Discrete Scale Invariance and Limit Cycle Physics

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Outline

- Physical System and Scale Invariance
- Singular Potentials and Renormalization
- Symmetry Breaking and Discrete Scale Invariance
- Binding Energies
- Scattering Amplitudes
- Summary and Conclusion

The System

- Attractive central potential – partial wave analysis
- Momentum space representation – integral equations
- Interested in low energy s-wave physics
- Dimensionless units

$$\hbar = 2m = 1 \qquad V(r) = \frac{g}{r^2}$$

Scale Invariance

- Scale invariance refers to the behavior of an observable under a change of scale transformation
- Hallmark of scale invariance is a power law behavior
- Our Hamiltonian is scale invariant

$$\mathcal{O}(\lambda x) = \mu \mathcal{O}(x)$$

$$\mathcal{O}(x) \sim x^{\alpha}$$

$$H\psi(\vec{r}) = E\psi(\vec{r})$$

$$H\psi(\lambda\vec{r}) = \lambda^2 E\psi(\lambda\vec{r})$$

The Problem

- Our potential is singular at the origin – too singular
- Schrodinger equation + normal boundary conditions do not lead to a unique physical solution
- Without additional information, the mathematics could represent an infinite number of physically distinct systems

The Problem

- Physical situation – unknown or complicated physical cutoff mechanism
- We are interested only in the low energy behavior of the system
- Schrodinger equation can give physical low energy results provided we impose a cutoff and “renormalize”

Solution – Momentum Cutoff

- Impose of high momentum cutoff
- Cutoff will regulate the high energy (short distance) divergence of the potential

$$\psi(p) = \int_0^\infty dq K(p,q)\psi(q)$$

$$\psi(p;\Lambda) = \int_0^\Lambda dq K(p,q)\psi(q;\Lambda)$$

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

Renormalization

- Observables at all energy scales are cutoff dependent $K(p,q) \rightarrow K(p,q;\Lambda)$
- Counterterm should make low energy observables cutoff independent
- Counterterm behavior as function of cutoff is called RG Flow

$$\Lambda f(\Lambda) = \frac{1 - 2\nu \tan\left(\nu \ln \Lambda + \beta\right)}{1 + 2\nu \tan\left(\nu \ln \Lambda + \beta\right)}$$

Symmetry Breaking

- The momentum cutoff breaks the continuous scale invariance of the Hamiltonian
- Anomaly – classical symmetry broken by quantization process

