Hadron Structure

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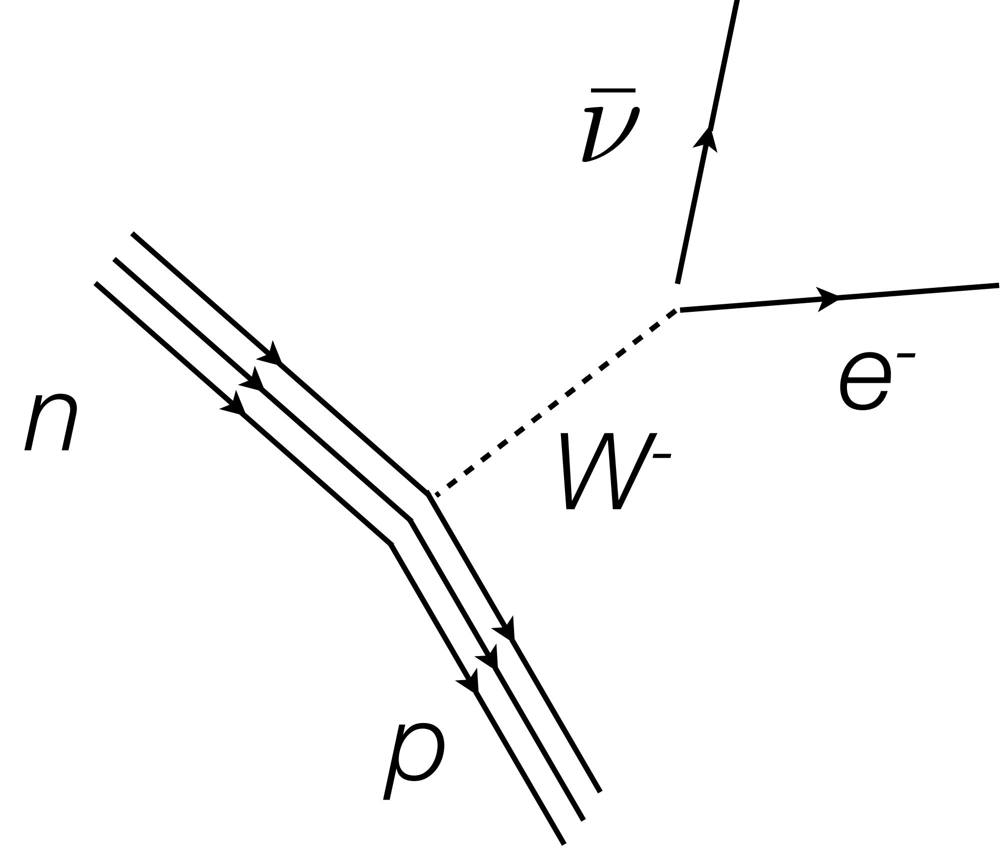
Lecture 3

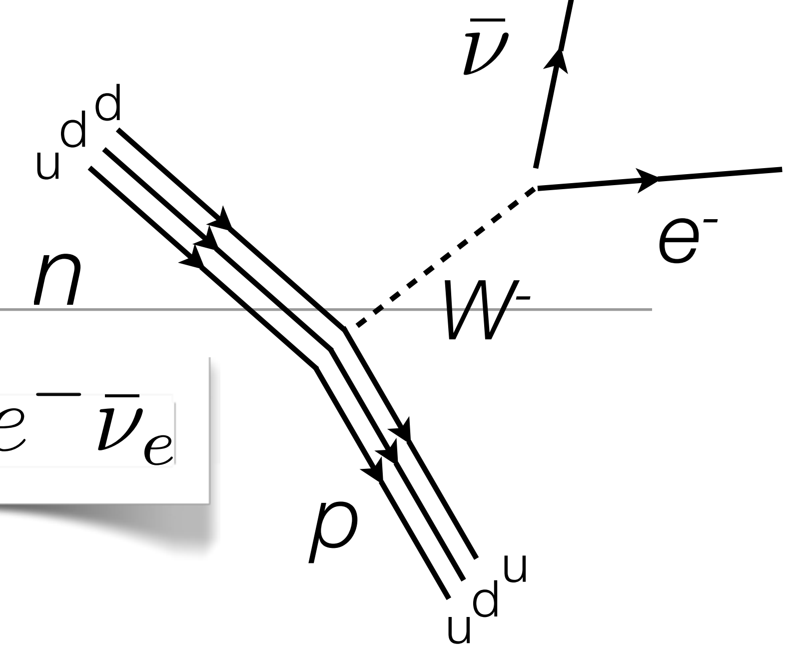
- Neutron beta decay
    - Nucleon axial charge, g_A
- Deep Inelastic Scattering
    - Structure Functions
    - Parton Model and Parton Distribution Functions
- Lattice techniques
    - Moments of PDFs from lattice 3pt functions
    - Renormalisation
    - Results for the momentum fraction



$$n \rightarrow p e^- \bar{\nu}_e$$

Neutron beta decay

Neutron beta decay



- Free neutrons are unstable $\tau \approx 15\,\mathrm{mins}$

$$n \to p e^- \bar{\nu}_e$$

- The most common way to study the weak interaction

- The decay rate is proportional the matrix element of the weak V-A current

$$\langle p(p',s')|(V_\mu - A_\mu)|n(p,s)\rangle = \bar{u}_p(p',s')\{\gamma_\mu f_1(q^2) + i\frac{\sigma_{\mu\nu}q^\nu}{2M} f_2(q^2) + \frac{q_\mu}{2M} f_3(q^2)$$
$$-[\gamma_\mu\gamma_5 g_1(q^2) + i\frac{\sigma_{\mu\nu}q^\nu}{2M}\gamma_5 g_2(q^2) + \frac{q_\mu}{2M}\gamma_5 g_3(q^2)]\}u_n(p,s)$$

- Here the momentum transfer is so small that we only need to consider the f_1 and g_1 terms

$$M_n - M_p \simeq 1.3\ \mathrm{MeV}$$

- By convention, we call $g_V = f_1(0) \quad g_A = g_1(0)$

- with g_V=1 according to the conserved vector current (CVC) hypothesis

- Adler-Weisberger relation predicts g_A=1.26

Neutron beta decay

- The decay rate for
  - a neutron at rest and with spin in the $\vec{s}_n$ direction
  - final e$^-$ and $\bar{\nu}_e$ with velocities $\vec{v}_e,\ \vec{v}_{\bar{\nu}}$

$$\frac{dR}{dp_e d\Omega_e d\Omega_{\bar{\nu}}} = \frac{G_F^2 |V_{ud}|^2}{(2\pi)^5} \left[\alpha + \beta \vec{v}_e \cdot \vec{v}_{\bar{\nu}} + \gamma \vec{s}_n \cdot \vec{v}_e + \delta \vec{s}_n \cdot \vec{v}_{\bar{\nu}}\right] p_e^2 (E_{\max} - E_e)^2$$

  - with $E_{\max} = M_n - M_p \simeq 1.3\ \text{MeV}$

$$\alpha = g_V^2 + 3g_A^2 \qquad \beta = g_V^2 - g_A^2$$
$$\gamma = 2(g_A g_V - g_A^2) \qquad \delta = 2(g_A g_V + g_V^2)$$

  - so even without neutron polarisation, we can determine $|g_A/g_V|$ through an accurate determination of the angular correlation between outgoing e$^-$ and $\bar{\nu}_e$
  - To determine the sign of g_A ⇨ spin-dependent measurement
  - Current best determination (PDG 2012) $g_A/g_V = 1.2701(25)$

Axial Form Factor

- The matrix element of the axial operator between neutron and proton states takes the general form

$$\langle p(p',\, s')|A^\mu(\vec{q})|n(p,\, s)\rangle \;=\bar{u}_p(p',\, s')\left[\gamma^\mu\gamma_5 G_A(q^2)+i\sigma^{\mu\nu}\gamma_5\frac{q_\nu}{2m}G_T(q^2)+\gamma_5\frac{q^\mu}{2M}G_P(q^2)\right]u_n(p,\, s)$$

axial FF

induced pseudoscalar FF

vanishes if charge symmetry assumed, $u_p=d_n$

- PCAC (Chiral symmetry)

$$\Rightarrow \quad \partial_\mu A^\mu \propto m_\pi^2 \xrightarrow{m_q\to 0} 0$$

- Not true for above matrix element, but if G_P has a pion pole $G_P(q^2)\to\frac{4Mf_\pi g_{\pi NN}(q^2)}{-q^2+m_\pi^2}$

- The matrix element satisfies PCAC if $MG_A(q^2)=f_\pi g_{\pi NN}(q^2)$

- At $q^2=0$ $\Rightarrow$ Goldberger-Treiman relation $Mg_A=f_\pi g_{\pi NN}$

Axial Charge, g_A

- The axial charge is defined as the value of the axial form factor at *q^2=0*

$$g_A = G_A(q^2 = 0)$$

- Ideal quantity for a benchmark lattice calculation of nucleon structure
  - Zero momentum

    ⇨ Statistically clean

  - Isovector

    ⇨ Disconnected contributions cancel

$$\langle p|\bar{u}\gamma^\mu\gamma^5 d|n\rangle = \langle p|\bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d|p\rangle$$

Determination of g_A on the Lattice

- Need access to the matrix element $\langle n|\bar{u}\gamma^\mu\gamma^5 d|p\rangle = \bar{u}(p',s')\gamma^\mu\gamma^5 u(p,s) g_A$

- from our three-point functions

$$G(t,\tau,\vec{p},\vec{p}\,') = \sum_{s,s'} e^{-E_{\vec{p}\,'}(t-\tau)} e^{-E_{\vec{p}}\tau} \Gamma_{\beta\alpha} \langle\Omega|\chi_\alpha(0)|N(p',s')\rangle \langle N(p',s')|\mathcal{O}(\vec{q})|N(p,s)\rangle \langle Np,s)|\overline{\chi}_\beta(0)|\Omega\rangle$$

- From yesterday, we know that after the spin-trace, our 3pt will be proportional to

$$F(\Gamma,\mathcal{J}) = \frac{1}{4}\mathrm{Tr}\left\{\Gamma\left(\gamma_4 - i\frac{\vec{p}\,'\cdot\vec{\gamma}}{E_{\vec{p}\,'}} + \frac{m}{E_{\vec{p}\,'}}\right)\mathcal{J}\left(\gamma_4 - i\frac{\vec{p}\cdot\vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}}\right)\right\}$$

- For $\mathcal{J} = \gamma_\mu\gamma_5$ we have $F(\Gamma_{\rm unpol},\gamma_4\gamma_5) = 0 \quad F(\Gamma_{\rm unpol},\gamma_i\gamma_5) = 0$

$$F(\Gamma_{\rm pol},\gamma_4\gamma_5) = \frac{-1}{2E_{\vec{p}}E_{\vec{p}\,'}}\left[(E_{\vec{p}}+m)\vec{p}\,'\cdot\vec{s} + (E_{\vec{p}\,'}+m)\vec{p}\cdot\vec{s}\right]$$

$$F(\Gamma_{\rm pol},\gamma_i\gamma_5) = \frac{i}{2E_{\vec{p}}E_{\vec{p}\,'}}\left[(E_{\vec{p}}+m)(E_{\vec{p}\,'}+m)\vec{s} + (\vec{p}\,'\cdot\vec{s})\vec{p} + (\vec{p}\cdot\vec{s})\vec{p}\,' - (\vec{p}\,'\cdot\vec{p})\vec{s}\right]_i$$

- When using the projector $\Gamma_{\rm pol} = \frac{1}{2}(1+\gamma_4)i\gamma_5\vec{\gamma}\cdot\vec{s}$ when computing the 3pt function

- Requires nucleon state to be polarised in, e.g. +z direction

Determination of g_A on the Lattice

- At zero momentum, $F(\Gamma_{\rm pol}, \gamma_i\gamma_5) = 2is_i$

- So g_A can be determined by choosing the direction of the axial current to be the same as the direction of the nucleon polarisation. E.g. use a 3pt function with

$$\Gamma_{\rm pol} = \Gamma_3 = \frac{1}{2}(1+\gamma_4)i\gamma_5\gamma_3 \qquad \mathcal{O} = \gamma_3\gamma_5$$

