Hadron Structure

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Lecture 1 - Recap

- Elastic scattering
  - Form factors
    - Surprises in their Q^2 dependence
    - Density distributions in a hadron
- Lattice techniques
  - Three-point functions via sequential source method
  - Extraction of matrix elements

Lecture 2 - all about Form Factors

- Extracting matrix elements from Lattice three-point functions
- Extracting form factors from matrix elements
- Lattice nucleon form factors
    - Compare with experiment
    - Investigation of systematic errors
    - Flavour dependence
- Lattice pion form factor
    - Twisted boundary conditions
- Other hadron form factors

Extracting Matrix Elements

- Recall hadronic form of the nucleon 3pt function

$$G_\Gamma(t,\tau,\vec{p},\vec{p}\,',\mathcal{O}) = \sum_{s,s'} e^{-E_{\vec{p}\,'}(t-\tau)} e^{-E_{\vec{p}}\tau} \Gamma_{\beta\alpha} \langle \Omega | \chi_\alpha(0) | N(p',s') \rangle \langle N(p',s') | \mathcal{O}(\vec{q}) | N(p,s) \rangle \langle Np,s | \overline{\chi}_\beta(0) | \Omega \rangle$$

- Need to remove time dependence and wave function amplitudes

Form a ratios with the nucleon 2pt function

$$G_2(t,\vec{p}) = \sum_s e^{-E_p t} \Gamma_{\beta\alpha} \langle \Omega | \chi_\alpha | N(p,s) \rangle \langle N(p,s) | \overline{\chi}_\beta | \Omega \rangle$$

- E.g.

$$R(t,\tau;\vec{p}\,',\vec{p};\mathcal{O}) = \frac{G_\Gamma(t,\tau;\vec{p}\,',\vec{p},\mathcal{O})}{G_2(t,\vec{p}\,')} \left[\frac{G_2(\tau,\vec{p}\,')G_2(t,\vec{p}\,')G_2(t-\tau,\vec{p})}{G_2(\tau,\vec{p})G_2(t,\vec{p})G_2(t-\tau,\vec{p}\,')} \right]^{\frac{1}{2}}$$

Extracting Matrix Elements

- Using the relation for spinors

$$\bar{u}(\vec{p},\sigma')\Gamma u(\vec{p},\sigma) = \mathrm{Tr}\Gamma(E\gamma_4 - i\vec{p}\cdot\vec{\gamma} + m)\frac{1}{2}\left(1 - \gamma_5\gamma_4\frac{\vec{p}\cdot\vec{s}}{EM} + i\gamma_5\frac{\vec{\gamma}\cdot\vec{s}}{m}\right)\delta_{\sigma\sigma'}$$

- We can write the two point function as

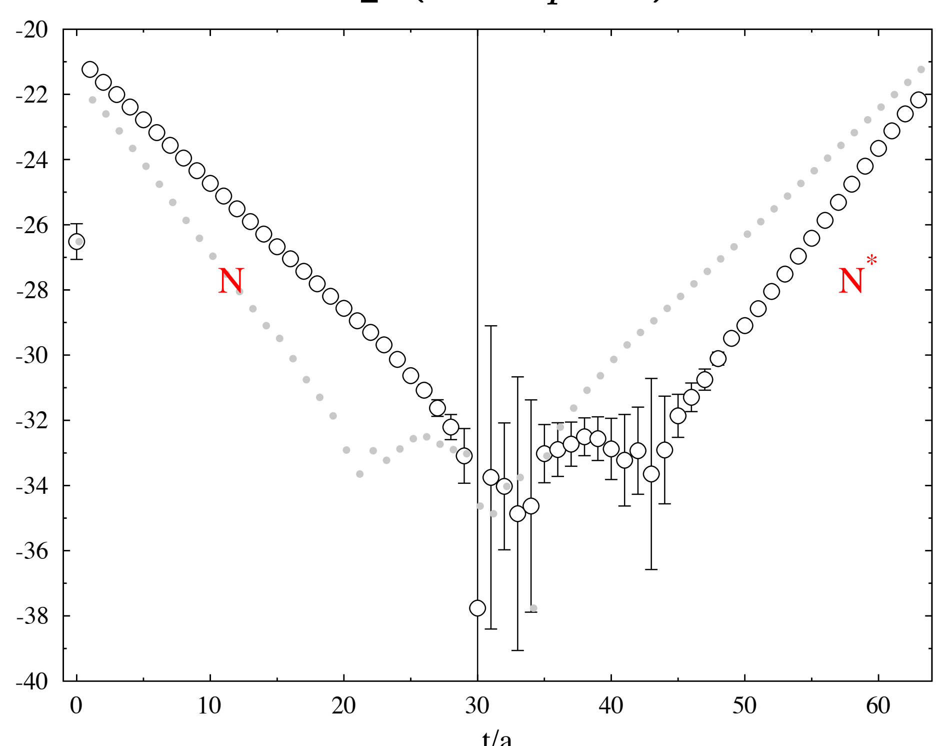
$$G_2(t,\vec{p}) = \sum_s \frac{\sqrt{Z^{\mathrm{snk}}(\vec{p})}\sqrt{\overline{Z}^{\mathrm{src}}(\vec{p})}}{2E_{\vec{p}}}\,\mathrm{Tr}\bar{u}(\vec{p},s)\Gamma u(\vec{p},s)[e^{-E_p t} + e^{-E'_{\vec{p}}(T-t)}]$$

+ *v*-spinor terms with opposite parity

- Use $\Gamma_4 = \frac{1}{2}(1+\gamma_4)$ to maximise overlap with positive parity forward propagating state

$$G_2(t,\vec{p}) = \sqrt{Z^{\mathrm{snk}}(\vec{p})\overline{Z}^{\mathrm{src}}(\vec{p})}\left[\left(\frac{E_{\vec{p}}+m}{E_{\vec{p}}}\right)e^{-E_p t} + \left(\frac{E'_{\vec{p}}+m'}{E'_{\vec{p}}}\right)e^{-E'_{\vec{p}}(T-t)}\right]$$

log(G₂)



Extracting Matrix Elements

- Similarly for the three-point function, if we express the nucleon matrix element under study as

$$\langle N(p',\, s')|\mathcal{O}(\vec{q})|N(p,\, s)\rangle = \bar{u}(p',\, s')\mathcal{J}u(p,\, s)$$

- E.g., for the EM current $\mathcal{O} = J^\mu$

$$\mathcal{J} = \gamma^\mu F_1(Q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2M}F_2(Q^2)$$

- Then we have

$$G_3(t,\tau;\vec{p}\,'\vec{p};\Gamma,\mathcal{O}) = \sqrt{Z^{\rm snk}(\vec{p}\,')\overline{Z}^{\rm src}(\vec{p})}F(\Gamma,\mathcal{J})e^{-E_{\vec{p}\,'}(t-\tau)}e^{-E_{\vec{p}}\tau}$$

- where

$$F(\Gamma,\mathcal{J}) = \frac{1}{4}\mathrm{Tr}\left\{\Gamma\left(\gamma_4 - i\frac{\vec{p}\,'\cdot\vec{\gamma}}{E_{\vec{p}\,'}} + \frac{m}{E_{\vec{p}\,'}}\right)\mathcal{J}\left(\gamma_4 - i\frac{\vec{p}\cdot\vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}}\right)\right\}$$

Example

- If we consider the particular case

$$\Gamma = \Gamma_{\rm unpol} \equiv \frac{1}{2}(1+\gamma_4),\ \mathcal{O} = J^\mu,\ \vec{p}\,' = \vec{p} \Rightarrow q = 0$$

- then the contribution from F_2 to the matrix element drops out (proportional to q)

$$\langle N(p',\,s')|J^\mu(0)|N(p,\,s)\rangle = \bar{u}(p',\,s')\gamma^\mu u(p,\,s) F_1(Q^2=0) + \bar{u}(p',s')i\frac{\sigma^{\mu\nu}q_\nu}{2M}u(p,s)F_2(Q^2=0)$$

q=0

- Euclideanisation $\gamma_0^{\rm M} = \gamma_4^{\rm E},\ \gamma_i^{\rm M} = -i\gamma_i^{\rm E}$ $\quad p_4^{\rm E} = ip_0^{\rm M} \equiv iE(\vec{p}),\ p_i^{\rm E} = -p_i^{\rm M}$

- $$\langle N(p',\,s')|\bar{q}\gamma_\mu^{\rm E} q|N(p,\,s)\rangle = \bar{u}(p',\,s')\gamma_\mu^{\rm E} u(p,\,s) F_1(Q^2=0) + \bar{u}(p',s')\frac{\sigma_{\mu\nu}^{\rm E} q_\nu^{\rm E}}{2M}u(p,s)F_2(Q^2=0)$$

q=0

Using the local vector current $J^\mu = \bar{q}\gamma^\mu q$

Example

$$F(\Gamma,\mathcal{J}) = \frac{1}{4}\mathrm{Tr}\left\{\Gamma\left(\gamma_4 - i\frac{\vec{p}\,'\cdot\vec{\gamma}}{E_{\vec{p}\,'}} + \frac{m}{E_{\vec{p}\,'}}\right)\mathcal{J}\left(\gamma_4 - i\frac{\vec{p}\cdot\vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}}\right)\right\}$$

- Then the three-point function is now

$$G_3(t,\tau;\vec{p}\,'\vec{p};\Gamma,\mathcal{O}) = \sqrt{Z^{\mathrm{snk}}(\vec{p}\,')\overline{Z}^{\mathrm{src}}(\vec{p})}F(\Gamma,\mathcal{J})e^{-E_{\vec{p}\,'}(t-\tau)}e^{-E_{\vec{p}}\tau}$$

- with

$$\mathcal{J} = \gamma^\mu F_1(Q^2)$$

- and

$\vec{p} = \vec{p}\,' = 0$

$$F(\Gamma_{\mathrm{unpol}},\gamma_4) = \frac{1}{2E_{\vec{p}}E_{\vec{p}\,'}}\left[(E_{\vec{p}}+m)(E_{\vec{p}\,'}+m)+\vec{p}\,'\cdot\vec{p}\right]$$

=2

$$F(\Gamma_{\mathrm{unpol}},\gamma_i) = \frac{-i}{2E_{\vec{p}}E_{\vec{p}\,'}}\left[(E_{\vec{p}}+m)\vec{p}\,'+(E_{\vec{p}\,'}+m)\vec{p}\right]$$

=0

Example

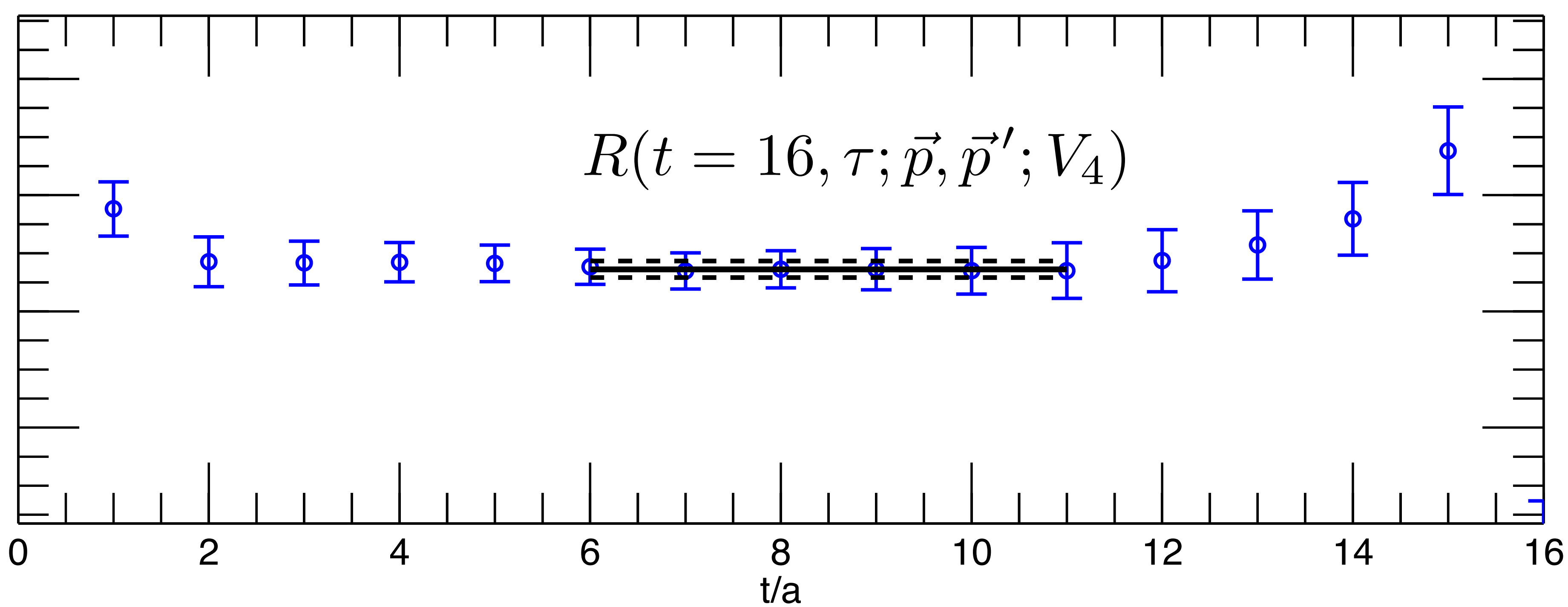
- So our ratio determines

$$R(t,\tau;\vec{p}',\vec{p};\mathcal{O}) = \frac{G_\Gamma(t,\tau;\vec{p}',\vec{p},\mathcal{O})}{G_2(t,\vec{p}')}\left[\frac{G_2(\tau,\vec{p}')G_2(t,\vec{p}')G_2(t-\tau,\vec{p})}{G_2(\tau,\vec{p})G_2(t,\vec{p})G_2(t-\tau,\vec{p}')}\right]^{\frac{1}{2}}$$

$$= \sqrt{\frac{E_{\vec{p}'}E_{\vec{p}}}{(E_{\vec{p}}+m)(E_{\vec{p}}+m)}}F(\Gamma,\mathcal{J}_{\mathcal{O}}(\vec{q})) \quad 0 \ll \tau \ll t \ll \frac{1}{2}T$$

$$= F_1(q^2=0)$$

$$\Gamma_{\mathrm{unpol}} = \frac{1}{2}(1+\gamma_4),\ \mathcal{O}=V_4\equiv\gamma_4,\ \vec{p}'=\vec{p}=0$$



Other Useful Combinations

Exercise: Prove them!

$$R(t,\tau;\vec{0},\vec{p};V_4,\Gamma_4) = F_1(q^2) - \frac{E_{\vec{p}}-M}{2M}F_2(q^2) = G_E(q^2)$$

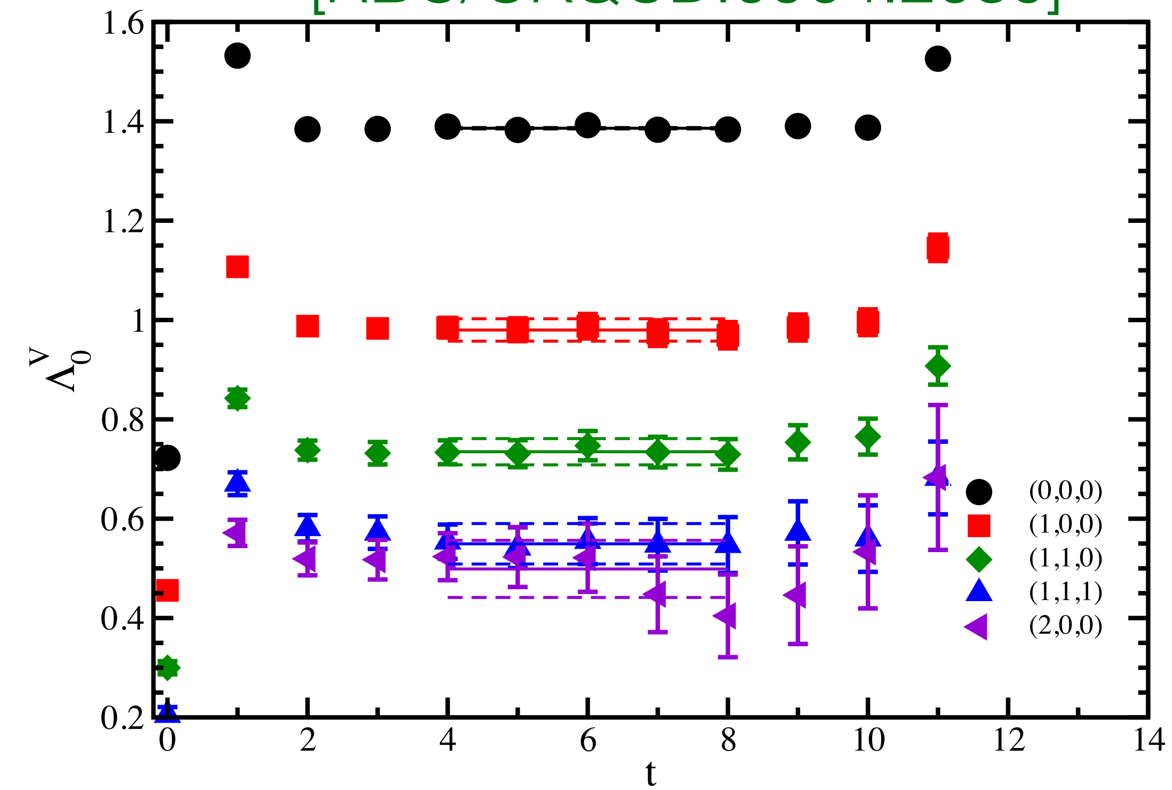
$$R(t,\tau;\vec{0},\vec{p};V_i,\Gamma_4) = -i\frac{q_i}{E+M}G_E(q^2)$$

$$R(t,\tau;\vec{0},\vec{p};V_i,\Gamma_j) = -i\epsilon_{ijk}\frac{q_k}{E+M}G_M(q^2)$$

$$\Gamma_4 \equiv \Gamma_{\text{unpol}}$$

$$\Gamma_j = \frac{1}{2}(1+\gamma_4)i\gamma_5\gamma_j$$

[RBC/UKQCD:0904.2039]