$$\psi(p) = \int_0^{\Lambda} dq K(p, q; \Lambda) \psi(q)$$

Discrete Scale Invariance

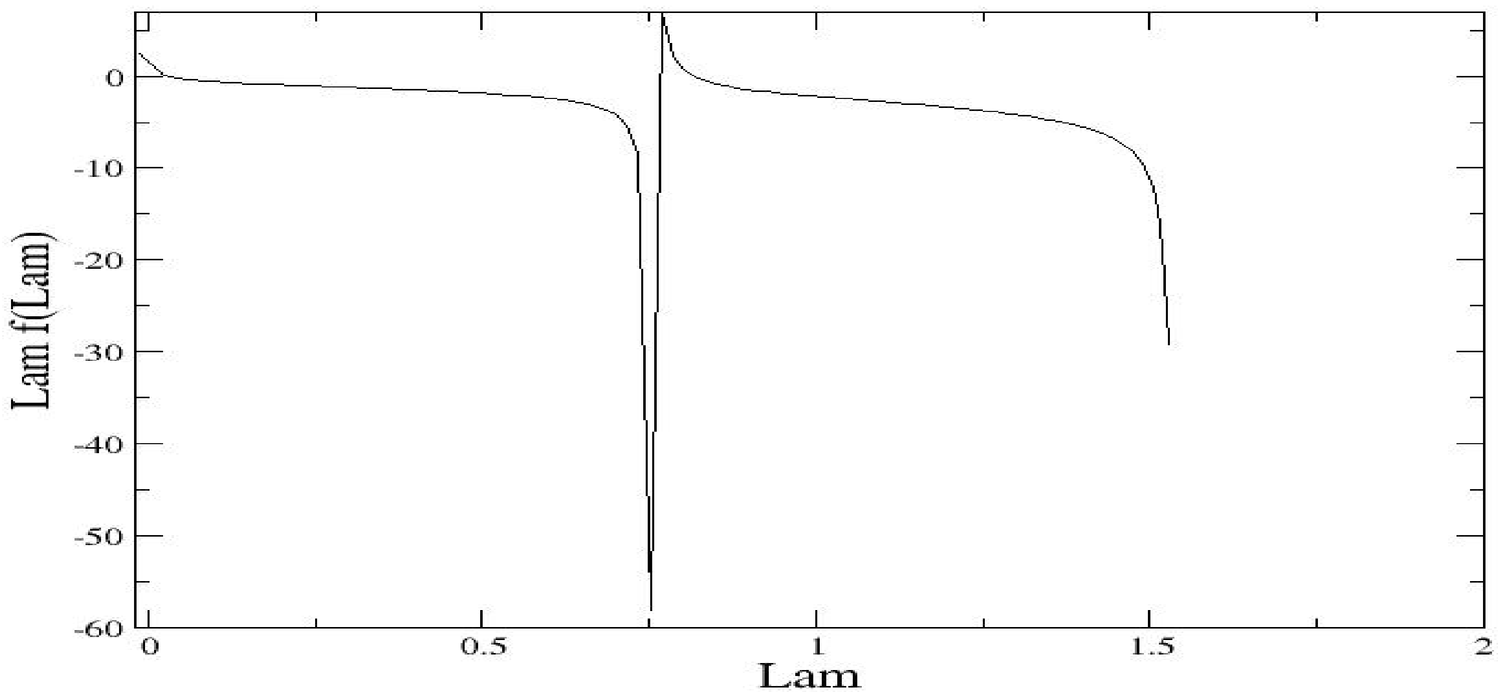
- Scale invariance except only a discrete set of scaling parameters allowed
- Hallmark of DSI is log-periodic behavior

