Integral Geometry: A new Frontier?

By Craig Webster

Advisor: Prof. Jerry Seidler

UW REU 2004

Outline

- *Introduction to Hadwiger's Theorem*
- *Physics Review Letters paper that gave Jerry the idea for this project*
- *First project*
  - *Generalize Hadwiger's Theorem*
  - *Or Apply It*
- *Second project: Simulated Annealing*
- *Conclusion*

Definitions

- **Valuation:** *A function that maps from the set of convex bodies to a real number, such that*

$$\mu(\emptyset) = 0,$$

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$

for all convex bodies, A and B.

Definitions [cont'd]

- **Euler characteristic, *χ:*** *Topological descriptor: qualitatively, number of distinct bodies.*

$$\chi = vertices - edges + faces = 1 - genus$$

Minkowski Functionals

- **Minkowski functionals, M_v:** *Morphological descriptors of a system.*

$$M_{\upsilon}(A) = \int \chi(A \cap E_{\upsilon}) d\mu(E_{\upsilon}),$$

where *A* is any convex body, *v* goes from 0 to the dimension of A, ω_v is the volume of a unit ball, E_v denotes the *v*-dimensional plane, and $d\mu(E_v)$ is a normalization factor.

Hadwiger's Theorem

Suppose that μ is a continuous rigid motion invariant valuation on the set of convex bodies. Then there exists $c_0, c_1, . . . c_n$ elements of the real numbers such that for all convex bodies L,

$$\mu(L) = \sum_{i=0}^{d} c_i M_i(L).$$

Reconstructing Complex Materials via Effective Grain Shapes[1]:

- Experimentally in 3D, Minkowski functionals appear to be a basis for conductivity – not linearly, though.
- Conductivity is not additive . . .
- Mathematical band-aid?

[1]C.H. Arns, et al., PRL, Vol. 91, Num. 21, 215506

Ghasp!

TWO PATHS

Generalize Hadwiger's Theorem to Include non-Additive Systems

Simplify Complicated Additive Systems Using Hadwiger's Theorem

Where to go?

- Not possible to generalize easily
- No Hadwiger-applicable additive systems in sight

Simulated Annealing: A new Frontier?

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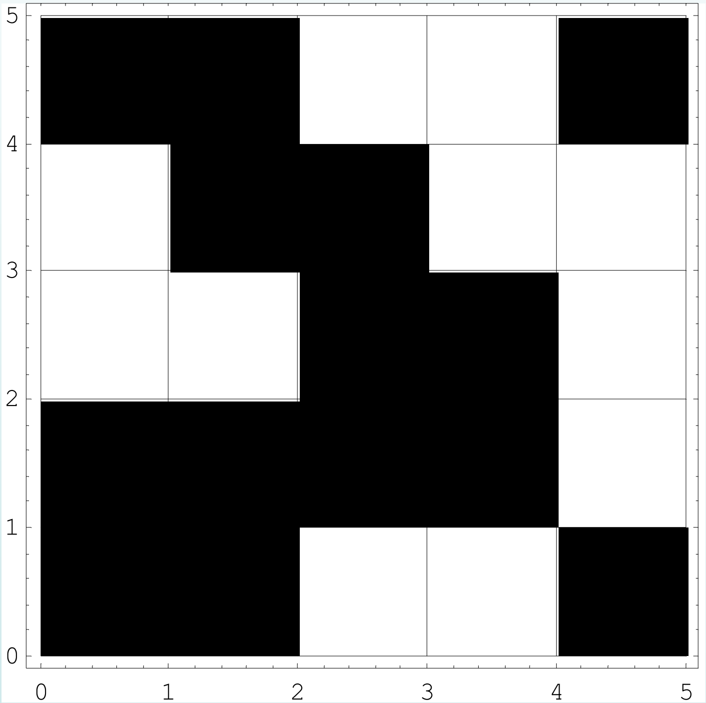
Simulated Annealing

A heuristic method used in statistical mechanics that finds the energetic minimum of a system.

Objectives

- Create a simulated annealing program that goes to target values (input by user) for the Minkowski functionals
- Determines whether the two-dimensional test of conductivity currently being done by another undergraduate student is including all of state space.