- Certain combinations of parameters and kinematics give access to the form factors

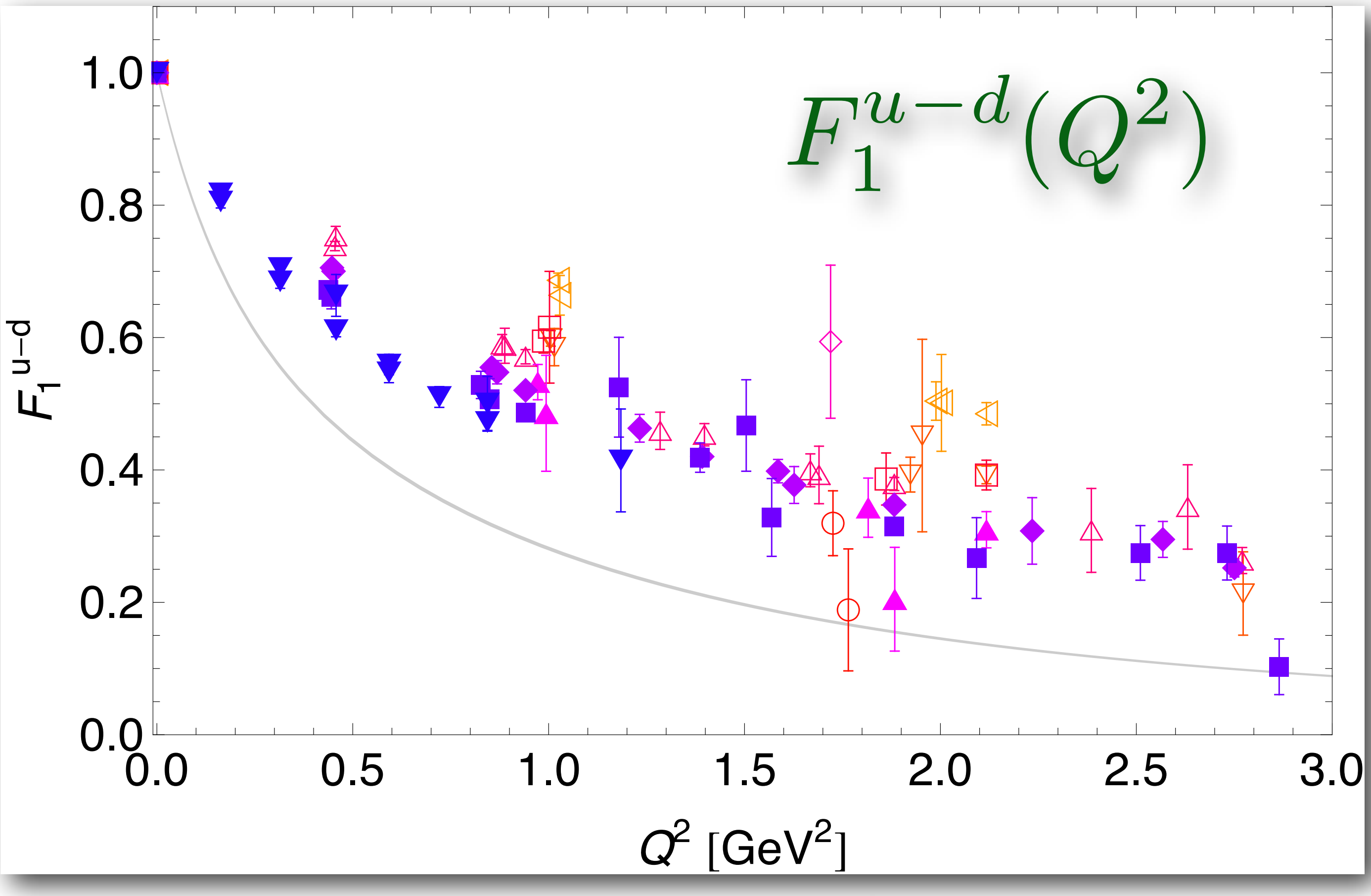
- It is possible to have several choices giving access to the form factors at a fixed Q^2

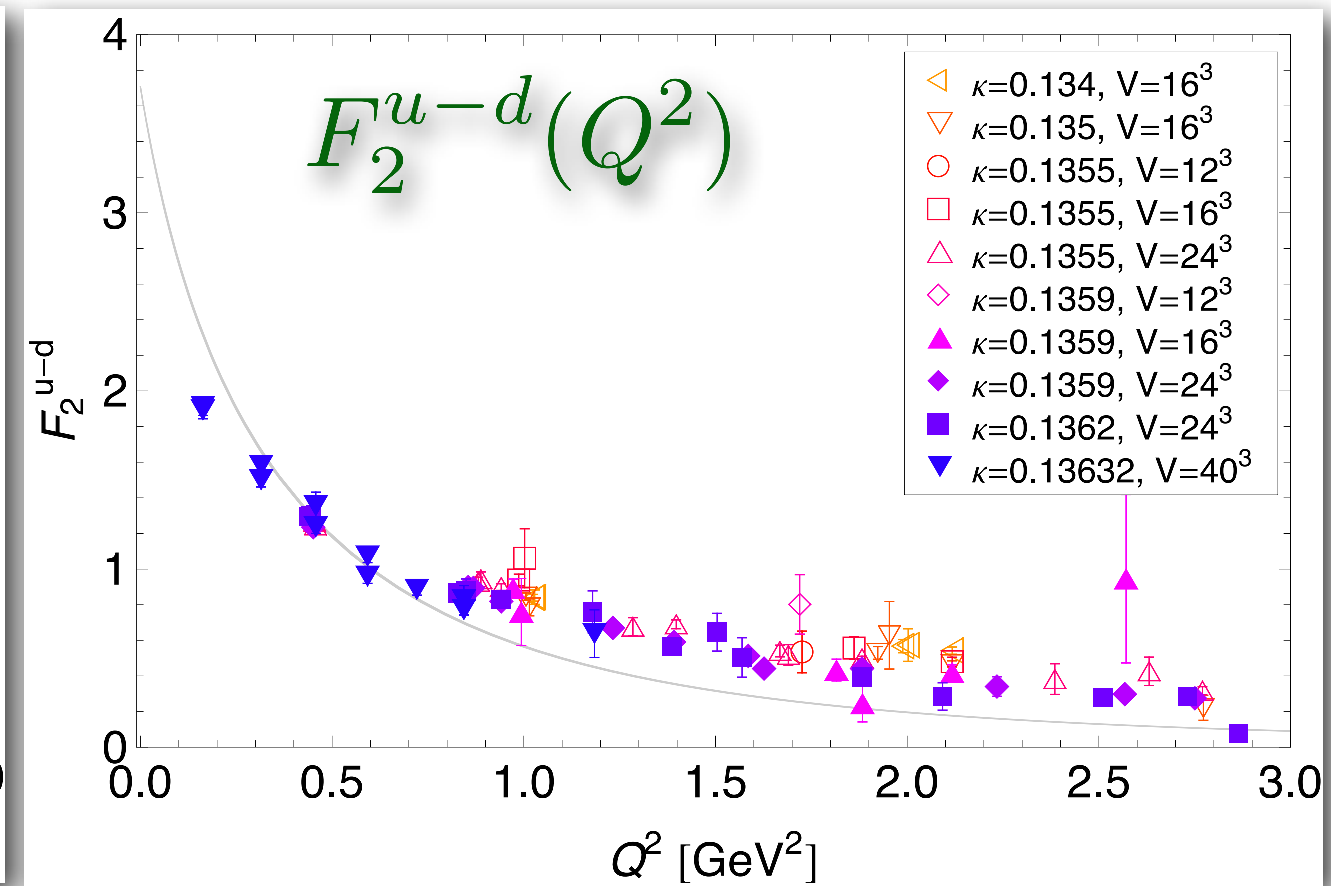
Overdetermined set of simultaneous equations that can be solved for

F_1, F_2 or G_E, G_M

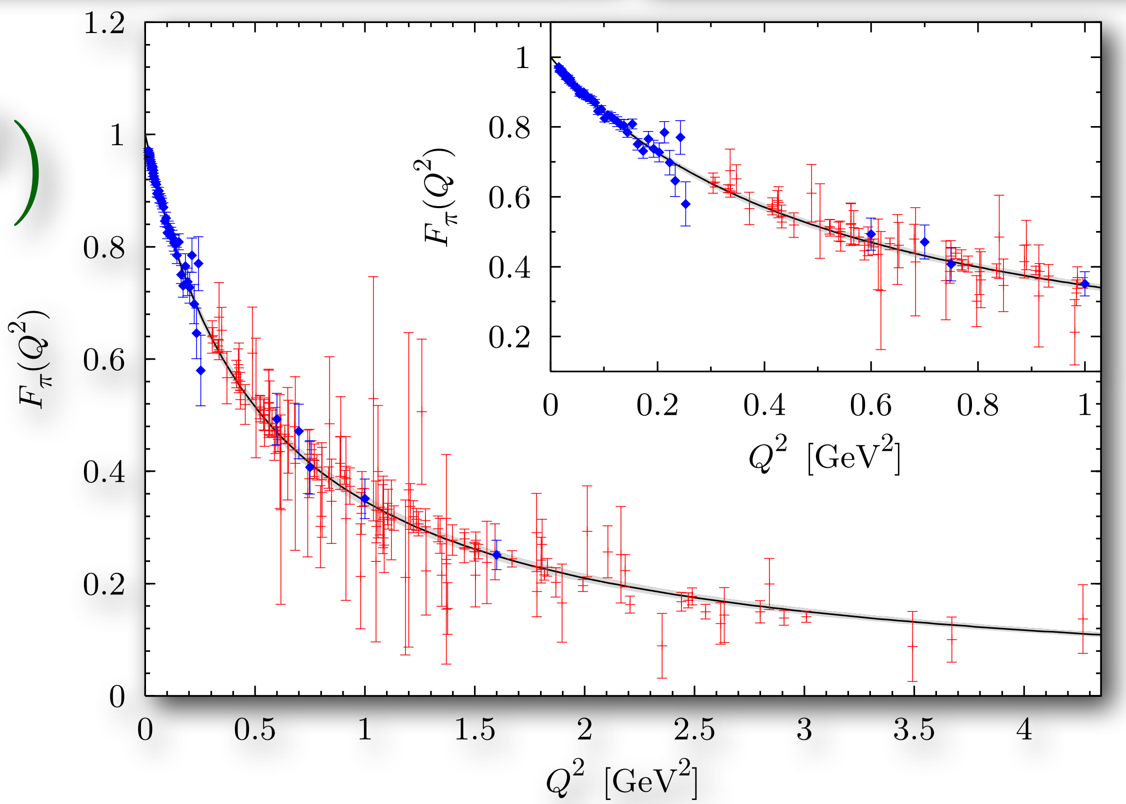
Typical Examples

More detailed look at lattice results to follow





$F_\pi(Q^2)$



QCDSF:
1106.3580
hep-lat/0608021

Some Recent Works

[Not an exhaustive list]

Nucleon

- Review: Ph. Hägler, 0912.5483
- QCDSF: 1106.3580
- ETMC: 1102.2208
- LHPC: 1001.3620
- RBC/UKQCD: 0904.2039
- CSSM: hep-lat/0604022

Pion

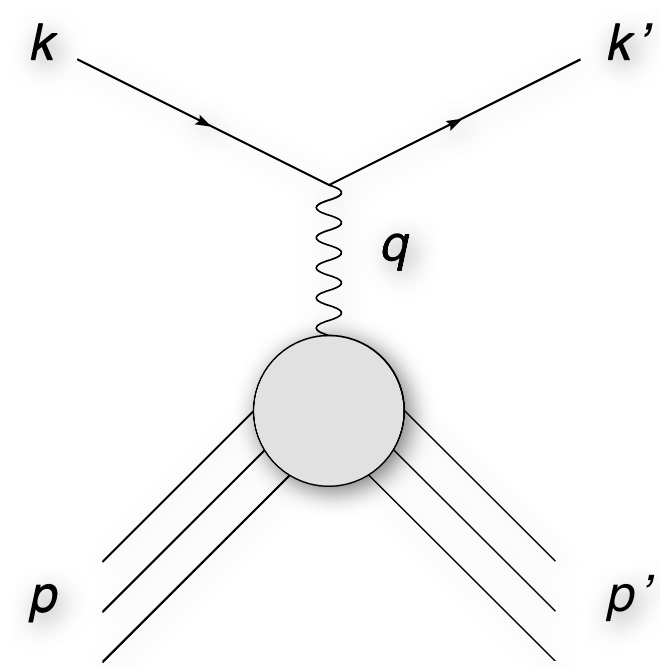
- Mainz: 1109.0196
- PACS-CS: 1102.3652
- JLQCD/TWQCD: 0905.2465
- ETMC: 0812.4042
- RBC/UKQCD: 0804.3971
- QCDSF: hep-lat/0608021

Electromagnetic Form Factors

Electromagnetic Form Factors

- Recall

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} |F(q^2)|^2$$



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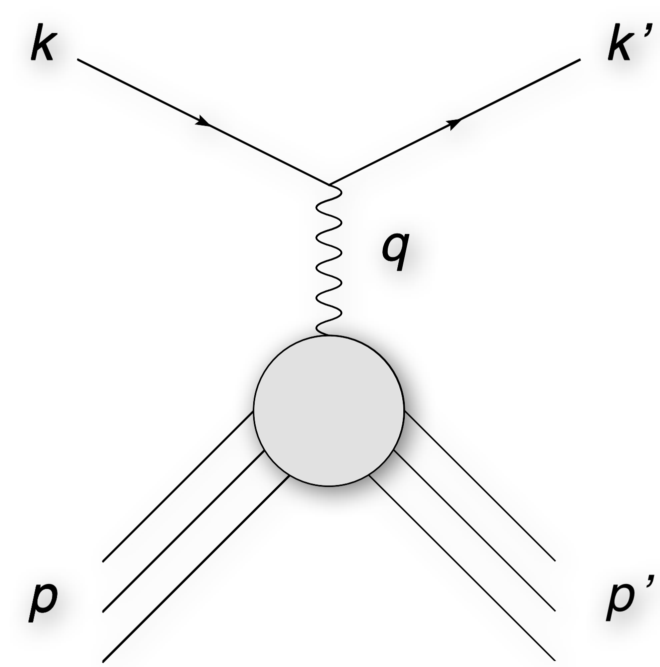

If a nucleon was a point-like object with no internal structure, a probe would simply measure its e.g. charge for all q^2

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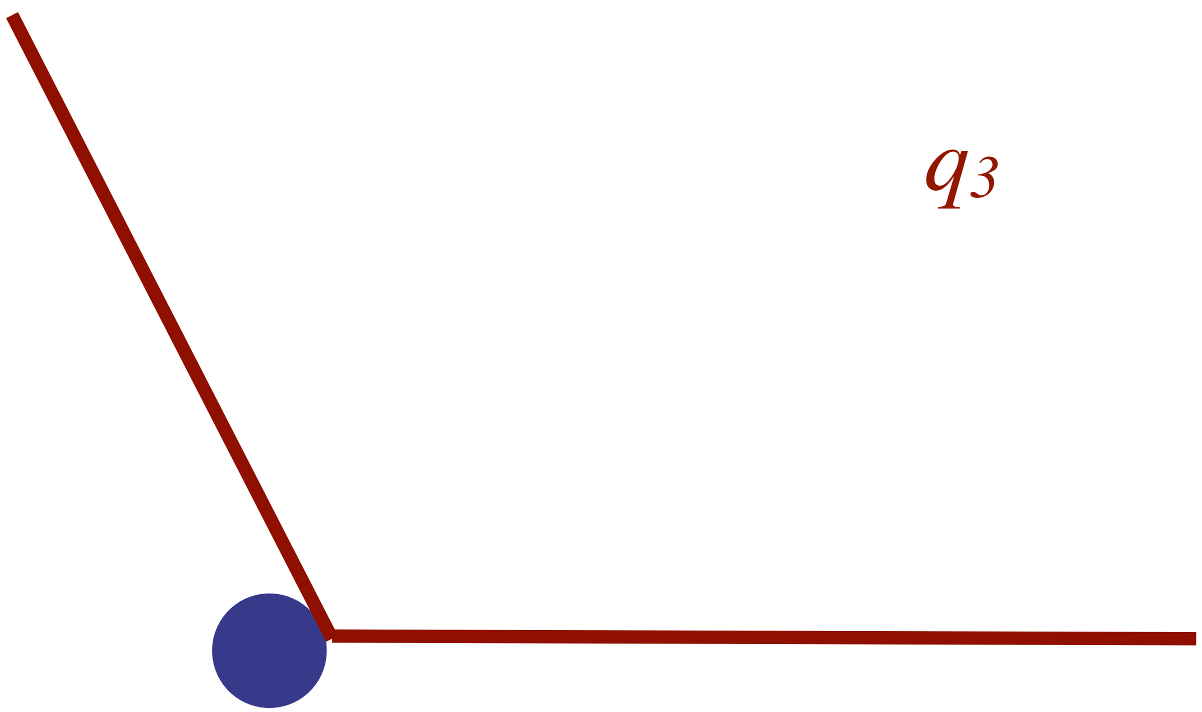

