Relativistic Electron Scattering off Nucleons

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Our research this summer concerns the measurement of nucleon structure by scattering electrons off a target nucleon. At nonrelativistic electron energies (~MeV), information about the nucleon structure is easily attainable from the available data. However, at higher energies (~GeV- energies at which current experiments are carried out), relativistic analysis is required, and the expressions are more complicated. Our goal has been to obtain an algorithm with which one can obtain information about the nucleon's structure from data taken at these energies.

Nonrelativistic scattering theory states that the number of electrons dN scattered into a solid angle $d\Omega$ at a solid angle Ω from the incident beam of current density J_{in} is given by the relation

$$dN = J_{in} \frac{d\sigma}{d\Omega} d\Omega \tag{1}$$

where $\frac{d\sigma}{d\Omega}$ is the differential cross section. If the scattered electron wavefunction is assumed to have the form

$$\psi_{scattered} = f(\Omega) \frac{e^{ikr}}{r} \tag{2}$$

where r is the distance from the target nucleon, the form factor can be shown to be of the form

$$\frac{d\sigma}{d\Omega} = \left| f(\Omega) \right|^2 \quad . \tag{3}$$

If then $\boldsymbol{k}_{\mathbf{i}}$ and $\boldsymbol{k}_{\mathbf{f}}$ denote the initial and final momenta of the electron, and the initial and final electron/nucleon states are given by

$$\begin{aligned} \left\langle \mathbf{r}_e, \mathbf{r}_p \middle| \Psi_i \right\rangle &= e^{i\mathbf{k}_i \bullet \mathbf{r}_e} \psi_i(\mathbf{r}_n) \\ \left\langle \mathbf{r}_e, \mathbf{r}_p \middle| \Psi_f \right\rangle &= e^{i\mathbf{k}_f \bullet \mathbf{r}_e} \psi_f(\mathbf{r}_n) \end{aligned} \tag{4}$$

It can further be shown that

$$\left\langle \Psi_f \middle| V^{\text{int}} \middle| \Psi_i \right\rangle = \frac{2\pi\hbar^2}{m} f(\Omega) \tag{7}$$

Now, eq. (3) reads:

$$\frac{d\sigma_{i\to f}}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left| \left\langle \Psi_f \middle| V^{\text{int}} \middle| \Psi_i \right\rangle \right|^2 = \frac{m^2}{4\pi^2\hbar^4} \left| \underline{T} \right|^2_{i\to f} \tag{8}$$

$T_{i\to f}$ is called the 'transition matrix," and it represents the probability of the system ending up in the state given by the second line of eq. (4). For an elastic collision, this can be written (for a coulomb interaction) as the product of two integrals:

$$= F(\mathbf{q})\int dv_s e^{i\mathbf{q}\cdot\mathbf{s}} \frac{e^2}{|\mathbf{s}|} \tag{9}$$

where F(**q**) is the Fourier transform of the nucleon charge density and is called the "form factor" and **q** is the momentum kick given to the nucleon. The form factor can be measured, and to obtain the nucleon density one has only to perform an inverse Fourier transform.

However, the relativistic expression is much more complicated. First, we introduce a set of relativistic coordinates invented by Dirac in 1949, known as light from coordinates. They are defined by the relations

$$\begin{aligned} p^{\pm} &\equiv p^0 \pm p^3 \\ \mathbf{p}_\perp &\equiv \left\langle p^1, p^2 \right\rangle \end{aligned} \tag{10}$$

where p^0 is E/c, p^3 is p_z, and $\mathbf{p}_\perp$ is in the x, y plane. Now, we assume that the

nucleon consists of two quarks, so that we can treat the nucleon as a two-body problem. The relative coordinates for two equal-mass quarks (particle 1 and 2) are then given found quantum field theory [1] as

$$\mathbf{p}_\perp = (1-x)\mathbf{p}_{\perp 1} - x\mathbf{p}_{\perp 2} \tag{11}$$

where

$$x \equiv \frac{p_1^+}{P^+} \tag{12}$$

is the "plus momentum fraction." If the scattered electron imparts a momentum q to quark 1 in the transverse plane, then the fully relativistic form factor is given by

$$\int \frac{dx d\mathbf{p}_\perp}{x(1-x)} \psi^*(\mathbf{p}_\perp + (1-x)\mathbf{q}, x)\psi(\mathbf{p}_\perp, x)\,) \tag{13}$$

This can be rewritten with the transverse momenta transformed to coordinate space, but the integral over x remains [1]:

$$F(q^2) = 2\pi \int dx db b\tilde{\rho}(x,b) J_0(qb(1-x) \tag{14}$$

Now, a simple inverse Fourier transform will no longer yield the nucleon density, and a more complicated algorithm is need to invert the equation.