- Our ratio from yesterday

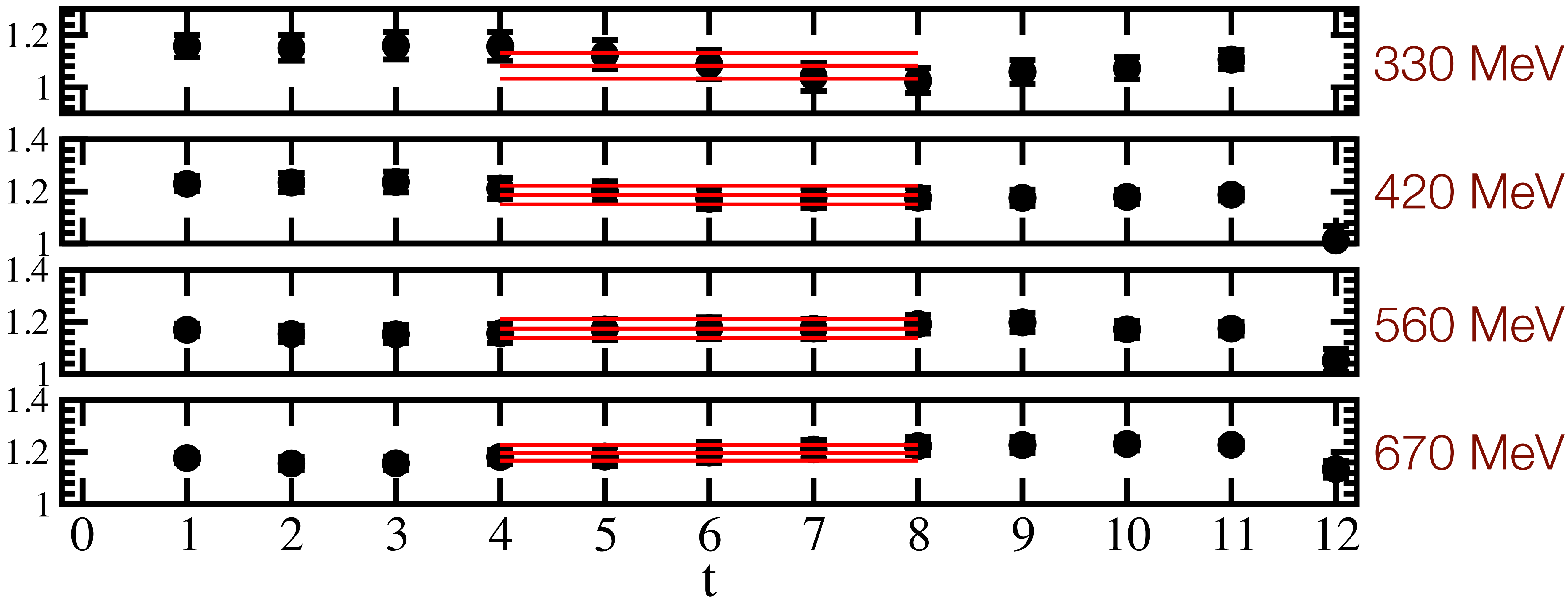
$$R(t,\tau;\vec{p}\,',\vec{p};\mathcal{O},\Gamma) \;=\; \frac{G_\Gamma(t,\tau;\vec{p}\,',\vec{p},\mathcal{O})}{G_2(t,\vec{p}\,')}\left[\frac{G_2(\tau,\vec{p}\,')G_2(t,\vec{p}\,')G_2(t-\tau,\vec{p})}{G_2(\tau,\vec{p})G_2(t,\vec{p})G_2(t-\tau,\vec{p}\,')}\right]^{\frac{1}{2}}$$

- now becomes

$$R(t,\tau;\vec{0},\vec{0};\gamma_3\gamma_5,\Gamma_3) \;=\; \frac{G_{\Gamma_3}(t,\tau;\vec{0},\vec{0},\gamma_3\gamma_5)}{G_2(t,\vec{0})} = ig_A$$

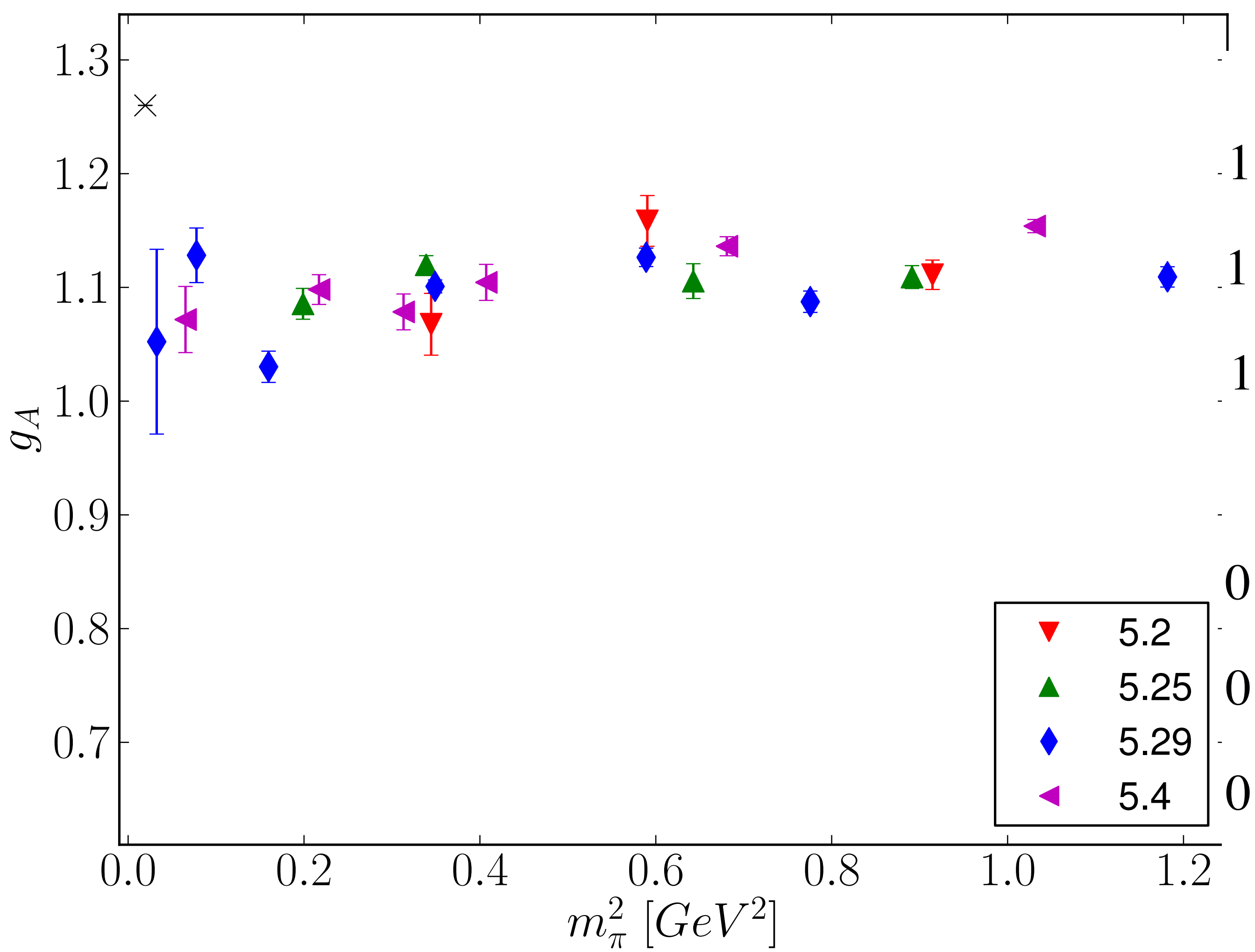
Determination of g_A on the Lattice

- Example of $$R(t,\tau;\vec{0},\vec{0};\gamma_3\gamma_5,\Gamma_3) = \frac{G_{\Gamma_3}(t,\tau;\vec{0},\vec{0},\gamma_3\gamma_5)}{G_2(t,\vec{0})} = i g_A$$
- from [RBC/UKQCD:0801.4016] at 4 different pion masses



Determination of g_A on the Lattice

QCDSF: 1101.2326



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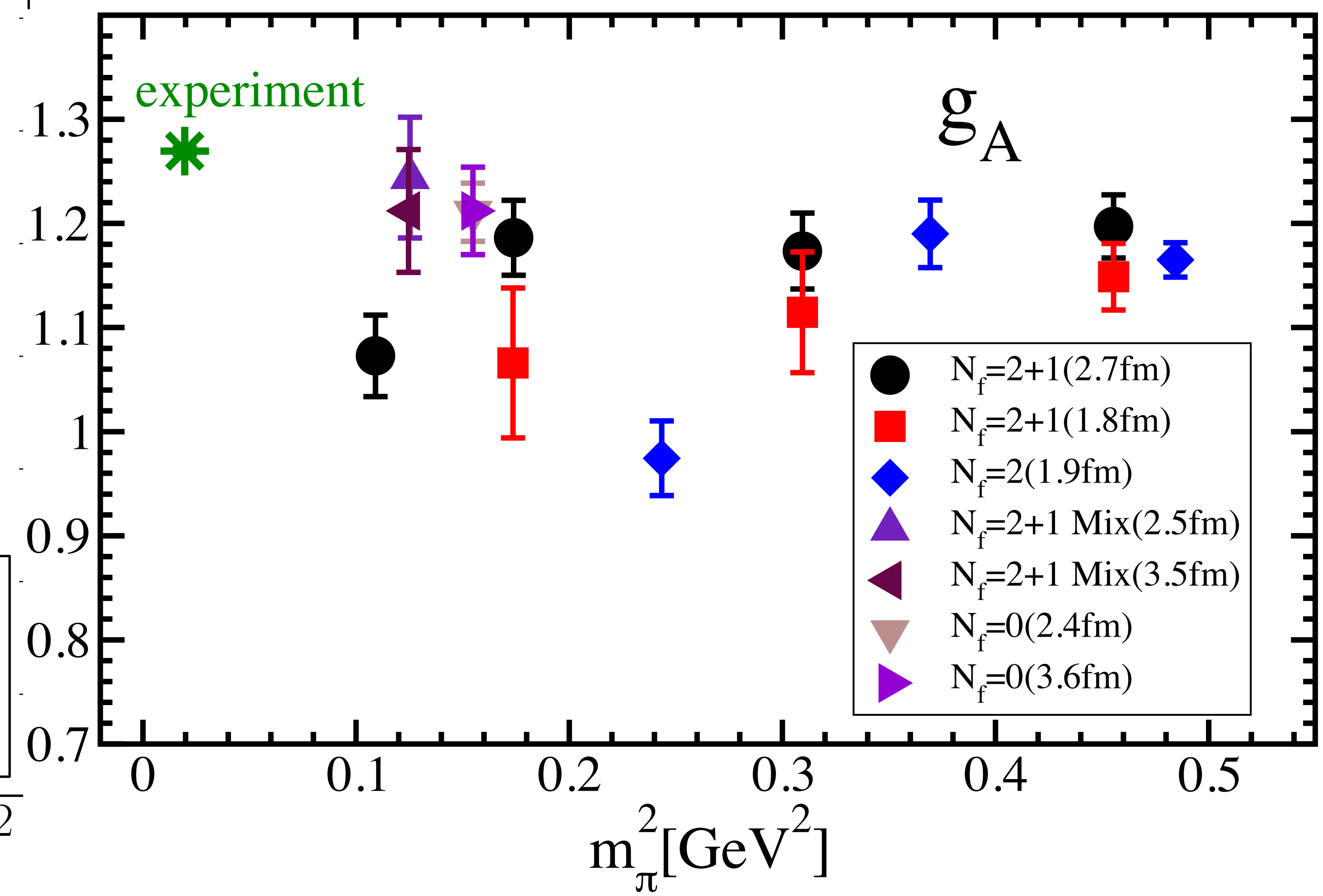


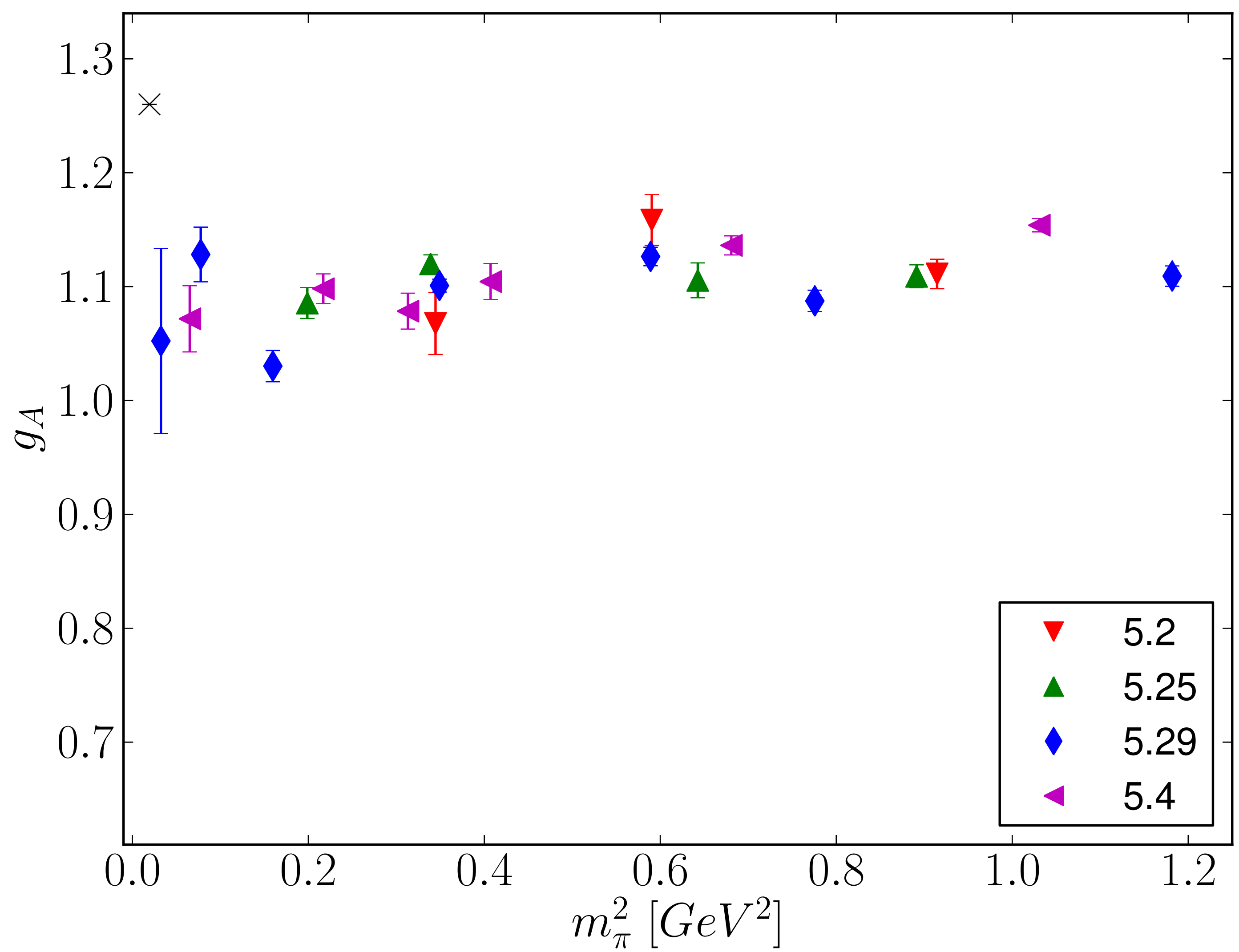
Results appear to undershoot by ~10%

Determination of g_A on the Lattice

- What about lattice systematic errors?
    - Finite lattice spacing
    - Large quark masses
    - Finite volume
    - Contamination from excited states

