# Relativistic Electron Scattering off Nucleons

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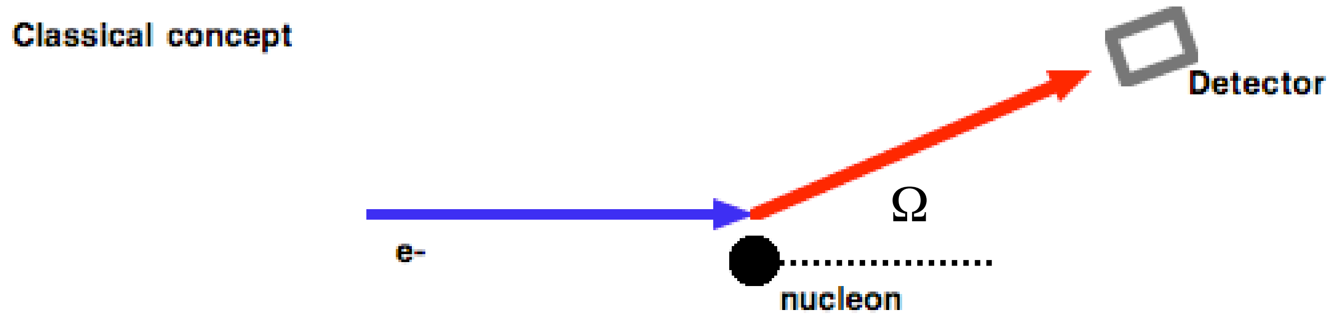
# Outline

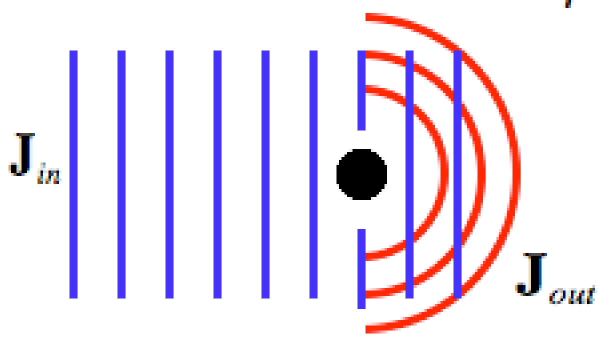
- Nonrelativistic scattering theory
- Relativistic scattering off nucleons
- The problem and our approach
- Results

# Electron Scattering




Quantum concept

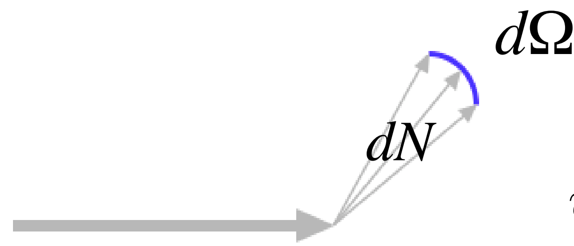
$$\Psi = \psi_{in} + \psi_{out} = Ae^{i\mathbf{k}_i \cdot \mathbf{r}} + A(\Omega)\frac{e^{ikr}}{r}$$


$$\mathbf{J}_{in} = \frac{h\mathbf{k}}{mA^2}\ , \quad \mathbf{J}_{out} = \frac{h}{2miA^2}(\psi^*\nabla\psi - \psi\nabla\psi^*) \approx \frac{h\mathbf{k}}{A^2 m r^2}\left|f(\Omega)\right|^2$$

# The Differential Scattering Cross Section

$$J_{in} = \frac{\hbar k}{mA^2}$$




$$\psi_{scattered} = f(\Omega)\frac{e^{ikr}}{r}$$

$$dN = J_{in}\frac{d\sigma}{d\Omega}d\Omega$$

where $\frac{d\sigma}{d\Omega}$ is the differential cross section

With the current densities from earlier, the differential cross section can be shown to be

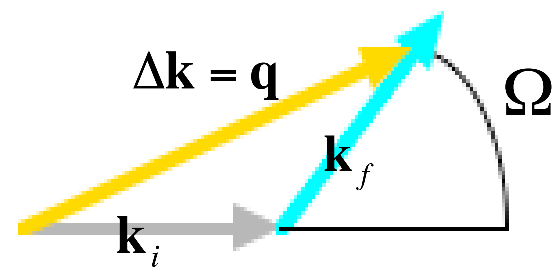
$$\boxed{\frac{d\sigma}{d\Omega} = \left|f(\Omega)\right|^2}$$

# The (nonrelativistic) Form Factor

Let $|\Psi_i\rangle$ and $|\Psi_f\rangle$ denote the states of the initial and final states of the e$^-$/nucleon system "before" and "after" interaction

$$\langle \mathbf{r}_e, \mathbf{r}_p | \Psi_i \rangle = e^{i\mathbf{k}_i \bullet \mathbf{r}_e} \psi_i(\mathbf{r}_n)$$

$$\langle \mathbf{r}_e, \mathbf{r}_p | \Psi_f \rangle = e^{i\mathbf{k}_f \bullet \mathbf{r}_e} \psi_f(\mathbf{r}_n)$$




