Analytic Parameterizations of Parton Distribution Functions

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Abstract

Proton parton distribution functions are necessary to calculate cross sections for interactions at pp and $p\bar{p}$ colliders. Since they are developed from experimental particle physics data, they have no natural analytic form. Here we present analytic parameterizations of proton parton distribution functions that are based upon those produced by CTEQ5. With these parameterizations we gain a better understanding of the behavior of parton distributions and provide a tool to quickly and accurately estimate cross sections. We then perform a calculation of the cross section for $gg \to t\bar{t}$ within $p\bar{p}$ collisions, both with our analytic parameterizations and with CTEQ distributions, and show that the results are nearly the same.

1 Introduction

A parton distribution function (*pdf*) gives the number of quarks and gluons inside of a hadron. Here we study *pdf*s for partons within protons and antiprotons [1]. Parton distributions are functions of three variables, so we will write them as *pdf* (i, x, μ). The parton index i corresponds to a gluon or the flavor of quark or antiquark. We will use the convention:

i	parton
1	up quark
2	down quark
0	gluon
-1	up antiquark
-2	down antiquark

The variable x is the fraction of the proton's momentum that is carried by the parton, so *pdf*$(i, x, \mu)dx$ is the total number of partons of index i with momentum fraction between x and $x + dx$. The parameter μ is the factorization scale.

[1] The *pdf*s for antiprotons can be found from the proton *pdf*s via the conjugation of quark and antiquark distributions.

Proton *pdf*s are necessary to calculate cross sections for interactions at pp and $p\bar{p}$ colliders. The total cross section for a particular interaction $p\bar{p} \to X$ is of the form

$$\sigma_{p\bar{p}\to X} = \sum_{i,j} \int_0^1 \int_0^1 dx_1 dx_2 pdf_1\left(i, x_1, \mu\right) pdf_2\left(j, x_2, \mu\right) \hat{\sigma}_{ij\to X}\left(z, \mu\right), \tag{1}$$

where $\hat{\sigma}_{ij\to X}$ is the partonic cross section and we have summed over all necessary combinations of partons i and j. If we change variables to $z = x_1 x_2$, this becomes

$$\sigma_{p\bar{p}\to X} = \sum_{i,j} \int_0^1 dz \int_z^1 \frac{dx}{x} pdf_1\left(i, x, \mu\right) pdf_2\left(j, \frac{z}{x}, \mu\right) \hat{\sigma}_{ij\to X}\left(z, \mu\right), \tag{2}$$

where the luminosity

$$\mathcal{L}\left(z\right) = \int_z^1 \frac{dx}{x} pdf_1\left(i, x, \mu\right) pdf_2\left(j, \frac{z}{x}, \mu\right), \tag{3}$$

is the convolution of the *pdf*s of the colliding hadrons.

The particular *pdf*s studied here are those developed by the Coordinated Theoretical-Experimental Project on QCD (CTEQ). A few of these distributions are shown in Figure 1. Since parton dis-

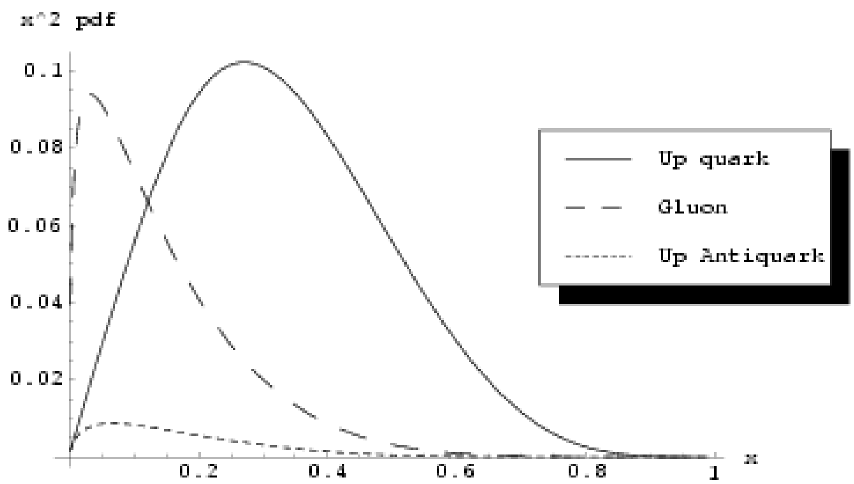

Figure 1: CTEQ5 parton distribution functions

tribution functions are determined by fitting to data from particle physics experiments, they have no natural analytic form. It is therefore necessary to look up and numerically evaluate these distributions each time a cross section calculation or estimate is to be made. In order to expedite this process and to be able to perform analytic calculations, we develop analytic parameterizations of CTEQ distributions at leading order in perturbation theory. With these functions we gain a better understanding of the behavior of parton distributions and the ability to quickly and accurately estimate cross sections for various interactions.