At first glance, this appears to be a Fredholm integral equation, and the solution is well known. The solution is a five-step process:

1.)Define: $I(B) \equiv \int \frac{qdq}{2\pi} J_0(qB) F(q^2)$

2.) Expand $\tilde{\rho}(x,b) = \sum_{n,m} a_{nm} H_m(b) L_n(x)$ where $L_n(x)$ and $H_m(b)$ are orthonormal and complete so that a_{nm} contains all the information about the density. Now,

$$BI(B) = \sum_{n,m} \int_{B/2}^{\infty} db\, a_{nm} H_m(b) L_n(1 - \frac{B}{b}) = \sum_{n,m} a_{nm} h_{mn}(B)$$

3.) Next, expand $h_{mn}(B) = \sum_p d_{nmp} H_p(B),$

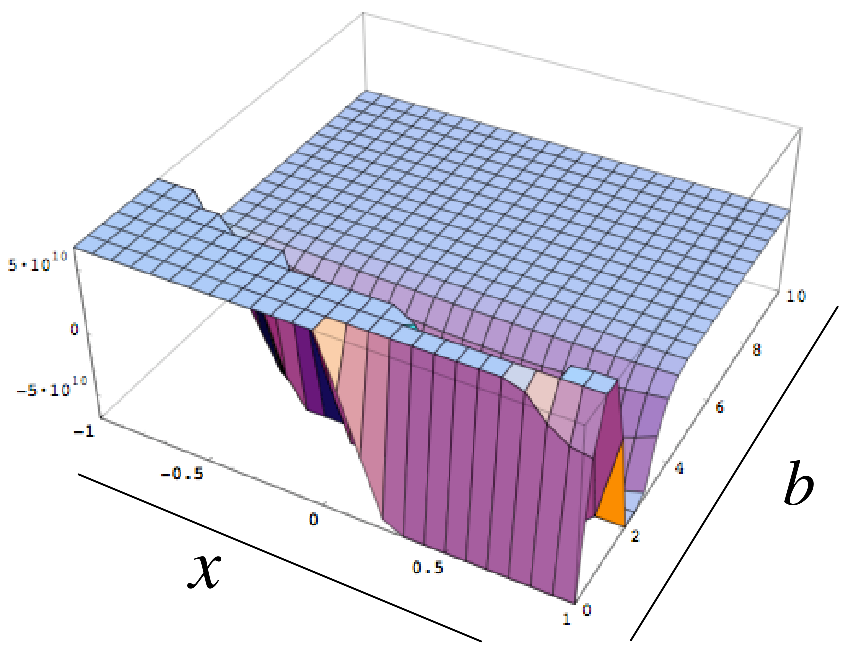
4.) Find the matrix *d* such that $\sum_{m,n} c_{qnm} d_{mnp} = \delta_{qp}$

5.) Then, expanding $BI(B) = \sum_r f_r H_r(B)$, it can be shown that $a_{nm} = \sum_r f_r c_{rnm}$

However, upon using a test function (a Gaussian) for F(q), it does not appear that there exists a unique solution for the matrix *d*. Therefore, for the same test function, we took several values of B and, using the relationship