Some math→

$$\langle \Psi_f | V^{\text{int}} | \Psi_i \rangle = \frac{2\pi\hbar^2}{m} f(\Omega)$$

$$\frac{d\sigma_{i\to f}}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left| \langle \Psi_f | V^{\text{int}} | \Psi_i \rangle \right|^2 = \frac{m^2}{4\pi^2\hbar^4} |\underline{T}|^2_{i\to f}$$

For a Coulomb interaction,

assume elastic collision, so $\psi_{n,i} = \psi_{n,f}$

$$T_{i\to f} = \left\langle \Psi_f \left| \frac{e^2}{|\mathbf{r}_e - \mathbf{r}_n|} \right| \Psi_i \right\rangle = \int dv_e dv_n \psi_{n,f}{}^* \psi_{n,i} \frac{e^2}{|\mathbf{r}_e - \mathbf{r}_n|} e^{i\mathbf{q}\cdot\mathbf{r}_e}$$

$$= \int dv_s dv_{r_n} \rho(\mathbf{r}_n) e^{i\mathbf{q}\cdot\mathbf{s}} \frac{e^2}{|\mathbf{s}|} e^{i\mathbf{q}\cdot\mathbf{r}_n} = F(\mathbf{q}) \int dv_s e^{i\mathbf{q}\cdot\mathbf{s}} \frac{e^2}{|\mathbf{s}|}$$

BIG IDEA: $F(\mathbf{q}) = \int dv_{r_n} \rho(\mathbf{r}_n) e^{i\mathbf{q}\cdot\mathbf{r}_n}$ is the "form factor"→

CAN BE MEASURED!

Now, we can obtain $\rho(\mathbf{r}_n)$ from the form factor by an inverse Fourier transform:

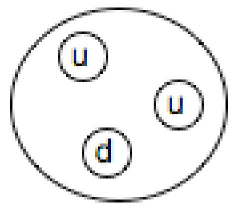
$$\boxed{\rho(\mathbf{r}_n) = \int d^3q F(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{r}_n}}$$

# More on the form factor…

For Rutherford scattering: $\rho(\mathbf{r}_n) \approx \delta^3(\mathbf{r}_n),$

$$F(\mathbf{q}) = \int \delta^3(\mathbf{r}_n) e^{i\mathbf{q}\bullet\mathbf{r}_n} d\mathbf{r}_n = 1$$

But modern experiment is concerned with the actual structure of a nucleon, so we need to worry about $\rho(\mathbf{r}_n)$




Helpful version of the form factor in momentum space:

$$F(\mathbf{q}) = \int d\mathbf{r}_n d\mathbf{r}_n' \psi^*(\mathbf{r}_n)\psi(\mathbf{r}'_n)\delta^3(\mathbf{r}_n - \mathbf{r}'_n) e^{i\mathbf{q}\bullet\mathbf{r}_n'}$$

$$= \int d\mathbf{p} \int d\mathbf{r}_n \psi^*(\mathbf{r}_n) e^{i(\mathbf{p}+\mathbf{q})\bullet\mathbf{r}_n} \int d\mathbf{r}_n' \psi(\mathbf{r}'_n) e^{-i\mathbf{p}\bullet\mathbf{r}_n'} = \int d\mathbf{p}\, \psi^*(\mathbf{p}+\mathbf{q})\psi(\mathbf{p})$$

# Light Front Coordinates

$$\mathbf{x} = \left\langle x^0, x^1, x^2, x^3 \right\rangle \qquad \mathbf{p} = \left\langle p^0, p^1, p^2, p^3{}_z \right\rangle$$

Dirac:

$$x^{\pm} \equiv x^0 \pm x^3 \qquad p^{\pm} \equiv p^0 \pm p^3$$

$$\mathbf{x}_\perp \equiv \left\langle x^1, x^2 \right\rangle \qquad \mathbf{p}_\perp \equiv \left\langle p^1, p^2 \right\rangle$$

when $x^+ = const.,\qquad \dfrac{dz}{dt} = -c$

Useful relationship:

$$m^2 = p^\lambda p_\lambda = p^0 p^0 - p^3 p^3 - \left|\mathbf{p}_\perp\right|^2 = p^+ p^- - \left|\mathbf{p}_\perp\right|^2$$

$$p^- = \frac{\left|\mathbf{p}_\perp\right|^2 + m^2}{p^+}$$

# Relative Coordinates

Assume equal nucleon masses and work in CM frame:

$$m_1 = m_2, \quad \text{so} \quad \mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$$

NONRELATIVISTIC:

$$\mathbf{p}_\perp = \frac{(\mathbf{p}_{\perp 1} - \mathbf{p}_{\perp 2})}{2}$$

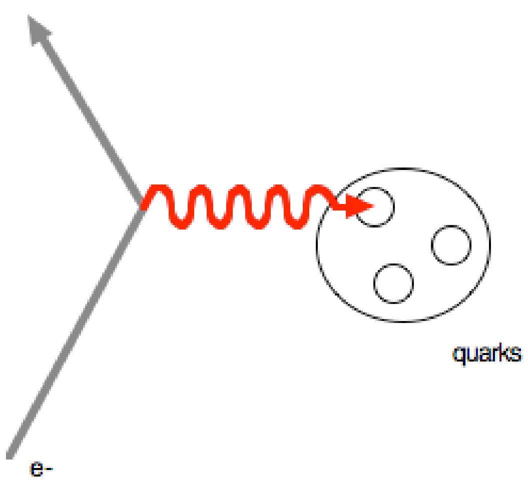
→

RELATIVISTIC:

