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Applications of Hadwiger's Theorem

I. Introduction

In the past twenty years, physicists have utilized the mathematical principles specified by integral geometry to analyze systems ranging from maps of the morphological structure of Cosmic Microwave Background to estimating percolation thresholds in porous materials [1, 2]. Myriad papers have acclaimed its utility in finding the higher-order morphological structure of objects with relatively little calculation [3, 4]. The only stipulation is that Hadwiger's theorem, the physically-useful component of integral geometry, requires the studied system be translation and rotation invariant and additive [5].

In 2003 the experimental paper "Reconstructing Complex Materials Via Effective Grain Shapes" suggested that the morphological information that governs Hadwiger-applicable systems also completely defines some non-additive systems, albeit in a non-linear fashion [6]. This is counterintuitive, since Hadwiger's Theorem only applies to additive systems. If the experimental implications of this finding can be described mathematically, the results have the potential to provide insight into a multitude of physical phenomena.

My project began by attempting to resolve the mathematical dilemma presented by the experimental results of 2003. I planned to analyze the proof of Hadwiger's theorem and determine whether it could be generalized to include non-additive systems. Along the way, I would look for systems in which to apply integral geometry; specifically those in which a complex system could be simplified by Hadwiger's theorem. If such a situation arose, the physical system could be described completely by its morphological structure, potentially an exciting result that would provide new insight into the system.

After understanding the proof to Hadwiger's theorem and attempting to apply it without success, I embarked upon a new direction in my research and created simulated

annealing program that could test the state space of various values of Minkowski functionals.

This paper will choronologically follow the research that I did in the summer of 2004. Section II will describe my study of Hadwiger's Theorem, including background information and my attempts to apply and generalize it. Then in Section III, I will make some concluding remarks.

II. Hadwiger's Theorem

Integral geometry is defined as the combinatorics of convex bodies. This branch of math has applications in both math and physics. Hadwiger's Theorem is one of the most physically useful aspects of integral geometry, since it supplies morphological descriptors for any additive, rotation-invariant system. Before presenting Hadwiger's Theorem, I will provide definitions and useful background material.

i. Background and Definitions

Denote K^n as the set of all compact convex subsets of R^n. The elements of K^n are convex bodies.

Convex Body: *An object such that any straight line segment drawn within the object is enclosed. Examples of convex bodies: squares, circles, ellipses. Examples of non-convex objects: conjoined circles, figure eights.*

Valuation: *A function that maps from a set of convex bodies to a real number that also satisfies the following conditions:*

$$\mu(\varnothing) = 0, \qquad [1]$$

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B), \qquad [2]$$

for convex bodies A and B. A function that satisfies the above conditions is referred to as additiveIn many systems energy could be considered a valuation, since energy is rigid-motion invariant, additive, and elicits a scalar value for a convex body in a given system.
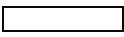
**Euler characteristic, χ**: *Topological descriptor: qualitatively, the number of distinct bodies.*

$$\chi = vertices - edges + faces = 1 - genus \qquad [3]$$

**Minkowski functionals, M_ν:** *Morphological descriptors of a system.*

$$M_\upsilon(A) = \int \chi(A \cap E_\upsilon)\, d\mu(E_\upsilon),$$

[4]

where A is any convex body, v goes from 0 to the dimension of A, ω_v is the volume of a unit ball, E_v denotes the v-dimensional plane, and $d\mu(E_v)$ is a normalization factor. The above equation can be read as the Euler characteristic of the intersection of a convex body with the v-dimensional plane taken over all orientations. Here is a simple example to illuminate this formula.

$A=$ ▭

E_2 *(2-d planes) over all orientations*

$M_2 = \int \chi(A \cap E_2) = $ *Euler (A)=1*

rep. of E_1 (lines) over all orientations
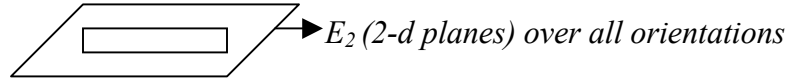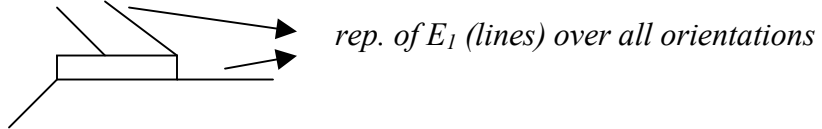
$M_1 = \int \chi(A \cap E_1) = $ *Boundary(A)=perimeter*

$M_0 = \int \chi(A \cap E_0) = $ *area (A)=Area of A, since E_0 is a point taken over all orientations*

