A point in favour of the canonical version of the momentum and angular momentum operators

Elliot Leader*

Blackett laboratory

Imperial College London

Prince Consort Road

London SW7 2AZ, UK

(Dated: June 29, 2012)

* e.leader@imperial.ac.uk

While it seems clear, following the Seattle Workshop, that one can work with any version of the momentum and angular momentum operators, provided only that one specifies clearly which version one is using, I'd like to present a little argument which suggests that the canonical version is superior from an intuitive point of view.

I. ADDITIVITY PROPERTIES OF THE ENERGY MOMENTUM DENSITY TENSORS

Energy momentum density tensors $T^{\mu\nu}(x)$ always contain a term of the form $g^{\mu\nu}\,\mathcal{L}$, where $\mathcal{L}$ is some piece of the lagrangian density. Since discussions of angular momentum typically only involve the elements T^{ij} with $i \neq j$, the latter terms are usually ignored. However in studying the Pauli-Lubanski vector for a particle moving along OZ one has to deal with J^{20} which involves T^{00} , so that the $g^{\mu\nu}$ terms are relevant. There is then some difference between the canonical and Belinfante versions of the energy momentum density as regards additivity.

The Belinfante tensor for the total system is given by

$$T^{\mu\nu}_{bel} = \frac{i}{4}[\bar{\psi}_l \gamma^\mu \overleftrightarrow{D}^\nu \psi_l + (\mu \leftrightarrow \nu)] - G^{\mu\beta}_a G^\nu_{a\beta} - g^{\mu\nu}\mathcal{L}_{qG} \tag{1}$$

where the QCD lagrangian density is

$$\mathcal{L}_{qG} = \left\{ -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \right\} + \left\{ \frac{1}{2}\bar{\psi}^l [\delta_{lm}\, i\,(\overrightarrow{\not\partial} - \overleftarrow{\not\partial}) - 2\,g t^a_{lm}\ \not{A}^a]\psi^m \right\} \tag{2}$$

where a, l, m are colour labels.
The quark and gluon parts of $T^{\mu\nu}$ are, *by definition*, taken to be[1]

$$T^{\mu\nu}_{bel}(\text{quark}) = \frac{i}{4}[\bar{\psi}_l \gamma^\mu \overleftrightarrow{D}^\nu \psi_l + (\mu \leftrightarrow \nu)] - g^{\mu\nu}\ \left\{ \frac{1}{2}\bar{\psi}^l [\delta_{lm}\, i\,(\overrightarrow{\not\partial} - \overleftarrow{\not\partial}) - 2\,g t^a_{lm}\ \not{A}^a]\psi^m \right\} \tag{3}$$

and

$$T^{\mu\nu}_{bel}(\text{gluon}) = -G^{\mu\beta}_a G^\nu_{a\beta} - g^{\mu\nu}\ \left\{ -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \right\}, \tag{4}$$

so that, manifestly, $T^{\mu\nu}_{bel}$ is additive

$$T^{\mu\nu}_{bel} = T^{\mu\nu}_{bel}(\text{quark}) + T^{\mu\nu}_{bel}(\text{gluon}). \tag{5}$$

---

[1] The $g^{\mu\nu}$ term in $T^{\mu\nu}_{bel}(\text{quark})$ happens to vanish via the equations of motion, but that is irrelevant to the present argument.

The situation with the canonical version is rather different. For the energy momentum tensor of the whole system we have

$$T^{\mu\nu}_{can} = \frac{i}{2}\bar{\psi}_l\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi_l - G^{\mu\beta}_a\partial^{\nu}A^a_{\beta} - g^{\mu\nu}\mathcal{L}_{qG} \tag{6}$$

but in this case there is a very natural definition of the separate quark and gluon pieces, which follows from Noether's theorem, namely,

$$T^{\mu\nu}_{can}(\text{quark}) = \frac{i}{2}\bar{\psi}_l\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi_l - g^{\mu\nu}\left\{\frac{1}{2}\bar{\psi}^l\delta_{lm}\, i\,(\overrightarrow{\not{\partial}} - \overleftarrow{\not{\partial}})\psi^m\right\} \tag{7}$$

$$T^{\mu\nu}_{can}(\text{gluon}) = -G^{\mu\beta}_a\partial^{\nu}A^a_{\beta} - g^{\mu\nu}\left\{-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\right\} \tag{8}$$

so that via Eq. (2)

$$T^{\mu\nu}_{can} = T^{\mu\nu}_{can}(\text{quark}) + T^{\mu\nu}_{can}(\text{gluon}) - g^{\mu\nu}\,\mathcal{L}_{int} \tag{9}$$

where

$$\mathcal{L}_{int} = -g\bar{\psi}^l\, gt^a_{lm}\;\not{A}^a\psi^m \tag{10}$$

describes the interaction between the quarks and the gluons.
Thus $T^{\mu\nu}_{can}$ is *not* additive, but rather than regarding this as a disadvantage it could be argued, for example, that Eq. (9) for the 00 component has a very attractive intuitive meaning, namely that the total energy density is the sum of the *free* quark and gluon energy densities plus the energy density of their interaction.

II. ACKNOWLEDGEMENT

I am grateful to Cédric Lorcé for discussions which helped clarify my understanding of the above.