$$\mathbf{p}_\perp = (1 - x)\mathbf{p}_{\perp 1} - x\mathbf{p}_{\perp 2}$$

where $x \equiv \frac{p_1^+}{P^+}$

Assume e⁻ interacts with only one quark in the target nucleon-




$$\mathbf{p}_{1\perp} \rightarrow \mathbf{p}_{1\perp} + \mathbf{q}$$

NONRELATIVISTIC: $\Delta\mathbf{p}_\perp = \frac{\mathbf{q}}{2}$

RELATIVISTIC: $\Delta\mathbf{p}_\perp = (1 + x)\mathbf{q}$

# The Relativistic Form Factor

| NONRELATIVISTIC | RELATIVISTIC (from QFT) |
| --- | --- |
| $F(q^2) = \int d\mathbf{p}\psi^*(\mathbf{p}_\perp + \frac{\mathbf{q}}{2})\psi(\mathbf{p})$ | $F(q^2) = \int \frac{dxd\mathbf{p}_\perp}{x(1-x)}\psi^*(\mathbf{p}_\perp + (1-x)\mathbf{q}, x)\psi(\mathbf{p}_\perp, x)$ |

This can be put back into spatial form in the perpendicular direction:

$$\psi(\mathbf{p}_\perp, x) = \int d^2\mathbf{x}_\perp \psi(\mathbf{x}_\perp, x)e^{-i\mathbf{p}_\perp \cdot \mathbf{x}_\perp}, \quad \psi(\mathbf{p}_\perp + (1-x)\mathbf{q}_\perp, x) = \int d^2\mathbf{x}_\perp \psi(\mathbf{x}_\perp, x)e^{-i(\mathbf{p}_\perp + (1-x)\mathbf{q}_\perp)\cdot \mathbf{x}_\perp}$$

$$F(q^2) = \int dxd^2\mathbf{x}_\perp \tilde{\rho}(x, \mathbf{x}_\perp)e^{i(1-x)\mathbf{q}\cdot\mathbf{x}}$$

$$= 2\pi \int dxdbb\tilde{\rho}(x,b)J_0(qb(1-x)) \qquad (b \equiv |\mathbf{x}_\perp|)$$

# Example of a form factor- the Hulthen potential

Consider a radially symmetric Hulthen potential: $V(r) = \frac{b^2 - a^2}{1 - e^{(b-a)r}}$

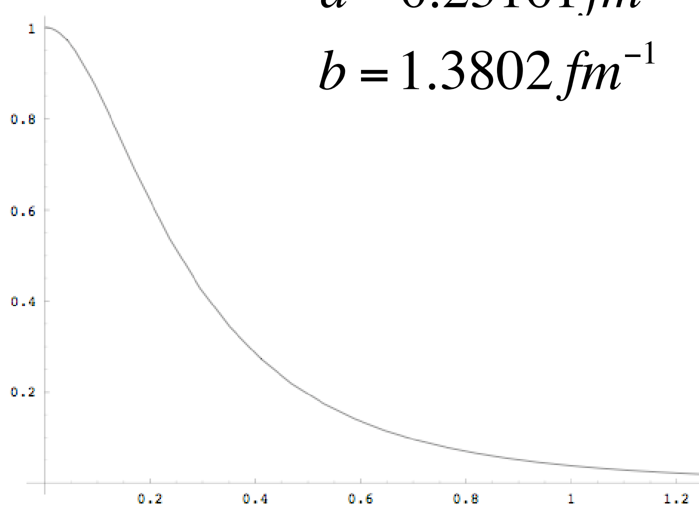
Solution: $\psi(r) \propto \frac{e^{-ar} - e^{-br}}{r}$

Deuteron parameters

$a = 0.23161\, fm^{-1}$

$b = 1.3802\, fm^{-1}$

F(q²)



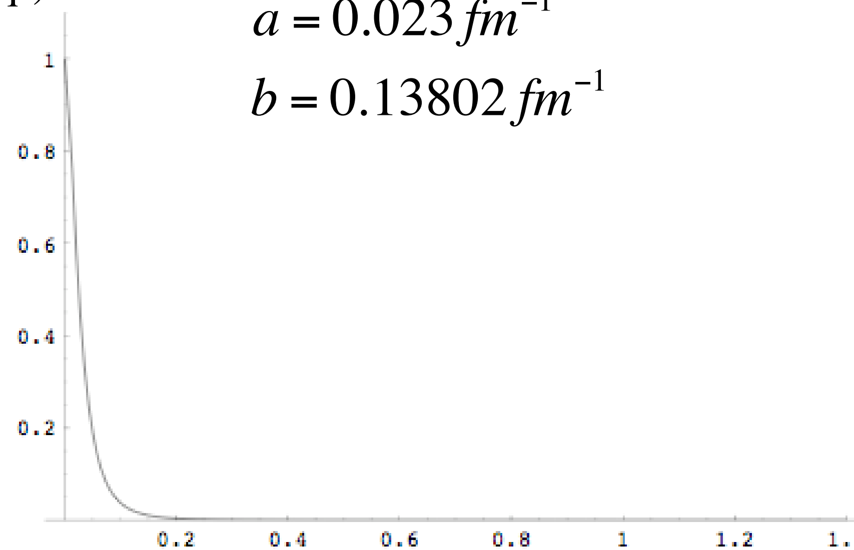

q (GeV)

Nucleon parameters

$a = 0.023\, fm^{-1}$

$b = 0.13802\, fm^{-1}$

F(q²)




q (GeV)

# The Problem

How do we get $\tilde{\rho}(x,b)$ from $F(q^2)$?