Determination of g_A on the Lattice

Lattice spacing dependence



- Different colours correspond to different lattice spacings $0.06\,\mathrm{fm} < a < 0.1\,\mathrm{fm}$

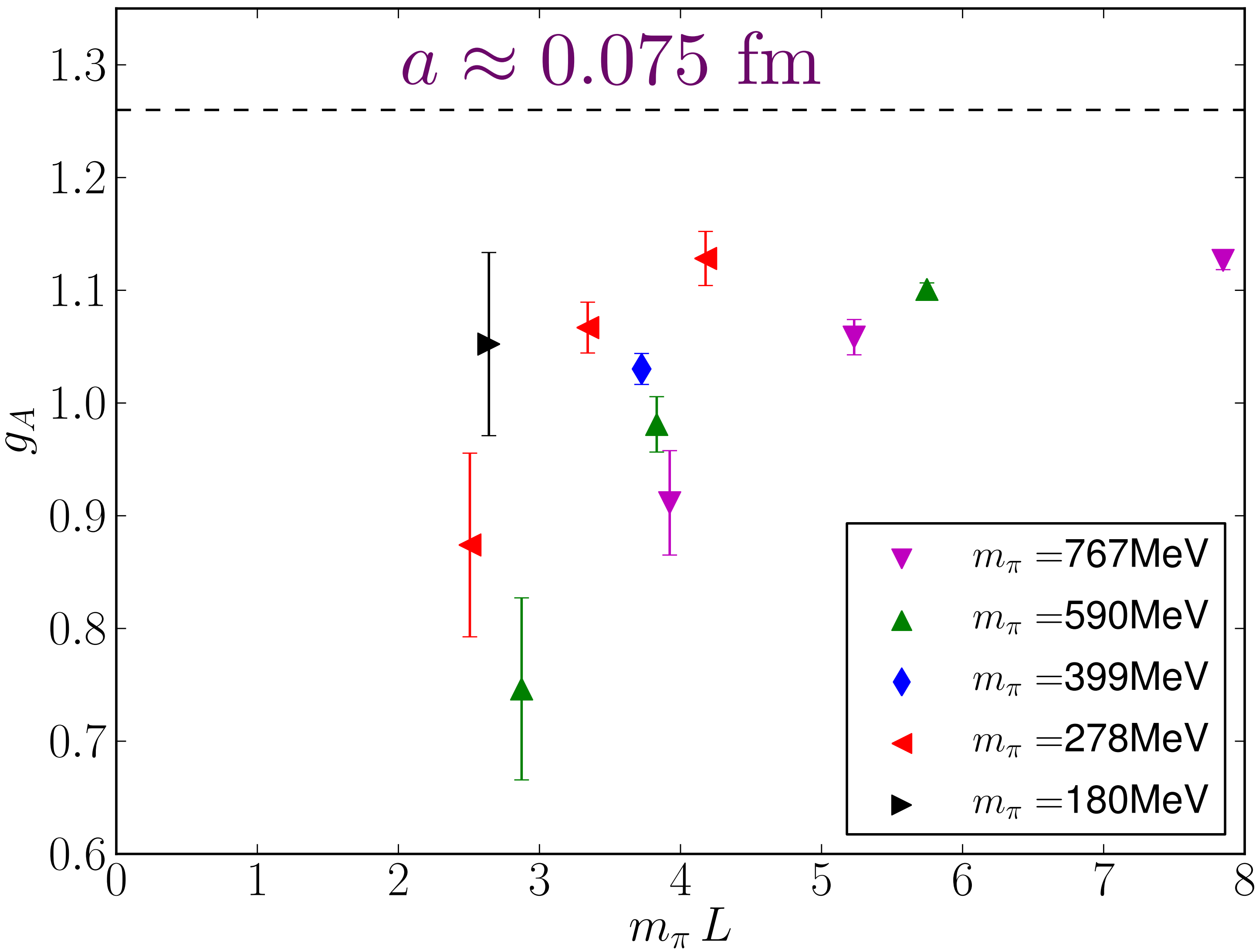
⇨ No obvious dependence on *a*

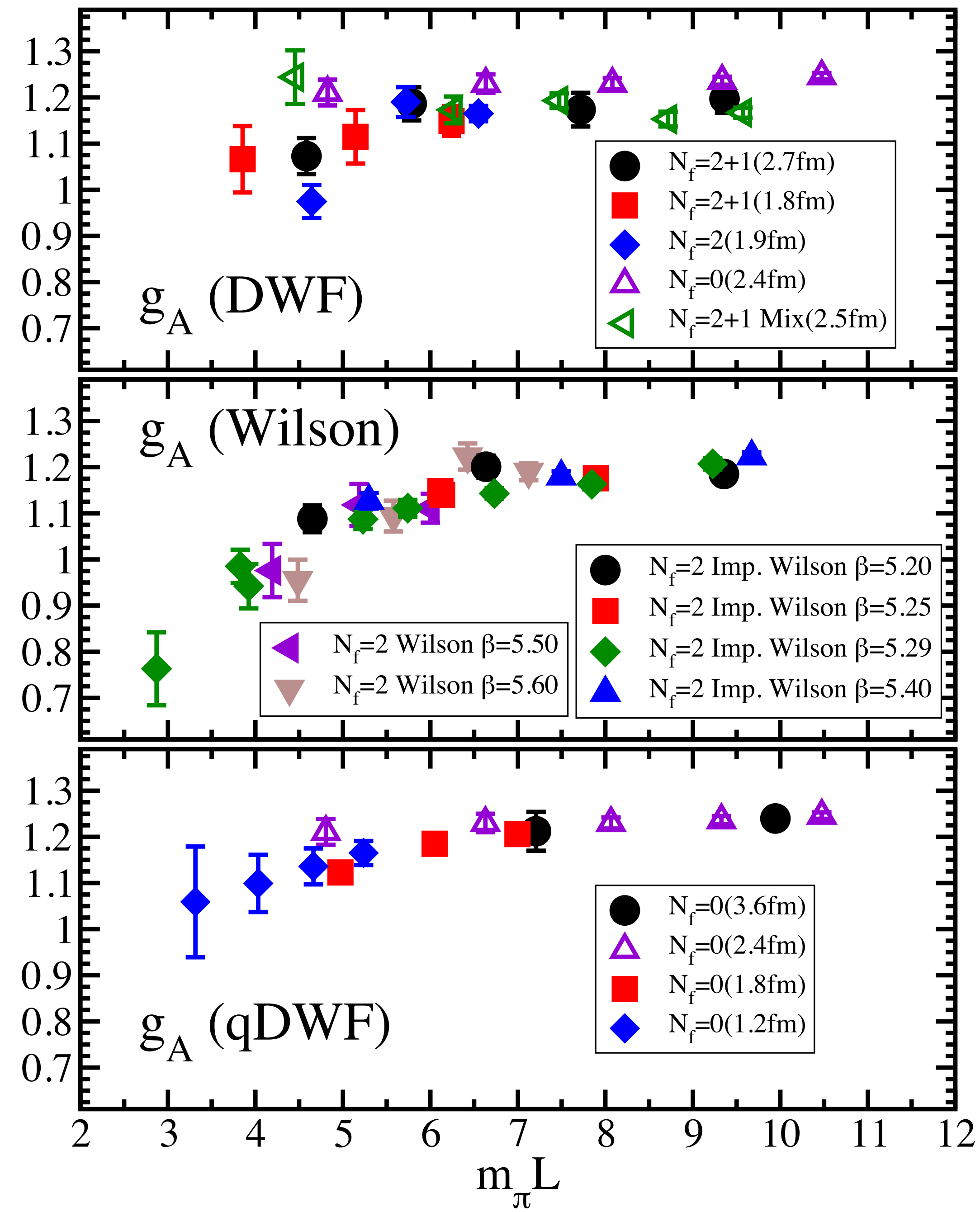
Determination of g_A on the Lattice

Lattice volume dependence

QCDSF: 1101.2326

RBC/UKQCD: 0801.4016

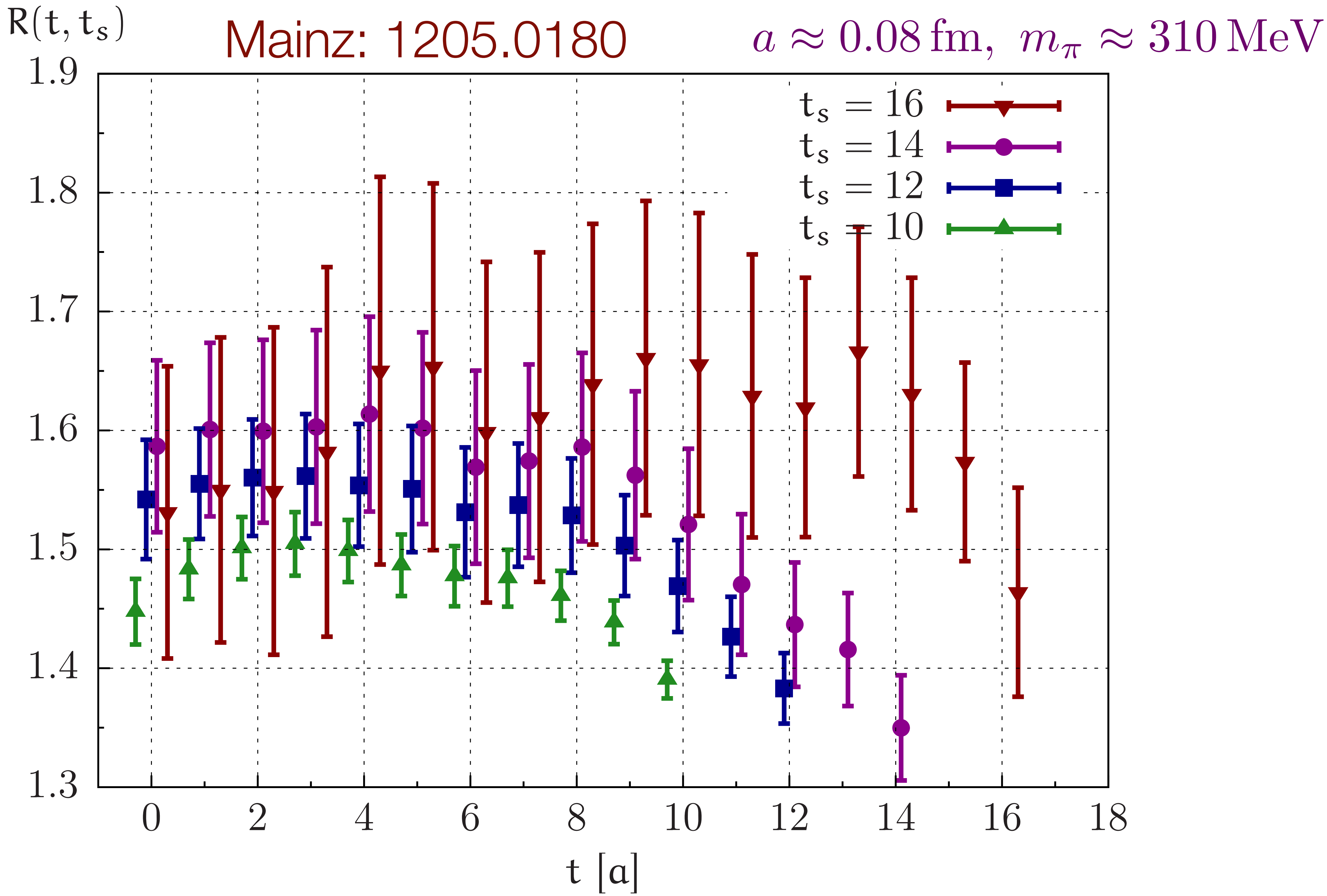




- Substantial finite size effects
- g_A suppressed on a finite volume
- See [CSSM: 1205.1608] for attempts to understand the source of this behaviour