2 Results

Our *pdf* parameterizations are found to be adequate representations of CTEQ5 [1] distributions when of the form

$$fit\left(x\right) = \frac{1}{n} x^{-b} \left(1 - x\right)^a, \tag{4}$$

where a, b, and n are parameters unique to each parton at a given factorization scale. In order to obtain accurate representations of the CTEQ *pdf*s while maintaining this simple form, it is necessary to fit over two regions of x. For each parton at a given μ, we have a low x parameterization that is accurate in the region $.001 \leq x \leq .5$, and a high x parameterization that is accurate in the region $.05 \leq x \leq .99$ [2]. The parameter values corresponding to partons at a factorization scale of $\mu = 175GeV$ for low x are shown in Table 1. For high x at $\mu = 175GeV$ these parameters are shown in Table 2.

parton	a	b	n
1	2.85	1.02	1.24
2	3.46	1.19	3.08
0	6.06	1.77	4.28
-1	6.26	1.50	19.46
-2	7.93	1.33	7.71

Table 1: Low x for $\mu = 175GeV$.

parton	a	b	n
1	4.11	0.52	0.38
2	5.08	0.65	0.79
0	5.81	1.79	4.71
-1	6.69	1.48	17.0
-2	6.75	1.88	27.6

Table 2: High x for $\mu = 175GeV$.

Plots of CTEQ5 *pdf*s at $\mu = 175GeV$ and our parameterizations of those functions (high x is shown in both) can be seen in Figures 2 and 3.

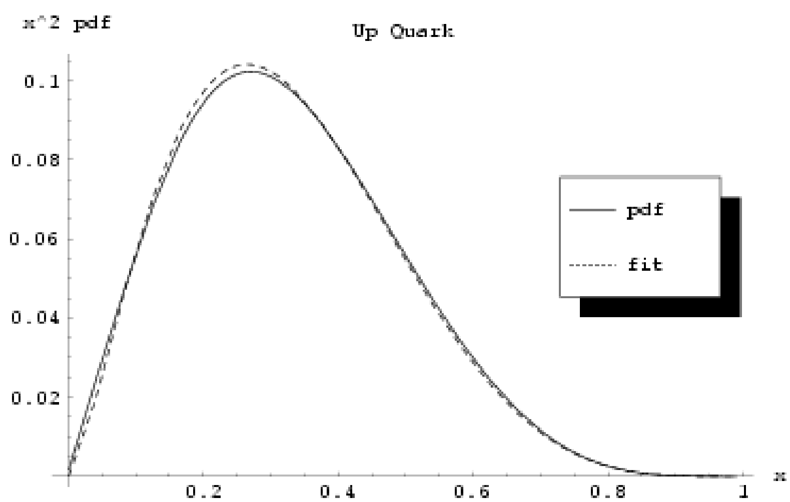

Figure 2: Up quark *pdf* and high x fit.

[2]For the antiquark distributions, the high x parameterizations are only accurate up to $x = .90$. This has negligible effects on the calculation of cross sections because there are almost no sea quarks at a momentum fraction above this value.

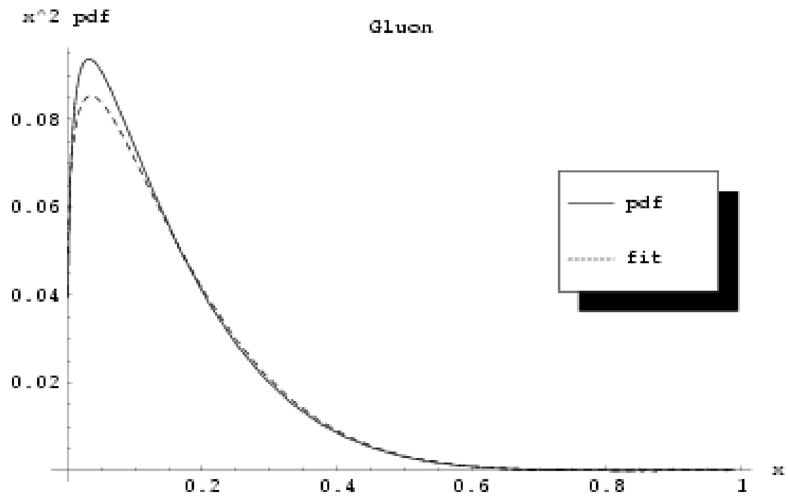

Figure 3: Gluon *pdf* and high x fit.