Answer- this appears to be a Fredholm Integral Equation, which has been solved before

1.) Define $I(B) \equiv \int \frac{qdq}{2\pi} J_0(qB) F(q^2)$

2.) Expand $\tilde{\rho}(x,b) = \sum_{n,m} a_{nm} H_m(b) L_n(x)$, so the goal now is to find $a_{nm}$

Then, $BI(B) = \sum_{n,m} \int_{B/2}^{\infty} db\, a_{nm} H_m(b) L_n(1 - \frac{B}{b}) = \sum_{n,m} a_{nm} h_{mn}(B)$

3.) Expand $h_{mn}(B) = \sum_{p} d_{nmp} H_p(B)$,

4.) Find (via Mathematica) $\sum_{m,n} c_{qnm} d_{mnp} = \delta_{qp}$

5.) If $BI(B) = \sum_{r} f_r H_r(B)$, then it can be shown: $a_{nm} = \sum_{r} f_r c_{rnm}$

***SO WE'VE SOLVED IT, RIGHT?***

# NOT QUITE……

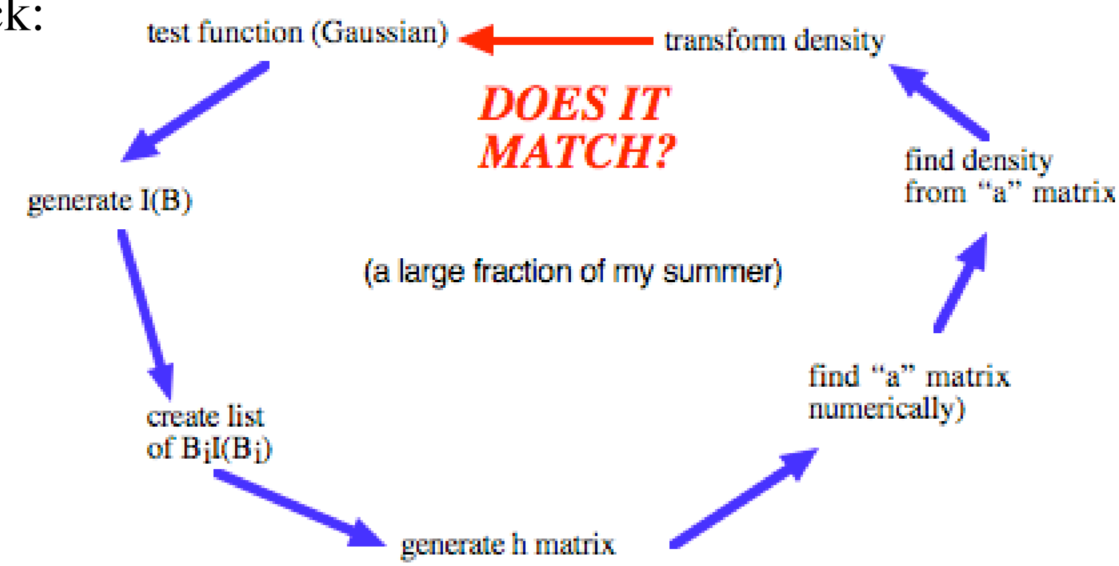
Not possible to find $C_{qnm}$ numerically

**NEW IDEA**: take several values of B and, using $BI(B) = \sum_{n,m} a_{nm} h_{mn}(B)$ create a matrix

$$I(B) = -\int_{B^2}^{\infty} dt e^{-t} = e^{-B^2}$$

$$\begin{pmatrix} B_1 I(B_1) \\ \dots \\ \\ B_T I(B_T) \end{pmatrix} = \begin{pmatrix} h_{00}(B_1) & h_{01}(B_1) & \dots & h_{NN}(B_1) \\ h_{00}(B_2) & & & \\ \dots & & & \\ h_{00}(B_T) & & & h_{NN}(B_T) \end{pmatrix} \begin{pmatrix} a_{00} \\ \dots \\ \\ a_{NN} \end{pmatrix}$$