$$\lambda_n = \lambda_0^n$$

$$\cos\left(\ln x\right)$$

Discrete Scale Invariance

- Counterterm is log periodic – RG flow has a limit



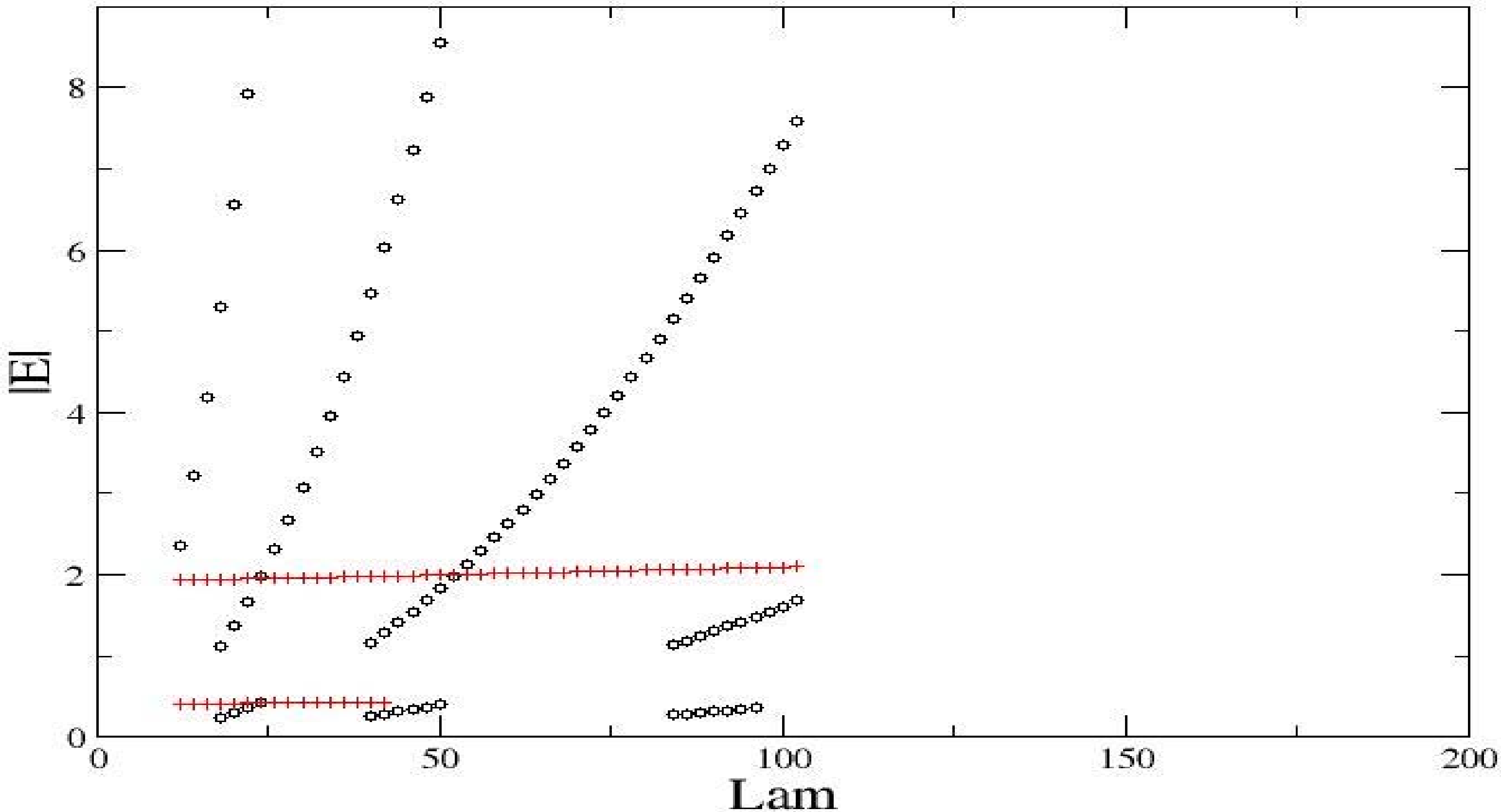
Binding Energies

- Governing equation is integral equation – solve numerically using Gauss-Legendre Quadrature and LU decomposition
- Binding energies are discrete scale invariant – ratio is constant

$$\det\left(1 - wK\right) = 0$$

1.516	2.193
3.113	2.053
6.530	2.098
13.903	2.129
29.914	2.152
64.872	2.169
141.659	2.184

Cutoff Dependence



Scattering Amplitudes

- Experiments measure scattering cross section
- Cross section can be calculated from knowledge of scattering amplitude

