# Renormalization, Symmetry Breaking, and Discrete Scale Invariance

Brian Swingle
*University of Washington Physics REU 2004*

## INTRODUCTION

One of the most basic applications of quantum mechanics is the study of a two body system with a spherically symmetric interaction. This problem is characterized simply by giving the mass of each particle as well as the inter-particle potential $V(r)$ which is a function of the inter-particle separation only. This problem can be reduced in a straightforward way to a simpler one particle problem by introducing the reduced mass $m = m_1 m_2/(m_1 + m_2)$. All that remains is then to solve the familiar one particle Schrodinger equation for a particle of mass $m$ in a force field given by $V(r)$ with the appropriate boundary conditions. As we shall see, our cavalier attitude towards this final step is not entirely justified.

We investigate a system interacting via a central potential $V(r) = c/r^2$, our basic goal being to determine the bound state spectrum and the scattering amplitudes of the system. Without giving too much away too soon, let us say that this naive goal is not physically realistic for our potential. It turns out that the $r^{-2}$ potential is a special transition case leading to a large class of so called singular potentials. Singular potentials are so called for their singular behavior near the origin $r = 0$ and one finds that in general these potentials, if not supplemented with additional information, lead to ill defined physical systems.

How does this singularity manifest itself in the physics of the potential $V = g/r^2$? When we put our potential into the Schrodinger equation and turn the mathematical crank, we find two linearly independent solutions as we would expect. However, we find that for a sufficiently attractive potential the usual boundary conditions (vanishing at infinity and regularity at zero) do not single out a unique physical solution. This means that without further information, the Schrodinger equation and the usual boundary conditions do not suffice to specify the physical properties of the $r^{-2}$ interaction.

In a true physical system, the $r = 0$ behavior of the Schrodinger equation will certainly not be valid. Since a particle must in general have momentum inversely proportional to $a$ to sense features of the potential on the order of $a$ (think de Broglie waves), at the very least relativistic effects will eventually be important as the particle begins to sense $r = 0$. This limit is actually much exaggerated, the $r^{-2}$ behavior of the potential is modified by any number of other effect in a true physical system. In general we find different physical systems will probably have quite different physical mechanism for ameliorating the $r = 0$ behavior of the potential.

Having seen that the $r = 0$ behavior of the potential is connected to high momentum (and thus high energy) modes, we might expect that low energy observables, low compared to the energy scale of new physics, might be independent of the specific cutoff mechanisms of different physical systems. In other words, we would expect physically a kind of universal behavior, at least among the low energy observables, for systems interacting via a $r^{-2}$ tail. We may thus calculate low energy observables using any particular cutoff we like, probably a computationally convenient one, so long as it regulates the $r = 0$ behavior of the potential or equivalently the high momentum modes.

The final piece of the puzzle amounts to erasing the cutoff dependence of low energy observables. Since naively imposing a momentum cutoff affects observables at all energy scales we turn to the machinery of renormalization theory to keep our low energy observables independent of the cutoff. In addition to the cutoff, renormalization theory demands that we also supply a counterterm within our governing equation that is a function of the cutoff. We choose the functional dependance of the counterterm such that the low energy observables of the theory remain fixed as we vary the cutoff.

The resulting theory is mathematically well defined because of the high momentum cutoff. The low energy observables calculated using our modified Schrodinger equation will be, by construction, independent of the cutoff. These low energy observables will be universal to all systems interacting via an $r^{-2}$ tail in the sense that they do not depend on the details of the cutoff mechanism but merely on the presence of such a mechanism.

## THE SYSTEM

An observable $\mathcal{O}(x)$ depending on a parameter $x$ is said to be scale invariant if

$$\mathcal{O}(\lambda x) = \mu \mathcal{O}(x) \quad (1)$$

The general solution to such an equation is a power law $\mathcal{O}(x) \sim x^\alpha$ where $\alpha = \log \mu / \log \lambda$. For a scale invariant system, the scaling factor $\lambda$ is arbitrary so that the system is invariant under any change of scale.