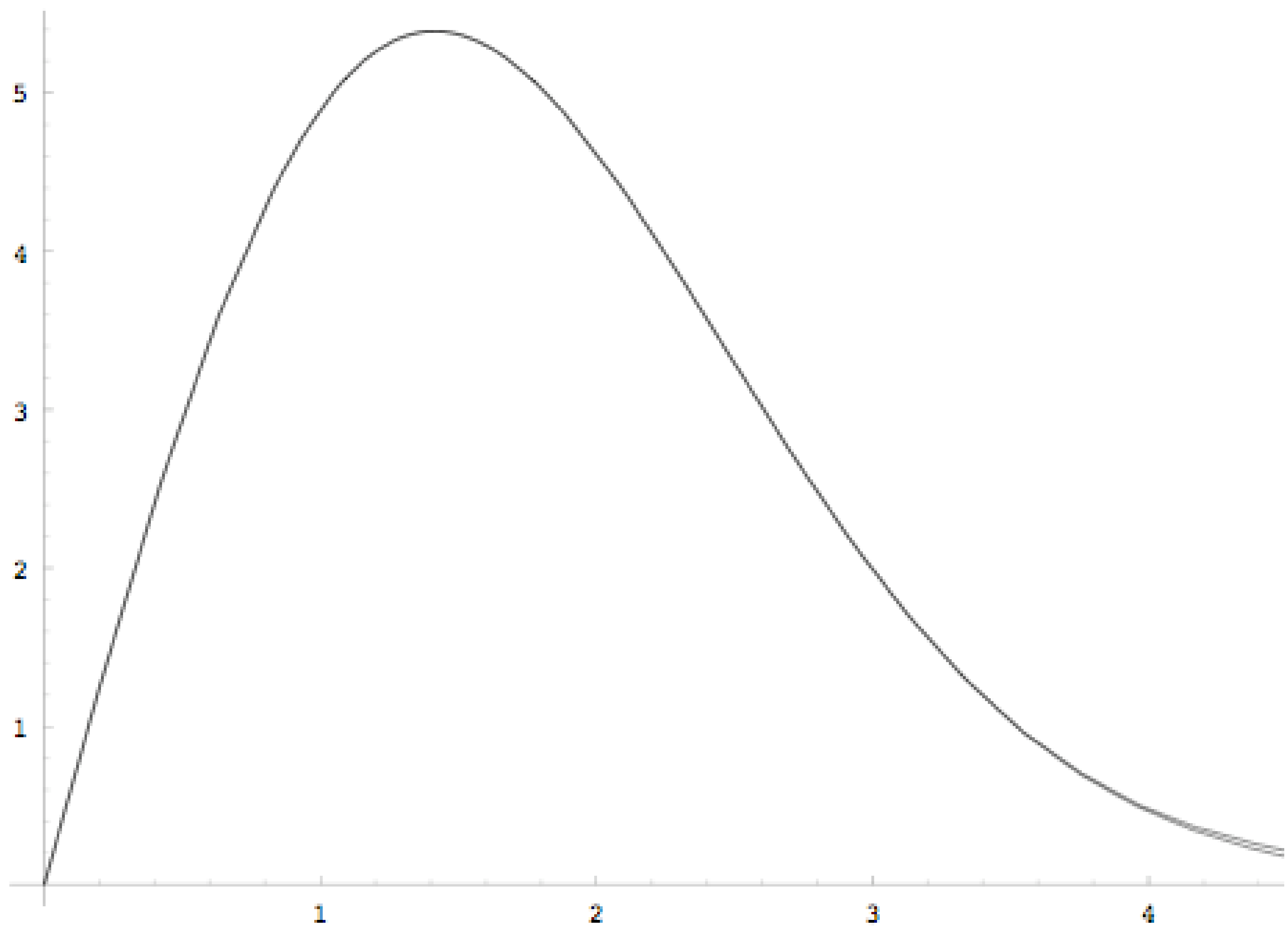
So we used a test function (a Gaussian) for the form factor to generate the nucleon density, then transformed it to see if we got the form factor back:



# Results (sort of)

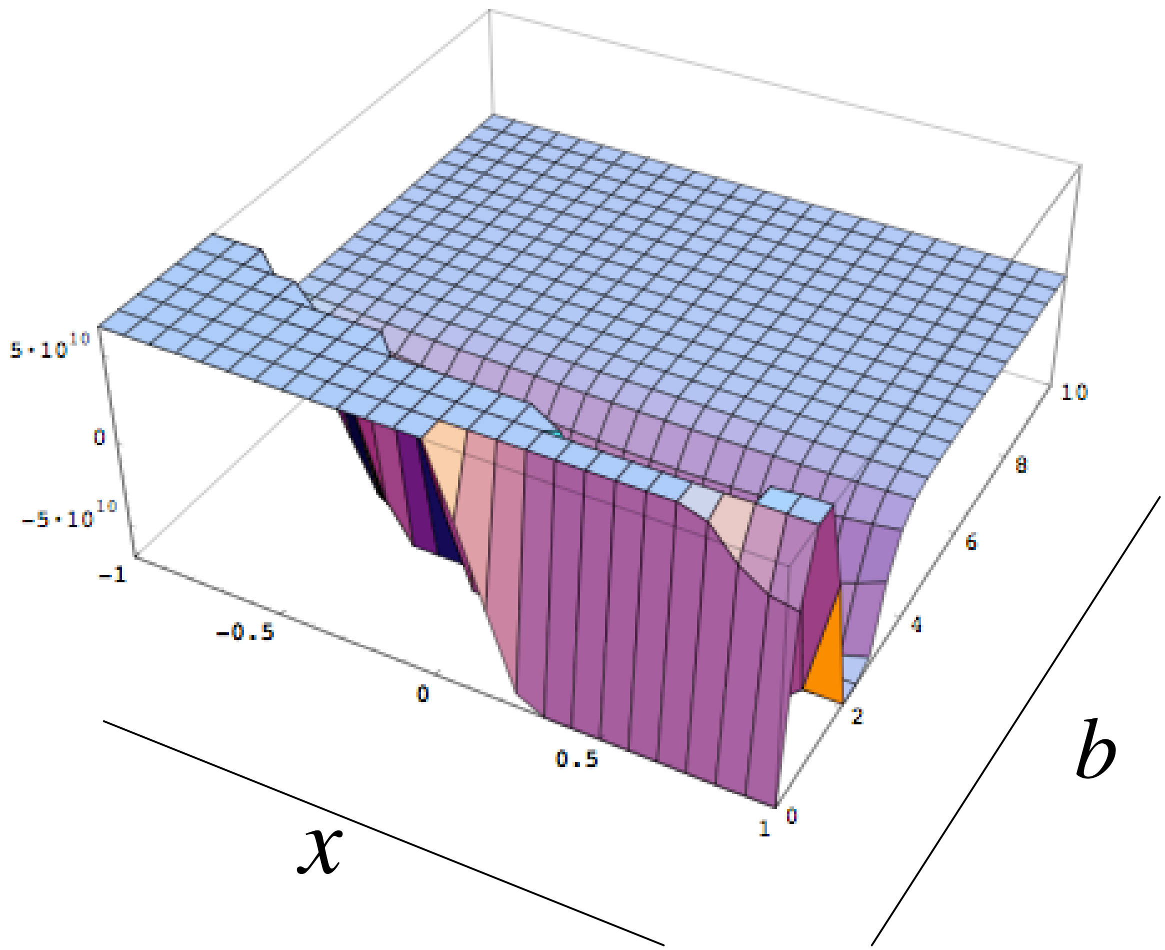
Approximating $BI(B)$ with a matrix:

$$BI(B) \approx a_{00}h_{00}(B) + a_{10}h_{01}(B) + \ldots + a_{NN}h_{NN}(B)$$



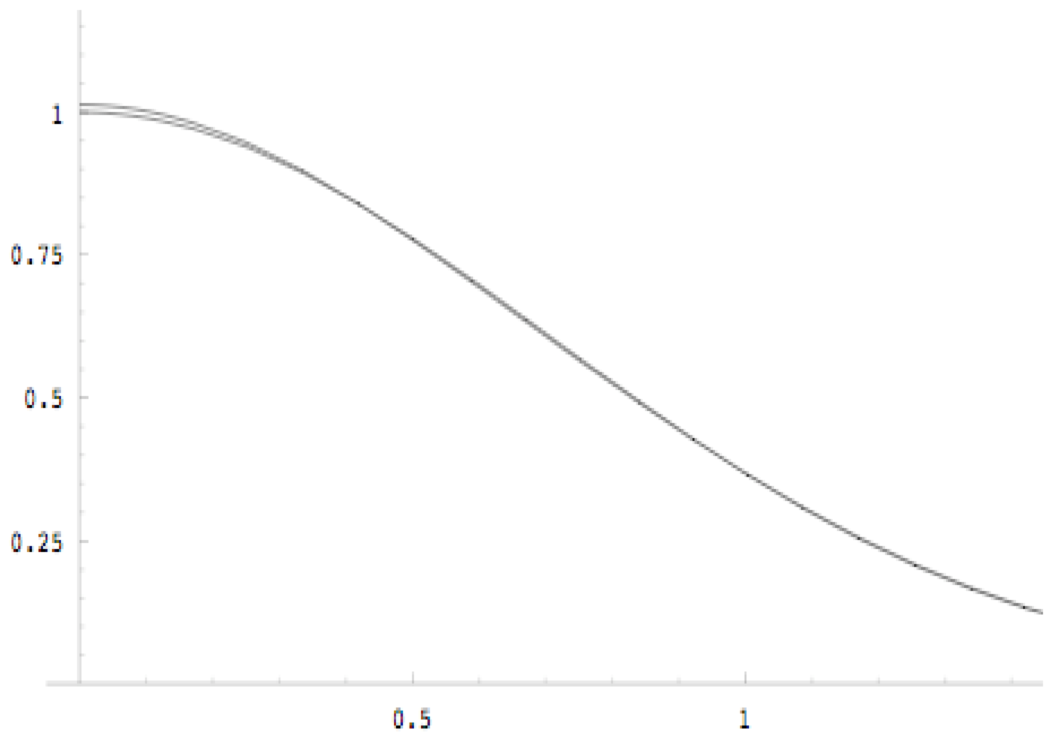
The density:

$\tilde{\rho}(x,b)$



Comparing the generated form factor with the original:

(after having divided out a factor of $4\pi^2$)