Determination of g_A on the Lattice

Excited state contamination



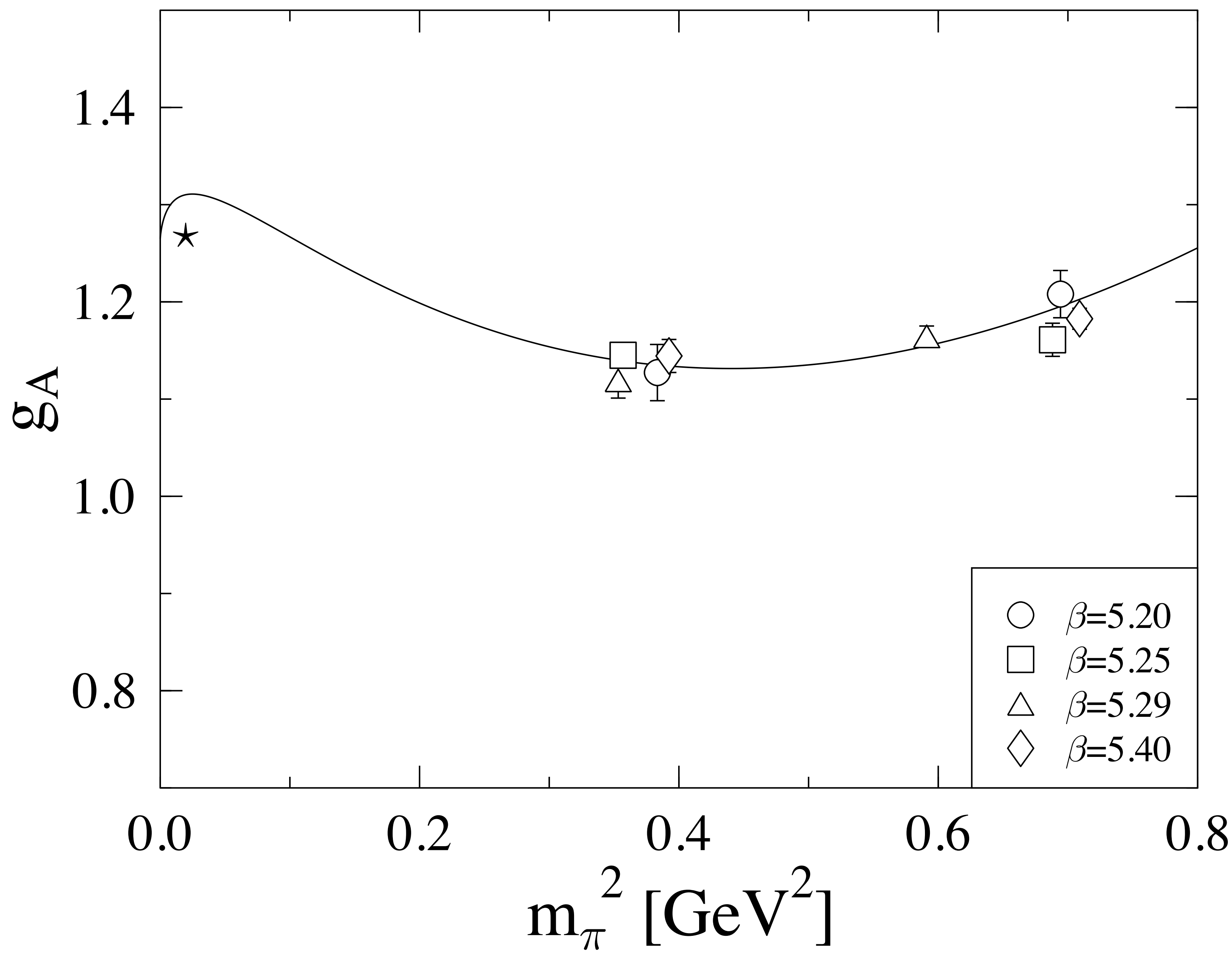
- Test for contamination from excited states by varying the location of the sink
- Evidence that excited state contamination suppresses g_A

Determination of g_A on the Lattice

Quark mass dependence

QCDSF: hep-lat/0603028

- HBChPT form suggests that an enhancement is expected in the infinite volume at light quark masses



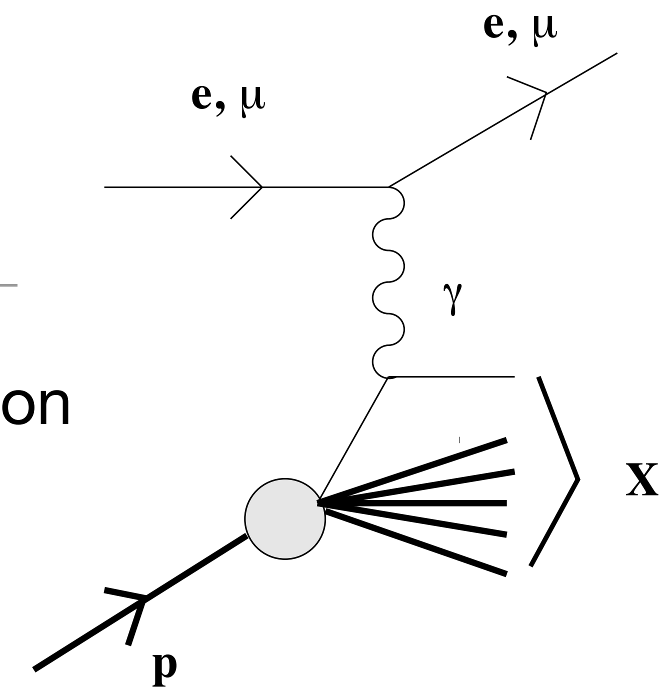
Determination of g_A on the Lattice

- The previous collection of results indicate that while g_A was hoped to a "simple" quantity to compute on the Lattice, this appears to be far from the case due to
    - Substantial finite size effects
    - Possible excited state contamination
    - Non-trivial quark mass dependence (interplay of Delta and N loops)
- But the removal of these effects all appear to shift g_A in the right direction

- Increased interest from several lattice collaborations
    - QCDSF PRD 74, 094508 (2006)
    - LHPC PRL 96, 052001 (2006)
- RBC/UKQCD PRL 100, 171602 (2008)
- ETMC PRD 83, 045010 (2011)
- CLS/Mainz arXiv:1106.1554 [hep-lat]

Deep Inelastic Scattering

Deep Inelastic Scattering



- A single quark in the nucleon is “knocked out” by a virtual photon
- The proton is “smashed” into many fragments
- Allows the extraction of the quark and gluon distributions in momentum space - Feynman parton distributions
- As on Monday for elastic scattering, start with the S-Matrix

$$S = (2\pi)^4 \delta^4(k + P - P' - k')\overline{u}(k')(-ie\gamma^\mu)u(k)\frac{-i}{q^2}\langle X|(ie)J^\mu|P\rangle$$

Inclusive

- Inclusive cross section

$$\frac{d\sigma}{d\Omega dE} = \frac{\alpha^2}{Q^4}\frac{E'}{E}\ell_{\mu\nu}W^{\mu\nu}$$

- with hadronic tensor

$$W_{\mu\nu} = \frac{1}{4\pi}\sum_X \langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle(2\pi)^4\delta^4(P + q - P_X)$$

Deep Inelastic Scattering

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_X \langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle(2\pi)^4\delta^4(P+q-P_X)$$

- Since the final states, *X*, are summed over, W only depends on
    - initial proton momentum, *P*
    - photon momentum, *q*
- Using Lorentz symmetry, parity and time reversal invariance, current conservation, can express this in terms of two invariant tensors

$$W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{M^2}\left(P^\mu - q^\mu\frac{(P\cdot q)}{q^2}\right)\left(P^\nu - q^\nu\frac{(P\cdot q)}{q^2}\right)$$

- W_1 and W_2 are the so-called structure functions of the proton and depend on two variables

$Q^2 = -q^2$ the 4-momentum transfer squared

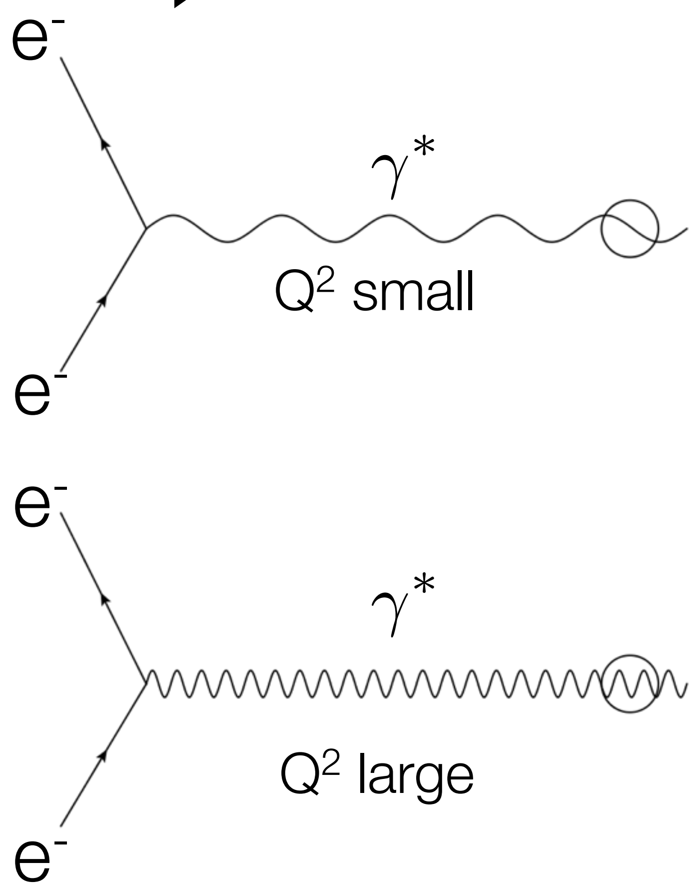
$\nu = \frac{P\cdot q}{M}$ the energy transferred to the nucleon by the scattering electron

Deep Inelastic Scattering

- A key factor for investigating the proton substructure is the wavelength of the probe

$$\lambda \sim \frac{1}{\sqrt{Q^2}}$$

- Large momentum transfer ⇨ high resolution



resolve:

proton

quark

Deep Inelastic Scattering

- Early SLAC data showed that W_1 and W_2 are nearly independent of Q^2 when plotted as a function of the dimensionless combination

$$x = -\frac{q^2}{2P \cdot q} = \frac{Q^2}{2M\nu}$$

the Bjorken scaling variable

- This is known as Bjorken scaling

Deep Inelastic Scattering

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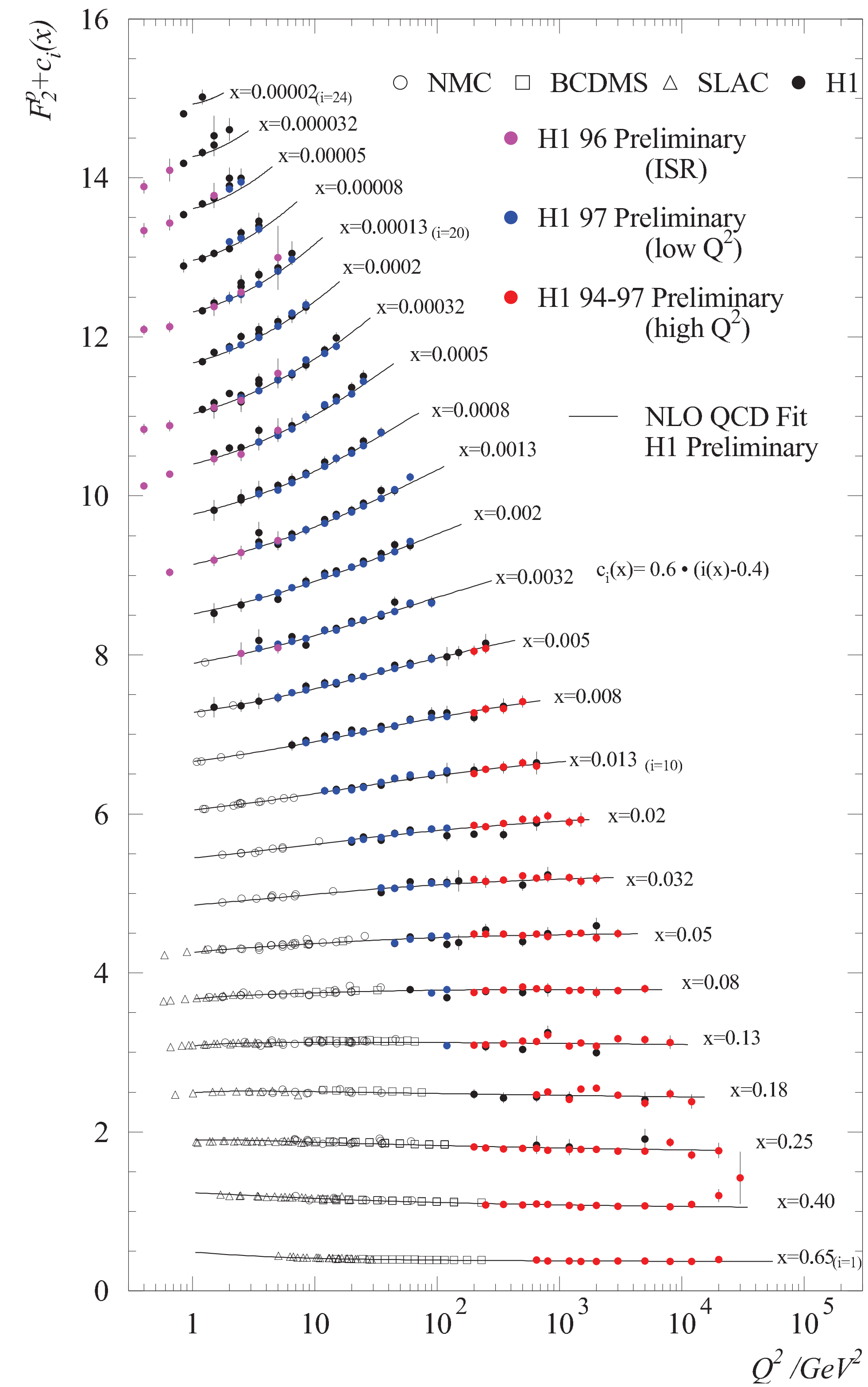

- Essentially x and Q^2 degrees of freedom

- Some scaling violations at small-x

Deep Inelastic Scattering

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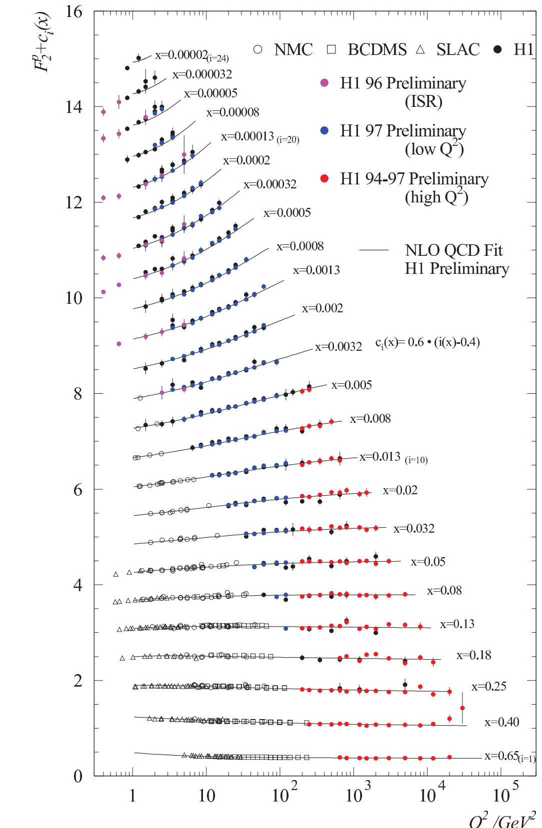