It may happen that a system is scale invariant only for a discrete set of scaling ratios $\lambda_n$. Such a system is no longer scale invariant, but rather discrete scale invariant. It is obvious that discrete scale invariance is a sub-symmetry of full scale invariance. The set of preferred scaling ratios $\lambda_n$ have the form $\lambda_n = \lambda_0^n$ where $\lambda_0$ is called the fundamental scaling ratio.

The Hamiltonian for a quantum mechanical particle in the presence of a $r^{-2}$ potential is given by

$$H = \frac{p^2}{2m} + \frac{g}{r^2} \tag{2}$$

and is scale invariant for all values of $g$. Concretely, if $\psi(\vec{r})$ is solution to the time independent Schrodinger equation with energy $E$ then $\psi(\lambda\vec{r})$ is a solution with energy $\lambda^2 E$. The existence of single solution allows an infinite family of other solutions to be constructed. For the $g = 0$ case in particular (a free particle), the continuous spectrum is well known.

We shall find that in the process of repairing the $r = 0$ singularity the scale invariance of the Hamiltonian is broken. This is a simple example of an anomaly, a classical symmetry that does not survive the process of quantization. To be precise, the full scale invariance is broken to discrete scale invariance with a fundamental scaling ratio depending only on the strength of the potential. The essential characteristics of discrete scale invariance are of course the discrete scaling and log periodic (like $\cos\log x$) behavior of observables.

We will be analyzing the quantum mechanics of this Hamiltonian in the momentum space representation. In this representation, the cutoff will be achieved by simply excluding modes with momentum greater than some value $\Lambda$. The renormalization counterterm will take the physical form of an additional short range potential. Of course the spherical symmetry of our problem means angular momentum is conserved, and in particular, allows a partial wave analysis to be carried through. The inherently low energy character of interesting observables means we should be interested in s-wave phenomenon.

## THEORY

We consider a particle of mass $m$ moving in a potential given by $V(r) = g/r^2$ and to simplify the analysis we use units so that $\hbar = 2m = 1$. If we are later interested in restoring the mass and $\hbar$ to our equations, we may do so by replacing $g$ with $2mg/\hbar^2$. We also make the definition $|E| = k^2$ with the sign of $E$ depending on whether we are considering bound states or scattering. The Schrodinger now reads

$$\pm k^2|\psi\rangle = \left[p^2 + \frac{g}{r^2}\right]|\psi\rangle \tag{3}$$

We are interested in the momentum space representation of the Schrodinger equation and so we adopt the convention

$$\langle\vec{x}|\vec{p}\rangle = (2\pi)^{-3/2}\exp\left(i\vec{p}\cdot\vec{x}\right) \tag{4}$$

so that with this convention the matrix elements of the potential are

$$\langle\vec{p}|V|\vec{q}\rangle = \frac{g}{4\pi}\frac{1}{|\vec{p}-\vec{q}|} \tag{5}$$

### Bound States

To study the bound state spectrum we set $E = -k^2$ so that in the momentum space representation the Schrodinger equation is given by

$$-k^2\psi(\vec{p}) = p^2\psi(\vec{p}) + \int d^3q\,\frac{g}{4\pi}\frac{1}{|\vec{p}-\vec{q}|}\psi(\vec{q}) \tag{6}$$

We are interested in s-waves so we assume that $\psi$ depends only on the magnitude of $\vec{p}$. Performing the angular integration and rewriting we find

$$-(k^2+p^2)\psi(p) = g\int_0^\infty dq\,q^2\left(\frac{\theta(p-q)}{p} + \frac{\theta(q-p)}{q}\right)\psi(q) \tag{7}$$

Finally, making the substitution $(k^2+p^2)\psi(p) = \phi(p)$ we can write our equation in its final form

$$\phi(p) = -g\int_0^\infty dq\frac{q^2}{k^2+q^2}\left(\frac{\theta(p-q)}{p} + \frac{\theta(q-p)}{q}\right)\phi(q) \tag{8}$$

