Fluctuations and observables: volume fluctuations, critical dynamics

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September 28, 2016

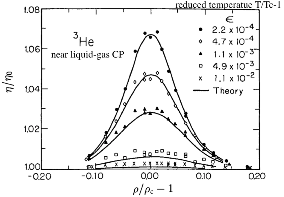
Outline

- Volume fluctuations: heuristic approach
- General properties of volume fluctuations
- Illustration based on Glauber MC
- Are ratios of susceptibilities volume independent? Answer with chiral model
- Role of “vacuum term” in chiral models
- Results with Functional Renormalization Group

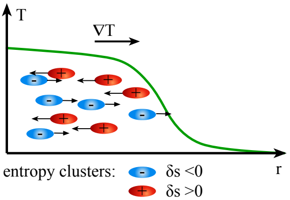
- …critical dynamics

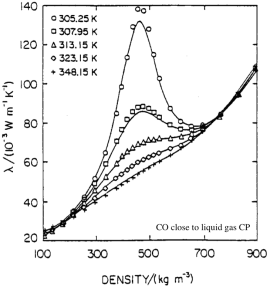
Signatures of phase transition and critical point I

- Dynamic:
  - Divergent shear and bulk viscosities at CP. Model H universality class
    $\eta \propto \xi^{\frac{1}{19}}$ and $\zeta \propto \xi^{2.8}$, where ξ is correlation length. Cavitation phenomena?
  - Divergent heat and charge conductivities $\lambda \propto \xi^{18/19}$. Possible observable: separation of entropy and charge clusters.
  - Spinodal decomposition at 1-st phase transition J. Randrup 2011; V.S. and D. Voskresensky 2011



Clusters with lower entropy tend to move along the temperature gradient; Clusters with higher entropy tend to move opposite to the temperature gradient.



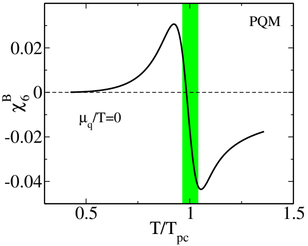


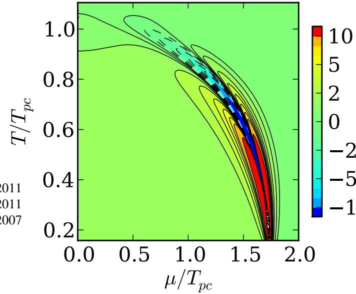
Experimental results from C. Agosta et al J. Low Temp. Phys. 67, 237 (1987). J. Luettmer-Strathmann et al J. Chem. Phys 103, 7482 (1995).

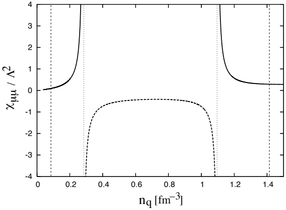
- Static:
  - O(4) crossover: universal sign structure of higher order cumulants, χ_6
  - Divergent cumulants of baryon number fluctuations at CP

$$\chi_n \propto \xi^{3\left(\frac{n\beta\gamma}{2-\alpha}-1\right)}$$

  - In vicinity of CP: universal sign structure of cumulants.
  - Divergent baryon number susceptibility at off-equilibrium first order phase transition





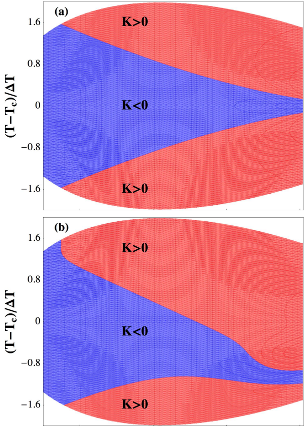


B. Friman, F. Karsch, K. Redlich, V. S. 2011
M. Stephanov 2011
C. Sasaki, K. Redlich, B. Friman 2007

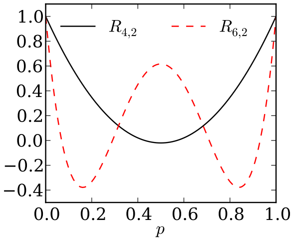
Listed predictions for an infinite static medium.
In HIC:

- Finite lifetime (S. Mukherjee, R. Venugopalan, Y. Yin 2015/2016)
- Finite size and anisotropy (G. Almasi and V.S. work in progress)
- Conservation laws (A. Bzdak, V. Koch, V.S. 2011)
- Fluctuations not related to interesting physics (V.S., B. Friman, K. Redlich 2012)

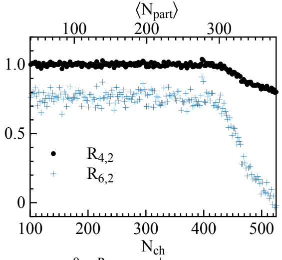
Modification due to finite life-time:



S. Mukherjee, R. Venugopalan, Y. Yin 2015



p is a fraction of measured baryons



$\mu = 0$ $\quad R_{n,m} = \chi_n/\chi_m$

Volume fluctuations: introduction 1

- Consider fixed V, where net baryon number B described by $P(B, V)$.
- n-th order moments of the net baryon number

$$\langle B^n \rangle_V = \sum_{B=-\infty}^{\infty} B^n P(B, V)$$

- Reduced cumulants, corresponding to the net baryon number fluctuations per unit volume. The first four reduced cumulants are

$$\kappa_1(T,\mu) = \frac{1}{V}\langle B \rangle_V, \qquad \kappa_2(T,\mu) = \frac{1}{V}\langle (\delta B)^2 \rangle_V,$$

$$\kappa_3(T,\mu) = \frac{1}{V}\langle (\delta B)^3 \rangle_V, \quad \kappa_4(T,\mu) = \frac{1}{V}\left[\langle (\delta B)^4 \rangle_V - 3\langle (\delta B)^2 \rangle_V{}^2\right],$$

where $\delta B = B - \bar{B}$ and $\bar{B} = \langle B \rangle_V$; $\kappa_n = T^3 \chi_n$.
κ_i are, to leading order, independent of volume V.

Formal approach in V.S., B. Friman and K. Redlich, 1205.4756

Volume fluctuations: introduction 2

- V dependence of moments follows

$$\langle B\rangle_V = \kappa_1 V, \qquad \langle B^2\rangle_V = \kappa_2 V + \kappa_1^2 V^2,$$

$$\langle B^3\rangle_V = \kappa_3 V + 3\kappa_2\kappa_1 V^2 + \kappa_1^3 V^3, \quad \langle B^4\rangle_V = \kappa_4 V + \left(4\kappa_3\kappa_1 + 3\kappa_2^2\right) V^2 + 6\kappa_2\kappa_1^2 V^3 + \kappa_1^4 V^4.$$

  The coefficients are from the Bell polynomials.
- Now allow for fluctuations of the volume $\mathcal{P}(V)$, corresponding moments $\langle V^n\rangle = \int V^n \mathcal{P}(V) dV$ and reduced cumulants of volume fluctuations, v_n.
- In presence of volume fluctuations the moments of the net baryon number are given by

$$\langle B^n\rangle = \int dV\, \mathcal{P}(V) \sum_{B=-\infty}^{\infty} B^n P(B, V) = \int dV\, \mathcal{P}(V) \langle B^n\rangle_V$$

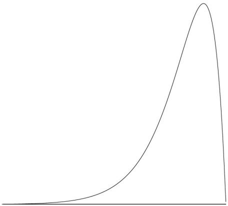
- Thus cumulants

$$c_1 = \kappa_1, \qquad c_2 = \kappa_2 + \kappa_1^2 v_2,$$

$$c_3 = \kappa_3 + 3\kappa_2\kappa_1 v_2 + \kappa_1^3 v_3, \quad c_4 = \kappa_4 + (4\kappa_3\kappa_1 + 3\kappa_2^2) v_2 + 6\kappa_2\kappa_1^2 v_3 + \kappa_1^4 v_4$$

$$c_1 = \kappa_1, \qquad c_2 = \kappa_2 + \kappa_1^2 v_2,$$
$$c_3 = \kappa_3 + 3\kappa_2\kappa_1 v_2 + \kappa_1^3 v_3, \quad c_4 = \kappa_4 + (4\kappa_3\kappa_1 + 3\kappa_2^2)v_2 + 6\kappa_2\kappa_1^2 v_3 + \kappa_1^4 v_4$$

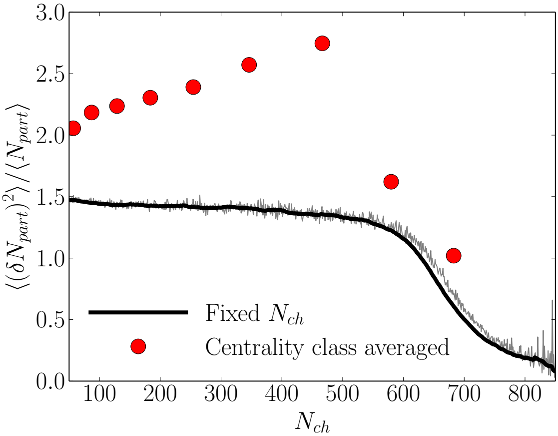
- $v_2 \geq 0 \rightsquigarrow c_2 \geq \kappa_2$
- c_3 receives contribution from v_3.
  For very central events (defined by N_{ch}) v_3 is negative: volume (or $S_\perp$) has an upper bound $V_{\max}$, while systems with lower volume may produce large N_{ch} owing to fluctuations.
  $c_3 < \kappa_3$ if $v_3 < -3v_2\kappa_2/\kappa_1^3$.