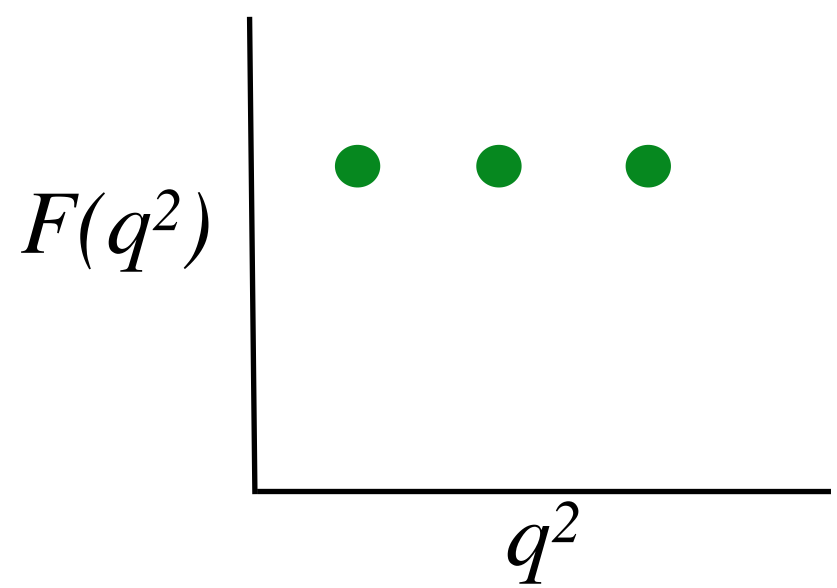

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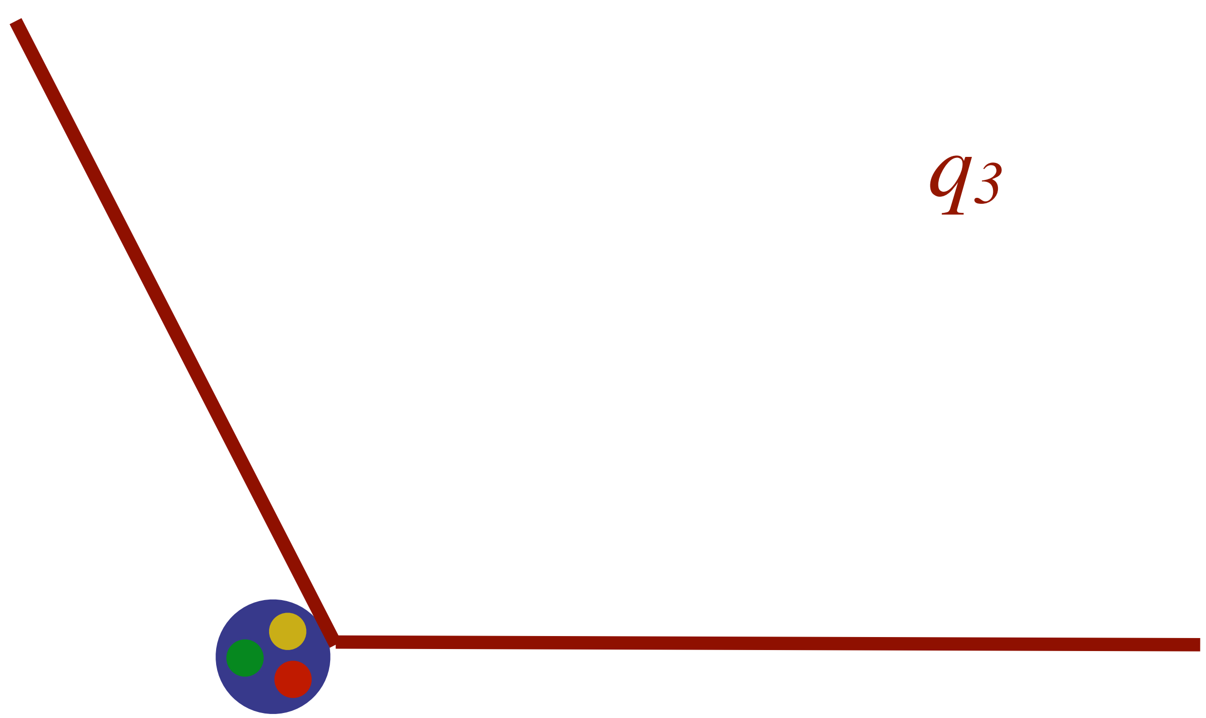
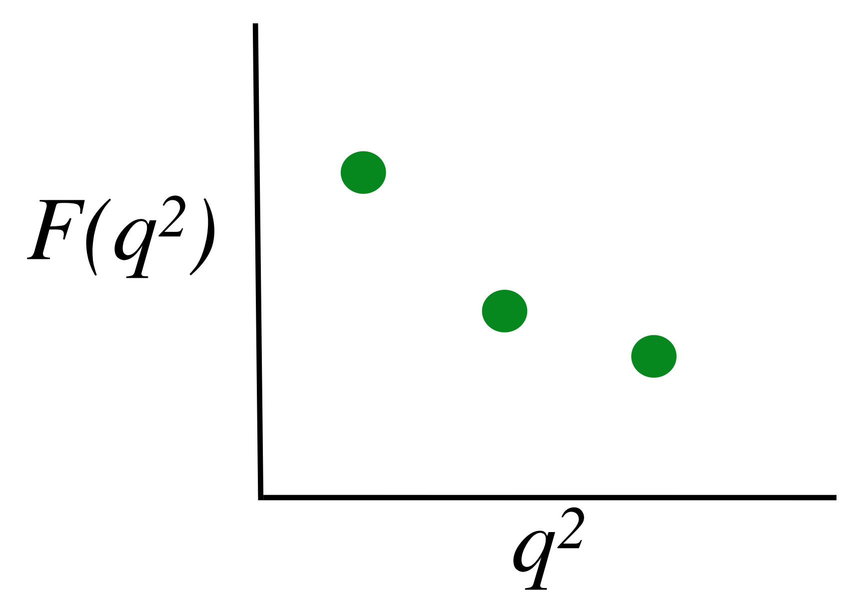
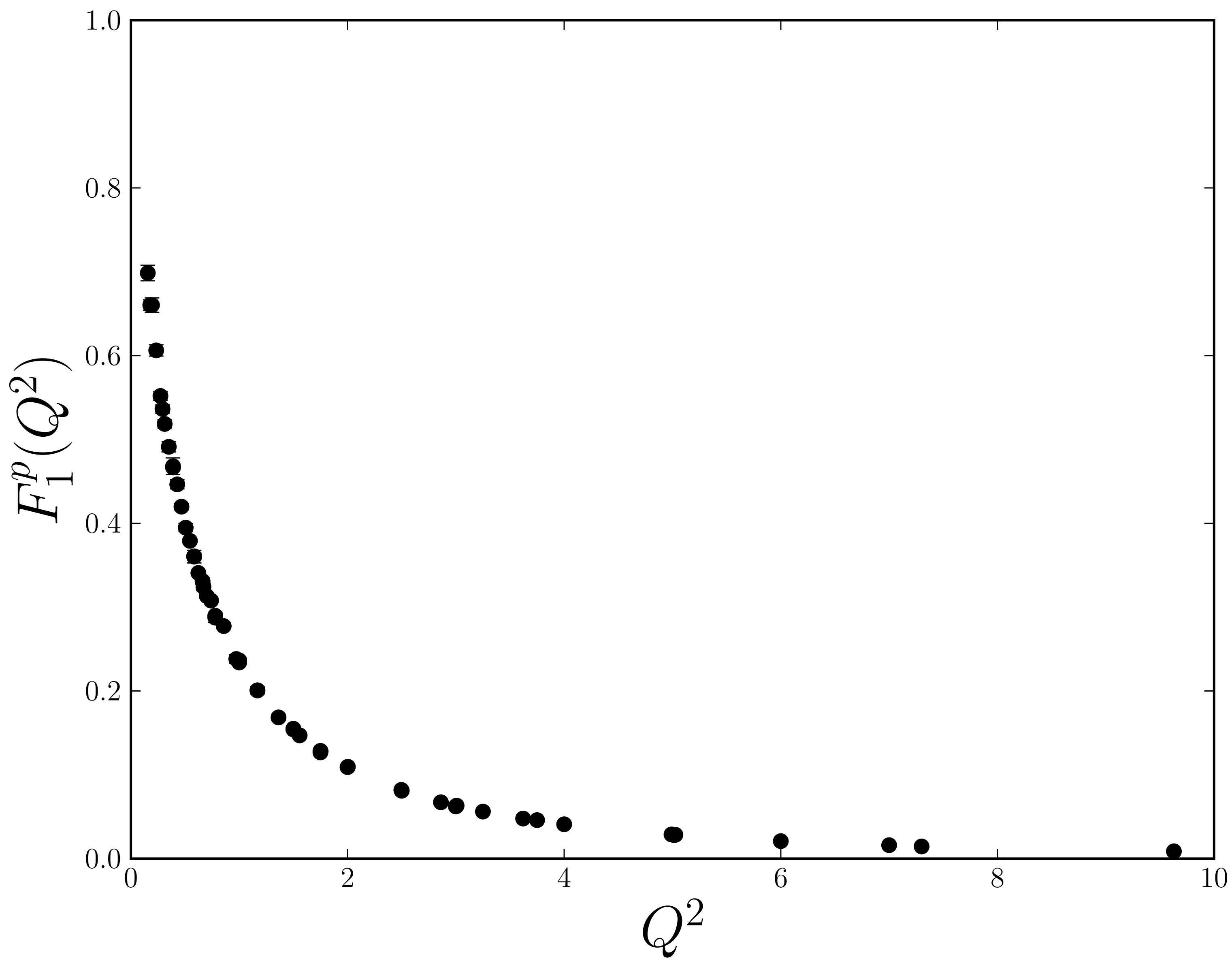


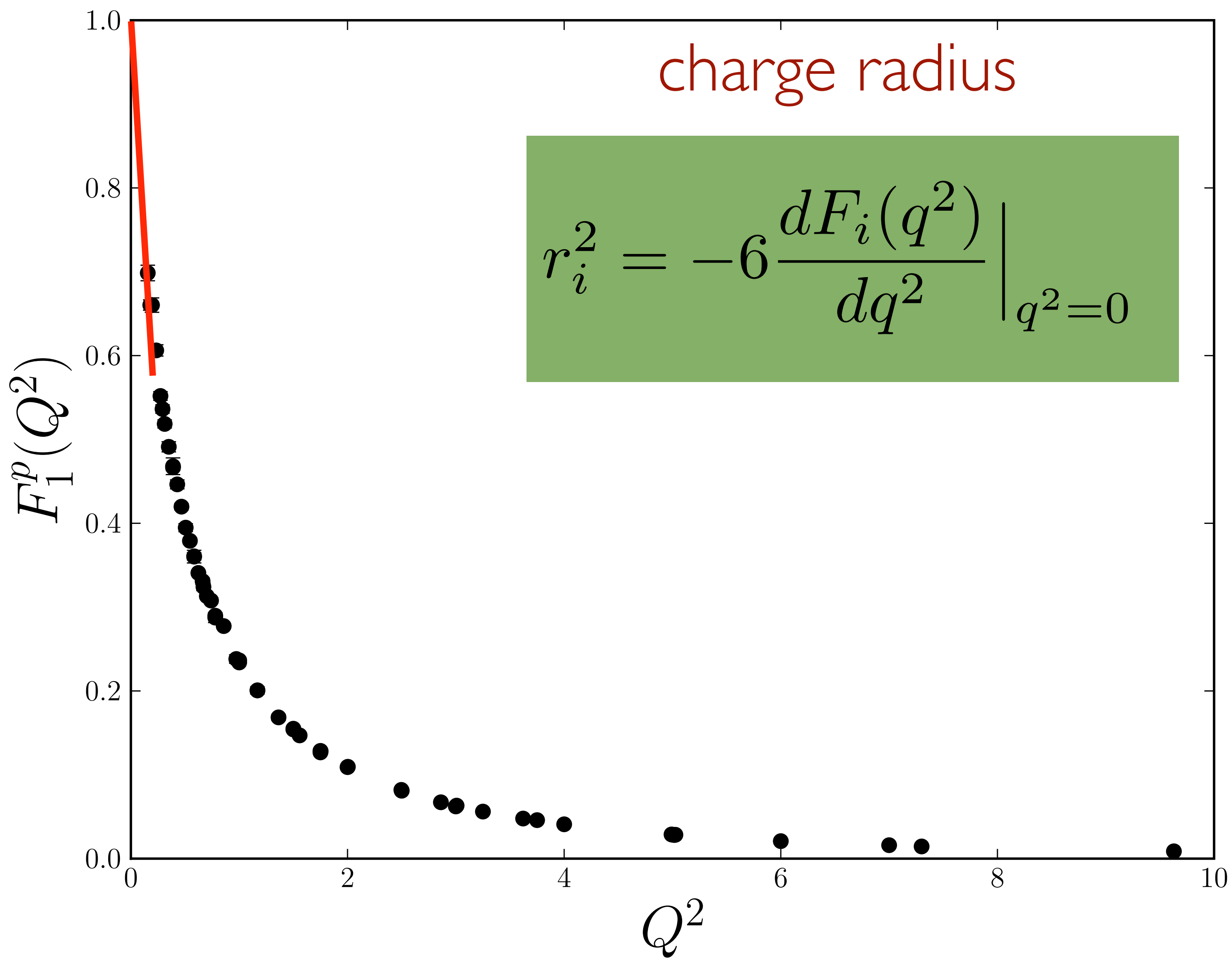


But it's not

Example: Proton F_1 Form Factor



Example: Proton F_1 Form Factor



Electromagnetic Form Factors

$$\langle p',\, s'|J^\mu(\vec{q})|p,\, s\rangle = \bar{u}(p',\, s')\left[\gamma^\mu F_1(q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2m}F_2(q^2)\right]u(p,\, s)$$

Electric charge	$F_1(0) = Q$
Anomalous magnetic moment	$F_2(0) = \kappa$
Magnetic moment	$F_1(0) + F_2(0) = \mu$

$$Q_p = 1,\ Q^n = 0$$

$$\mu_p = 2.79\mu_N,\ \mu_n = -1.91\mu_N$$

Radii: $$r_i^2 = -6\frac{dF_i(q^2)}{dq^2}\Big|_{q^2=0}$$

$q^2 > 0$: "Look inside" hadron

Scaling of Form Factors

From dimensional counting *[Brodsky & Farrar, 1973]*

$$F_1 \propto \frac{1}{Q^4} \quad \text{(dipole?)}$$

$$\frac{F(0)}{(1+Q^2/M^2)^p}$$

$$F_2 \propto \frac{1}{Q^6} \quad \text{(tripole?)}$$

for $Q^2 > \zeta_{\rm pQCD}$

$$Q^2 \frac{F_2}{F_1} \propto \text{const}$$

$$\frac{G_E}{G_M} \propto \text{const}$$

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$$Q \frac{F_2}{F_1} \propto$$