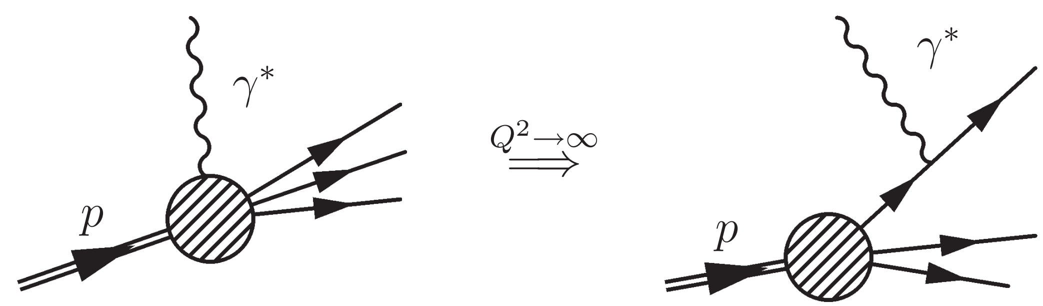
$$x = -\frac{q^2}{2P\cdot q} = \frac{Q^2}{2M\nu}$$

- This is known as Bjorken scaling

- Bjorken limit: x-fixed, $Q^2 \to \infty$

- This lead Feynman to introduce the "parton model"

⇒ inelastic *e-p* scattering is a sum of the elastic scatterings of the electron on free partons with the proton



any particle with no internal structure

- Picture valid for a fast moving nucleon, as in DIS

Parton Model

- In the Bjorken limit, one defines the functions

$$F_1(x) = \lim_{Q^2 \to \infty} W_1(Q^2, \nu)$$

$$F_2(x) = \lim_{Q^2 \to \infty} \frac{\nu}{M} W_2(Q^2, \nu)$$

- And in Feynman's parton model, the structure functions are sums of the parton densities constituting the proton, f_i

$$F_1(x) = \frac{1}{2} \sum_i e_i^2 f_i(x)$$

$$F_2(x) = x \sum_i e_i^2 f_i(x)$$

- f_i is the probability that the struck parton, i, carries a fraction, x, of the proton momentum and is called a parton distribution function (PDF)

- Total probability must be 1, so

$$\sum_i \int_0^1 dx \, x f_i(x) = 1$$

Deep Inelastic Scattering & Parton Model

- Results from DIS tell us that the fraction of the nucleon momentum carried by the quarks $\int dx\, xq(x)$ is only about 50%

$q(x) = f_q(x)$

⇒ gluons must play an important role in the structure of the nucleon

- In fact, much of our knowledge about QCD and the structure of the nucleon has been derived from Deep Inelastic Scattering experiments:

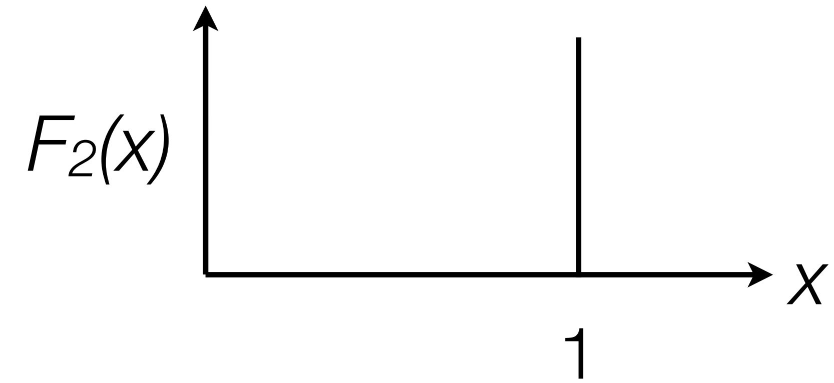
  - 2 up and 1 down valence quarks with electric charge 2/3 and -1/3 in the proton

  - the number of quarks is infinite because $\int dx\, q(x)$ does not seem to converge

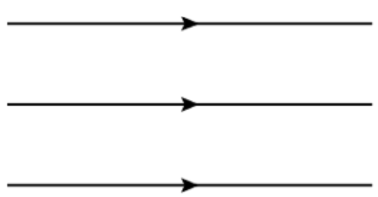
⇒ infinite number of quark and antiquark pairs

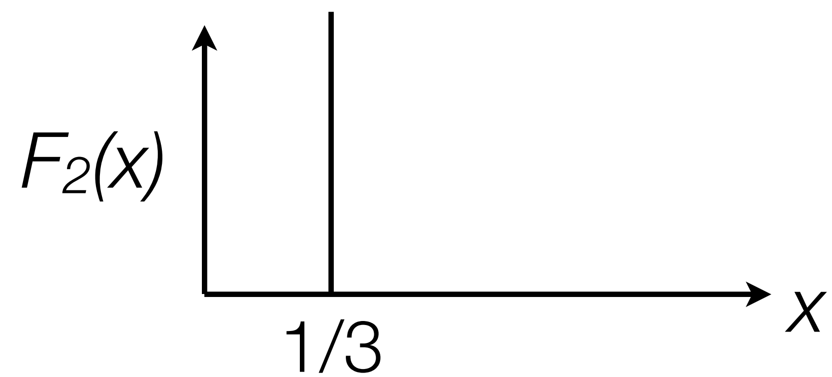
Parton Distribution Functions

- Point nucleon (a single quark carries all momentum): F_2 is a delta function at $x=1$

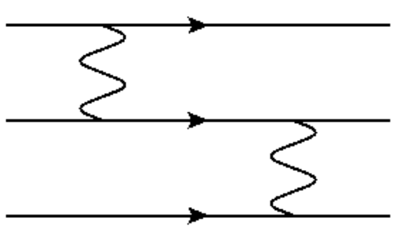


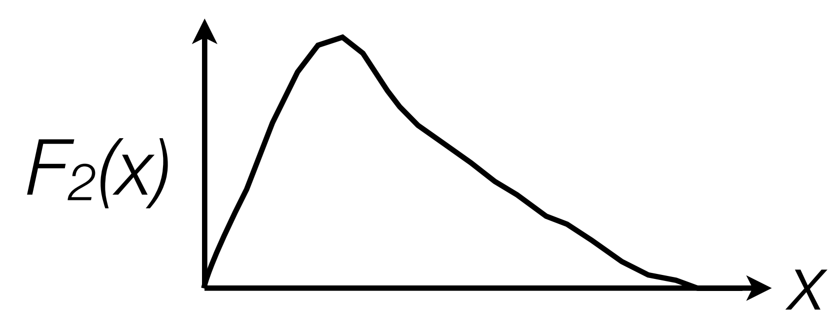
- Nucleon with 3 quarks (share the momentum)



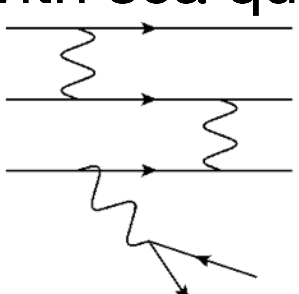


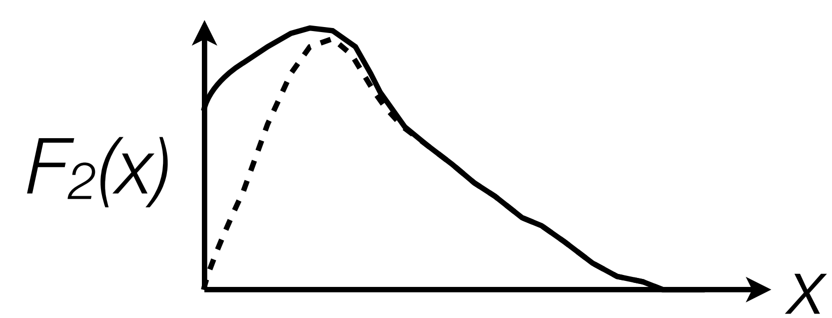
- Three interacting quarks (smeared out)





- With sea quarks (when a quark emits a q-$\bar{q}$ pair, they all have lower x than the original quark)





Parton Distribution Functions

- Proton contains 2 u and 1 d quarks termed “valence” quarks $u_v(x),\ d_v(x)$
- It is possible that a valence quark radiates a gluon which then turns into a q-q̄ pair which are termed “sea” quarks $u_s(x),\ d_s(x),\ s_s(x)$
- Can now write the proton and neutron structure functions as (ignoring heavy quarks)

$$F_2^p(x) = x\left\{\frac{4}{9}[u^p(x)+\bar{u}^p(x)] + \frac{1}{9}[d^p(x)+\bar{d}^p(x)] + \frac{1}{3}[s^p(x)+\bar{s}^p(x)]\right\}$$
$$F_2^n(x) = x\left\{\frac{4}{9}[u^n(x)+\bar{u}^n(x)] + \frac{1}{9}[d^n(x)+\bar{d}^n(x)] + \frac{1}{3}[s^n(x)+\bar{s}^n(x)]\right\}$$

- where total PDF of a quark is $q := q_v + q_s$
- Under isospin flip $u \leftrightarrow d$ and $p \leftrightarrow n$, assuming *charge symmetry* means

$$u(x) \equiv u^p(x) = d^n(x)$$
$$d(x) \equiv d^p(x) = u^n(x)$$

Parton Distribution Functions

- Further, we assume that (u,d,s) occur with equal probability in the sea

$$S := u_s = \bar{u}_s = d_s = \bar{d}_s = s_s = \bar{s}_s$$

- To obtain

$$F_2^p(x) = x\left\{\frac{1}{9}[4u_v(x) + d_v(x)] + \frac{4}{3}S(x)\right\}$$

$$F_2^n(x) = x\left\{\frac{1}{9}[4d_v(x) + u_v(x)] + \frac{4}{3}S(x)\right\}$$

- Expect at low $x \ll 1$ the sea quarks to dominate and

$$\frac{F_2^n}{F_2^p} \to 1$$

- while at high $x \to 1$ the valence quarks will dominate (and $u_v(x) > d_v(x)$)

$$\frac{F_2^n}{F_2^p} \to \frac{1}{4}$$

[2 up vs 1 down valence quarks in the proton]

Parton Distribution Functions

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$$S := u_s = \bar{u}_s = d_s = \bar{d}_s = s_s = \bar{s}_s$$

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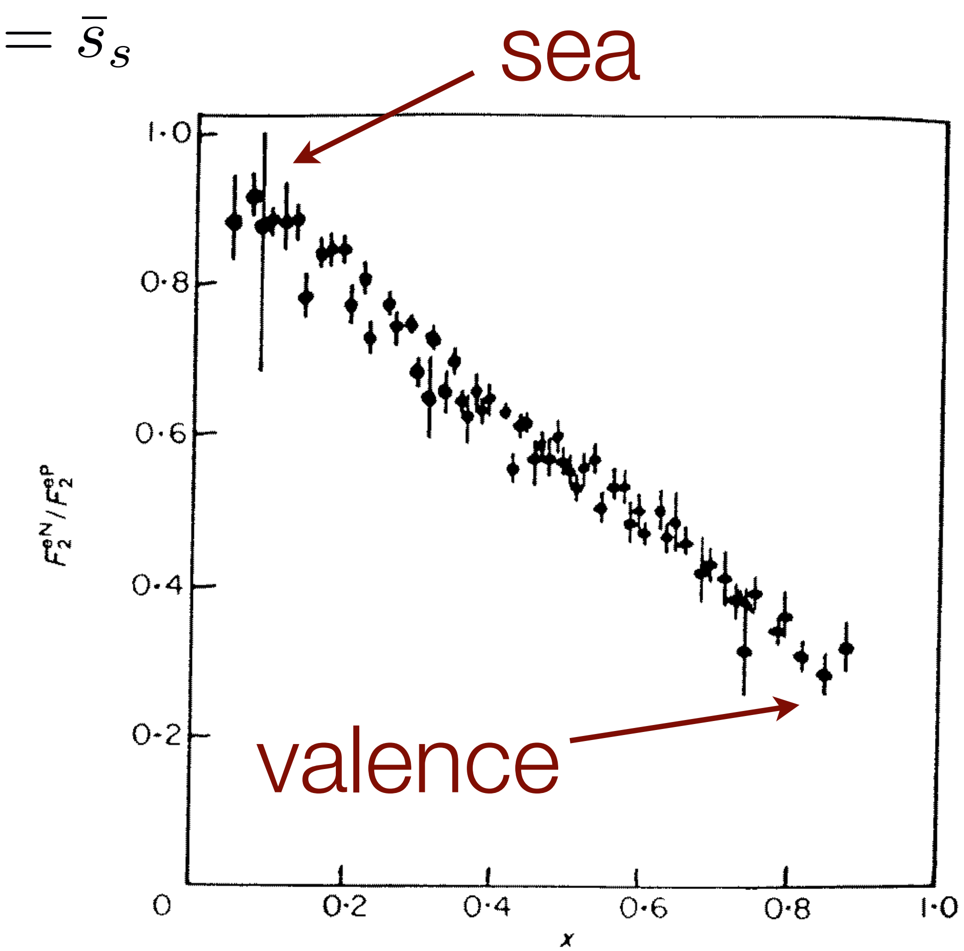
$$\frac{F_2^n}{F_2^p} \to 1$$