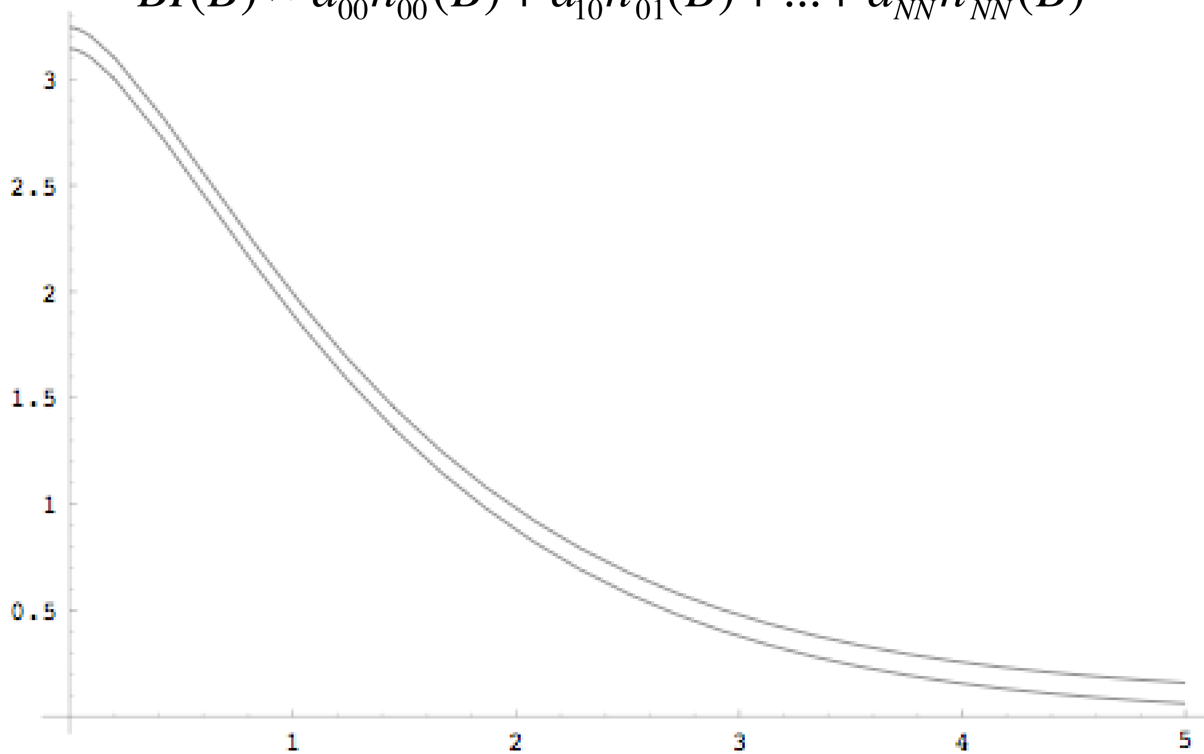

***FORM FACTORS APPEAR TO MATCH***

Using a different test form factor:

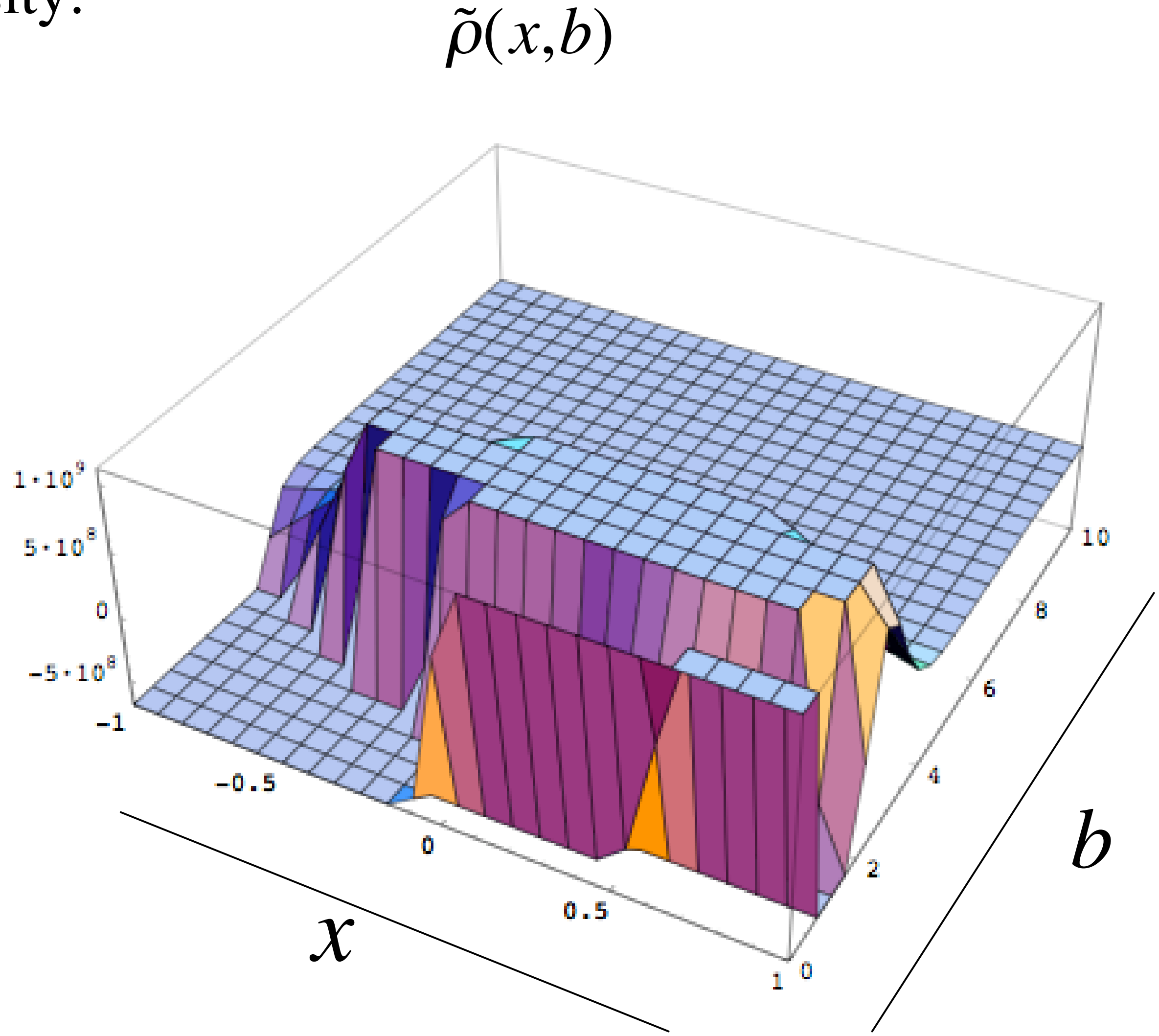
$$f(q^2) = \frac{1}{(1+q^2)^2} \rightarrow BI(B) = 2\pi B K_1(B)$$