Main Idea of Program



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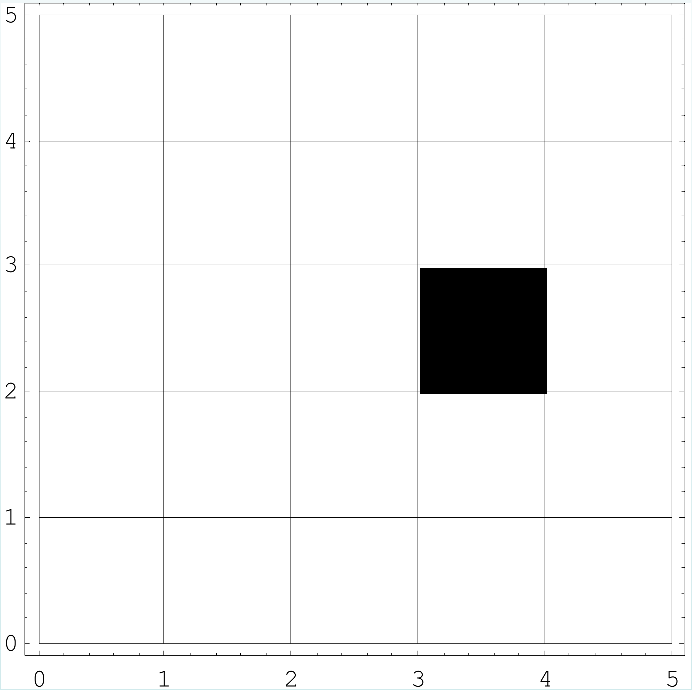


Random two-phase 5 x 5 graph
Target Area = 24
Target Bound = 4
Target Euler = -1

Annealed to target values of:
Area = 24
Boundary = 4
Euler = -1

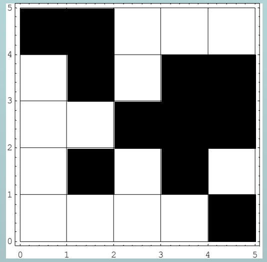
Equations

$$MockEnergy = c_0 \cdot (V_0 - V_{0,tar})^2 + c_1 \cdot (V_1 - V_{1,tar})^2 + c_2 \cdot (V_2 - V_{2,tar})^2$$

$$p(T) = \frac{e^{-\frac{E_{white}}{kT}}}{e^{-\frac{E_{black}}{kT}} + e^{-\frac{E_{white}}{kT}}}$$

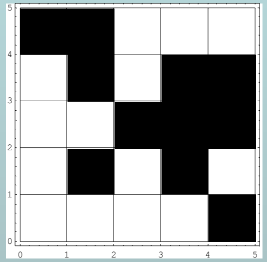
Algorithm

Graph A



1. Create a random, two-phase, two-dimensional graph, denoted Graph A.

Graph A



Mock Energy A = 14180

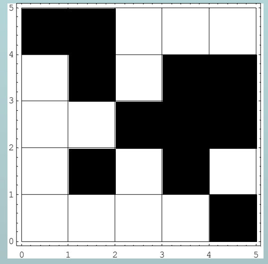
Algorithm

1. Create a random, two-phase, two-dimensional graph, denoted Graph A.
2. Calculate the Minkowski functionals and mock energy.

Algorithm

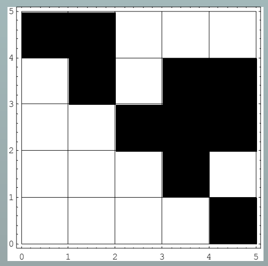
1. Create a random, two-phase, two-dimensional graph, denoted Graph A.
2. Calculate the Minkowski functionals and mock energy.
3. Invert one pixel to create Graph B.

Graph A



Mock Energy A = 14180

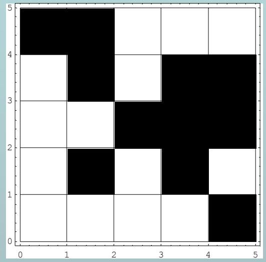
Graph B



Algorithm

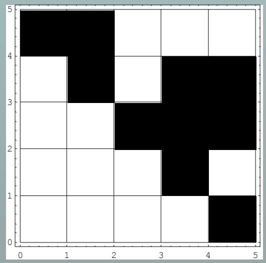
1. Create a random, two-phase, two-dimensional graph, denoted Graph A.
2. Calculate the Minkowski functionals and mock energy.
3. Invert one pixel to create Graph B.
4. Calculate the Minkowski functionals and mock energy B.