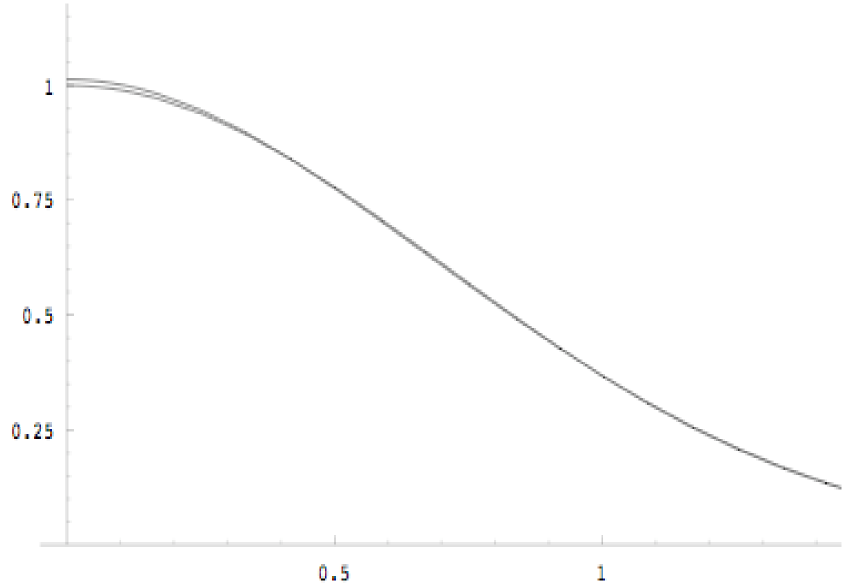
$$BI(B) = \sum_{n,m} a_{nm} h_{mn}(B) \tag{15}$$

we approximated the first elements of a_{nm}. To test this, we obtained the nucleon density and the form factor from the test form factor with the algorithm to see if we would get the test function back. The density is shown below:



At first glance, there are a number of problems with this result. First, it is astronomically larger in places than one would expect. Further, it is negative at other points. However, as our form factor is an arbitrary function, there is nothing mathematically that would prohibit the density from being negative. Indeed, when we insert the density into eq. (14)

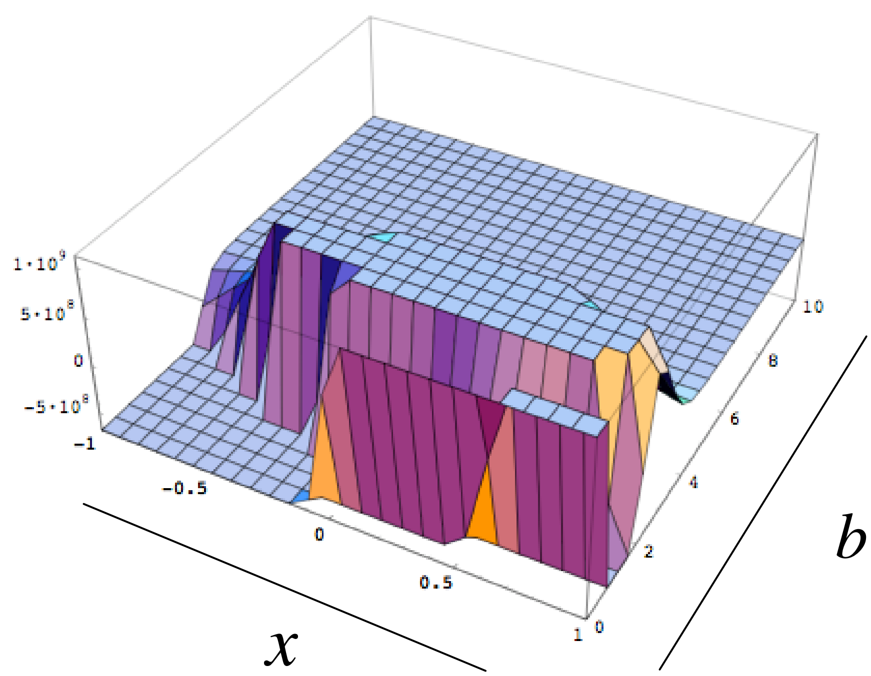
to obtain the form factor, we appear to get the correct result, save a mysterious factor of 4π:



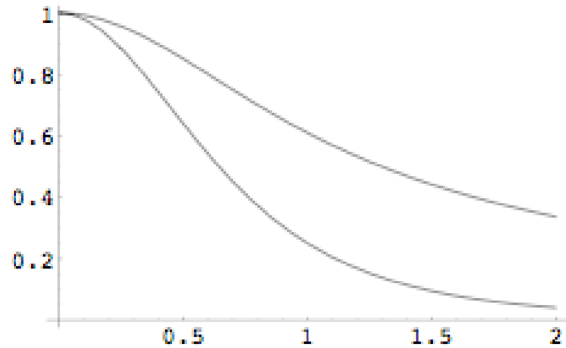
Here, both the original form factor and the form factor carried through the algorithm are pictured. When carried through the algorithm, a second test form factor,

$$f(q^2) = \frac{1}{(1+q^2)^2} \rightarrow BI(B) = 2\pi B K_0(B) \tag{16}$$

yields a comparable density function:



However, when we transform the density, we do not obtain the original form factor. After renormalizing the resulting function to one at q=0, we obtain:



Clearly, they do not match. However, we were able to guess a density,

$$\tilde{\rho}(x,b) = (1-x)4b^2 e^{-\frac{4b^2}{2}}, \tag{17}$$

that, when transformed as in eq. (14) yields a Gaussian. Inserting this into the expression for I(B) from the first step of the algorithm, we have

$$I(B) = \int \frac{dx}{(1-x)^3} 4B^2 e^{-\frac{4B^2}{2(1-x)^2}} . \tag{18}$$

However, with the substitution

$$t = \frac{4B^2}{(1-x)^2} \tag{19}$$

we have

$$\mathrm{I(B)} = -\int_{B^2}^{\infty} dte^{-t} = e^{-B^2}. \tag{20}$$

However, integration by parts reveals the same integral can be written:

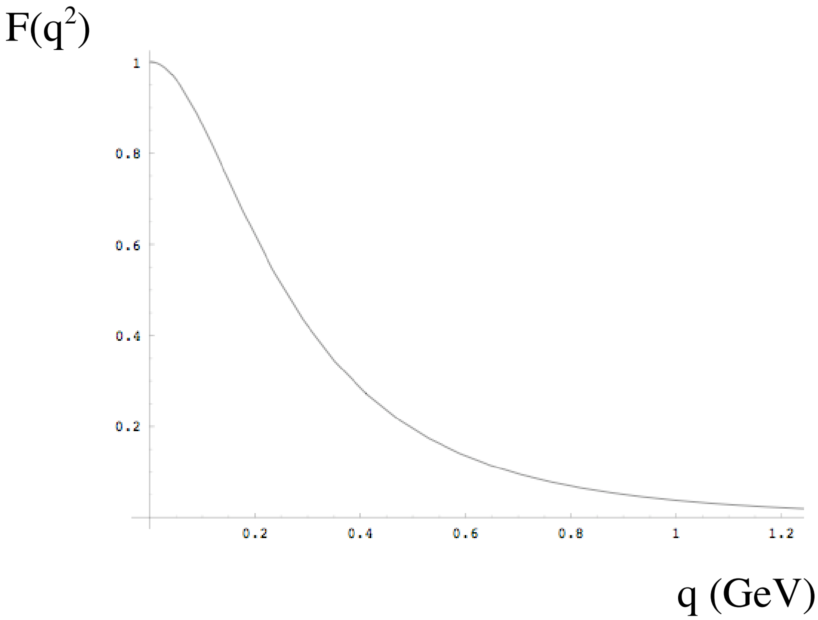
$$I(B) = \int_{B^2/2}^{\infty} dte^{-t} = \int_{B^2/2}^{\infty} dt(\frac{B^2}{2} e^{-t} + te^{-t}) \tag{21}$$

Therefore, the integral equation in eq. (14) does not have a unique kernel. Some other method must be devised to interpret data.

However, we may still calculate the form factor from the potential. For example, this has been done [2] for the ground state Klein-Gordon solution to the Hulthen potential,

$$V(r) = \frac{b^2 - a^2}{1 - e^{(b-a)r}} \tag{22}$$

yielding a form factor:



for the parameters a=0.23161fm^{-1} and b=1.3802 fm^{-1}. This can be interpreted as the probability of an elastic collision where a momentum q is exchanged.

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[1] G.A.Miller./ Prog. Part. Nucl. Phys. **45** (20002) 83-155
[2] B.C. Tiburzi and G. A. Miller, Phys. Rev. C, **63** (2001) 044014