- while at high $x \to 1$ the valence quarks will dominate (and $u_v(x) > d_v(x)$)

$$\frac{F_2^n}{F_2^p} \to \frac{1}{4}$$

[2 up vs 1 down valence quarks in the proton]

Parton Distribution Functions

- Recall the momentum sum rule including all partons

$$\sum_i \int_0^1 dx\, x f_i(x) = 1$$

- But *e-p* scattering experiments find the light quark contributions to be

$$\int dx x[u(x) + \bar{u}(x)] \approx 0.36$$

$$\int dx x[d(x) + \bar{d}(x)] \approx 0.18$$

⇒ Almost half of the proton momentum is carried by electrically neutral partons

- Repeating the experiments with neutrinos indicates that these partons do not interact weakly

⇒ Missing momentum carried by gluons

- Need for inclusion of gluons in the parton model also evidenced by scaling violations at finite Q^2

Structure Functions

- Much of our knowledge about QCD and the structure of the nucleon has been derived from Deep Inelastic Scattering experiments, e.g. $lN \to lX \ \ (\nu N \to \mu^- X)$

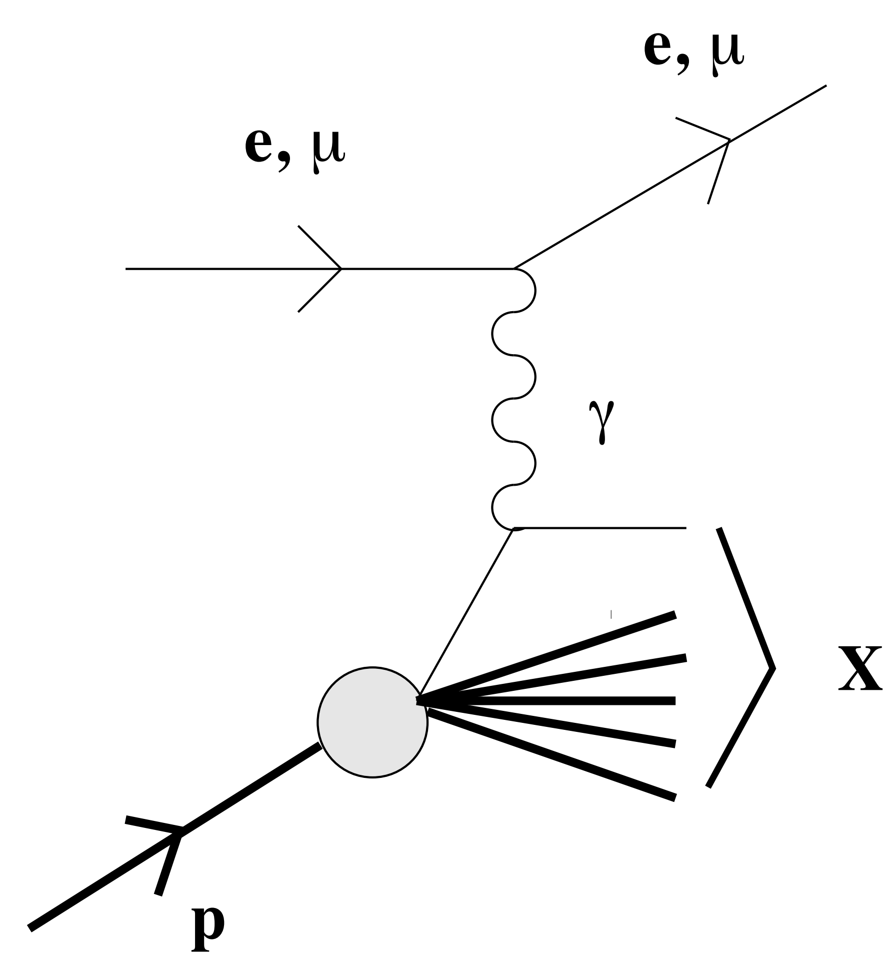
- The cross section is determined by the structure functions:

- F_1, F_2 when summing over beam and target polarisations

- F_3 when using neutrino beams $(\gamma \to W^+)$

- g_1, g_2 when both the beam and target are suitably polarised

- h_1 transversity - need Drell-Yan type processes



Moments of Structure Functions

- Relate partonic structure of hadrons and QCD via moments

renormalisation scale

$$\int_0^1 dx x^{n-2} F_2(x, Q^2) = E^{\mathcal{S}}_{F_2;v_n}(M^2/Q^2, g^{\mathcal{S}}) v_n^{\mathcal{S}}(M) + \underbrace{O(1/Q^2)}_{\text{higher twist}}$$

Wilson coefficients
calculated in perturbation theory

proton (forward) matrix elements
nonperturbative quantities - compute on the lattice

- similar relations for other structure functions

$F_1/F_2/F_3 \leftrightarrow v_n$	unpolarised
$g_1 \leftrightarrow a_n$	polarised
$g_2 \leftrightarrow a_n - d_n$	
$h_1 \leftrightarrow h_n$	transversity

Moments of Structure Functions

- Moments are obtained from forward (*q=0*) matrix elements of local operators

$$\langle N(p,s')|\mathcal{O}_q^{\{\mu_1\cdots\mu_n\}}|N(p,s)\rangle = 2\bar{u}(p,s')v_n^{(q)}\; p^{\{\mu_1}\cdots p^{\mu_n\}}u(p,s)$$

- where {...} indicates symmetrisation of indices and the subtraction of traces

⇒ twist-2 operators

dominating contribution in the deep inelastic (large Q^2) limit

- $\mathcal{O}_q^{\mu_1\cdots\mu_n} = \overline{q}\,\gamma^{\mu_1}\,\overleftrightarrow{D}^{\mu_2}\ldots\overleftrightarrow{D}^{\mu_n}\,q$

$$\overleftrightarrow{D} = \tfrac{1}{2}(\overrightarrow{D} - \overleftarrow{D})$$

- and similarly for the moments of polarised structure functions

$$\langle N(p,s')|\mathcal{O}_q^{5;\{\mu_1\cdots\mu_n\}}|N(p,s)\rangle = \bar{u}(p,s')\frac{a_{n-1}^{(q)}}{n+1}s^{\{\mu_1}p^{\mu_2}\cdots p^{\mu_n\}}u(p,s)$$

$$\mathcal{O}_q^{5;\mu_1\cdots\mu_n} = \overline{q}\,\gamma^{\mu_1}\gamma^5\,\overleftrightarrow{D}^{\mu_2}\ldots\overleftrightarrow{D}^{\mu_n}\,q$$

Moments of PDFs

- Interpretation in terms of moments of parton distribution functions *q(x)*

$$v_n^{(q)} = \int_0^1 dx\, x^{n-1}\left(q(x) + (-1)^n \bar{q}(x)\right) = \langle x^{n-1} \rangle_q$$

- $q(x)$ $(\bar{q}(x))$ "probability" to find a quark (antiquark) with momentum fraction *x*

- Polarised:

$$a_n^{(q)} = 2\int_0^1 dx\, x^n\left(\Delta q(x) + (-1)^n \Delta\bar{q}(x)\right) = 2\langle x^n \rangle_{\Delta q}$$

- with $\Delta q(x) = q_+(x) - q_-(x)$ and $q_+(x)$ $(q_-(x))$ "probability" of finding a quark with momentum fraction *x* and helicity equal (opposite) to that of the proton

- In particular $$\frac{1}{2} a_0^{(q)} = \langle 1 \rangle_{\Delta q} = \Delta q$$
  - is the fraction of the nucleon spin carried by quarks of flavour *q*

- and $g_A = \Delta u - \Delta d$

Moments of Polarised Structure Functions

- The moments of the polarised structure functions are

$$2\int_0^1 \mathrm{d}x\, x^n g_1(x,Q^2) = \frac{1}{2}\sum_{f=u,d} e^{(f)}_{1,n}(\frac{\mu^2}{Q^2}, g(\mu))a^{(f)}_n(\mu) \qquad + \text{ higher twist}$$

$$2\int_0^1 \mathrm{d}x\, x^n g_2(x,Q^2) = \frac{1}{2}\frac{n}{n+1}\sum_{f=u,d} [e^{(f)}_{2,n}(\frac{\mu^2}{Q^2}, g(\mu))d^{(f)}_n(\mu) - e^{(f)}_{1,n}(\frac{\mu^2}{Q^2}, g(\mu))\, a^{(f)}_n(\mu)] \qquad + \text{ higher twist}$$

twist-3 but not power suppressed

- In addition, the moments of transversity *h(x)* are related to matrix elements of the operators $\mathcal{O}_q^{\sigma;\mu\nu\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)\overline{q}\; i\sigma_{\mu\nu}\, \overleftrightarrow{D}^{\mu_1}\ldots\overleftrightarrow{D}^{\mu_n}\, q$

  - "probability" weighted by quark transverse-spin projection relative to the nucleon's transverse-spin direction

- Lowest moment gives the *tensor charge* δq

Operators

- Minkowski $\Rightarrow$ Euclidean - replace the Lorentz group by the orthogonal group $O(4)$
- Discrete space-time - reduce to the hypercubic group $H(4) \subset O(4)$
- *H*(4) is finite $\Rightarrow$ mixings [hep-lat/9602029]
- Using the following operators reduces mixings

$$
\begin{aligned}
\mathcal{O}_{v_{2\mathrm{a}}} &= \mathcal{O}^{\{14\}} \\
\mathcal{O}_{v_{2\mathrm{b}}} &= \mathcal{O}^{\{44\}} - \frac{1}{3}\left(\mathcal{O}^{\{11\}} + \mathcal{O}^{\{22\}} + \mathcal{O}^{\{33\}}\right) \\
\mathcal{O}_{v_3} &= \mathcal{O}^{\{114\}} - \frac{1}{2}\left(\mathcal{O}^{\{224\}} + \mathcal{O}^{\{334\}}\right) \\
\mathcal{O}_{v_4} &= \mathcal{O}^{\{1144\}} + \mathcal{O}^{\{2233\}} - \mathcal{O}^{\{1133\}} - \mathcal{O}^{\{2244\}}
\end{aligned}
$$

v_{2a} and v_{2b} different representation of the same continuum operators

Extracting Moments

- Recall from yesterday, we can write the lattice three-point function as

$$G_3(t,\tau;\vec{p}\,'\vec{p};\Gamma,\mathcal{O}) = \sqrt{Z^{\rm snk}(\vec{p}\,')\overline{Z}^{\rm src}(\vec{p})}F(\Gamma,\mathcal{J})e^{-E_{\vec{p}\,'}(t-\tau)}e^{-E_{\vec{p}}\tau}$$

- where

$$F(\Gamma,\mathcal{J}) = \frac{1}{4}{\rm Tr}\left\{\Gamma\left(\gamma_4 - i\frac{\vec{p}\,'\cdot\vec{\gamma}}{E_{\vec{p}\,'}} + \frac{m}{E_{\vec{p}\,'}}\right)\mathcal{J}\left(\gamma_4 - i\frac{\vec{p}\cdot\vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}}\right)\right\}$$

- and

$$\langle N(p',\,s')|\mathcal{O}(\vec{q})|N(p,\,s)\rangle = \bar{u}(p',\,s')\mathcal{J}u(p,\,s)$$

- so using the operator for v_{2a} as an example $\mathcal{O}^{\rm M}_{v_{2a}} = \mathcal{O}^{\rm M}_{\{01\}} = \frac{i}{4}\bar{q}\left(\gamma_0^{\rm M}\overleftrightarrow{D}_1 + \gamma_1^{\rm M}\overleftrightarrow{D}_0\right)q$

$$\frac{i}{4}\langle N(p,s')|\bar{q}\left(\gamma_0^{\rm M}\overleftrightarrow{D}_1+\gamma_1^{\rm M}\overleftrightarrow{D}_0\right)q|N(p,s)\rangle = v_2^{(q)}\,\frac{1}{2}\bar{u}(p,s')\left(\gamma_0^{\rm M}p_1+\gamma_1^{\rm M}p_0\right)u(p,s)$$

- Euclideanisation

$$\gamma_0^{\rm M} = \gamma_4^{\rm E},\;\; \gamma_i^{\rm M} = -i\gamma_i^{\rm E} \qquad p_4^{\rm E} = ip_0^{\rm M} \equiv iE(\vec{p}),\;\; p_i^{\rm E} = -p_i^{\rm M} \qquad D_4 = -iD^{({\rm M})0} \;\; D_i = -D^{({\rm M})i}$$

$$\frac{i}{4}\langle N(p,s')|\bar{q}\left(\gamma_4^{\rm E}\overleftrightarrow{D}_1+\gamma_1^{\rm E}\overleftrightarrow{D}_0\right)q|N(p,s)\rangle = v_2^{(q)}\,\frac{1}{2}\bar{u}(p,s')\left(-\gamma_4^{\rm E}p_1-i\gamma_1^{\rm E}E_N(\vec{p})\right)u(p,s)$$

Extracting Moments

- Taking $\Gamma = \Gamma_{\text{unpol}} \equiv \frac{1}{2}(1+\gamma_4)$

$$F(\Gamma, \mathcal{J}) = \frac{1}{4}\mathrm{Tr}\left\{\Gamma_{\text{unpol}}\left(\gamma_4 - i\frac{\vec{p}\cdot\vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}}\right)\left(-\gamma_4 p_1 - i\gamma_1 E_N(\vec{p})\right)\left(\gamma_4 - i\frac{\vec{p}\cdot\vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}}\right)\right\}$$