JLab $\frac{G_E}{G_M} \propto$

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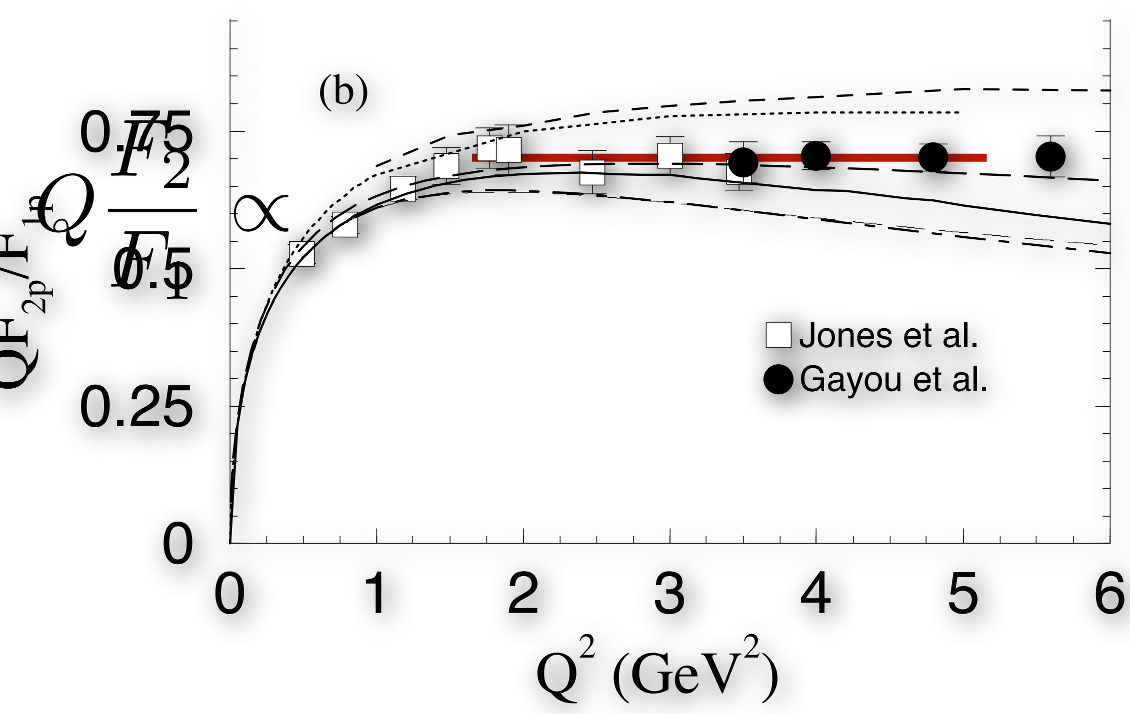
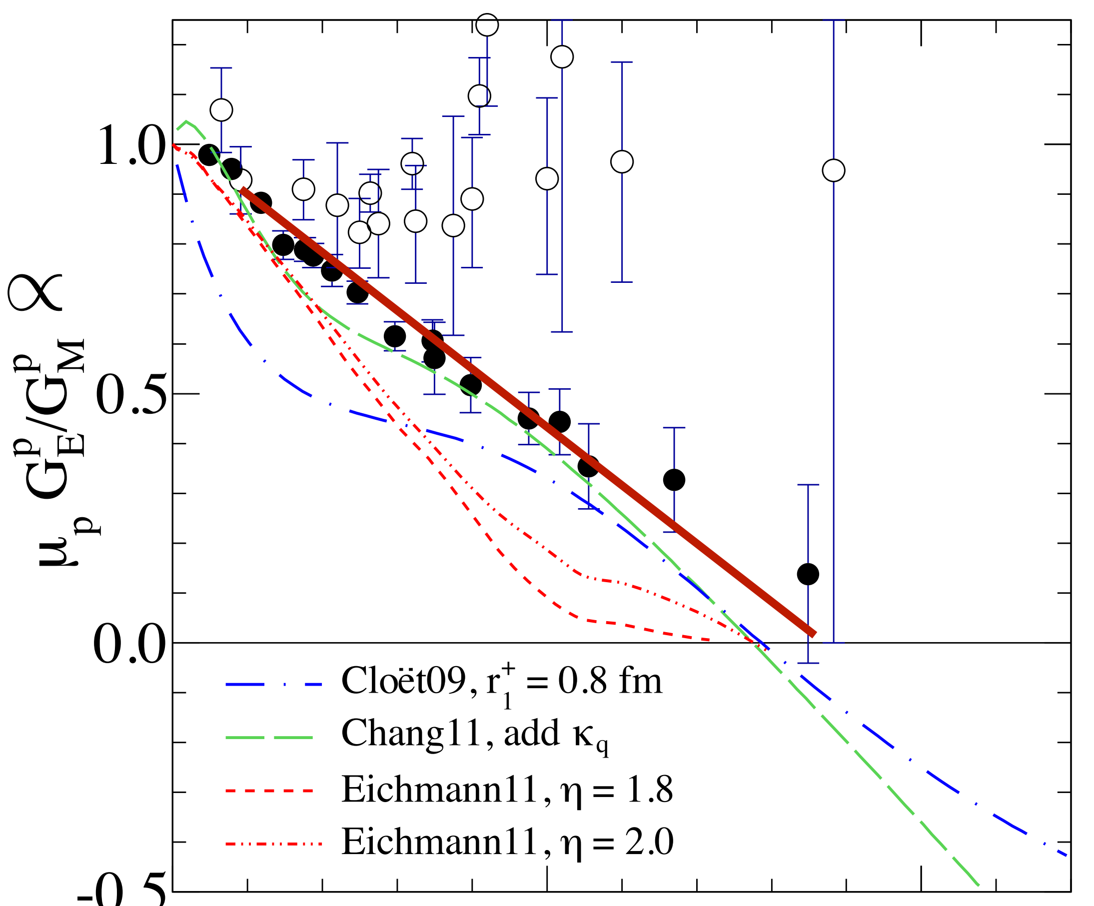


JLab $\frac{G_E}{G_M} \propto$



Q^2 Parameterisation

- Sachs form factors reasonably described by a dipole

$$G_E^p(Q^2) = \frac{1}{(1+Q^2/M_D^2)^2}$$

$$G_M^{p,n}(Q^2) = \frac{\mu^{p,n}}{(1+Q^2/M_D^2)^2}$$

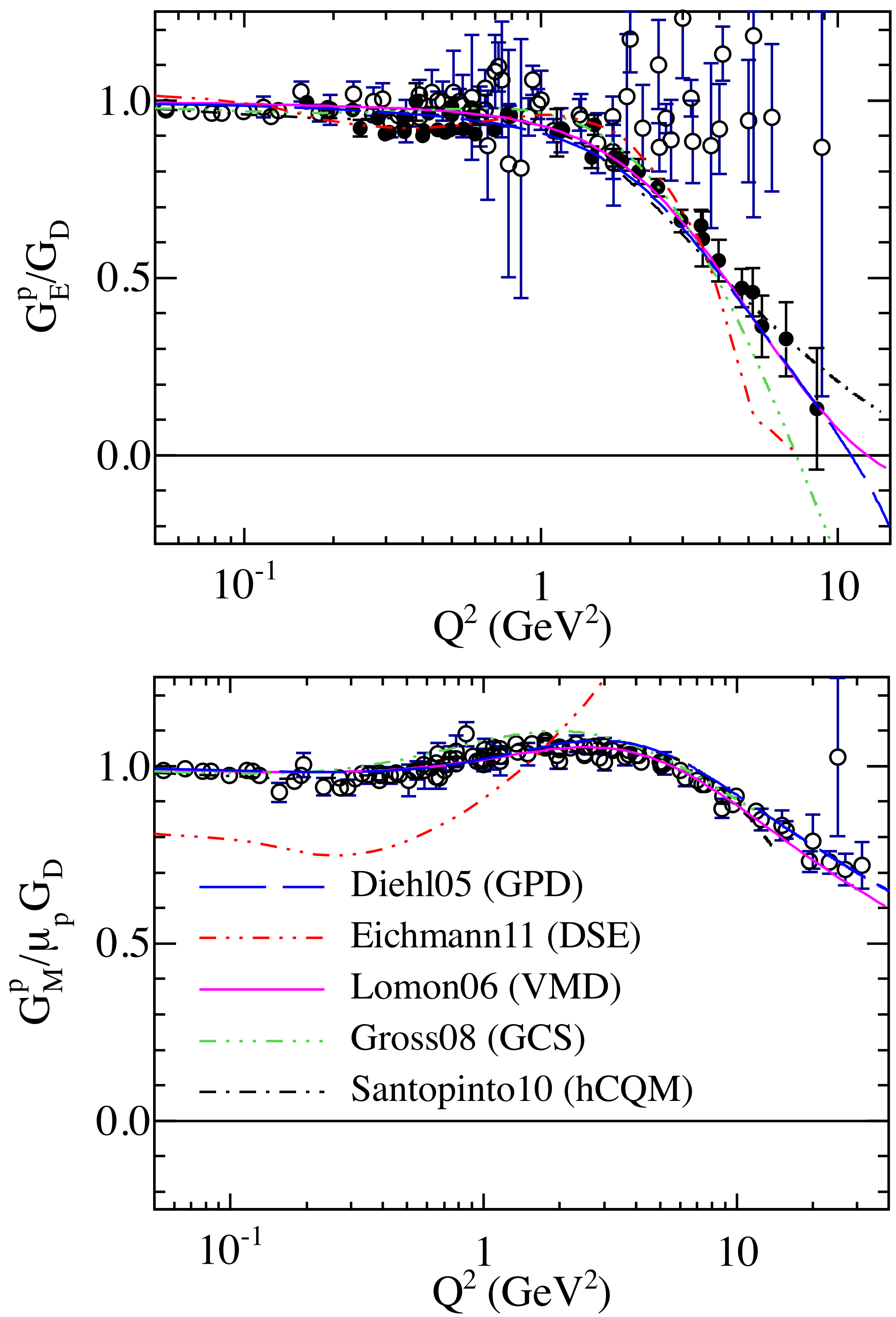
- with

$$M_D \approx 0.71\,\mathrm{GeV}$$

$$\mu^p = 2.79\,\mu_N$$

$$\mu^n = -1.91\,\mu_N$$

- But deviations seen, particularly at large Q^2



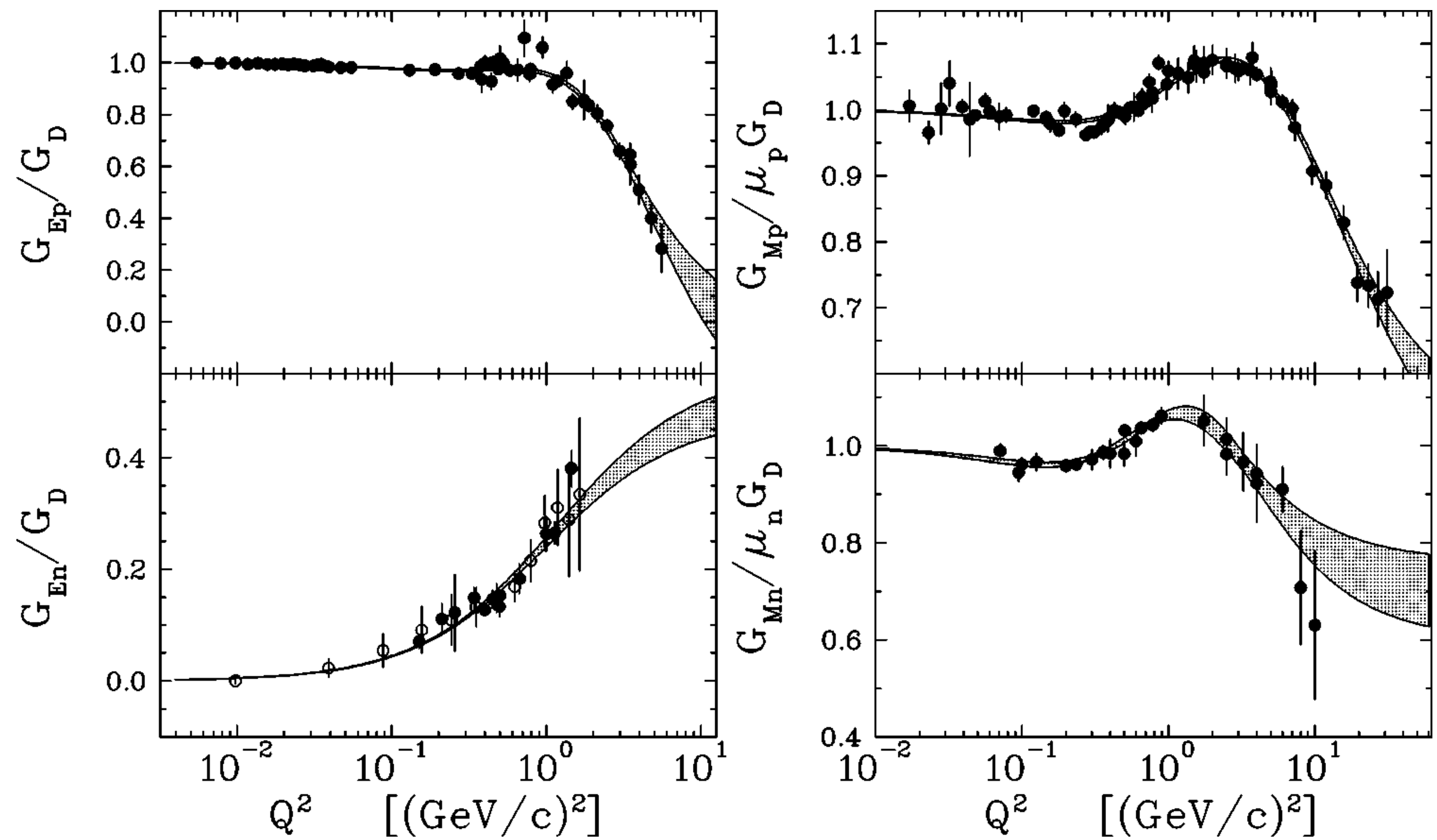
Q^2 Parameterisation

[Phys. Rev. C 66, 065203 (2002)]

- Kelly proposed a simple parameterisation for the form factors

$$G(Q^2) \propto \frac{\sum_{k=0}^{n} a_k \tau^k}{1 + \sum_{k=1}^{n+2} b_k \tau^k} \qquad \tau = Q^2/4M^2$$

- with n=1 and a_0=1 for $G_M^{p,n}(Q^2),\ G_E^p(Q^2)$



[Recent work: Cloët & Miller, 1204.4422]

Form Factor Radii & Magnetic Moments

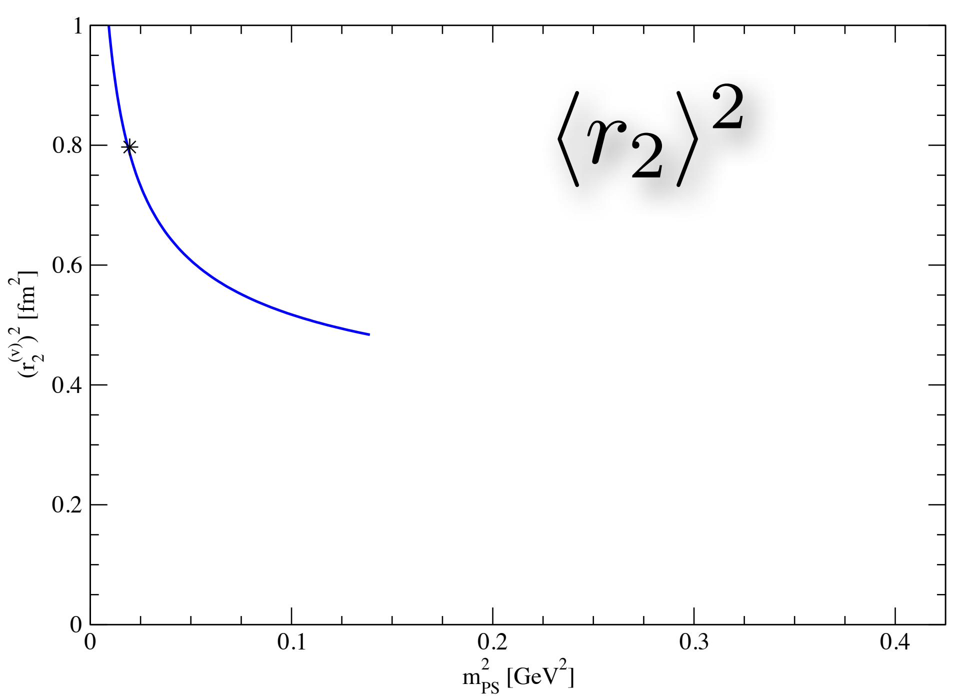
Search for non-analytic behaviour predicted by Chiral Perturbation Theory

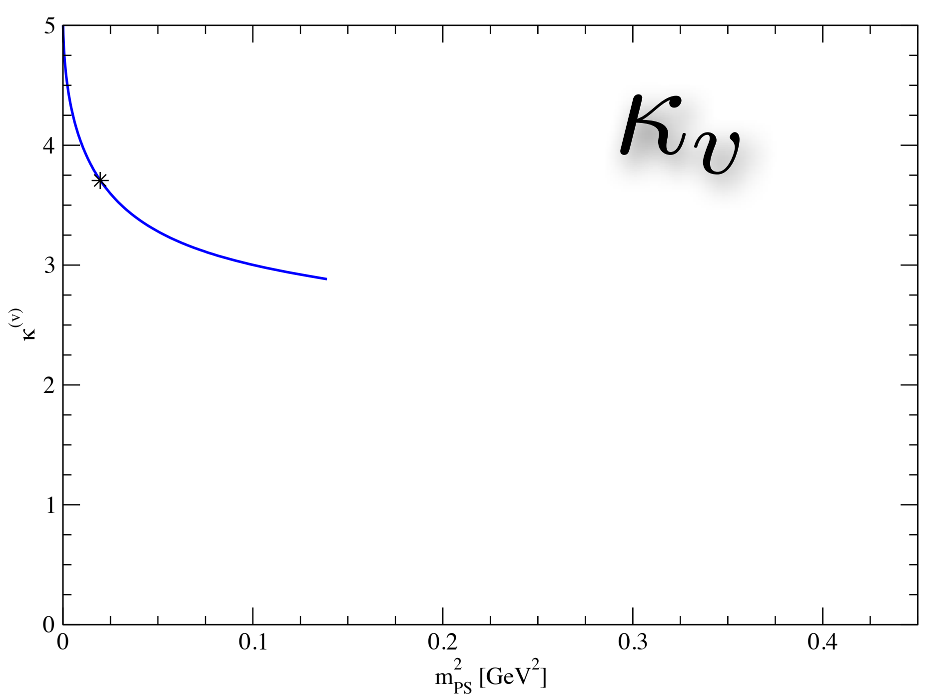
[see lectures by B. Tiburzi]