For zero energy, the equation may be solved analytically. Because the equation is scale invariant at zero energy, we know that if $\phi(p)$ is a solution then so is $\phi(\lambda p)$. As with any scale invariant system, we make the ansatz $\phi(p) = p^\alpha$. Plugging into the above equation with $k = 0$ and integrating we find

$$p^\alpha = -g\left[\frac{p^\alpha}{\alpha+1} - \frac{p^\alpha}{\alpha}\right] \tag{9}$$

and after a bit of manipulation, $\alpha$ is determined by the quadratic equation

$$\alpha(\alpha+1) - g = 0 \tag{10}$$

with solutions given by

$$\alpha_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + g} \tag{11}$$

We find that below the critical coupling $g_c = -1/4$, both exponents $\alpha_\pm$ become complex, so that above this critical coupling constant the system is well determined while below the critical coupling constant the system is ill defined. Setting $g = -(1/4 + \nu^2)$ we see that both exponents $\alpha_\pm$ lead to valid solutions in the presence of the standard boundary conditions. This means that any linear combination of $p^{\alpha_\pm}$ is a solution of the Schrodinger equation with boundary conditions. However, this ambiguity is critically important because the bound state spectrum actually depends on the phase relationship between the two solutions.

As already discussed our solution is to cutoff the system at some momentum $\Lambda$, a procedure which is equivalent to altering the short distance $r = 0$ behavior of the potential. This achieved with our equations above by simply replacing $\infty$ with $\Lambda$ in the upper limit of integration. Having cutoff the system, we must now subtract out the cutoff dependence of the low energy observables.

The renormalization is accomplished by altering the integral kernel in Eq. (8); we add to the present kernel a constant counterterm that depends on $\Lambda$. Our new working equation is

$$\phi(p) = -g \int_0^\Lambda dq \frac{q^2}{k^2+q^2} \left( \frac{\theta(p-q)}{p} + \frac{\theta(q-p)}{q} + f(\Lambda) \right) \phi(q) \tag{12}$$

Our goal will now be to choose $f(\Lambda)$ in such a way that low energy observables ($E < \Lambda^2$) are approximately cutoff independent.

We parameterize our physical $k = 0$ solution in terms of the parameter $\beta$ as

$$\phi(p;\beta) = \mathcal{N} \left( e^{i\beta} p^{\alpha_+} + e^{-i\beta} p^{\alpha_-} \right) \tag{13}$$

thus $\beta$ gives directly the phase relationship between our two independent solutions. Specification of the physical system will amount to specification of $\beta$.

By requiring that $\phi(p;\beta)$ satisfy Eqn. (12) we find, after a modest bit of algebra, the following for the counterterm

$$f(\Lambda) = \frac{1}{\Lambda} \frac{(1 + 2\nu \tan(\nu \ln \Lambda + \beta))}{(1 - 2\nu \tan(\nu \ln \Lambda + \beta))} \tag{14}$$

We notice immediately that the dimensionless coupling constant $F(\Lambda) = \Lambda f(\Lambda)$ is a log-periodic function of the cutoff $\Lambda$, an early sign of discrete scale invariance. Fig. (1) is a graph of the $\Lambda$ dependence of $f(\Lambda)$, and in the language of renormalization theory we call such a graph the Renormalization Group Flow. We now want to apply our results to the case $k \neq 0$ and study the cutoff dependence of observables.

While exact solutions for the Schrodinger equation in our problem exist, they are not particularly physically illuminating. Instead, we solve our fundamental equation numerically. The procedure is first to approximate the integral equation as a finite set of linear equations. This is done by discretizing the interval of integration and approximating the integral using Gauss-Legendre quadrature. This yields the following schematic system of linear equations

$$\phi(p_i) = \sum_j K(p_i, p_j) w_j \phi(p_j) \tag{15}$$

where the $w_j$ are the weights generated by Gauss-Legendre quadrature.

