu - p' g^{\mu\nu} \quad , \quad T^{\mu\nu}_Z = g(b,y) y^\mu_\alpha \partial^\nu Z^\alpha + g_{\mu\nu} F(b,y)$$

$$\rho' = -F(b,y) \quad , \quad p' = b \frac{dF(b,y)}{db} \quad , \quad u^\mu = \epsilon^{\mu\nu\alpha\beta} \epsilon^{IJK} \partial_\nu \phi_I \partial_\alpha \phi_J \partial_\beta \phi_K$$

(Note that in general ρ', p' depend on y and are not the canonical ones)

$$g(b,y) = -b \, dF(X)/dX|_{X = b\left(1 - c y_{\mu\nu} y^{\mu\nu}\right)}$$

Note that $T^{\mu\nu}_Z$ and Belinfante term break diffeomorphism symmetry!

Polarization and vorticity conservation

When polarization is not dynamical ($y_{\mu\mu}$ constant), vorticity conservation arises as a non-local Noether current of the diffeomorphism invariance of the theory, specifically

$$\oint_{\Omega} dx_i u^i \frac{dF(b)}{db} = -\int_0^1 d\tau \int d^3x \frac{\partial L}{\partial(\partial_0 \phi^I)} \frac{d\Omega^I}{d\tau} \delta^3\left(\phi^J - \Omega^J(\tau)\right)$$

LHS Vorticity defined along closed loop Ω

RHS Noether current of the diffeomorphism moving ϕ^I along closed path Ω in terms of parameter τ

$$\zeta_{\Omega}^I(\phi^J) = -\int_0^1 d\tau \frac{d\Omega^I}{d\tau} \delta^3\left(\phi^J - \Omega^J(\tau)\right)$$

Polarization and vorticity conservation

If polarization is not zero, the fact that the equation above only moves around ϕ_I and not $y^{\mu\nu}$ breaks the symmetry, by an amount

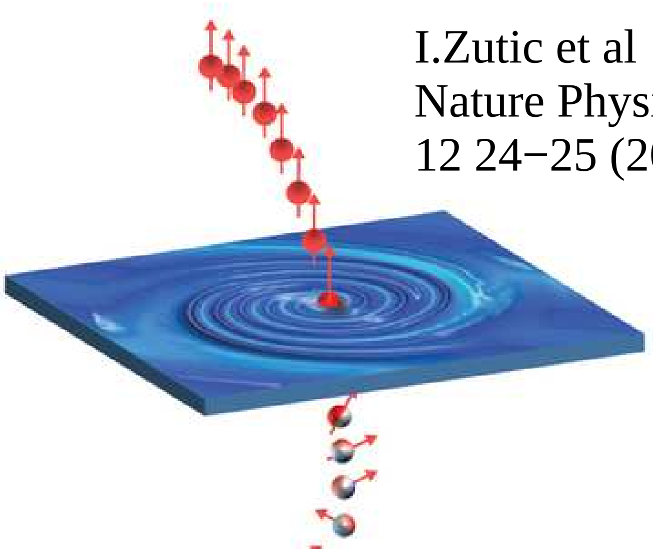
$$\frac{dy^{\mu\nu}}{d\tau} = \int d^3x \partial^\alpha y^{\mu\nu} \partial_\alpha \phi^I \delta^3 \left(\phi_J - \Omega_J(\tau)\right)$$

Hence, over a closed path we expect vorticity conservation to break down by an amount

$$\frac{d}{dt} \oint_\Omega dx_i u^i \frac{dF(b)}{db} = \dot{y}_{\alpha\beta} \frac{dL}{\partial(\partial_\mu y^{\alpha\beta})} \partial^\mu \zeta_\Omega(\phi^J) \equiv \frac{1}{2} g(b,y) \dot{y}^2 \int_0^1 \frac{d\Delta^{\alpha\beta}}{d\tau} \frac{\partial\Omega}{\partial\Delta^{\alpha\beta}} d\tau$$

$$\frac{d}{dt}\oint_{\Omega} dx_i u^i \frac{dF(b)}{db} = \frac{1}{2} g(b,y)\dot{y}^2 \int_0^1 \frac{d\Delta^{\alpha\beta}}{d\tau}\frac{\partial \Omega}{\partial \Delta^{\alpha\beta}} d\tau$$

The LHS is in principle a calculable but non-local quantity representing the transfer between local polarization and non-local vorticity degrees of freedom, the relativistic ideal hydrodynamic equivalent of



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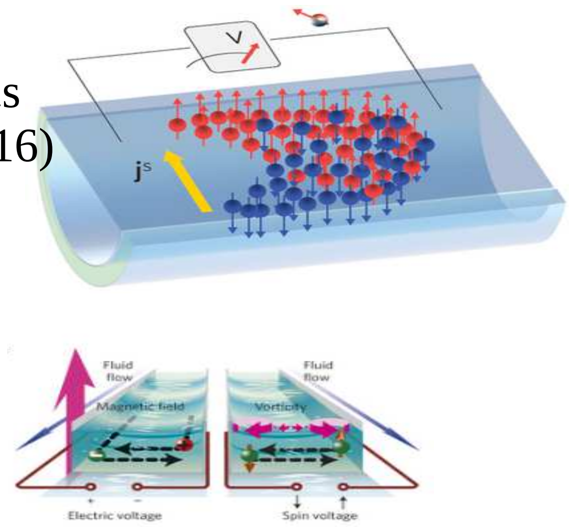

Instead of a conclusion: further steps

Write down ideal hydrodynamic limit of gas with polarization

Develop it Linearization, updated Kelvin's theorem, Cooper-Frying etc

Understand the effect of the interplay between polarization and vorticity. How "dissipative" is it?

Until this is done and incorporated in a numerical simulation, treat any prediction of polarization related to hydro with extreme caution!