- Form factor radii: $$r_i^2 = -6 \frac{dF_i(q^2)}{dq^2}\Big|_{q^2=0}$$

- Magnetic moment μ/ anomalous magnetic moment κ

$$\mu = 1 + \kappa = G_m(0)$$





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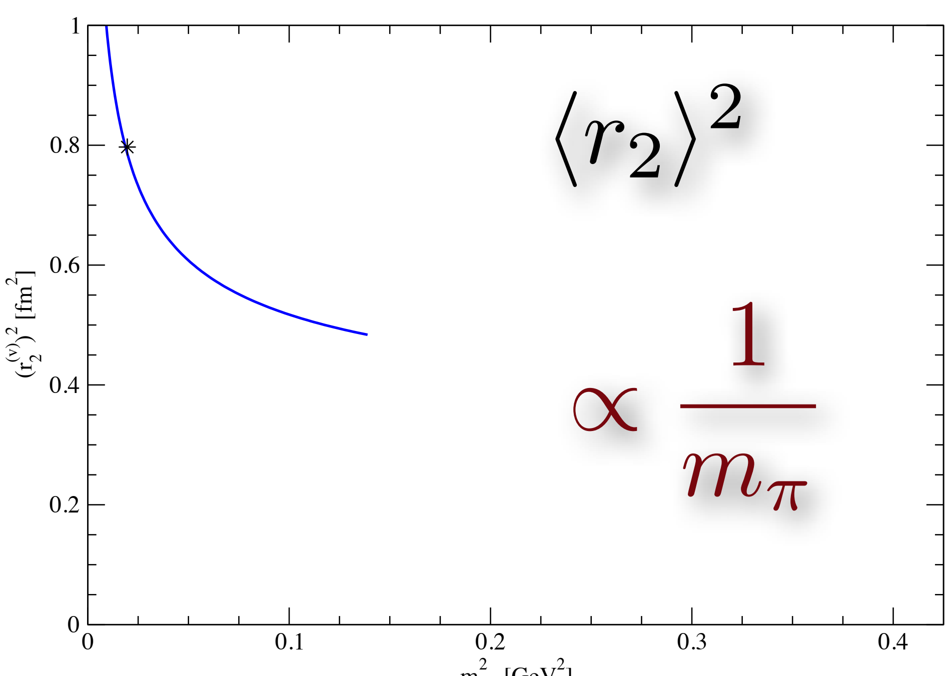
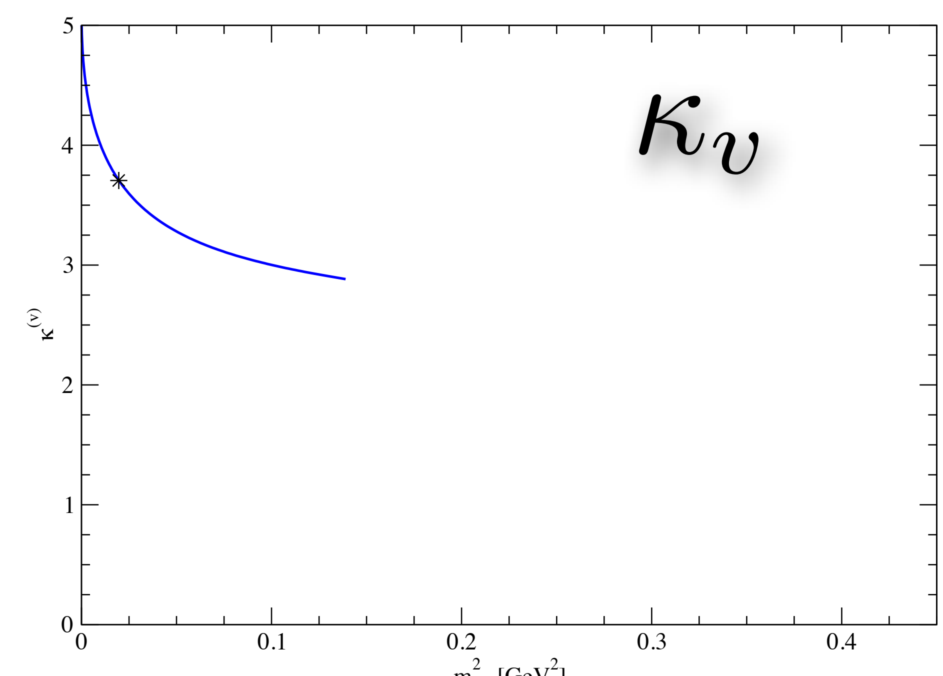




Lattice Nucleon Form Factors

Systematics of a Lattice Calculation

- In the following slides, we will be looking at lattice results for the EM form factors of the proton which can be compared with the experimental results

- We need to be careful of systematic errors that could affect our results

    - Finite lattice spacing

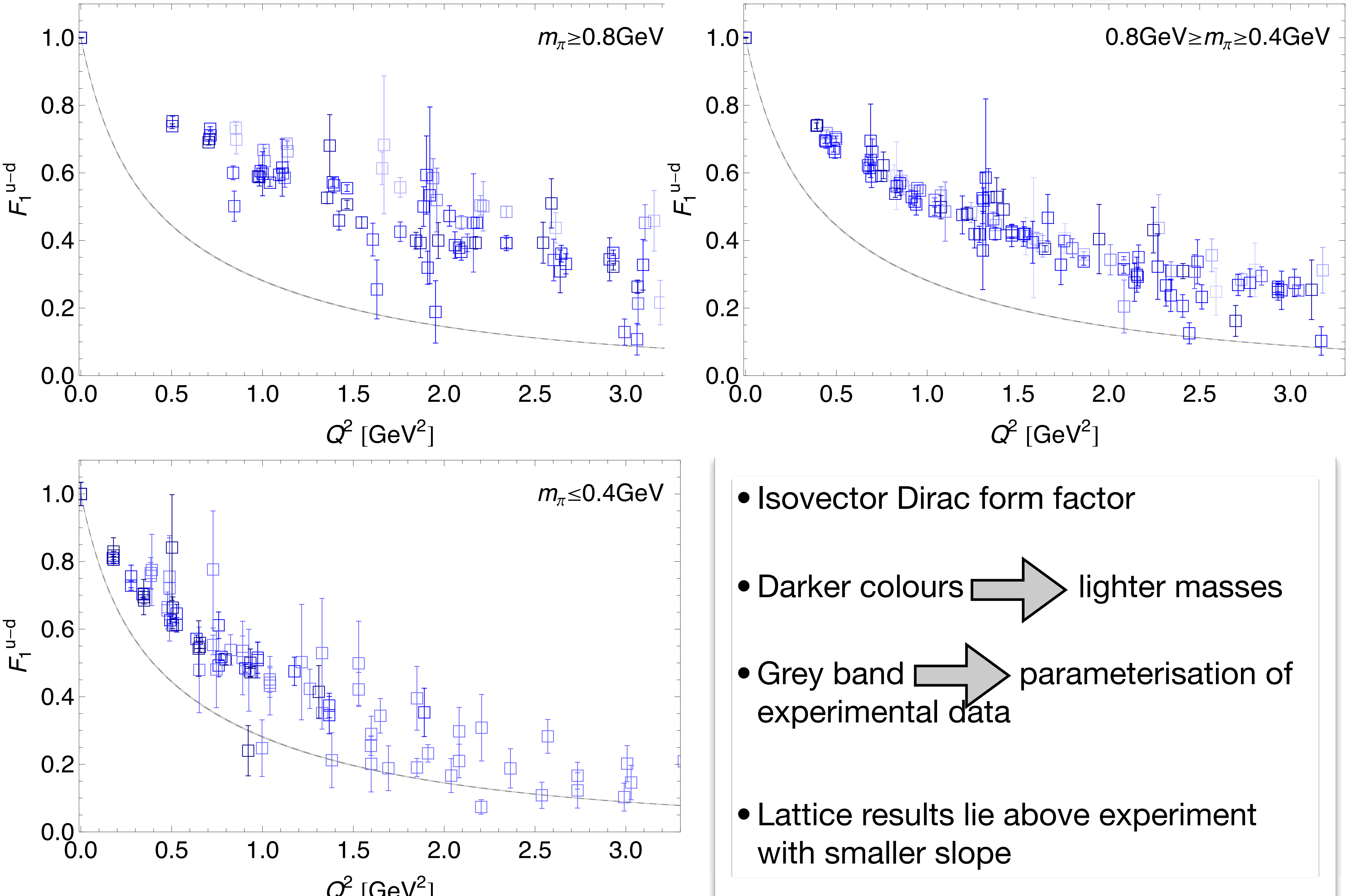
    - Large quark masses

    - Finite volume

    - Contamination from excited states

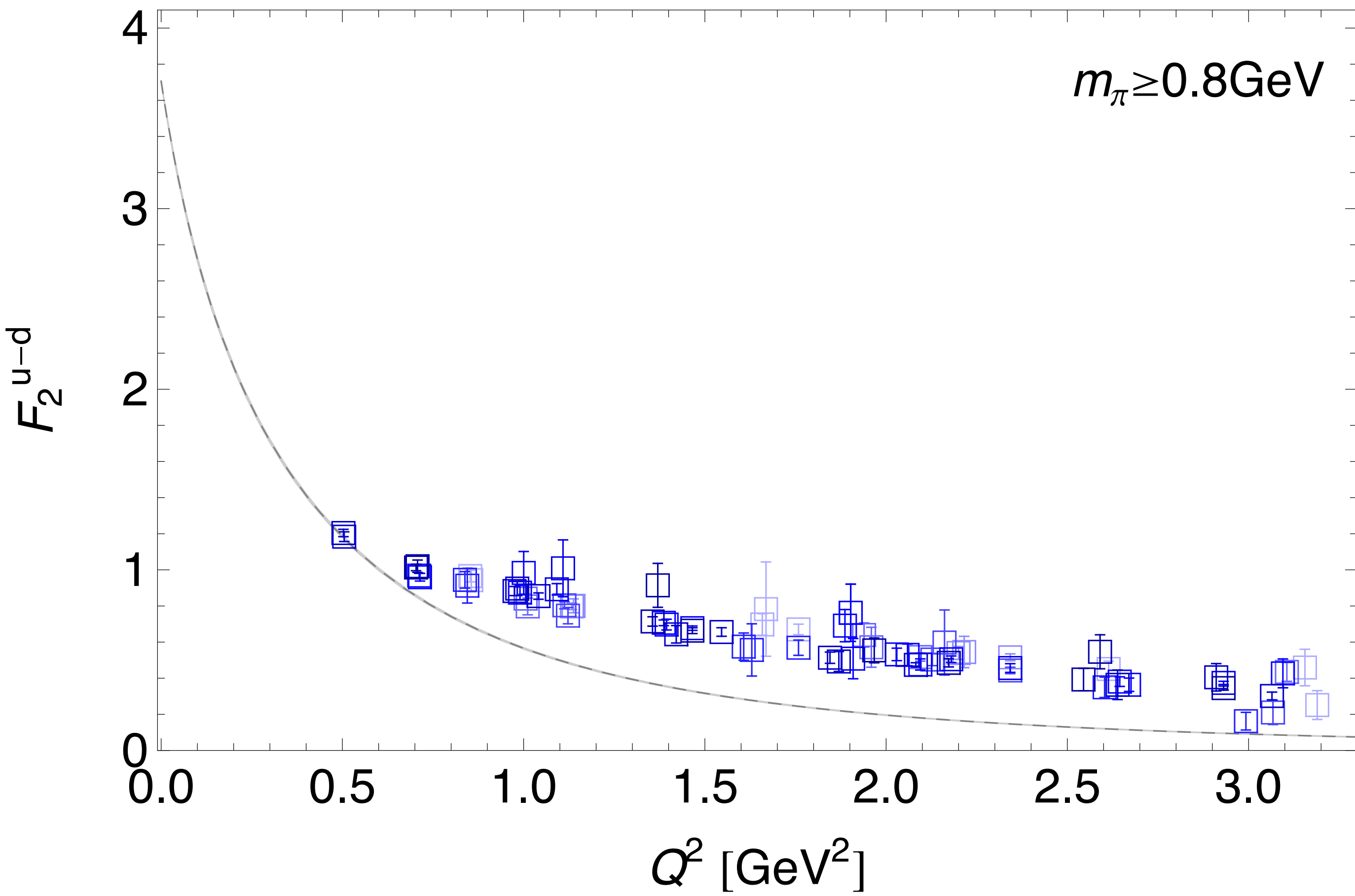
- Will focus on recent results from QCDSF PRD 84, 074507 (2011) [arXiv:1106.3580]

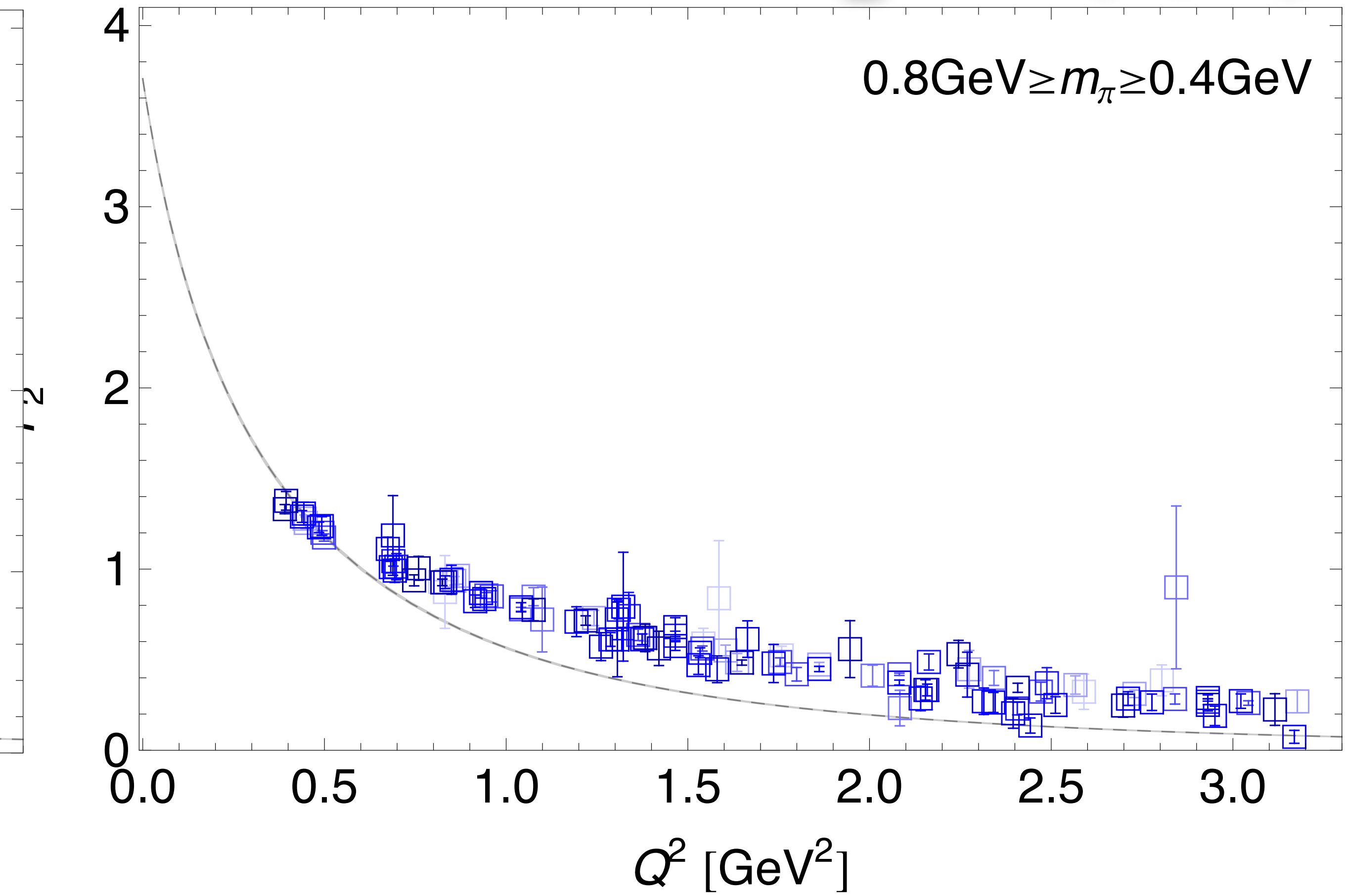
Comparison With Experiment $F_1^{u-d}(Q^2)$

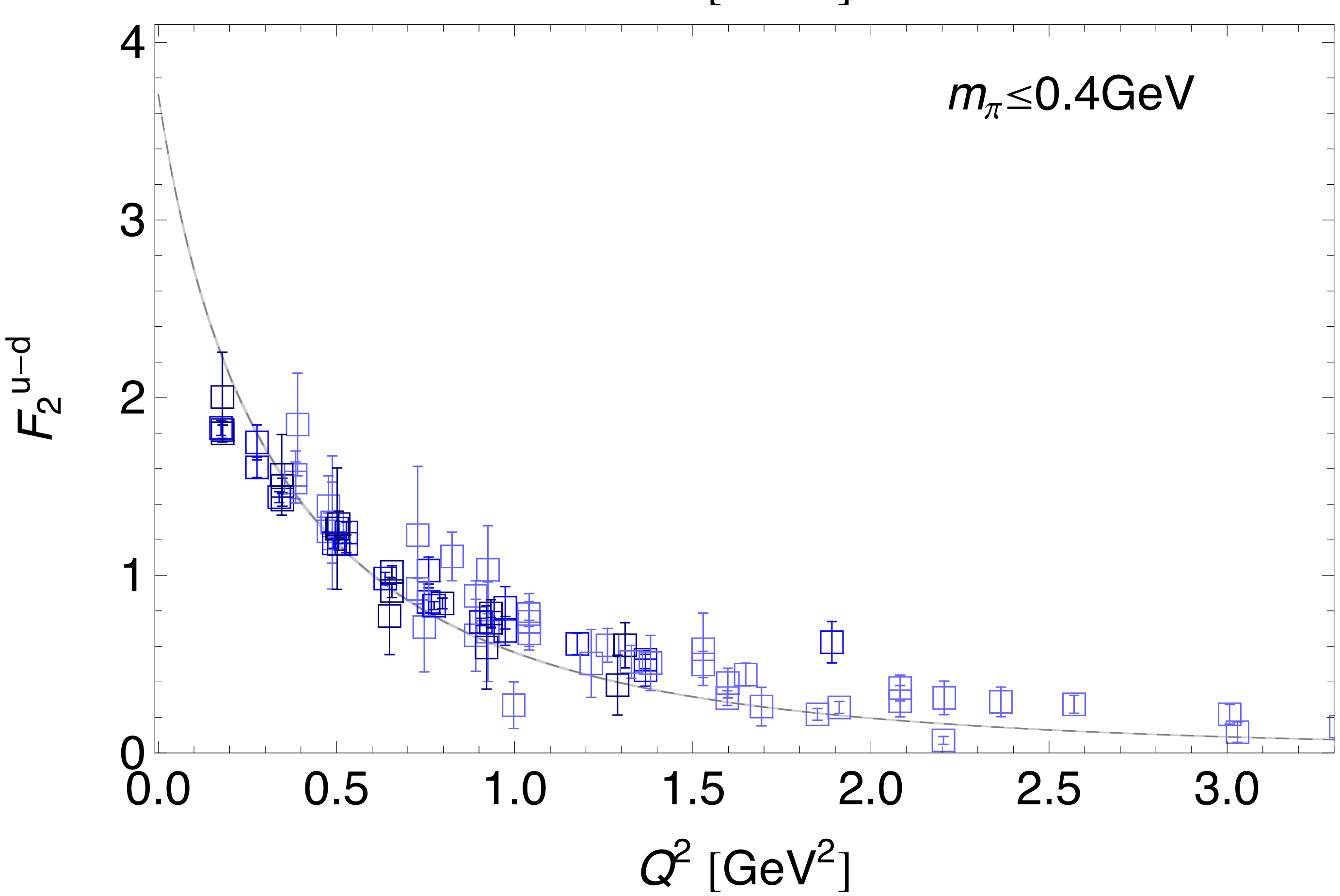


- Isovector Dirac form factor
- Darker colours → lighter masses
- Grey band → parameterisation of experimental data
- Lattice results lie above experiment with smaller slope