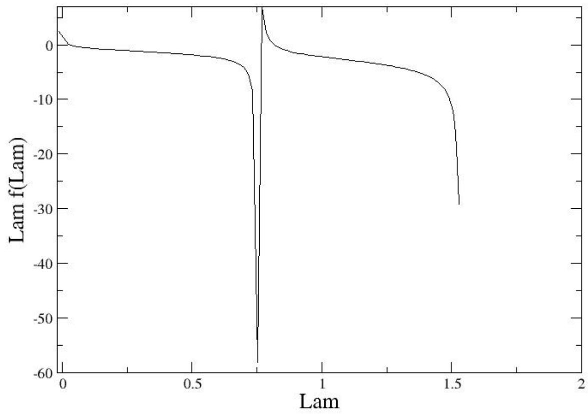


FIG. 1: Renormalization Counter Term

In the present case, we are interested not so much in the bound states themselves but rather their energies. In other words, we want to know for what $k$ values Eq. (12) has a solution. Since the system is homogenous, the answer, from elementary linear algebra, is that the determinant of the coefficient matrix must vanish. Since the kernel $K$ depends on $k$, the equation

$$\det(1 - wK) = 0 \tag{16}$$

will determine the allowed $k$ values, the bound state spectrum. When considering results, we will be interested primarily in the cutoff dependence of the observables as well as remnants of the original scale invariance.

**Scattering**

In the scattering problem we set the energy to be positive $E = k^2 > 0$ in contrast to the bound state case. One of the primary quantities of interest to the experimentalist is the scattering cross section. Theoretically, the cross section is calculated from the on shell scattering amplitude which we will denote $T(k)$. We would thus like to calculate scattering amplitudes for our physical system and study their cutoff dependence.

The on shell scattering amplitude is a special case of the more general (but unphysical) off shell scattering amplitude. The off shell scattering amplitude is a function of two momenta $T(p, k)$ but due to energy conservation a physical process must have $p = k$. As an aside, our interest in s-wave phenomenon enables us to disregard the angular dependence of the scattering, or more formally, we are calculating only the $\ell = 0$ partial wave contribution. The on shell scattering amplitude is given by $T(k) = T(k,k)$, the "diagonal elements". Nevertheless, the full off shell amplitude enters in calculations of the on shell amplitude and so cannot be neglected.

The equation obeyed by the scattering amplitude is not the actual Schrodinger equation but a derived equation

$$T = V + VG^{+}T \tag{17}$$

where $G^{+} = 1/(E - H_0 + i\epsilon)$. When written in the momentum space representation, we again obtain an integral equation as before except this time the equation is inhomogeneous and is soluble for all $k$.

Our basic equation takes the form

$$T(p,k) = V(p,k) + \int_0^{\Lambda} dq q^2 V(p,q) \frac{1}{k^2 - q^2 + i\epsilon} T(q,k) \tag{18}$$

where $V(p,q)$ represents the fourier transform the potential, integrated over angles. The $+i\epsilon$ is our pole prescription, telling how we navigate around the singularity in the kernel at $q = k$. In fact, this new singularity, not present in the bound state case, will have to be dealt with carefully.

In order to treat the singularity, we use the standard interpretation of $G^{+}$ as

$$\frac{1}{k^2 - q^2 + i\epsilon} = \frac{Pr}{k^2 - q^2} - i\pi\delta(k^2 - q^2) \tag{19}$$

where the $Pr$ denotes the principal value. We implement the principal value in the most direct way, by insuring that our grid always places $k$ symmetrically between the two nearby grid points. Since the on shell scattering amplitude is the interesting quantity, we add an extra point to our grid representing $k$, but this point is treated asymmetrically in the sense that it is not part of the Gauss-Legendre points used to evaluate the integral in our equation.

The numerics proceed as before, using Gauss-Legendre quadrature we reduce the integral equation to a set of coupled linear equations. The structural difference between the present case and the bound state problem is that we are now solving a inhomogeneous system. Unlike in the bound state case, we have solutions for all energy, and we would like to know the actual amplitude. The solution we obtain is the the full off shell amplitude $T(p,k)$ evaluated at the various grid points. The physical object is the on shell amplitude $T(k,k)$, and as in the bound state case we shall primarily be interested in the cutoff behavior of the amplitude as well as remnants of scale invariance.