Hadwiger's Theorem: *Any additive, rotation and translation invariant valuation, μ, of a convex body from the set of convex bodies can be written as a superposition of the d+1 Minkowski functionals. Mathematically, that is*

$$\mu(L) = \sum_{i=0}^{d} c_i M_i,$$

[5]

where $L \in K^n$.

ii. Applying Hadwiger's Theorem to Non-Additive Systems

Now that we have a working definition of Hadwiger's Theorem, it is evident that the theorem would be extremely difficult to generalize to non-additive systems. The additive definition of valuation is an essential aspect of the theorem. One would need to

create another branch of integral geometry that dealt specifically with *non-additive* functions that map from convex bodies to real numbers. After searching through the relevant literature, no such branch exists. Because the experimental data shows that the intrinsic volumes from Hadwiger's theorem are useful in non-additive systems, generalizing integral geometry would be exciting. However, it became apparent that it would not feasible for a summer project, so I instead attempted to apply Hadwiger's theorem to physical situations.

iii. Applications of Hadwiger's Theorem

       To illustrate the manner in which physicists use Hadwiger's theorem, I have outlined a general example. The first step in utilizing Hadwiger's Theorem is to find a rigid motion invariant system that is also additive. The beauty of the theorem is that it has the potential to explain complicated situations efficiently and accurately. There are however not many fresh systems that satisfy the three conditions set out by Hadwiger, which was a major problem for my project, as will be discussed later.

       Let us suppose that we have a physical system and valuation that both satisfy Hadwiger's theorem. We choose a valuation in the system that is both rigid-motion invariant and additive. In three dimensions, we can now write the energy of the system as

$$\mu(K)=c_0 V_0+c_1 V_1+c_2 V_2+c_3 V_3, \qquad\qquad [6]$$

where K is a convex body that we wish to study, V_0 is volume, V_1 is surface area, V_2 is boundary curvature, and V_3 is Euler characteristic. The physicist determines the value of the energy experimentally using previously-known Minkowski functionals. From there, one can calculate the c_i's. Once the c_i's are known, equation (6) becomes extremely useful in either determining the energy from the intrinsic volumes, vice-versa, or some combination of the two. With very little calculation, integral geometry allows us to completely determine a given system.

       Specific applications abound and are found readily in current literature in cosmology and statistical mechanics [1, 2, 7]. One such example given by K. R. Mecke in [2] exemplifies how integral geometry can be used to calculate the percolation

thresholds in porous materials. The percolation threshold is the critical volume density of pores that percolates. The goal of integral geometry in terms of analyzing porous materials is to describe macroscopic transport properties, such as the diffusion constant of the material via the morphology of the pores. Thus far, the most successful theory in finding transport properties of objects relies upon the observation that the Euler characteristic goes to zero near the critical threshold. Unlike the two-point correlation function which offers no information about the topological structure of a system, integral geometry yields the Euler characteristic and has thus become a more commonly used in this area. A plethora of other applications exist, and, since I wouldn't be able to generalize Hadwiger's theorem, my advisor and I began looking for systems that would satisfy Hadwiger's theorem and benefit from a simple description dependent on its morphological characteristics.

iv. Finding a system

My advisor and I came up with a variety of ideas for systems that could be explored with integral geometry such as the Ising Model, the torsion pendulum, and the shape of ice crystals in clouds. We were unsuccessful, however, in finding a way to extend integral geometry to these systems. They all violated one of the three requirements (rotational invariance, translational invariance, additivity) to satisfy Hadwiger's theorem. In essence, the beauty and simplicity of Hadwiger's theorem could not be applied to complex systems because complex systems are not often additive and rigid-motion invariant.

The utility of the theorem in other areas led us to believe that it could be useful in situations that have yet to be explored. But the conditions to apply the theorem are so stringent that we could not find a physical system outside of those that have been studied to apply integral geometry. So until we found a system that was Hadwiger-applicable, I had no project to work on. Thus, my advisor suggested that I create a simulated annealing program and that is the research project with which I will finish the summer.

III. Concluding Remarks

I began my project studying the details of integral geometry with the hope that I could either generalize Hadwiger's theorem or simplify a complex system using integral geometry. Neither of those two ideas worked. The first was unreasonably difficult, since I would have had to create a new branch of mathematics that generalized valuations to include non-additive systems. The second idea seemed reasonable, but we could not find a system that satisfied Hadwiger's theorem and was complex enough to study. My advisor therefore suggested that I create a simulated annealing program, which I am in the process of refining and testing.

Works Cited

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