Comparison With Experiment $F_2^{u-d}(Q^2)$







- Isovector Pauli form factor
- Darker colours → lighter masses
- Grey band → parameterisation of experimental data
- Lattice results lie above experiment with smaller slope

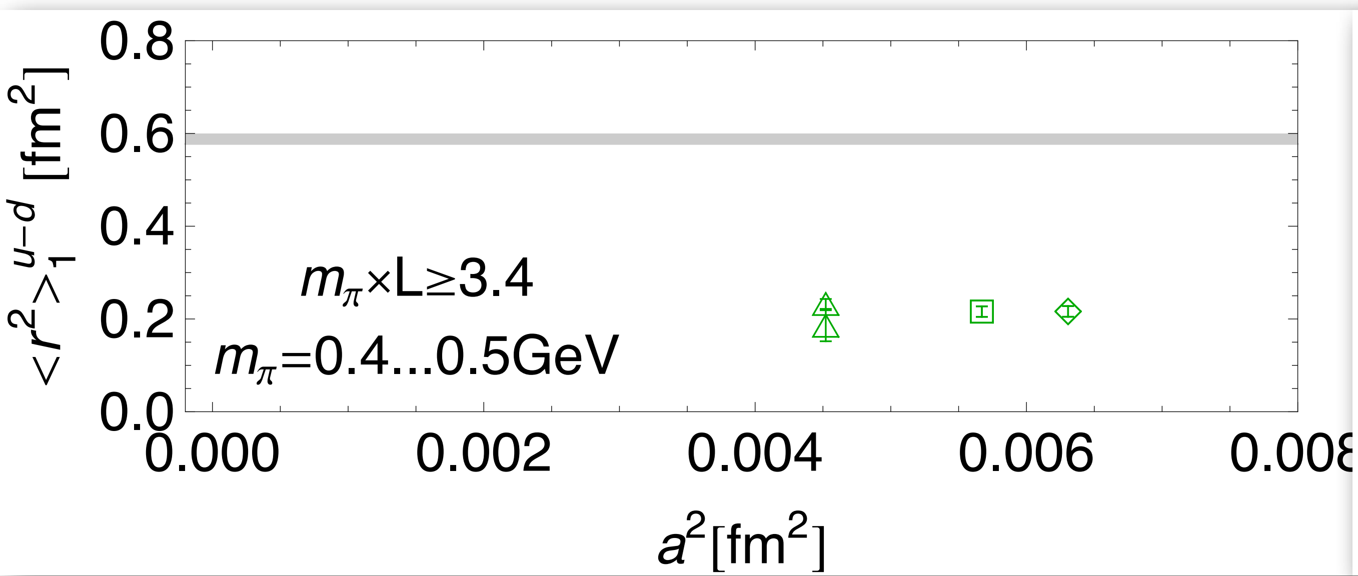
Systematic Errors

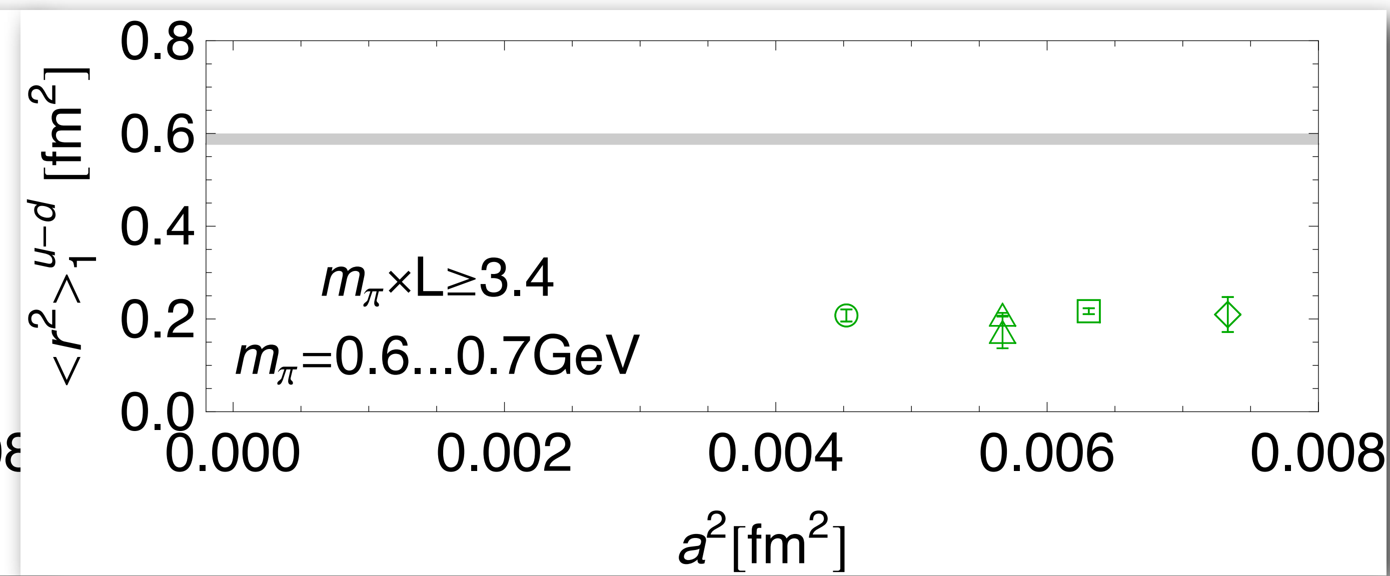
Lattice Spacing

- Scan available datasets for bins with constant m_π, but with 3 or more different lattice spacings, *a*

- Plot results as a function of a^2

Grey band: parameterisation of experimental data





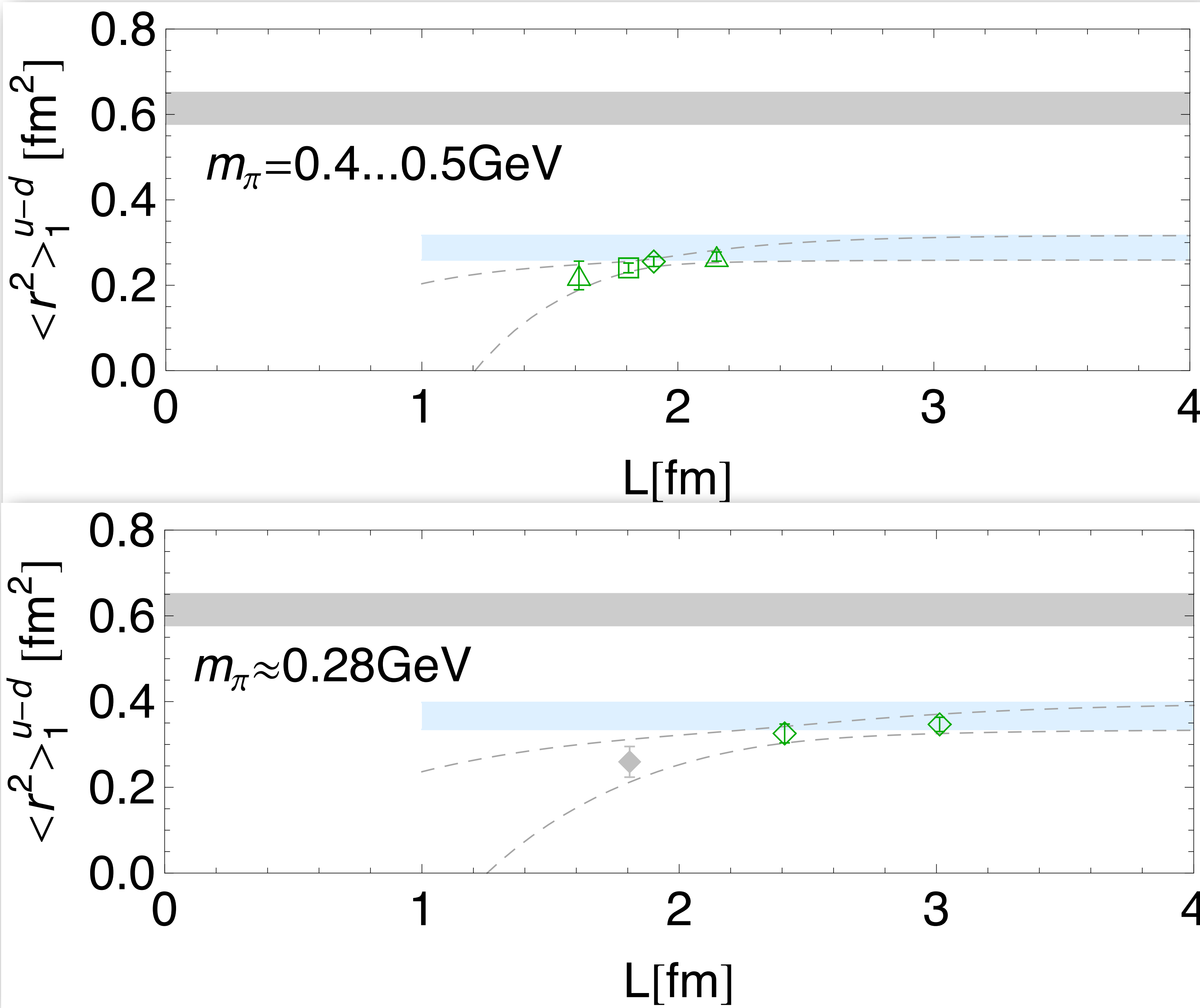
No visible dependence on *a*

Systematic Errors

Volume

- Scan datasets for bins with constant m_π but with 2 or more spatial volumes, *L*
- Plot results as a function of *L*

Grey band: parameterisation of experimental data



- Small volume correction accounted for by exponential factor

$$a + b\,e^{(-m_\pi L)}$$

Systematic Errors

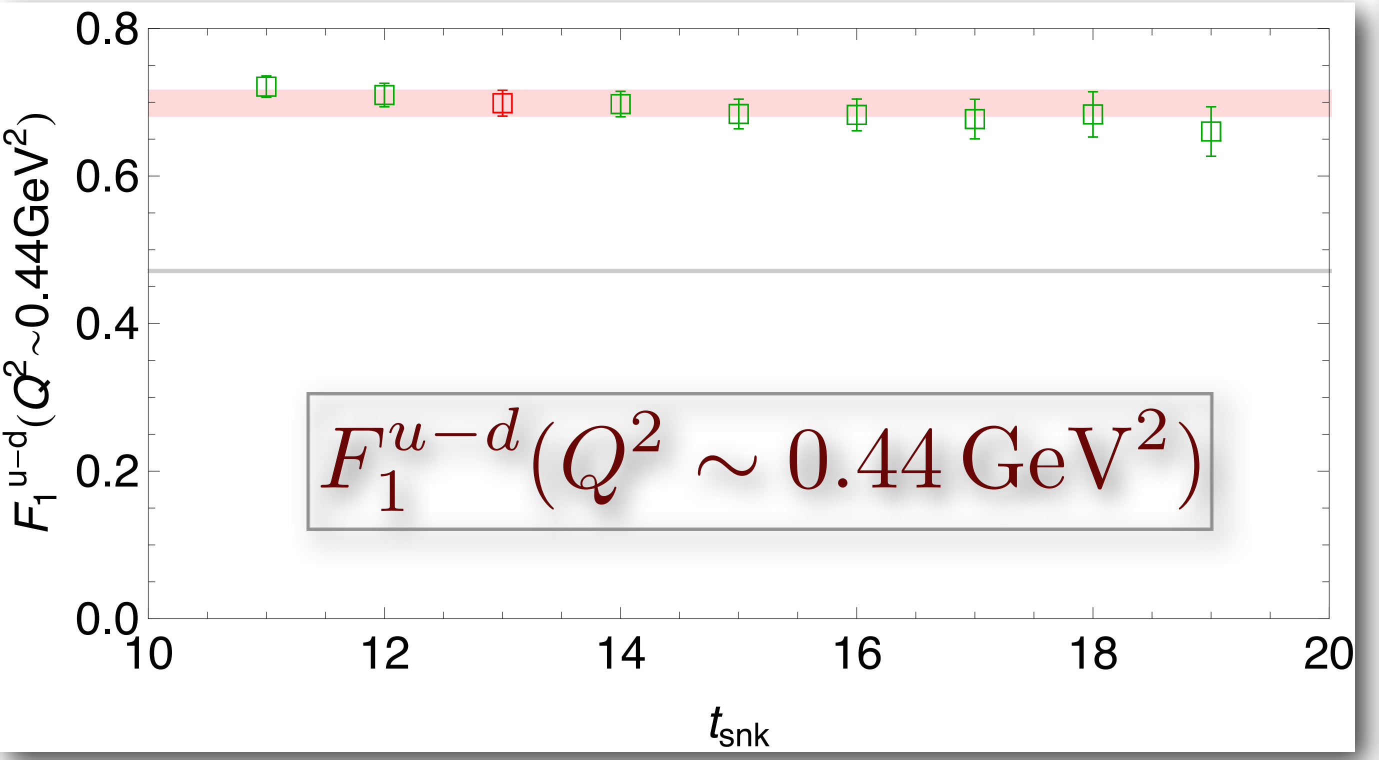
Excited State Contamination

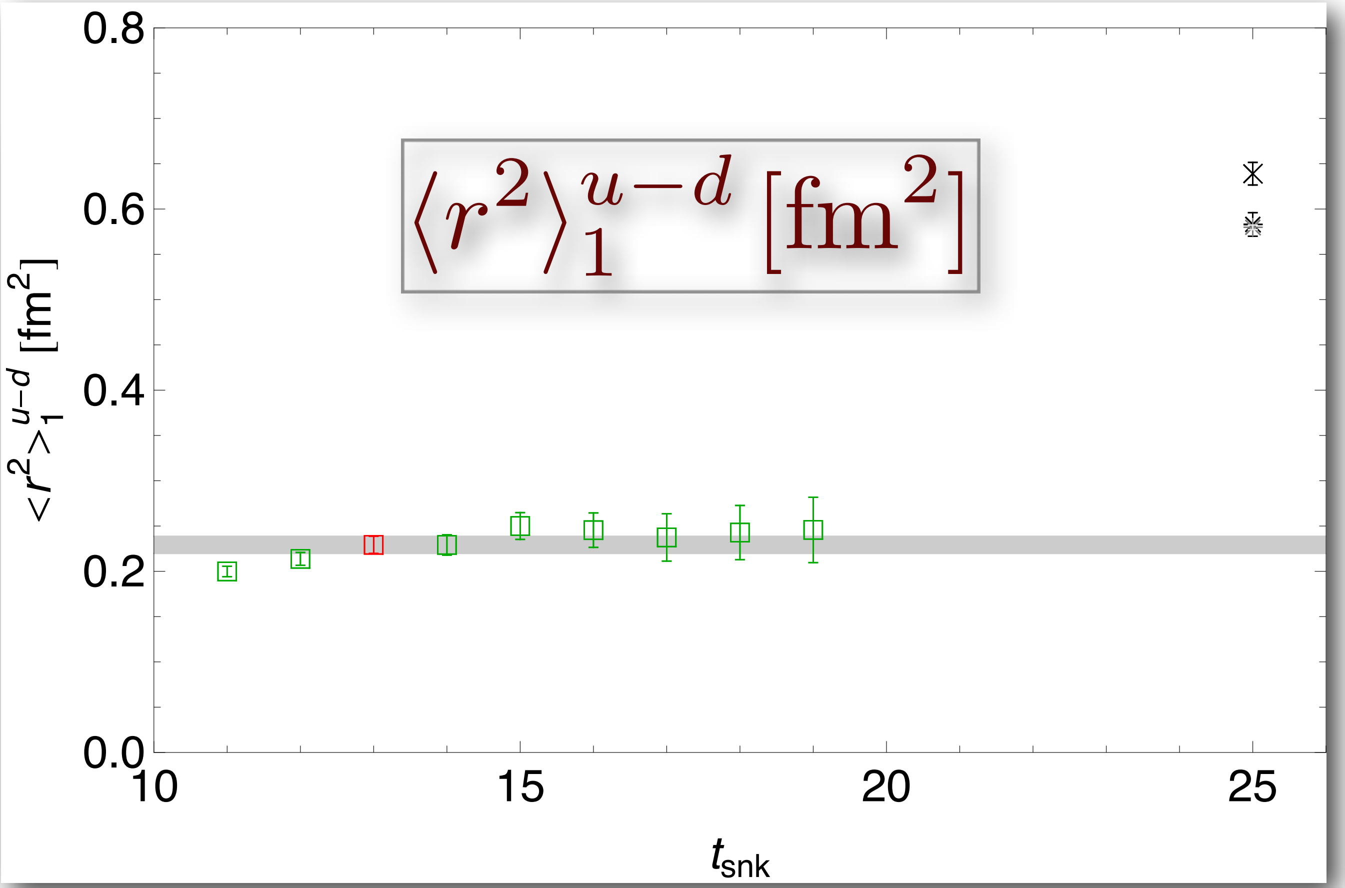
- For small values of Euclidean time, effects from excited states may adversely affect the extraction of physical observable from the lattice, e.g.