TABLE I: Binding Energies

| $\nu = 8$ | $E_{n+1}/E_n$ | $\exp(2\pi/\nu)$ |
|---|---|---|
| $\|E\| = 1.516$ | 2.193 | 2.193 |
| $\|E\| = 3.113$ | 2.053 | |
| $\|E\| = 6.530$ | 2.098 | |
| $\|E\| = 13.903$ | 2.129 | |
| $\|E\| = 29.914$ | 2.152 | |
| $\|E\| = 64.872$ | 2.169 | |
| $\|E\| = 141.659$ | 2.184 | |

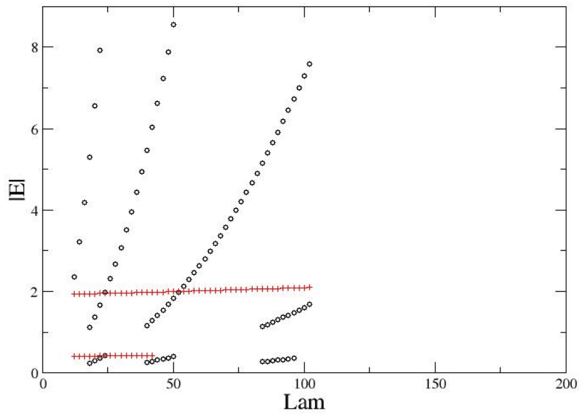


FIG. 2: Renormalized (Red), Not Renormalized (Black)

## RESULTS

### Bound State Energies

Some computed bound state energies for $\nu = 8$ are listed in Table 1, and in the same table we list the ratio of two adjacent binding energies. Notice how the ratio is approximately constant, this is a consequence of the discrete scale invariance. The theoretical scaling ratio is $\exp(2\pi/\nu)$, a value which is in good agreement with the actual calculated ratios. This is strong evidence that although the system was initially scale invariant, that symmetry has been broken to discrete scale invariance.

Instead of a continuum of bound states we have only those energies that satisfy the discrete scaling law $E = \lambda_E^n E_0$. Of course this scaling law does not determine the multiplicative constant, a number which actually depends on $\beta$. We have argued that different values of $\beta$

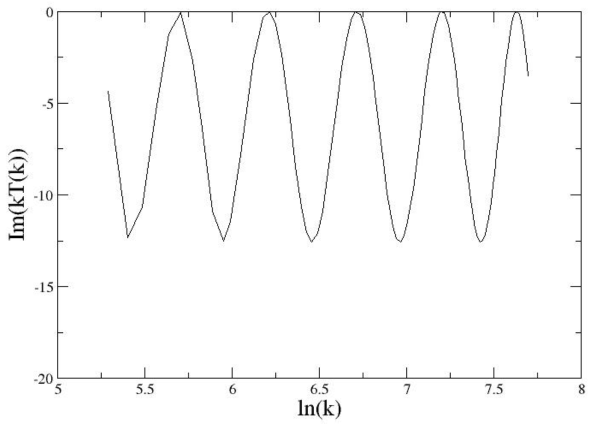


FIG. 3: On Shell Scattering Amplitude

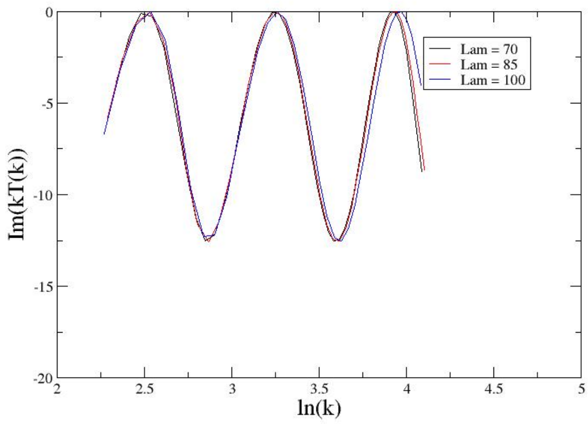


FIG. 4: Scattering Amplitude (Renormalized)

correspond to different physical systems, and here we see that freedom displayed explicitly in the observables of the theory. Physically, an experimentalist specifies $\beta$ by giving the theorist a single measured bound state energy from which the theory predicts the value of other bound state energies, at least those that are sufficiently shallow.