- At zero μ only c_n with $n \geq 4$ are modified, i.e. $c_4 = \kappa_4 + 3\kappa_2^2 v_2$. Thus $c_4 \geq \kappa_4$.
- At nonzero μ, c_4: competing contributions with opposite signs

Volume fluctuations: illustrations for v_2

- Assumption: $V \propto N_{\rm part}$
- Glauber + Negative binomial; Au-Au $\sqrt{s} = 200$

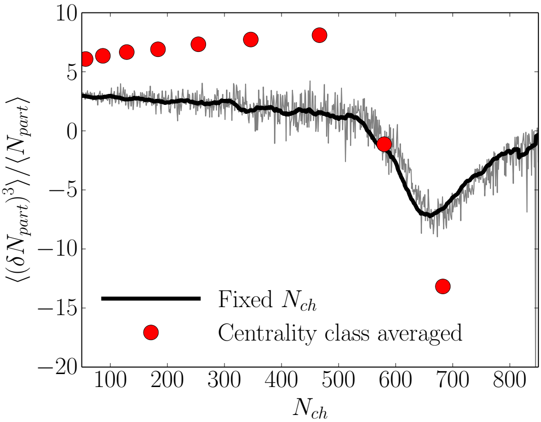


- Suppression of fluctuations at very large and very small (not shown) N_{ch}
- Centrality binning increases volume fluctuations and introduces non-monotonicity

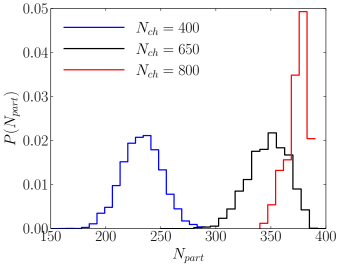
V.S., B. Friman and K. Redlich, 1205.4756

Points: centrality classes from right to left 0-5%, 5-10%, 10-20%, 20-30%, …

Volume fluctuations: illustrations for ν_3

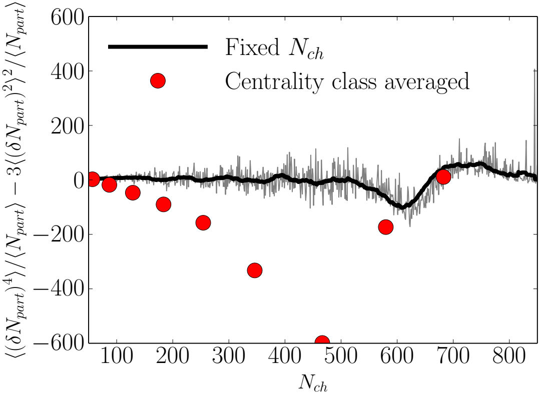


- Negative skewness for central collisions; negative and large skewness for 0-5% centrality bin.
- Skewness changes sign at about $N_{ch} \sim 500$



Points: centrality classes from right to left 0-5%, 5-10%, 10-20%, 20-30%, …

B. Friman, K. Redlich, V.S., work in progress



- Changes sign
- Centrality binning effect is strong

Points: centrality classes from right to left 0-5%, 5-10%, 10-20%, 20-30%, …

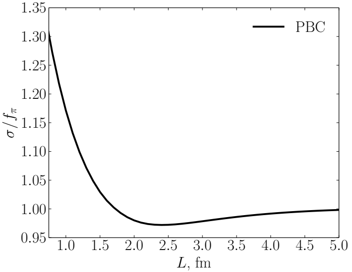
B. Friman, K. Redlich, V.S., work in progress

Kurtosis: opposite sign contributions may conspire to non-monotonic energy dependence. More studies are needed.

B. Friman, K. Redlich, V.S., work in progress

Volume fluctuations: are κ_n volume independent?

- Central assumption: $\langle \delta B^n \rangle$ are volume independent.
- This is true for large volumes if surface effects can be neglected.
- Volume dependence of chiral condensate at $T = 0$



Main ingredients:

- Functional Renormalization Group + Quark-meson model
- Equilibrium calculations in a box L^3 for physical pion mass.

- Expectation: cumulants are more sensitive

Gabor Almasi, V.S. to appear soon

On chiral model

- Polyakov-loop extended Quark Meson model

$$\mathcal{L} \quad = \quad \bar{\psi}\left[i\gamma^{\mu}D_{\mu} - g(\sigma + i\gamma_5\tau\cdot\pi)\right]\psi + \frac{1}{2}(\partial_{\mu}\sigma)^2 + \frac{1}{2}(\partial_{\mu}\pi)^2 - U(\sigma,\pi) - \mathcal{U}(\ell,\ell^*)$$

- Mean-field approximation: integrate out fermion fields & disregard fluctuations of mesonic field, $\Omega = \Omega_q - U(\sigma,\pi) - \mathcal{U}(\ell,\ell^*)$

$$\Omega_q = -\frac{1}{V}\mathrm{Tr}\log\left(i\gamma^{\mu}D_{\mu} - g\sigma\right) = \Omega_v + \Omega_T =$$

$$-2N_fN_c\int\frac{d^4p}{(2\pi)^4}\log(p^2+(g\sigma)^2)$$

$$-2TN_f\sum_a\int\frac{d^3p}{(2\pi)^3}\log\left(1+e^{-(E-\mu)/T+2\pi i q_a/3}\right) + \text{antiq.}$$