Since *pdf*s evolve in shape with changing factorization scale, these parameterizations produce accurate cross section calculations only when the processes involved require a scale of approximately $175GeV$. In order to be able to make accurate calculations over a larger range of μ, parameterizations of CTEQ5 *pdf*s at $\mu = 500GeV$ are found as well. These have the same form as before, and their values for low and high x are shown in Tables 3 and 4.

parton	a	b	n
1	1.76	1.16	2.19
2	3.61	1.22	3.37
0	6.52	1.78	4.49
-1	6.40	1.54	22.0
-2	7.76	1.40	9.96

Table 3: Low x for $\mu = 500GeV$.

parton	a	b	n
1	4.21	0.57	0.43
2	5.17	0.70	0.91
0	5.94	1.82	5.53
-1	6.95	1.44	16.2
-2	7.11	1.78	23.4

Table 4: High x for $\mu = 500GeV$.

An evaluation of the accuracy of these parameterizations and examples of their use can be found in the next section.

3 Calculations

Due to the simple form of the fit functions, their use in the calculation of luminosities and cross sections is straightforward. The luminosity, given by Equation 3, becomes

$$\begin{aligned}
\mathcal{L}(z) &= \int_z^1 \frac{dx}{x}\frac{1}{n_1}x^{-b_1}(1-x)^{a_1}\,\frac{1}{n_2}\left(\frac{z}{x}\right)^{-b_2}\left(1-\left(\frac{z}{x}\right)\right)^{a_2} \\
&= \int_z^1 \frac{dx}{z^{b_2}n_1n_2}x^{(b_2-b_1-a_1-1)}(1-x)^{a_1}(x-z)^{a_2} \\
&= \frac{(1-z)^{1+a_1+a_2}}{z^{(b_1+a_2+1-b_2)}}\frac{\Gamma(1+a_1)\Gamma(1+a_2)}{\Gamma(2+a_1+a_2)}F\left(1+a_2, 1+a_2+b_1-b_2; 2+a_1+a_2; \frac{z-1}{z}\right),
\end{aligned}$$

where F is just a hypergeometric function, whose values are known. This simple analytic form of the luminosity is a nice feature of these parameterizations that allows for quick and accurate estimations.

3.1 Cross section calculation

We will now perform a calculation of the cross section for $gg \to t\bar{t}$ within $p\bar{p}$ collisions both with the CTEQ5 distributions and with our parameterizations in order to evaluate their accuracy. The partonic cross section for this interaction is

$$\hat{\sigma}_{gg\to t\bar{t}}(z,\mu) = \frac{\alpha_s^2}{m_t^2}\frac{\pi\beta\rho}{24NV}\left[3\eta(\beta)\left(\rho^2+2V(\rho+1)\right)+2(V-2)(1+\rho)+\rho\left(6\rho-N^2\right)\right], \quad (5)$$

where

$$\begin{aligned}
\eta(\beta) &= \frac{1}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-2, \\
\beta &= \sqrt{1-\rho}, \\
\rho &= \frac{4m_t^2}{\hat{s}},
\end{aligned} \quad (6)$$

m_t is the mass of the top quark, $\hat{s} = sz$ is the partonic center of mass energy squared and $V = N^2 - 1 = 8$ is the dimension of the gauge group, in this case $SU(3)$ [2]. We take $m_t = 175GeV$ and to calculate an interaction at the Tevatron, we take $s = (1.96TeV)^2$. The part of the total cross section due to gg annihilation is then

$$\sigma_{gg\to t\bar{t}} = \int_{\frac{4m_t^2}{s}}^1 \mathcal{L}(z)\,\hat{\sigma}_{gg\to t\bar{t}}(z,\mu)\,dz. \quad (7)$$

Calculating this cross section with CTEQ5 gluon distributions at $\mu = 175GeV$ in the luminosity function, we get $\sigma_{gg\to t\bar{t}} = 0.381pb$. Since we are integrating over a region of x where our high x fits are accurate, we use our high x gluon distribution parameterization at $\mu = 175GeV$ in the luminosity function to calculate a cross section of $\sigma_{gg\to t\bar{t}} = 0.396pb$. The fractional difference between the cross section for $gg \to t\bar{t}$ using CTEQ5 distributions and our parameterizations is only 3.9%. This accuracy is typical of the fit functions for the other parton distributions as well.

4 Conclusions

By finding analytic parameterizations of parton distribution functions, we now have a tool to quickly and accurately estimate cross sections without having to return each time to the CTEQ distributions. Since the functions are of a simple form, we obtain an analytic representation of the luminosity in terms of known functions. As indicated in the example calculation of the cross section for $gg \to t\bar{t}$, the parameterizations are sufficiently accurate to estimate the cross sections of various interactions, and can be used easily.

5 Acknowledgements

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[2] M. Mangano, Two Lectures on Heavy Quark Production in Hadronic Collisions, hep-ph/9711337 (1997).