The limits on our bound state calculations are both numerical and physical. The physical limit comes from the cutoff $\Lambda$, since we explicitly thrown away all modes with momentum $p > \Lambda$ we do not expect to find a bound state any deeper than $|E| = \Lambda^2$. The second limit is numerical, the energy scaling law makes it clear that there is an accumulation point of bound states at $E = 0$ however with only a finite number of calculations we can only find a finite number of these infinite low energy bound states. With these limits in mind, the remaining question is whether our renormalization procedure has left the low energy observables cutoff independent.

Fig. (2) contains two sets of bound state energies plotted as functions of the cutoff $\Lambda$ with and without the renormalization included. The difference between the two data sets is dramatic, with the renormalization in place the energies small compared to $\Lambda^2$ are essentially cutoff independent where as if we exclude the counterterm a marked cutoff dependence is present for all energies.

### Scattering Amplitudes

Having seen the discrete scale invariance manifest itself in the bound state spectrum we might be curious what effects are present in the scattering problem. Fig. (3) is a plot of the on shell scattering amplitude $kT(k,k)$ plotted against $\log k$. The scattering amplitude is obviously periodic in $\log k$ which is a hallmark of discrete scale invariance, the log-periodicity comes from the complex exponents associated with the discrete scaling ratio. Note also that in accord with the basic correspondence $E\ k^2$ the scaling ratio we found for $E$ in the bound state spectrum is the square of the scaling ratio for $k$.

The other question we must ask is whether the scattering amplitude we calculate depends on the cutoff for $k \ll \Lambda$. Figs. (4) and (5) show plots of $T(k)$ for different values of $\Lambda$ with and without the counterterm present. Just as in the bound state case, the difference between the two cases is marked. In first plot, the scattering amplitudes for different values of $\Lambda$ almost overlap until the argument $k$ approaches $\Lambda$, where as in the second plot the scattering amplitudes change dramatically as a function of $\Lambda$ for all $k$.

## CONCLUSION

We have seen how a momentum cutoff leads to a physically well defined bound state spectrum and scattering problem. The full scale invariance of the Hamiltonian is broken to a discrete scale invariance with fundamental scaling ratio $\exp(\pi/\nu)$. Both the bound state spectrum and the scattering amplitude show behavior characteristic of discrete scale invariance, discrete scaling and log periodicity respectively. The renormalization successfully renders the low energy observables effectively cutoff independent.

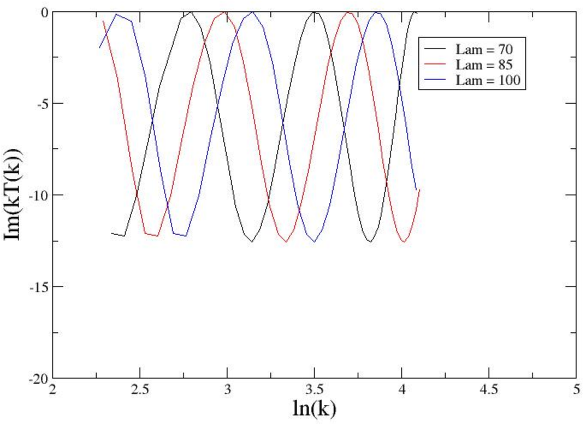


FIG. 5: Scattering Amplitude (Not Renormalized)

In addition to successfully calculating cutoff independent low energy observables, we have also found a simple example of discrete scale invariance. The system is pedagogically interesting in that one can go explicitly from the full scale invariance through the symmetry breaking to the final discrete scale invariance. The discrete scale invariance is indicative of a limit cycle in the Renormalization Group Flow signaled by the log periodicity of the dimensionless coupling $F(\Lambda)$. This system is thus a very simple example of an anomaly as well as an exactly soluble example of discrete scale invariance.