- Ω_v is divergent; this is not a good reason to ignore Ω_v

V. S. and B. Friman et al, 1005.3166; R. Pisarski and V. S., 1604.00022

On chiral model: vacuum term

- To extract finite contribution: dimensional regularization

$$\Omega_v = -\frac{N_c N_f}{8\pi^2} g^4 \sigma^4 \log\left(\frac{g\sigma}{\Lambda_{dr}}\right).$$

- Similar contribution, but for bosons (change of overall sign), first analyzed by S. Coleman and E. Weinberg in PRD 7, 1888, 1973: "Fluctuation induced first-order phase transition"

$$V = \frac{\lambda}{4!}\varphi_c{}^4 - \tfrac{1}{2}B\varphi_c{}^2 - \frac{1}{4!}C\varphi_c{}^4 \boxed{+\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\varphi_c{}^2}{2k^2}\right)},$$

$$V = \frac{\lambda}{4!}\varphi_c{}^4 + \tfrac{1}{2}B\varphi_c{}^2 + \frac{1}{4!}C\varphi_c{}^4 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c{}^2 \boxed{+\frac{\lambda^2\varphi_c{}^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_c{}^2}{2\Lambda^2}-\frac{1}{2}\right)}$$

- Gross-Neveu model, PRD 10, 3235, 1974: "Dynamical symmetry breaking in asymptotically free field theories";
  1+1 dimensional NJL model

$$V(\sigma_c,\sigma_0,g) = \tfrac{1}{2}\sigma_c{}^2 + \boxed{\frac{\lambda}{4\pi}\sigma_c{}^2\left[\ln\left(\frac{\sigma_c}{\sigma_0}\right)^2}-3\right].$$

- For $m_q/T = g\sigma/T \ll 1$, the expansion of thermal part

$$\Omega_T \simeq N_c N_f T^4 \left\{ -\frac{7\pi^2}{180} + \frac{1}{12}\frac{m_q^2}{T^2} + \boxed{\frac{1}{8\pi^2}\frac{m_q^4}{T^4}\left[\log\left(\frac{m_q}{\pi T}\right) + \gamma_E - \frac{3}{4}\right]} - \frac{7\zeta(3)}{192\pi^4}\frac{m_q^6}{T^6} + \mathcal{O}\left(\frac{m_q^8}{T^8}\right)\right.$$

- Close to chiral phase transition, thermal part brings

$$\Omega_T = \boxed{+}\frac{N_c N_f}{8\pi^2} g^4 \sigma^4 \log\left(\frac{g\sigma}{\pi T}\right) + \cdots$$

- Compare to vacuum part

$$\Omega_v = \boxed{-}\frac{N_c N_f}{8\pi^2} g^4 \sigma^4 \log\left(\frac{g\sigma}{\Lambda_{dr}}\right)$$

- **Exact cancellation of $\boxed{\sigma^4 \log(\sigma)}$ dependence close to transition!**

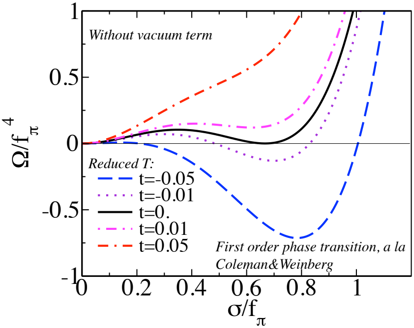
V. S. and B. Friman et al, 1005.3166

On chiral model: is Ω_v important?

- Lets neglect Ω_v.
- Close to chiral transition, Ω_T receives contribution $\boxed{\sigma^4 \log \sigma < 0}$.

W/o vacuum term, Landau theory

$$\Omega = \frac{1}{2}A(T - T_c)\sigma^2 + \frac{1}{4}B\sigma^4\,(1 + C \log(\sigma/\sigma_0))$$



- C.f. Coleman-Weinberg's fluctuation induced first-order phase transition:

$$V = \frac{\lambda}{4!}\varphi_c{}^4 + \tfrac{1}{2}B\varphi_c{}^2 + \frac{1}{4!}C\varphi_c{}^4 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c{}^2 + \boxed{\frac{\lambda^2\varphi_c{}^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_c{}^2}{2\Lambda^2} - \frac{1}{2}\right)}$$

- Shift of transition temperature

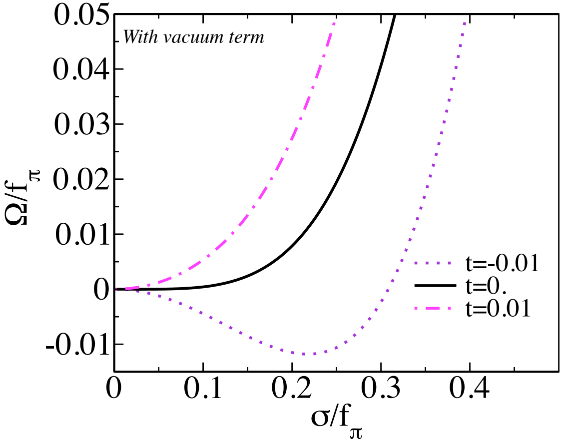
$$T_{\rm PT} = T_c + \frac{BC}{4A}\#^2$$

- Most important: first-order phase transition instead of expected second order.