Approximating $BI(B)$ with a matrix:

$$BI(B) \approx a_{00}h_{00}(B) + a_{10}h_{01}(B) + \ldots + a_{NN}h_{NN}(B)$$

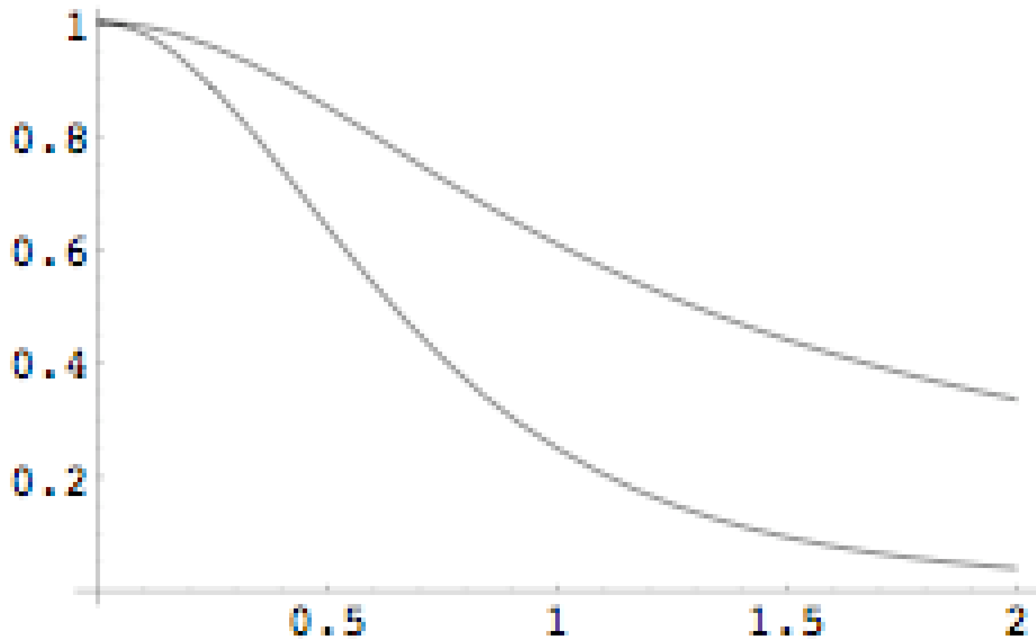


The density:



Comparing the generated form factor with the original:

(after having divided out by a factor of about 26)




***FORM FACTORS DO NOT MATCH***

So when all else fails we do what logically comes next…

# We guess the answer

Recall: we have a test form factor $F(q^2) = e^{-q^2}$, $I(B) = \int \frac{d^2q}{4\pi^2} e^{-iq\bullet B} f(q) = e^{-B^2}$,

and $I(B) = \int \frac{dx}{(1-x^2)^2} \tilde{\rho}(x, \frac{B}{1-x})$

So guess: $\tilde{\rho}(x,b) = (1-x) 4b^2 e^{-\frac{4b^2}{2}}$,

$$I(B) = \int \frac{dx}{(1-x)^3} 4B^2 e^{-\frac{4B^2}{2(1-x)^2}} \longrightarrow t \equiv \frac{4B^2}{(1-x)^2} \longrightarrow \mathrm{I(B)} = -\int_{B^2}^{\infty} dt e^{-t} = e^{-B^2}$$

BUT

$$\mathrm{I(B)} = -\int_{B^2}^{\infty} dt e^{-t} = \int_{B^2}^{\infty} dt (\frac{B^2}{2} e^{-t} + t e^{-t})$$

The form factor has multiple density solutions- can't be inverted!

# Conclusion

The relativistic form factor CANNOT be inverted to find the density

# Acknowledgments


- Dr. Miller
- The NSF and UW