- So in this case our ratio will be

$$R(t,\tau;\vec{p},\vec{p};\mathcal{O}_{\{14\}},\Gamma_{\text{unpol}}) = \frac{G_{\Gamma_{\text{unpol}}}(t,\tau;\vec{p},\vec{p},\mathcal{O}_{\{14\}})}{G_2(t,\vec{p})} = i p_1 v_2^{(q)}$$

- and for v_{2b}

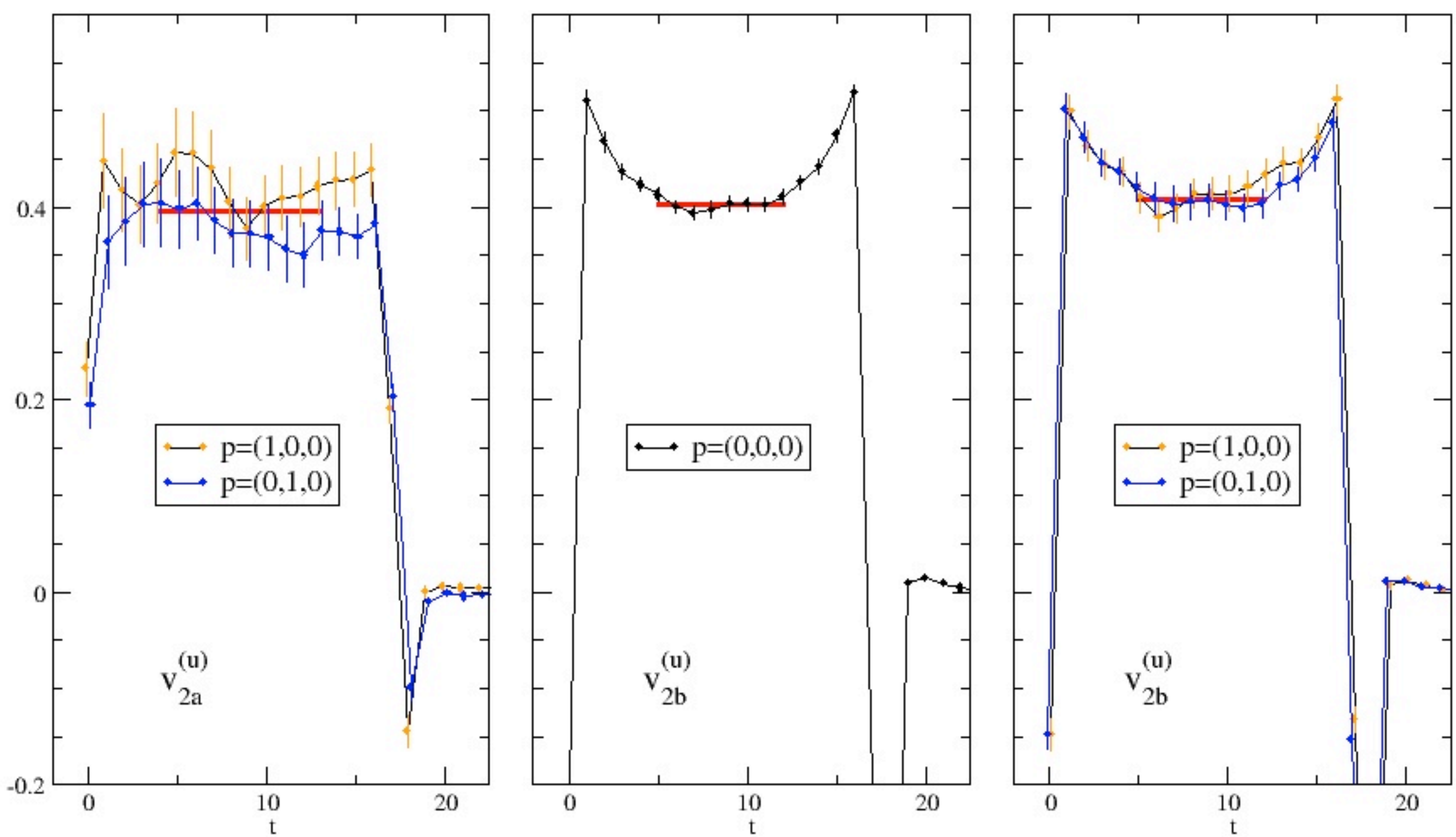
$$R_{\Gamma_4}(t,\tau;\vec{p},\vec{p};\mathcal{O}_{44}) = -\frac{E_{\vec{p}}^2 + \frac{1}{3}\vec{p}^{\,2}}{E_{\vec{p}}}\langle x\rangle$$

- Exercise: work out the corresponding ratio for the polarised case

$$\mathcal{O}_q^{5;\{43\}} \qquad \Gamma_{\text{pol}} = \Gamma_3 = \frac{1}{2}(1+\gamma_4)i\gamma_5\gamma_3$$

Ratios for $v_2 = \langle x \rangle$

- Excellent agreement for the two different representations of the same operator



Operator Renormalisation

- A huge field in it's own right and deserves its own set of lectures. (see e.g. R.Sommer [hep-lat/0611020])

- Renormalise bare lattice operators in scheme *S* and at scale *M*

$$\mathcal{O}^{\mathcal{S}}(M) = Z_{\mathcal{O}}^{\mathcal{S}}(M)\mathcal{O}_{bare}$$

- If there are more operators with
  - same quantum numbers
  - same or lower dimension

$$\mathcal{O}_i^{\mathcal{S}}(M) = \sum_j Z^{\mathcal{S}}_{\mathcal{O}_i\mathcal{O}_j}(M,a)\mathcal{O}_j(a)$$

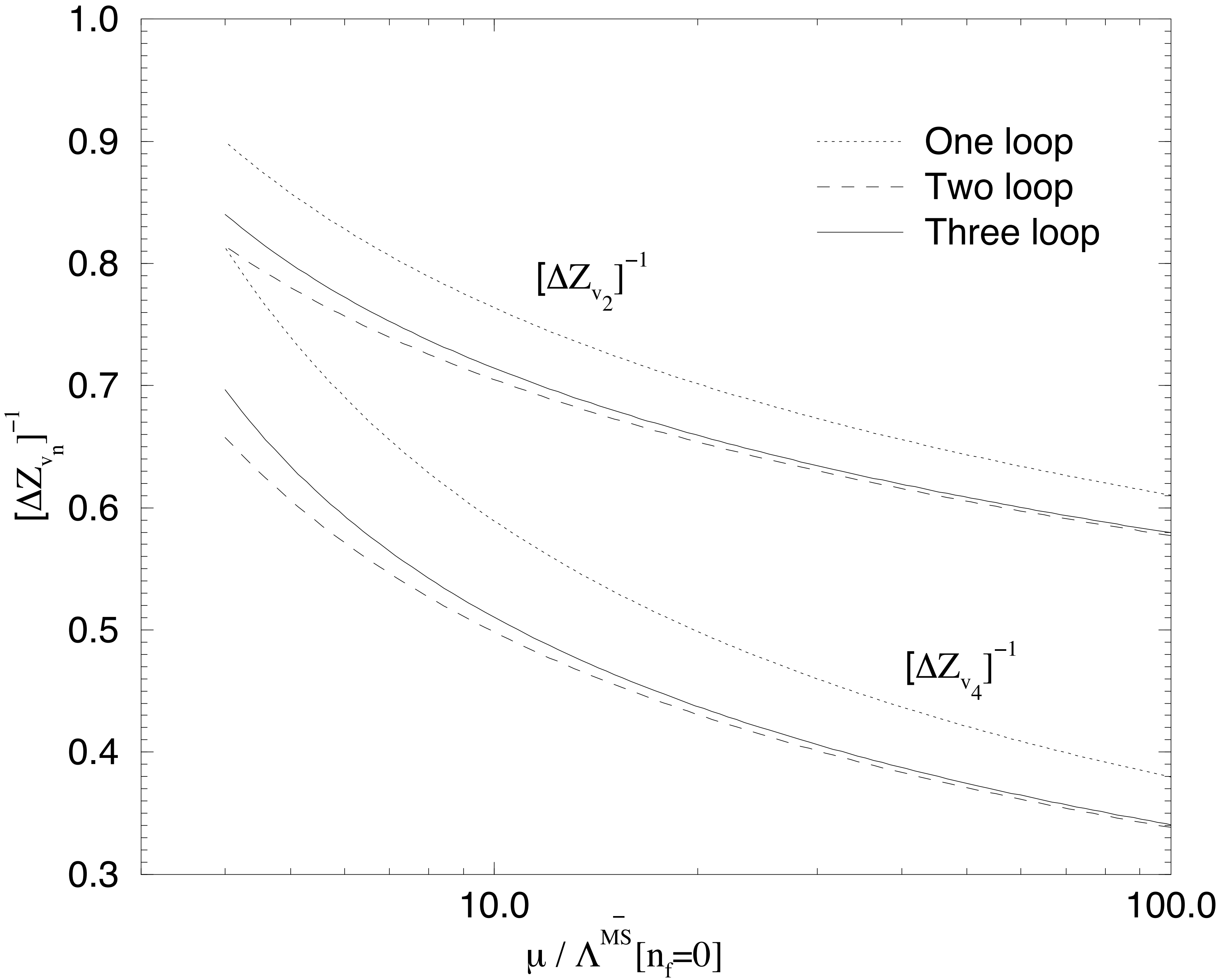
- Renormalisation Group Invariant quantities are defined as

$$\begin{aligned}\mathcal{O}^{\mathrm{RGI}} = Z_{\mathcal{O}}^{\mathrm{RGI}}\mathcal{O}_{bare} &= \Delta Z_{\mathcal{O}}^{\overline{MS}}(\mu)\mathcal{O}^{\overline{MS}}(\mu) \\ &= \Delta Z_{\mathcal{O}}^{MOM}(p)\mathcal{O}^{MOM}(p) \\ &= \Delta Z_{\mathcal{O}}^{\square}(a)\mathcal{O}(a)\end{aligned}$$

Independent of scale

$$[\Delta Z_{\mathcal{O}}^{\mathcal{S}}(M)]^{-1} = \left[2b_0 g^{\mathcal{S}}(M)^2\right]^{-\frac{d_0}{2b_0}} \exp\left\{\int_0^{g^{\mathcal{S}}(M)} d\xi \left[\frac{\gamma^{\mathcal{S}}(\xi)}{\beta^{\mathcal{S}}(\xi)} + \frac{d_0}{b_0\xi}\right]\right\}$$

Operator Renormalisation



Operator Renormalisation

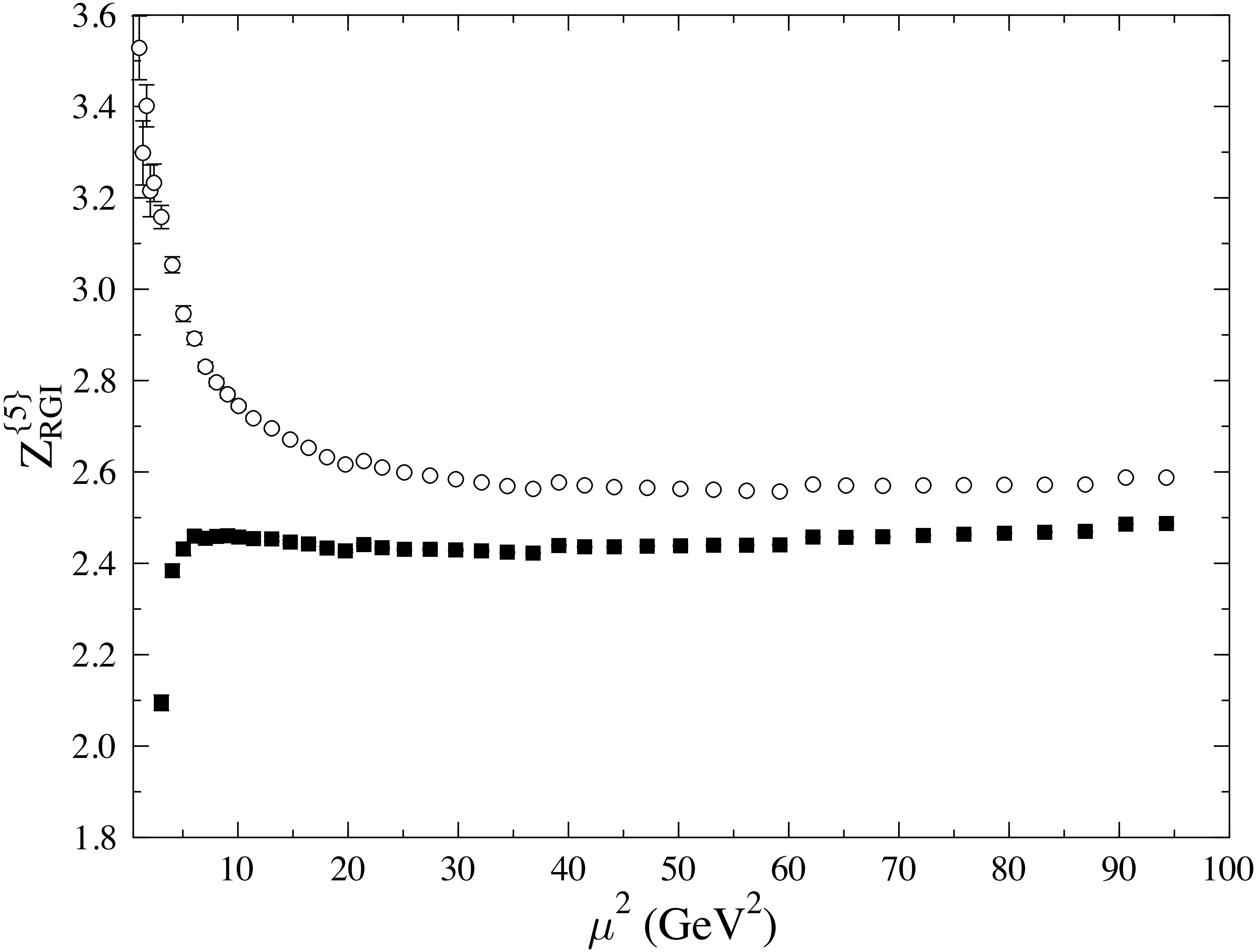
- Perturbative renormalisation:
  - Regard the lattice as a scheme
  - One loop perturbation theory

$$Z_{\mathcal{O}}^{\mathcal{S}}(M,g) = 1 - \frac{g^2}{16\pi^2} C_F \left[\gamma_{\mathcal{O};0} \ln(M) + B_{\mathcal{O}}^{\mathcal{S}}\right] + \dots$$