V. S. and B. Friman et al, 1005.3166

On chiral model: is Ω_v important?

- Lets restore Ω_v.
- With vacuum term, Landau theory $\Omega = \frac{1}{2}A(T - T_c)\sigma^2 + \frac{1}{4}B\sigma^4$

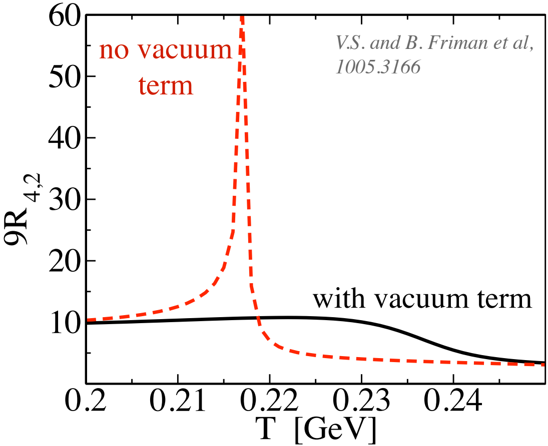


- Ω_v appears to be important in chiral limit; but for *physical pion mass?!*

V. S. and B. Friman et al, 1005.3166

On chiral model: is Ω_v important at m_π?

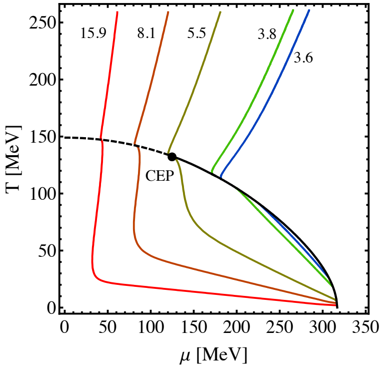
- Crossover for both models
- Transition properties are however very different: Lee-Yang zeros in finite volume, type of singularities in complex chemical potential plane, etc
- Baryon number fluctuations, kurtosis, $R_{4,2} = \chi_4/\chi_2 = \kappa\sigma^2$:



physical pion mass; zero chemical potential

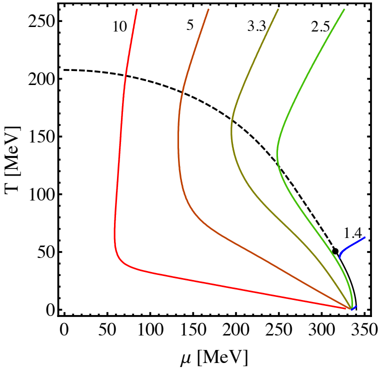
On chiral model: is Ω_v important at m_π?

Without vacuum term:



"Focusing" of isentropic due to remnant of first order phase transition in chiral limit

With vacuum term:

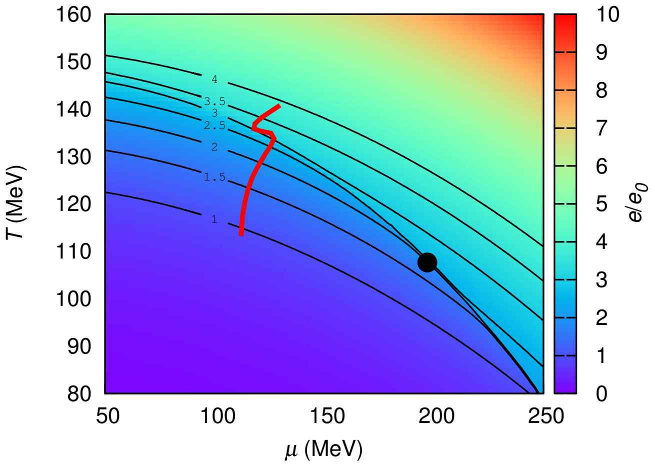


Expected behavior: isentropes are tangential to continuation of first order phase transition

E. Nakano et al, 0907.1344

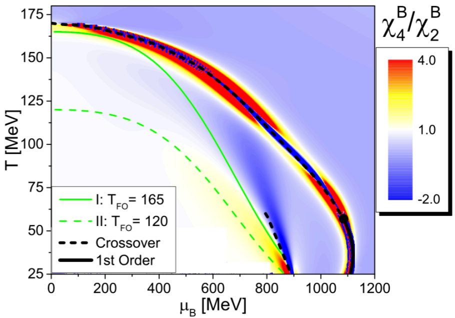
On chiral model: is Ω_v important at m_π?