$$\sigma(k) \sim |T(k)|^2$$

$$T(k) \sim \frac{1}{k \cot \delta - ik}$$

$$G^{+} = \frac{1}{E - H_0 + i\epsilon}$$

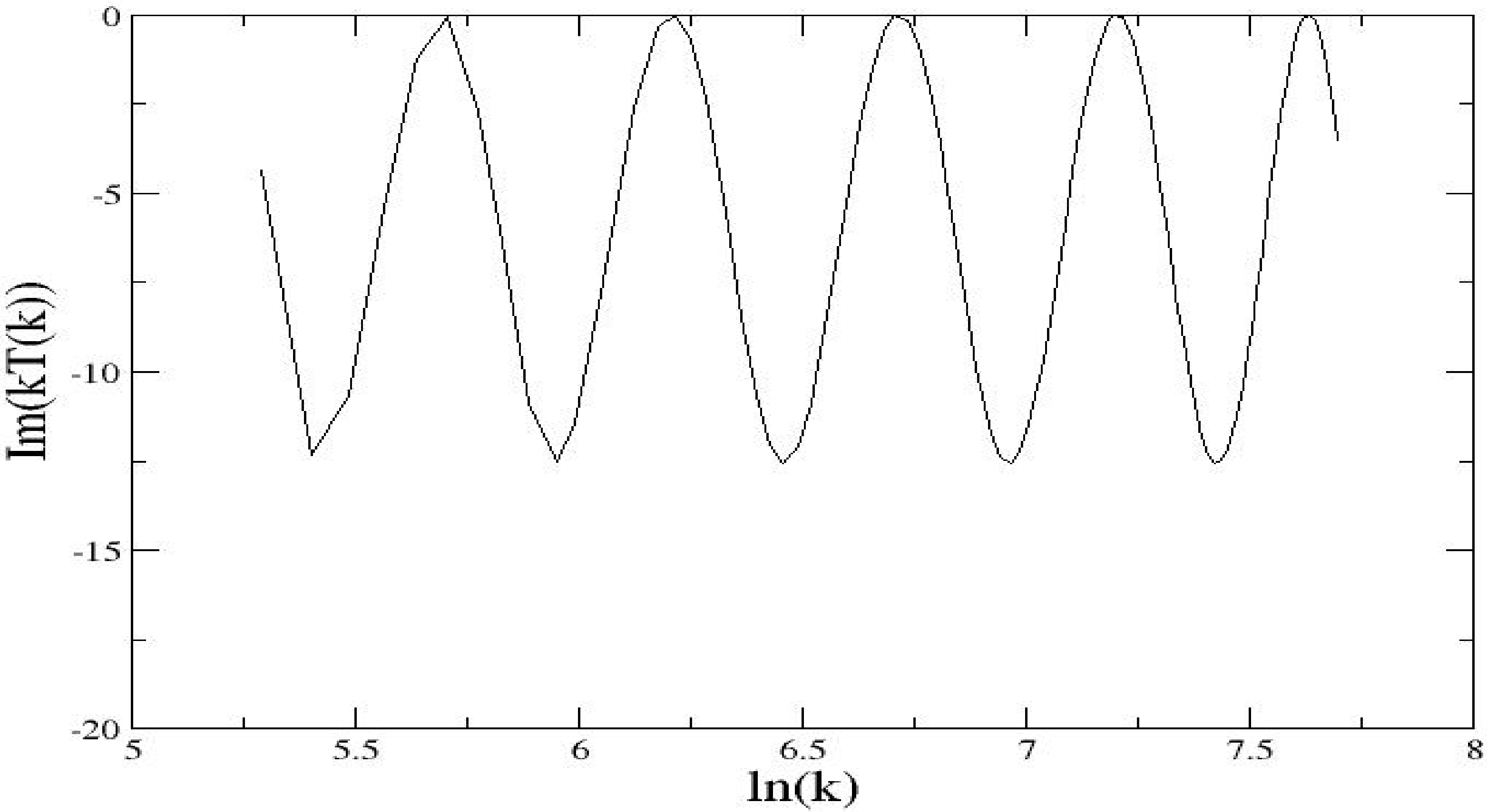
$$T = V + VG^{+}T$$

Scattering Amplitude

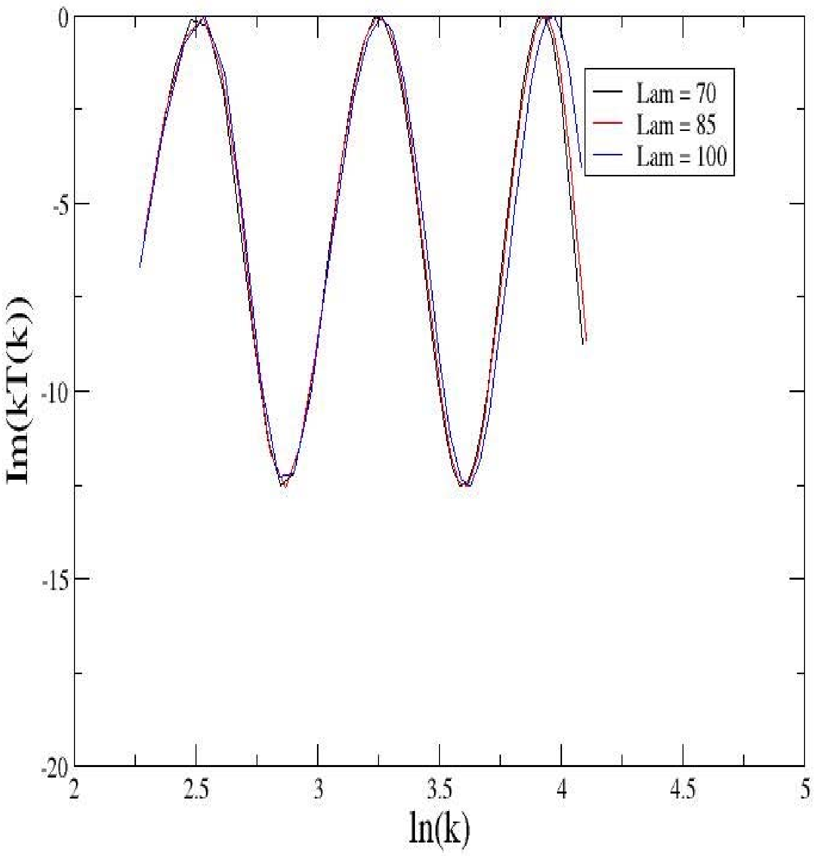
- Use Gauss-Legendre Quadrature and LU decomposition
- Integral kernel now has a singularity at the particles actual energy
- Singularity must be dealt with by implementing the principal value of the integral kernel