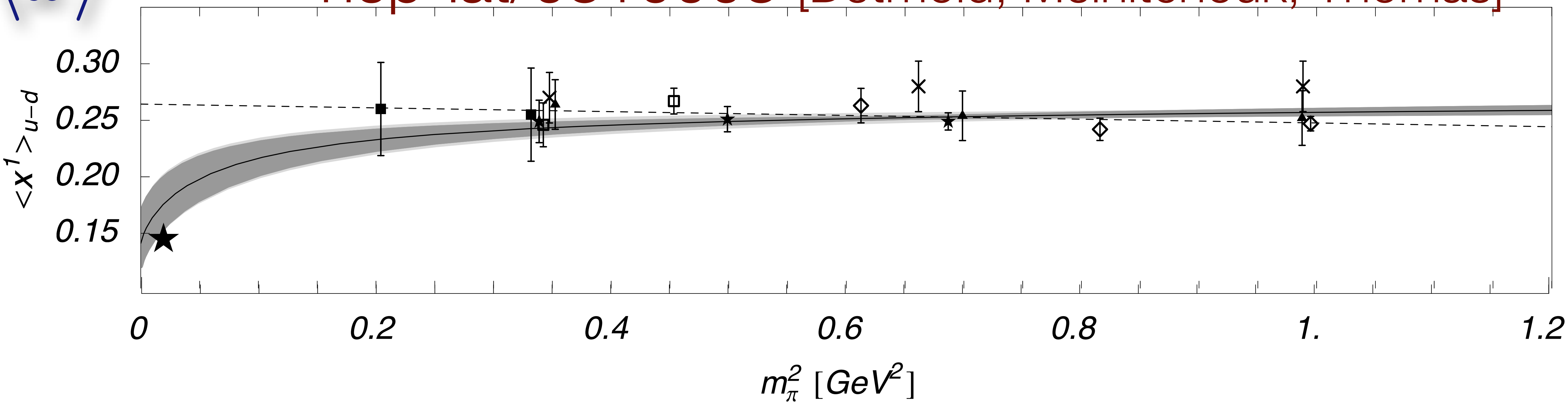
- Non perturbative renormalisation:
  - Schrödinger functional [ALPHA, hep-lat/9512009] - SF scheme
    - Gauge & quark fields take on specific values at the boundary of the space-time region (Dirichlet boundary conditions) -- a background field
  - Rome-Southampton Method [Martinelli et al., hep-lat/9411010]
    - Mimics (continuum) perturbation theory in a (RI')-MOM scheme

Operator Renormalisation

$$\Delta Z_{\mathcal{O}}^{RI'-MOM}(p) Z_{\mathcal{O}}^{RI'-MOM}(p, g_0)$$



$\langle x \rangle$ hep-lat/0310003 [Detmold, Melnitchouk, Thomas]



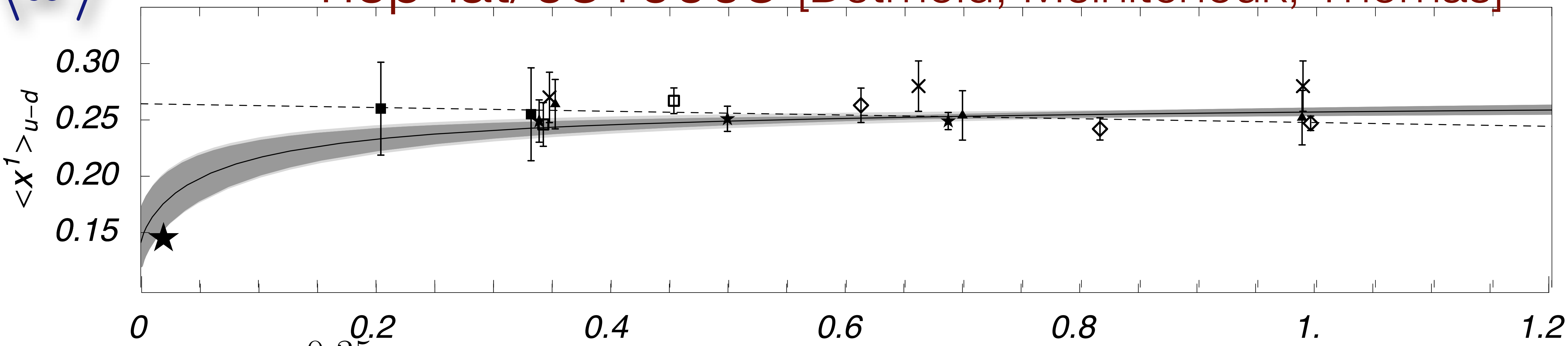
- First moment of the (isovector) nucleon parton distribution function

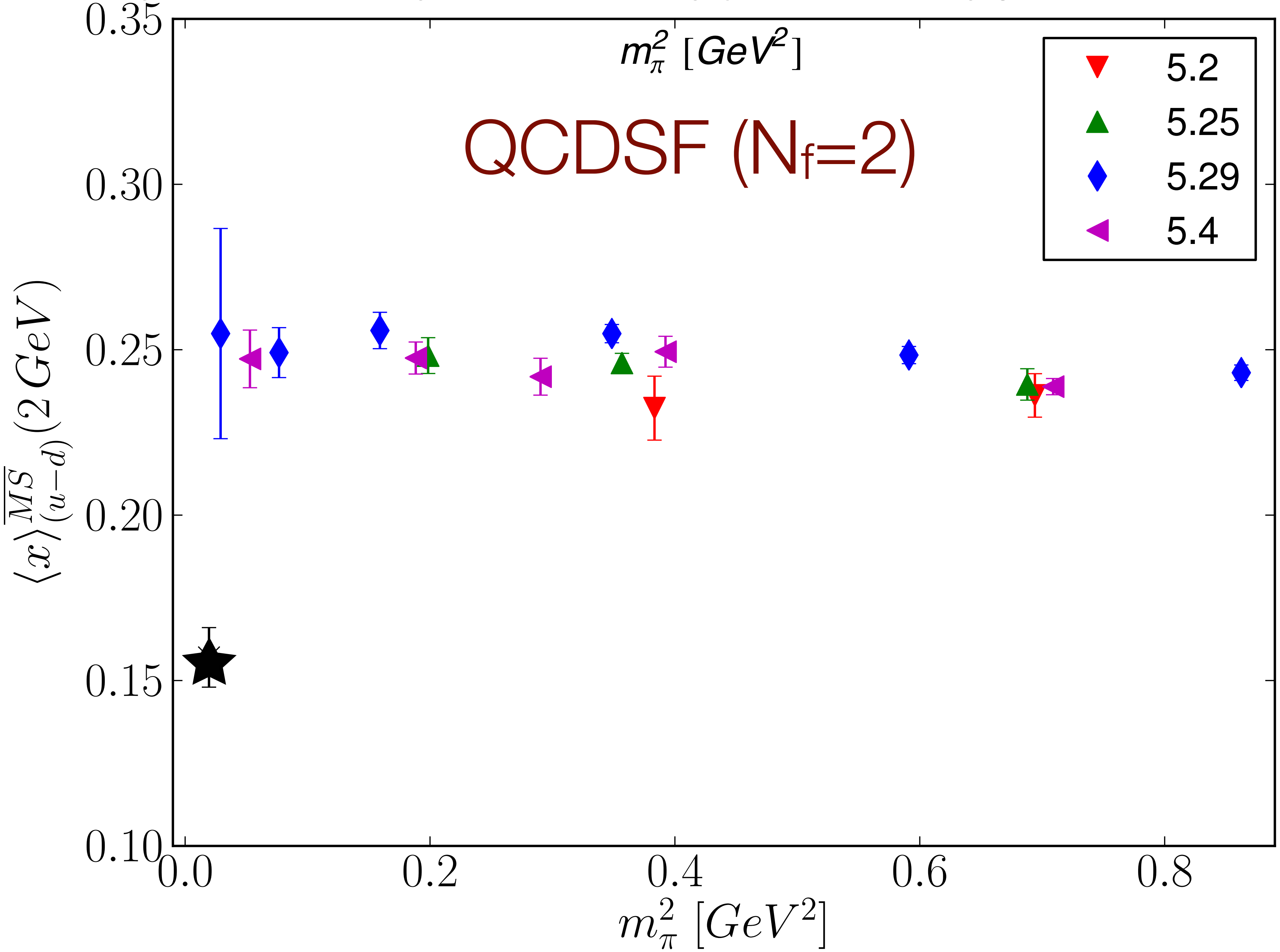
$$\langle x \rangle_\mu^{u-d} = \int_0^1 dx\, x(u(x,\mu) - d(x,\mu)) + \int_0^1 dx\, x(\bar{u}(x,\mu) - \bar{d}(x,\mu))$$

- Notorious for producing lattice results ≈ 2x too large for isovector nucleon
    - What are the possible systematic errors that could account for this
        - Quenching? Chiral physics? Finite volume effects?

$\langle x \rangle$

hep-lat/0310003 [Detmold, Melnitchouk, Thomas]

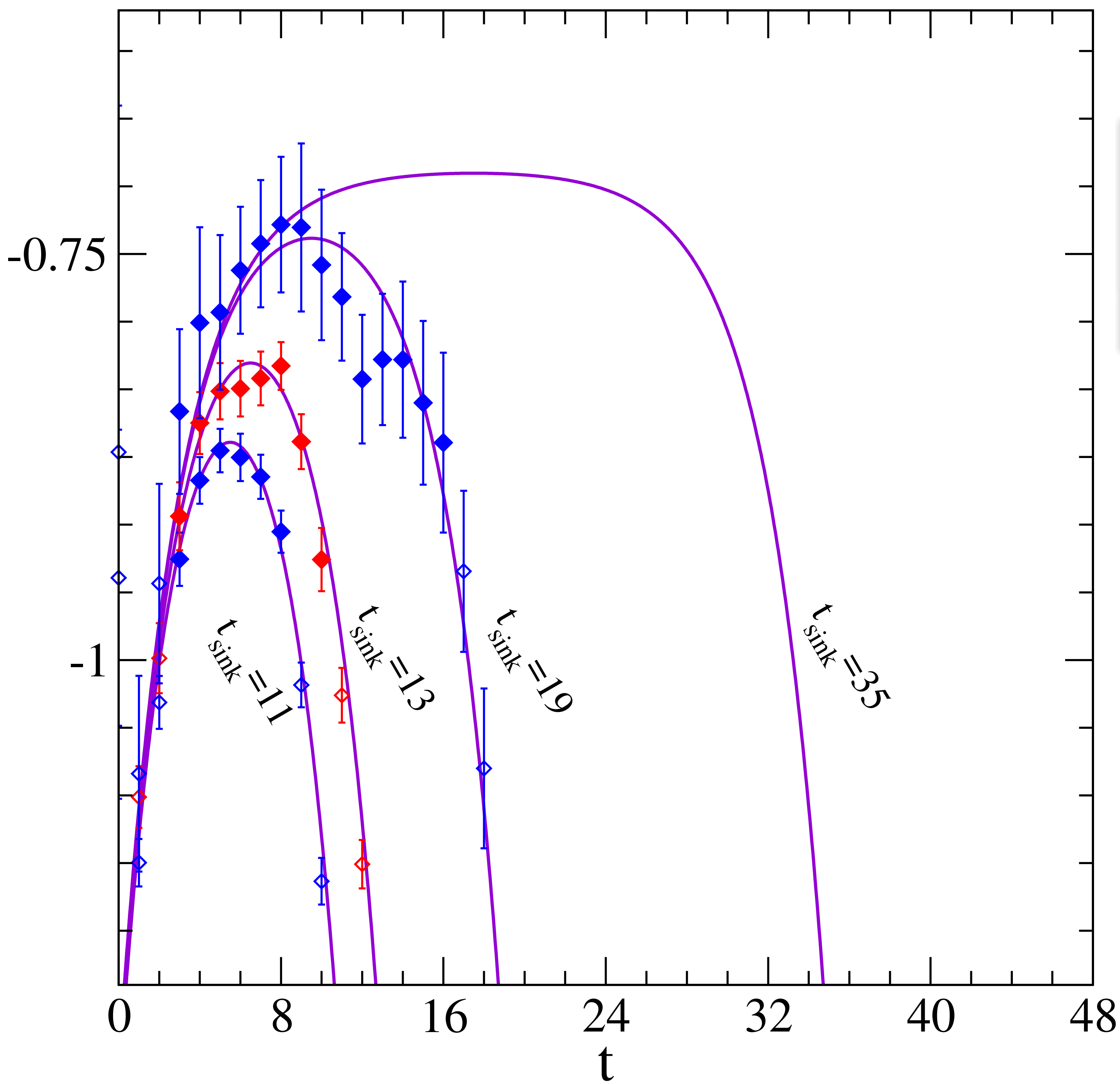




Excited State Contamination?

QCDSF

$a \approx 0.75\,\mathrm{fm},\;\; m_\pi \approx 650\,\mathrm{MeV}$



Evidence for severe excited state contamination!

- However ratios of lattice results look good, e.g.

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