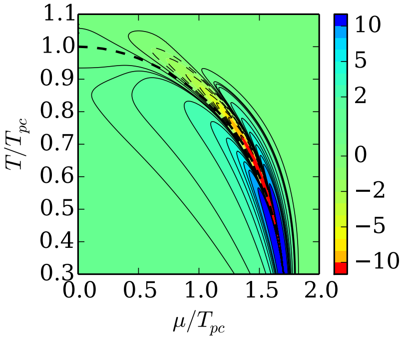
C. Herold, M. Nahrgang and Co, e.g. 1601.04839



Unfortunately these calculations do not include vacuum term…

On chiral model: is Ω_ν important at m_π?





V. S., QM 2012

Is the PQM model from J. Steinheimer, J. Randrup, V. Koch, PRC 89, 034901, 2014 ?

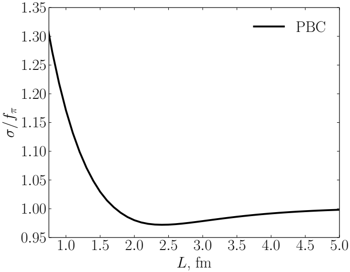
On chiral model: missing vacuum term in literature

Vacuum term was also overlooked in numerous papers

- ...
- O. Scavenius, A. Mocsy, I. N. Mishustin and D. H. Rischke, PRC 64, 045202 (2001)
- B.-J. Schaefer and J. Wambach, PRD 75, 085015 (2007)
- E. S. Bowman and J. I. Kapusta, PRC 79, 015202 (2009)
- U. S. Gupta and V. K. Tiwari, arXiv:0911.2464
- D. Nickel, PRD 80, 074025 (2009)
- T. Kahara and K. Tuominen, PRD 80, 114022 (2009)
- J. I. Kapusta and E. S. Bowman, NPA 830, 721C (2009)
- ...

Volume fluctuations: are κ_n volume independent?

- Central assumption: $\langle \delta B^n \rangle$ are volume independent.
- This is definitely true for large volumes if surface effects can be neglected.
- Volume dependence of chiral condensate at $T = 0$



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Gabor Almasi, V.S. to appear soon

Functional renormalization group

The general flow equation for the scale-dependent effective action Γ_k. Γ_k: loosely speaking modes with momenta greater than k are integrated out, k is IR cut-off.

$$\partial_k \Gamma_k[\Phi,\psi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_{kB}\left(\Gamma_k^{(2,0)}[\Phi,\psi]+R_{kB}\right)^{-1}\right\} - \mathrm{Tr}\left\{\partial_k R_{kF}\left(\Gamma_k^{(0,2)}[\Phi,\psi]+R_{kF}\right)^{-1}\right\}$$

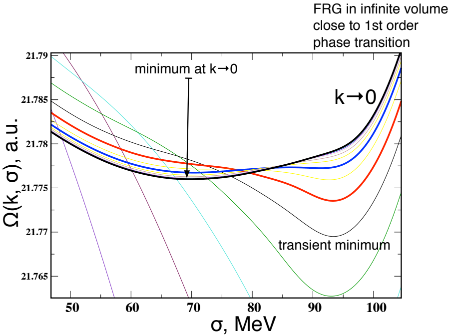
The flow equation for the PQM model in infinite volume

$$\partial_k \Omega(k,\rho \equiv \frac{1}{2}[\sigma^2+\pi^2]) = \frac{k^4}{12\pi^2}\left\{\frac{3}{E_\pi}\left[1+2n_B(E_\pi;T)\right] + \frac{1}{E_\sigma}\left[1+2n_B(E_\sigma;T)\right] - \frac{4N_f N_c}{E_q}\left[1-N(\ell,\ell^*;T,\mu_q)-\bar{N}(\ell,\ell^*;T,\mu_q)\right]\right\}$$

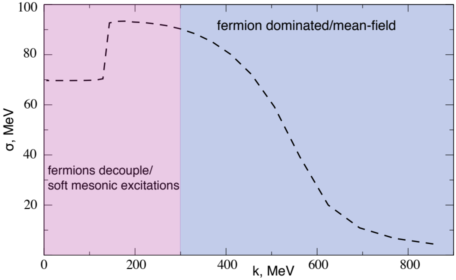
$n_B(E;T)$ is the boson distribution functions
$N(\ell,\ell^*;T,\mu_q)$ are fermion distribution function modified owing to coupling to gluons
E_σ and E_π are the functions of k, $\partial\Omega/\partial\rho$ and $\rho\partial^2\Omega/\partial\rho^2$
$E_q = \sqrt{k^2+2g\rho}$

FRG defines $\Omega(k,\rho;T,\mu_Q,\mu_B)$.
Physically relevant quantity is the thermodynamical potential
$\bar{\Omega}(T,\mu_Q,\mu_B) \equiv \Omega(k\to 0,\rho\to\rho_0;T,\mu_Q,\mu_B)$, where ρ_0 is the minimum of Ω. V.S. et al 1004.2665

Scale-dependent effective potential:



Minimum of scale-dependent effective potential:



Soft mesonic excitations drive evolution at very small k.