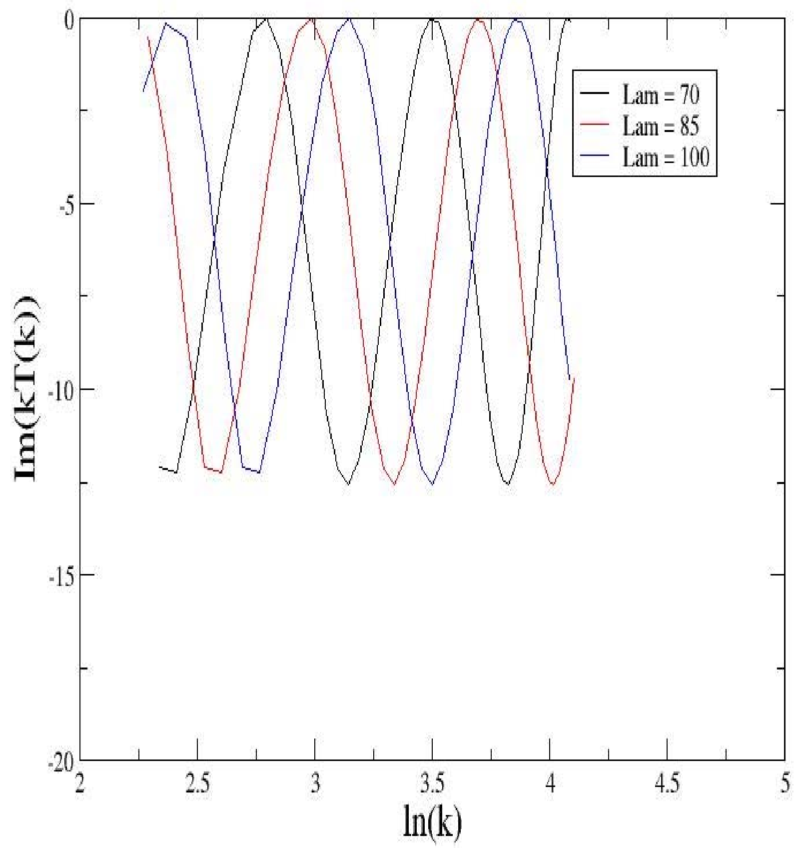
$$\frac{1}{E - H_0 + i\epsilon} = \frac{Pr}{E - H_0} - i\pi\delta(E - H_0)$$

Scattering Results



Scattering Results





Summary and Conclusions

- Pedagogically and physically interesting singular system with scale invariance that is pathological
- Cutoff plus renormalization cures the sickness
- Symmetry braking to discrete scale invariance – RG flow has a limit cycle
- Low energy observables are cutoff independent