$$C_{2pt}(t) = A_0 e^{-M_0 t} + A_1 e^{-M_1 t} + \ldots$$

- Require distances between source (*t=0*)- operator insertion (τ) - sink (t_{snk}) >>1

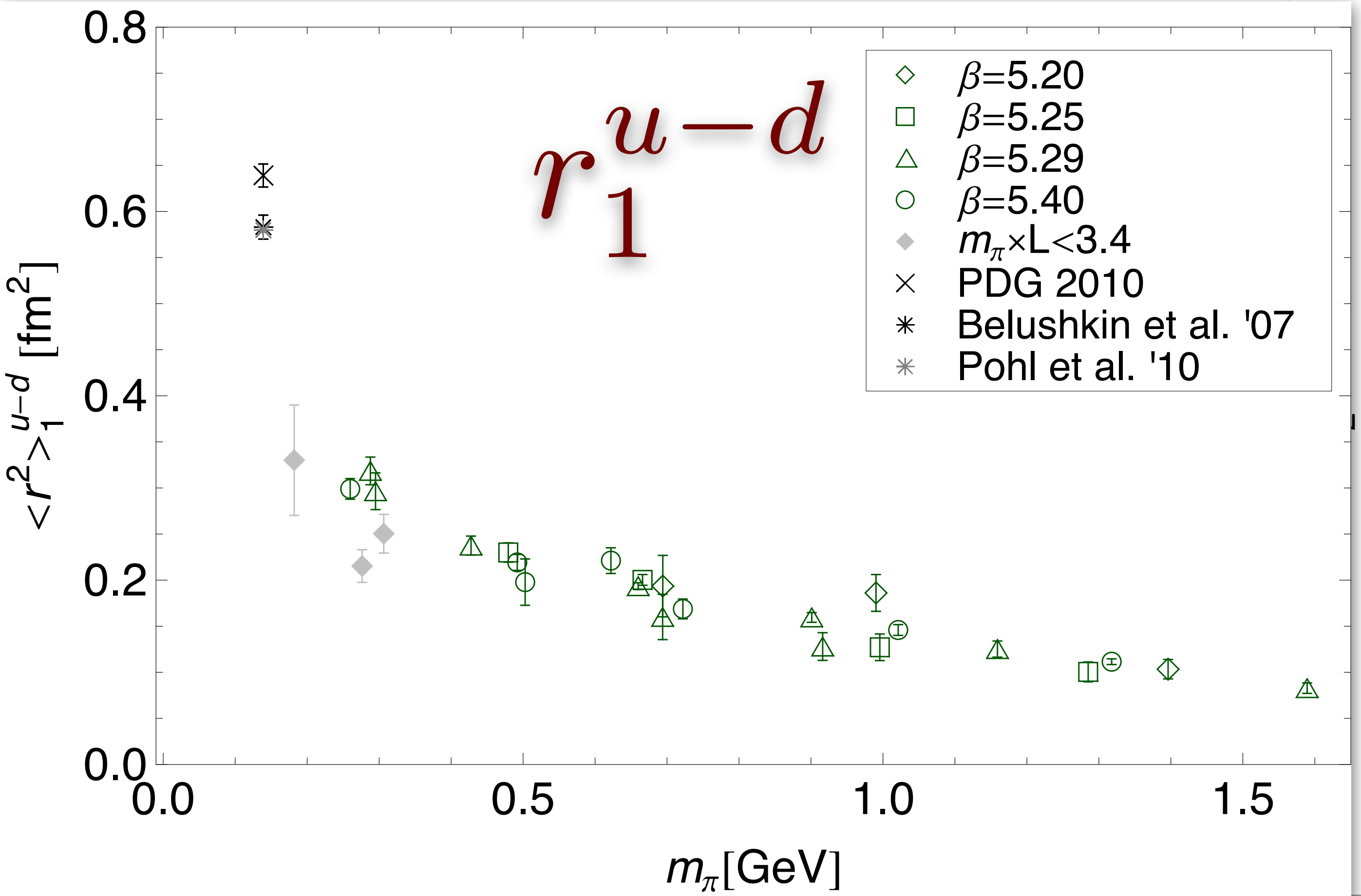
- Simulate with multiple t_{snk}'s on a single dataset to test the validity of our original choice $t_{snk}=13$

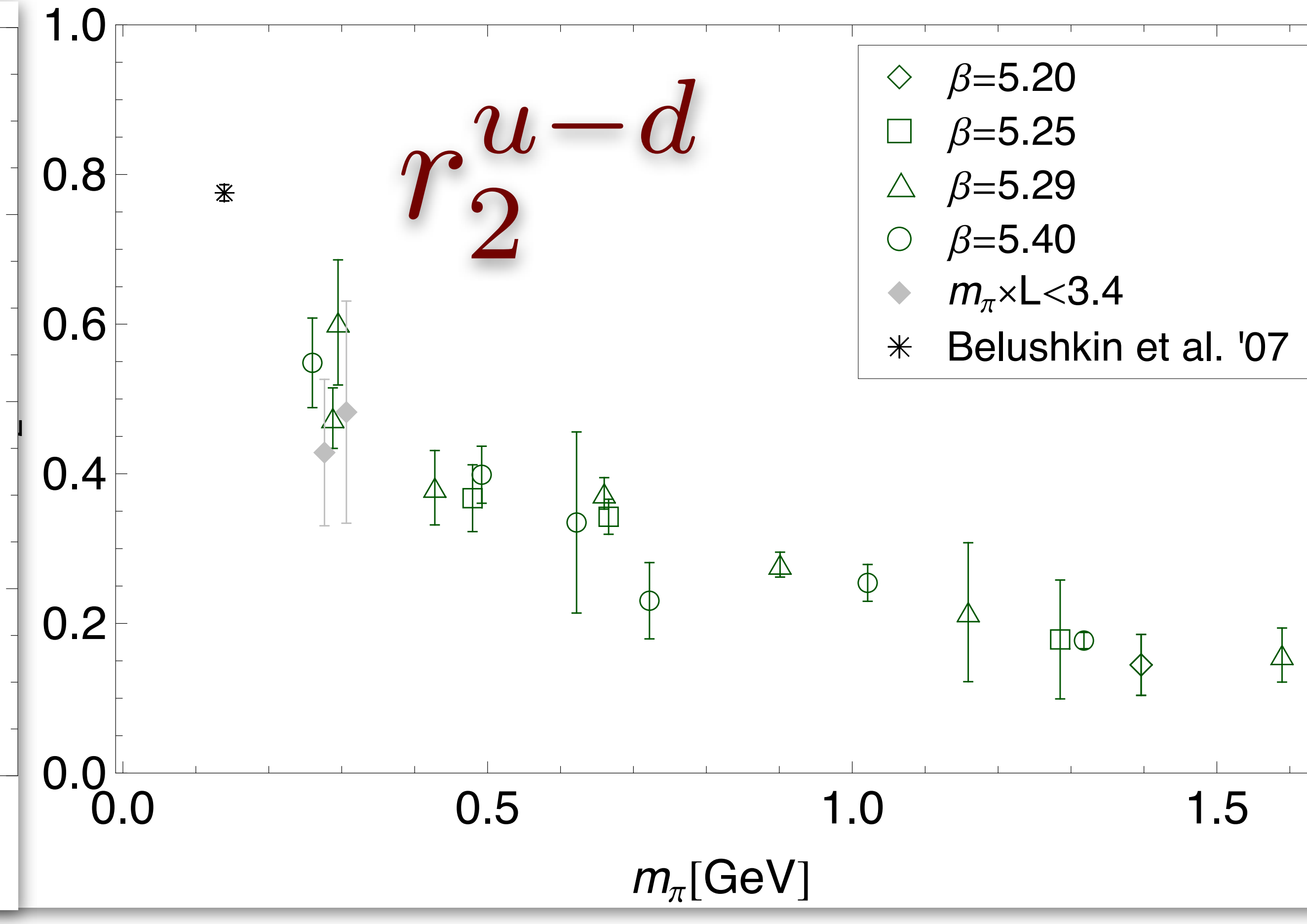


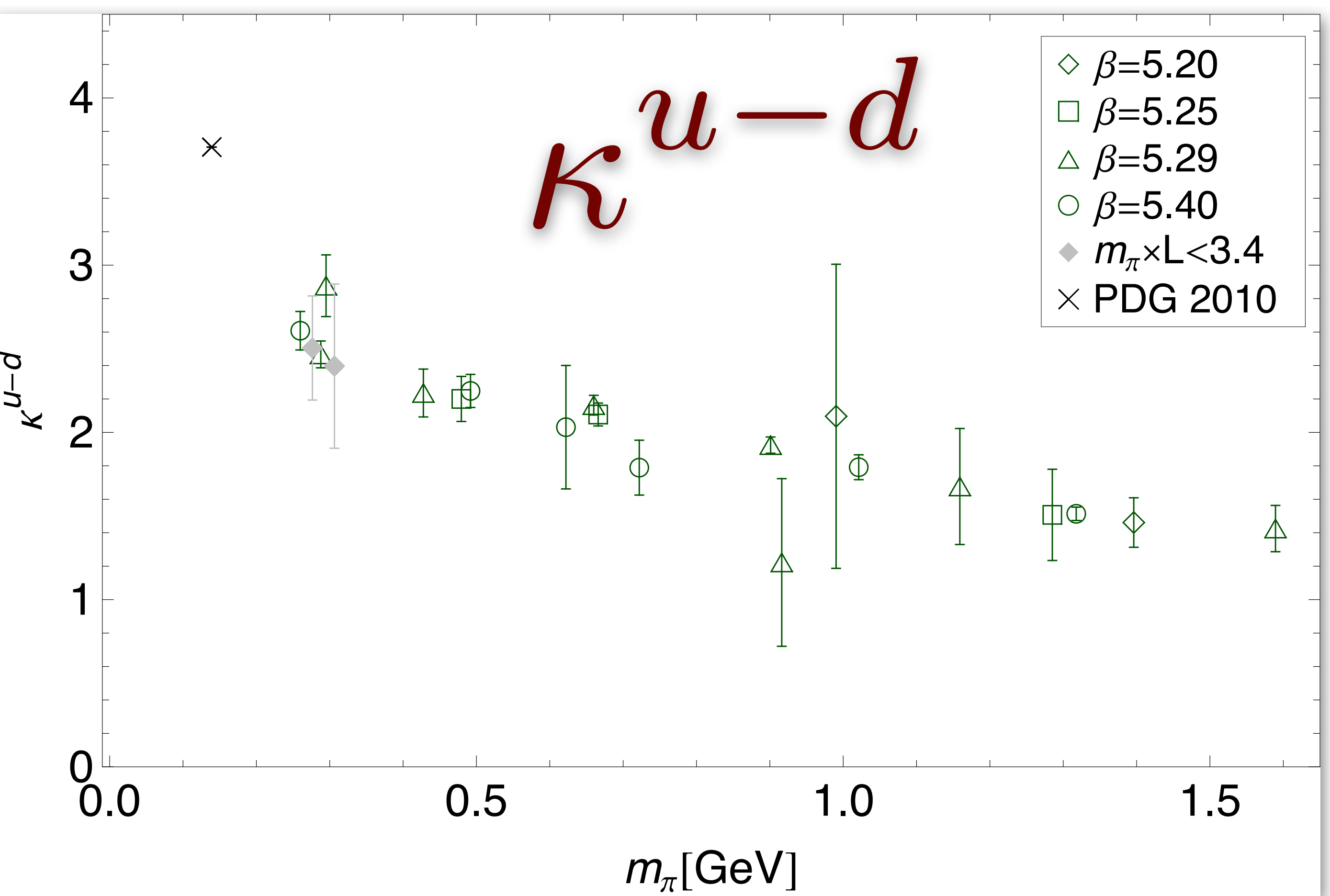


Systematic Errors

- Systematics appear to be under control
    - Finite lattice spacing √
    - Large quark masses
    - Finite volume √
    - Contamination from excited states √
- Remaining discrepancy must come from unphysical quark masses



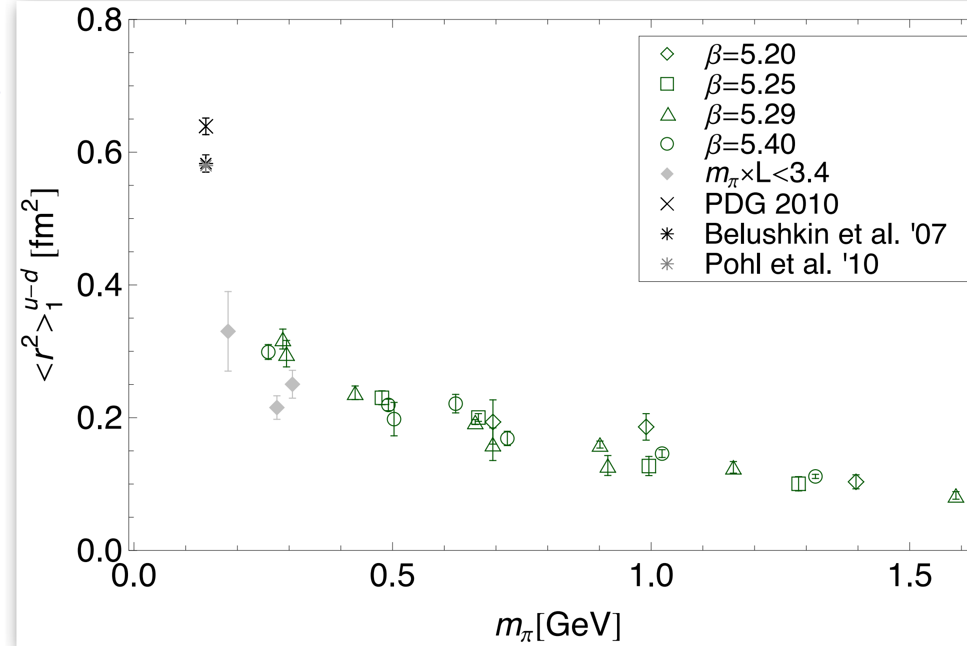




- Isovector Dirac radius (squared)
- Isovector Pauli radius (squared)
- Isovector anomalous magnetic moment
- Dirac radius: different experimental values

Light Quark Mass Dependence

- Radii suppressed at large masses and small volumes
- Hint of sharp rise at small masses
- r_2 approaching experimental result
- κ^{u-d} shows clear curvature at small masses

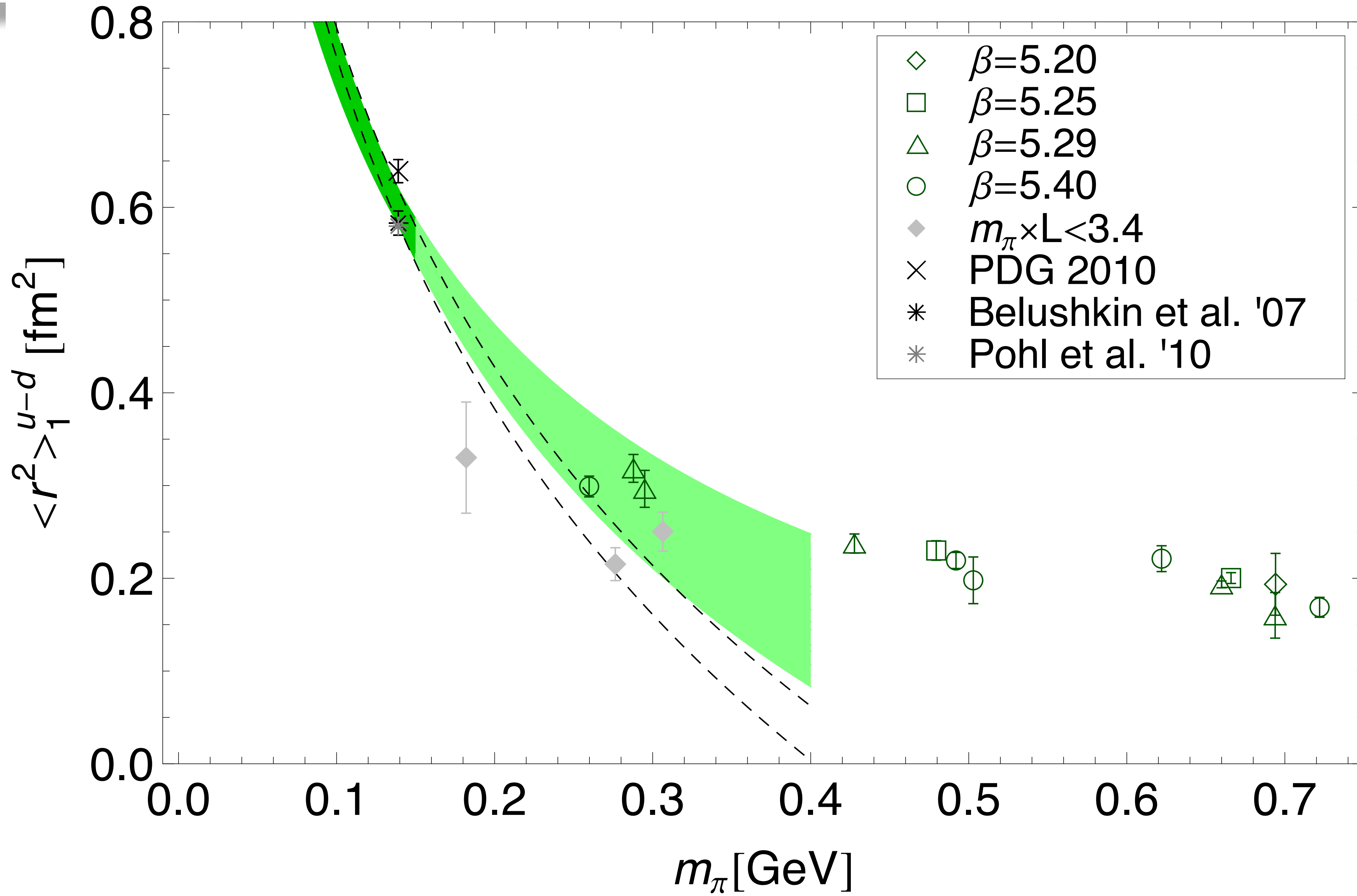


- Can the remaining discrepancy be due to the (still) unphysically large quark masses?
- Contact with ChPT?