The general flow equation for the effective action

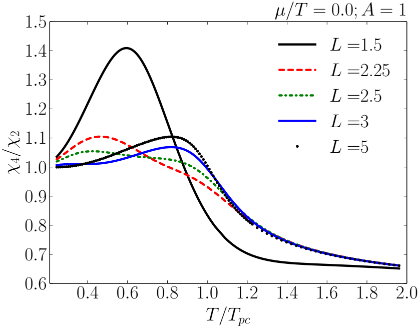
$$\partial_k \Gamma_k[\Phi,\psi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_{kB}\left(\Gamma_k^{(2,0)}[\Phi,\psi] + R_{kB}\right)^{-1}\right\} - \mathrm{Tr}\left\{\partial_k R_{kF}\left(\Gamma_k^{(0,2)}[\Phi,\psi] + R_{kF}\right)^{-1}\right\}$$

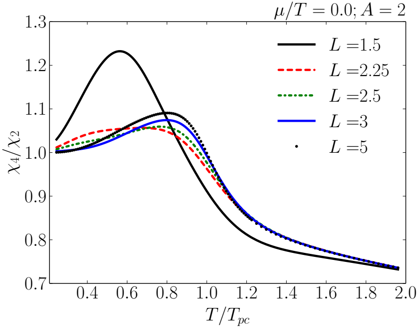
- The flow equation involve summation over modes $\sum_{n_1,n_2,n_3}$.
- All principal components were considered in 2010, but it was not until 2016 when we were able to solve the technical issues and the flow equation based on Chebyshev collocation pseudo spectral lattice.
- With parallel realization of code, it takes seconds to compute thermodynamics to very high precision at a given T and μ.
- This become possible due to support of **Gabor Almasi** by HGS-HIRE and NTG BNL.

Gabor Almasi, V. S. to appear soon

Results: PQM + FRG in finite model $\mu = 0$

- Anisotropy coefficient $A = L_{\|}/L_{\perp}$





Gabor Almasi, V. S. to appear soon

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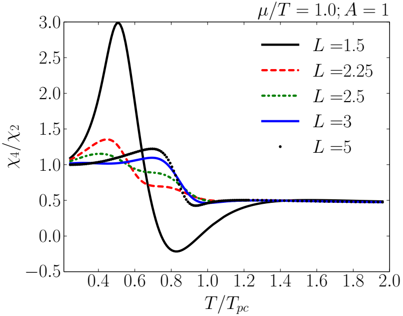

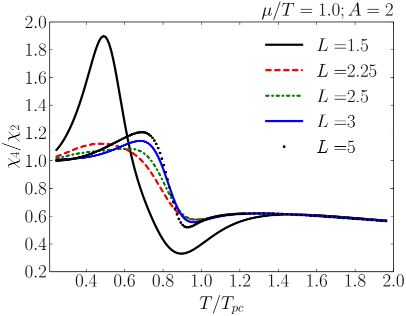

Approximation of volume independence of κ_n breaks down at about $L = 3 - 4$ fm: this makes analysis of volume fluctuations significantly more complicated and tractable only in a fully dynamical model.

Gabor Almasi, V. S. to appear soon

Conclusions

- Any sourse of fluctuations may strongly affect higher order cumulants
- Volume fluctuaions: some properties are model independent, i.e. negative v_3.
- In general, v_n are non-monotonic functions of centrality and energy
- Ratios of cumulants depend on volume for $L \sim 3$ fm
- Chiral model: the vacuum term should not be omitted.

Discussion Session

From hydrodynamics to model H

Navier-Stokes equation, continuity equation, and equation for heat transport:

$$mn\left[\partial_t u_i + (\mathbf{u}\nabla)u_i\right] = -\nabla_i P + \nabla_k \left[\eta\left(\nabla_k u_i + \nabla_i u_k - \frac{2}{d}\delta_{ik}\mathrm{div}\mathbf{u}\right) + \zeta\delta_{ik}\mathrm{div}\mathbf{u}\right]$$

$$\partial_t n + \mathrm{div}(n\mathbf{u}) = 0$$

$$T\left[\frac{\partial s}{\partial t} + \mathrm{div}(s\mathbf{u})\right] = \mathrm{div}(\kappa\nabla T) + \eta\left(\nabla_k u_i + \nabla_i u_k - \frac{2}{d}\delta_{ik}\mathrm{div}\mathbf{u}\right)^2 + \zeta(\mathrm{div}\mathbf{u})^2.$$

- Slowest mode only

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- Relate δT and δP
- Add appropriate noise
- Consider only small deviatons $\tilde{s} = \tilde{s}_c + \delta\tilde{s}$ and expand EoS near $\tilde{s}_c$