Graph A



Mock Energy A = 14180

Graph B

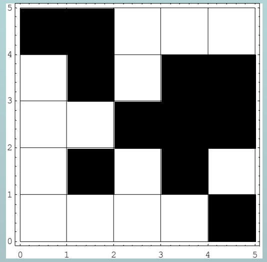


Mock Energy B = 11320

Algorithm

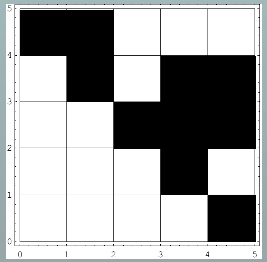
1. Create a random, two-phase, two-dimensional graph, denoted Graph A.
2. Calculate the Minkowski functionals and mock energy.
3. Invert one pixel to create Graph B.
4. Calculate the Minkowski functionals and mock energy B.
5. Find probability, *p,* and compare *p* to a random number.

Graph A



Mock Energy A = 14180

Graph B



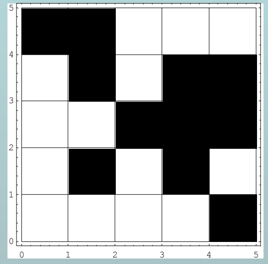
Mock Energy B = 11320

Step 5 : p = 1;1>0.6523

Algorithm

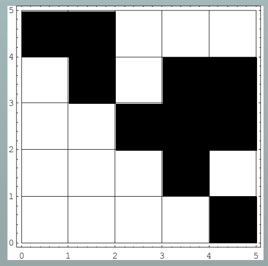
1. Create a random, two-phase, two-dimensional graph, denoted Graph A.
2. Calculate the Minkowski functionals and mock energy.
3. Invert one pixel to create Graph B.
4. Calculate the Minkowski functionals and mock energy B.
5. Find probability, *p,* and compare *p* to a random number.
6. Choose A or B.

Graph A



Mock Energy A = 14180

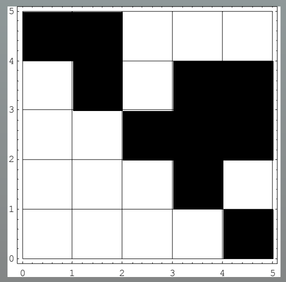
Graph B



Mock Energy B = 11320

Step 5 : p = 1;1>0.6523

Choose Graph B

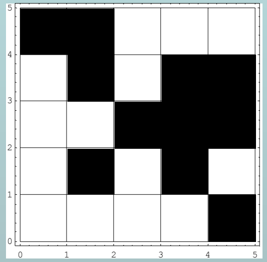


Algorithm

1. Create a random, two-phase, two-dimensional graph, denoted Graph A.
2. Calculate the Minkowski functionals and mock energy.
3. Invert one pixel to create Graph B.
4. Calculate the Minkowski functionals and mock energy B
5. Find probability, *p,* and compare *p* to a random number.
6. Choose A or B.
7. Lather, Rinse, Repeat [*Subtract "tempStep" from T, and return to (2)*].

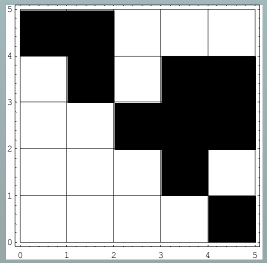
Follow these steps until T reaches 0.

Graph A



Mock Energy A = 14180

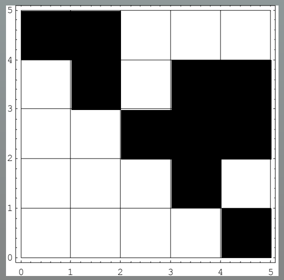
Graph B



Mock Energy B = 11320

Step 5 : $p = 1; 1 > 0.6523$

Choose Graph B



My Status

- Created simulated annealing program
- Did a qualitative test of the parameters
- Still left to do:
  - Determine how changing T affects annealing
  - In mock energy, determine optimal c_i’s
  - Test some target values

Conclusion

- No mathematical foundation to generalize Hadwiger’s Theorem
- Hadwiger’s Theorem is a tool, not a weapon
- For the next week and a half, run simulated annealing program to generate an in-depth understanding of the various scenarios corresponding to Minkowski functionals

Thanks to

- Dr. Jerry Seidler
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- REU Participants