From hydrodynamics to model H

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From Hohenberg-Halperin , Rev. of Mod. Phys. 49, 1977:

$$\frac{\partial\psi}{\partial t} = \lambda_0\nabla^2\frac{\delta F}{\delta\psi} - \boxed{g_0}\nabla\psi\cdot\frac{\delta F}{\delta\mathbf{j}} + \underbrace{\theta}_{\text{noise}}, \qquad \text{heat transport} \quad (5.1a)$$

mode-mode coupling

$$\frac{\partial\mathbf{j}}{\partial t} = \underbrace{\boxed{\mathcal{T}}}_{\text{transverse projector}}\cdot\left[\bar{\eta}_0\nabla^2\frac{\delta F}{\delta\mathbf{j}} + \boxed{g_0}(\nabla\psi)\frac{\delta F}{\delta\psi} + \underbrace{\zeta}_{\text{noise}}\right], \qquad \text{Navier-Stokes} \quad (5.1b)$$

$$F = F_0 - \int d^d x\{h(\mathbf{x},t)\psi + \mathbf{A}(\mathbf{x},t)\cdot\mathbf{j}\}, \qquad (5.1c)$$

$$F_0 = \int d^d x\{\underbrace{\tfrac{1}{2}r_0\psi^2}_{\text{EoS}} + \overbrace{\boxed{\tfrac{1}{2}(\nabla\psi)^2}}^{\text{surface tension}} + \underbrace{u_0\psi^4}_{\text{EoS}} + \tfrac{1}{2}\mathbf{j}^2\}, \qquad (5.1d)$$

Careful analysis in Vasil'ev "The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics"

Volume fluctuations is one of effects that may change cumulants. Higher order cumulants are most sensitive.

Assumptions

- volume fluctuations are not critical (independent of baryon fluctuations)

$$P(V,B) = \mathcal{P}(V)P(B)$$

- other thermodynamic quantities do not fluctuate

The last assumption can be justified at high energies ($\mu_B \to 0$).
If system thermalizes, the finite temperature is independent of initial one.

- Cumulant generating functions (CGF) for both fluctuations

$$\mathrm{CGF}^B(t) = \ln \sum_{B=-\infty}^{\infty} P(B) \exp(B\,t)$$

$$\mathrm{CGF}^V(s) = \ln \int_0^{\infty} dV\, \mathcal{P}(V) \exp(Vs)$$

- The additivity of cumulants and thermodynamic principles imply that

$$\mathrm{CGF}^B(t) = V \cdot \zeta^B(t),$$

where ζ^B is a volume-independent function.

Volume fluctuations

Aim is to obtain cumulants with volume fluctuations. These cumulants are obtained from the cumulant generating function

$$\phi^B(t) = \ln \int dV\, \mathcal{P}(V) \sum_B P(B) e^{Bt}$$

But

$$\sum_B P(B) e^{Bt} = e^{V\zeta^B(t)}$$

and consequently

$$\phi^B(t) = \ln \int dV \mathcal{P}(V) e^{V\zeta^B(t)}$$

From comparison with definition of cumulant generating function of volume fluctuations:

$$\phi^B(t) = \mathrm{CGF}^V\left(\zeta^B(t)\right)$$

Corresponding reduced cumulants are given by Taylor expansion of about $t = 0$

$$c_n = \frac{1}{\langle V \rangle} \frac{d^n}{dt^n} \phi^B(t)\Big|_{t=0}$$

Volume fluctuations: cumulants

Let κ_n are the cumulants for baryon fluctuations and v_n are for volume fluctuations ($\delta X = X - \langle X \rangle$)

$$\kappa_1 = \frac{1}{V}\langle B \rangle, \quad \kappa_2 = \frac{1}{V}\langle (\delta B)^2 \rangle, \quad \kappa_4 = \frac{1}{V}\left[\langle (\delta B)^4 \rangle - 3\langle (\delta B)^2 \rangle^2\right]$$

$$v_1 = \frac{1}{V}\langle V \rangle_V, \quad v_2 = \frac{1}{V}\langle (\delta V)^2 \rangle, \quad v_4 = \frac{1}{V}\left[\langle (\delta V)^4 \rangle - 3\langle (\delta V)^2 \rangle^2\right]$$

Then wanted cumulants of baryon number fluctuations including volume fluctuations are given by

$$\begin{aligned}
c_1 &= \kappa_1 \\
c_2 &= \kappa_2 + {\kappa_1}^2 v_2 \\
c_3 &= \kappa_3 + 3\kappa_2\kappa_1 v_2 + {\kappa_1}^3 v_3 \\
c_4 &= \kappa_4 + (4\kappa_3\kappa_1 + 3{\kappa_2}^2)v_2 + 6\kappa_2{\kappa_1}^2 v_3 + {\kappa_1}^4 v_4 \\
&\ldots
\end{aligned}$$