- Popular expressions from *Phys. Rev. D71, 034508 (2005)* (SSE)
- But are they valid up to $m_\pi < 300\ \mathrm{MeV}$?
- Check by: Varying unknown parameters over a “reasonable” range and extrapolate up from the chiral limit with the only constraint provided by the experimental point

Dirac Radius

- Rapidly decreasing isovector Dirac ms radius as pion mass increases
- Overlap with the lattice data points at $m_\pi \approx 250 \ldots 300$ MeV
- Similar observations for Pauli radius and anomalous magnetic moment
- Isoscalar r_1 indicates form not valid past physical pion mass



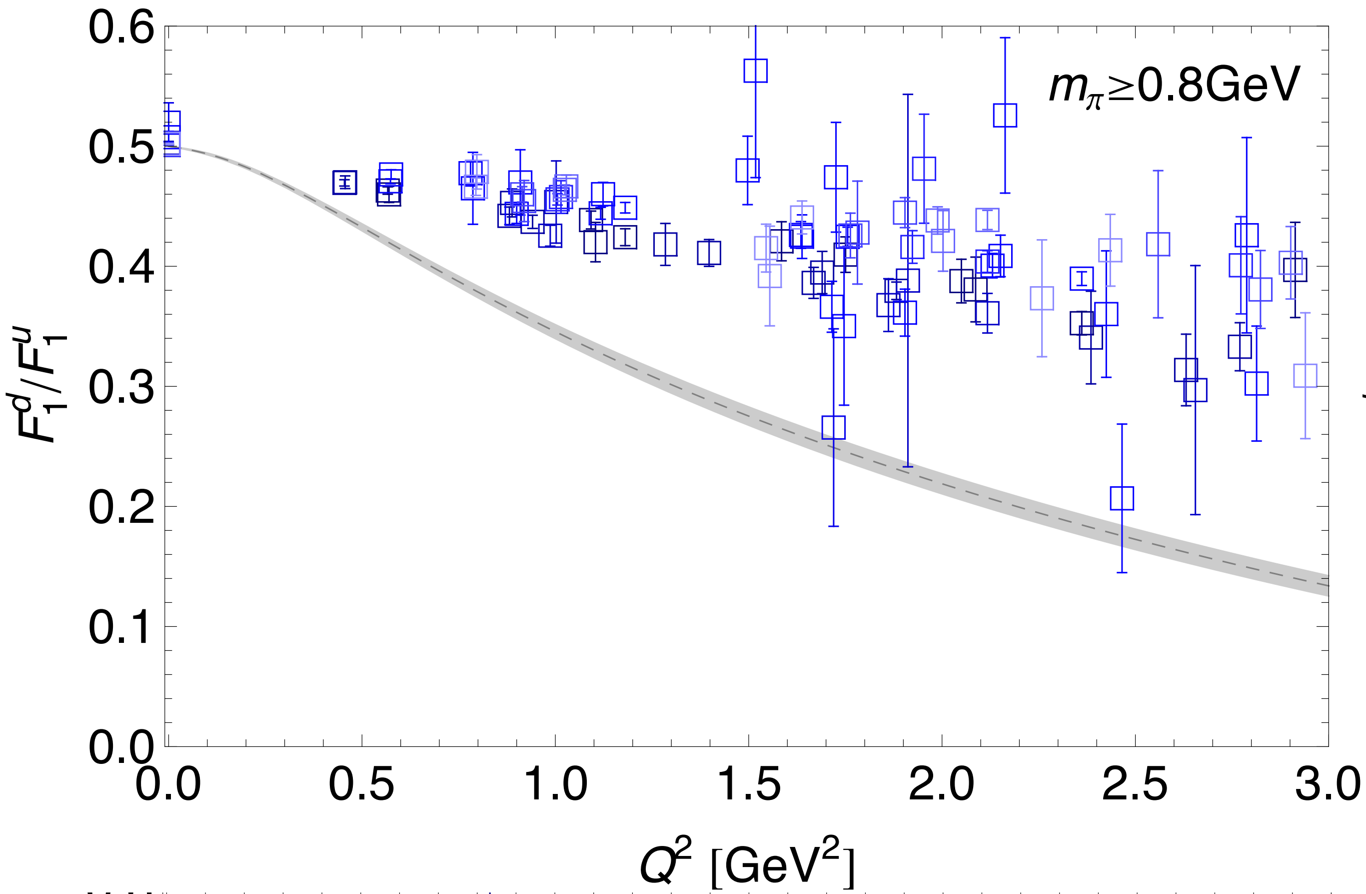
Flavour Distribution

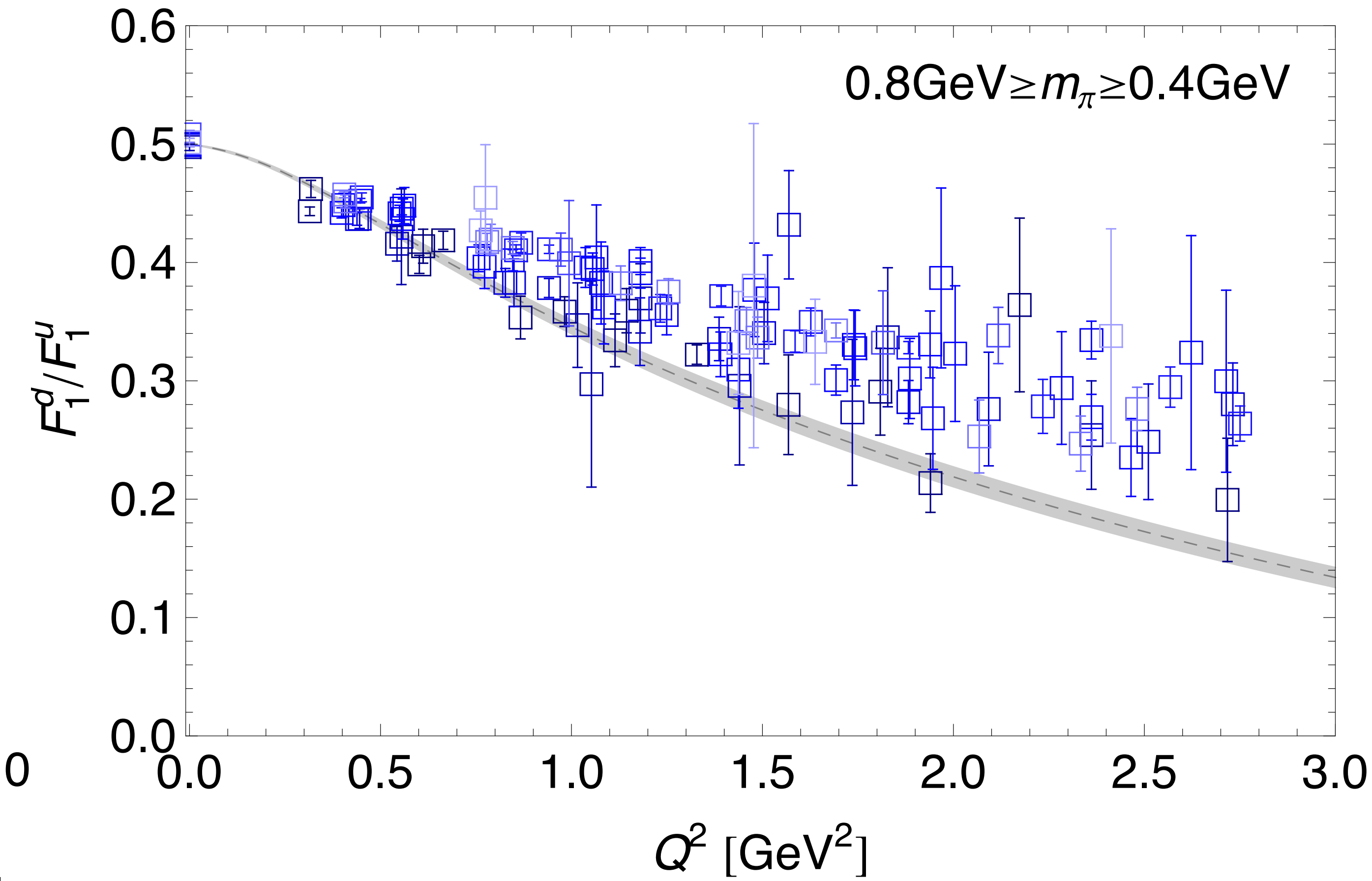
- Individual flavour contributions not accessible directly in experiment
- Must be derived from a combination of proton and neutron form factors
    - (assuming charge symmetry $u^p = d^n$)

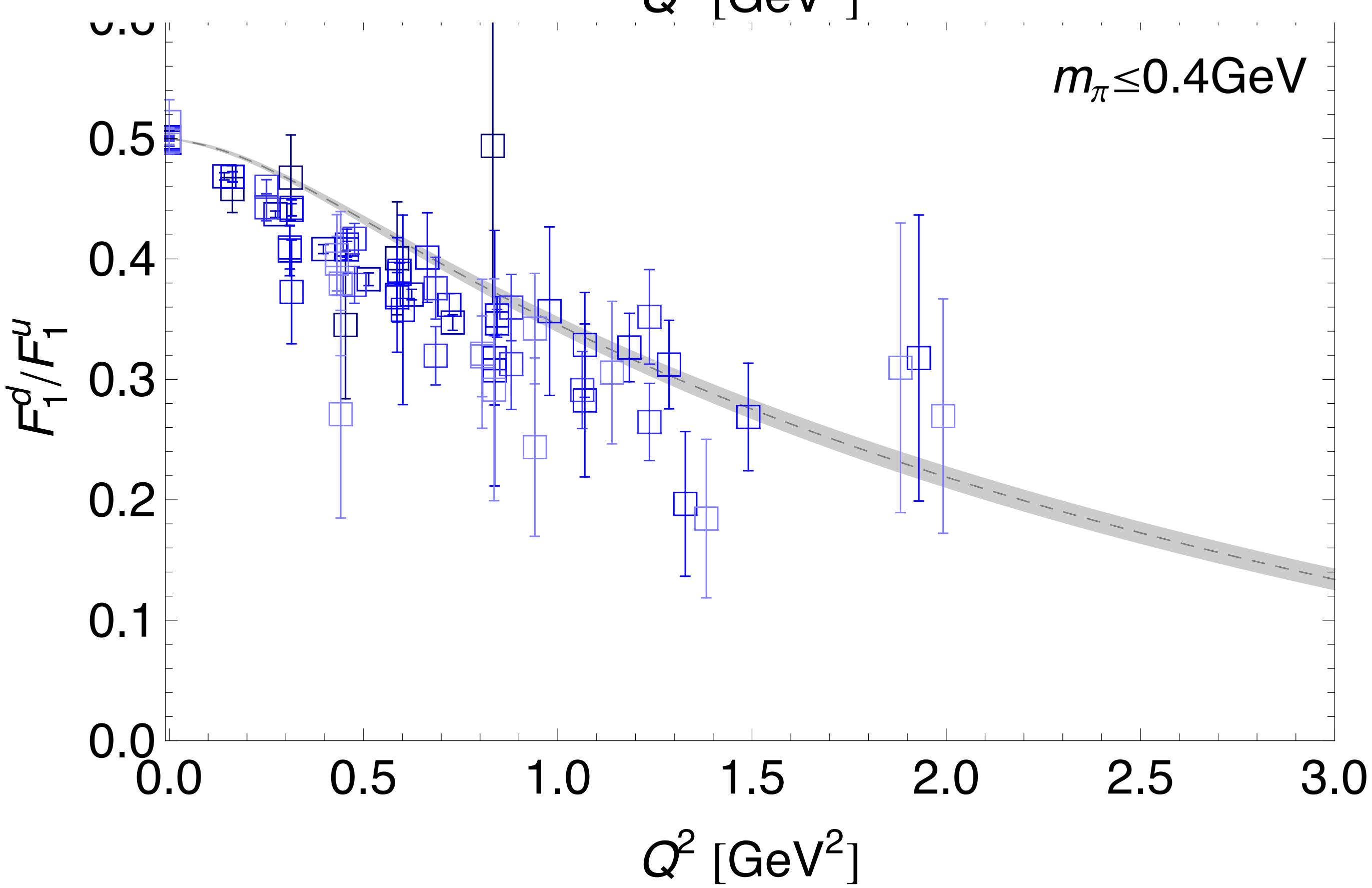
$$
\begin{aligned}
F^p &= \frac{2}{3}F_u^p - \frac{1}{3}F_d^p \\
F^n &= -\frac{1}{3}F_u^p + \frac{2}{3}F_d^p
\end{aligned}
$$

- On the lattice we compute the individual quark contributions directly

Flavour Distribution





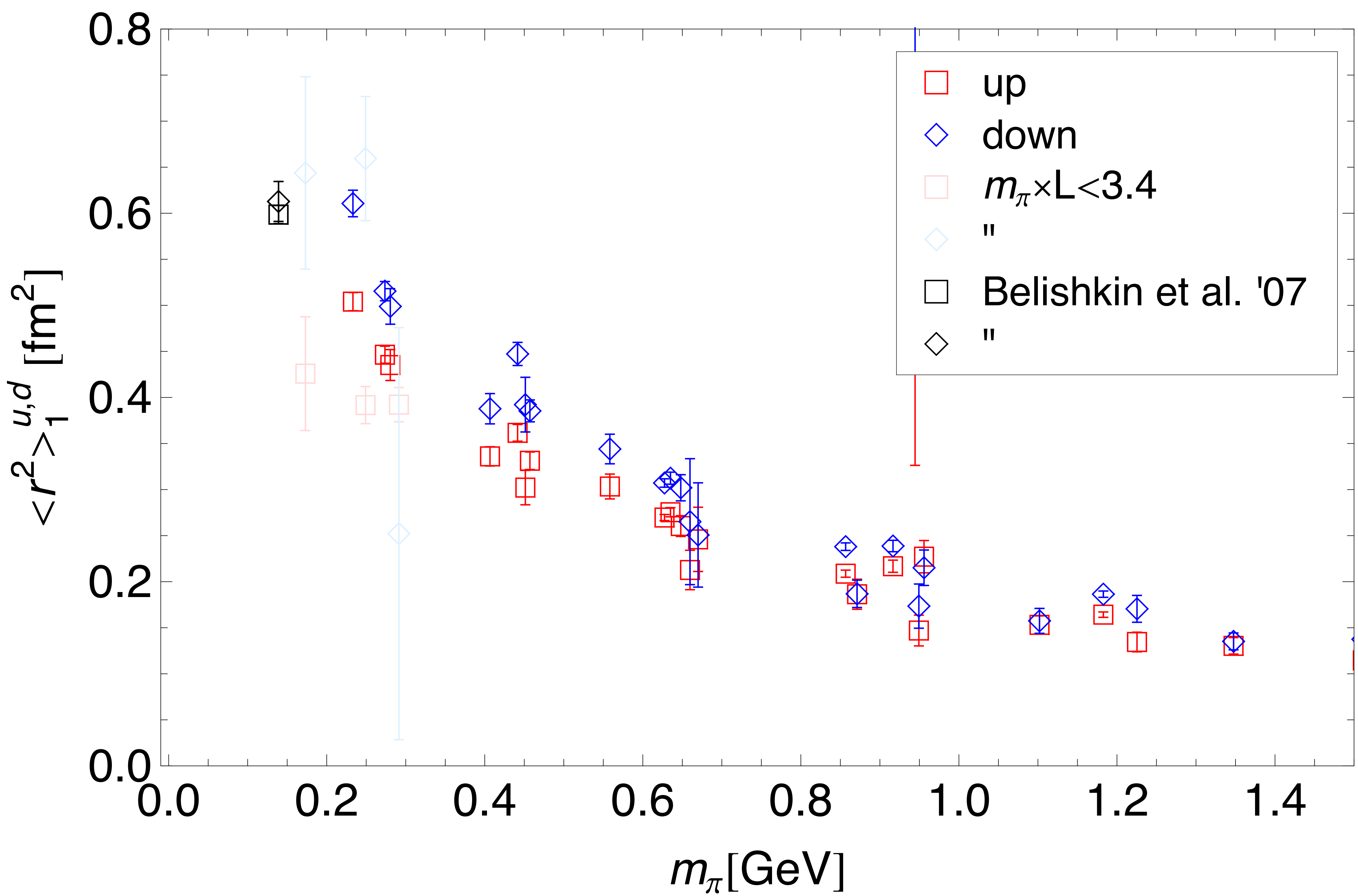


- d-quark contribution to $F_1(Q^2)$ falls off faster than the u-quark contribution
- Effect is enhanced at lighter quark masses

Flavour Distribution

$r^{d}_{1,2} > r^{u}_{1,2}$

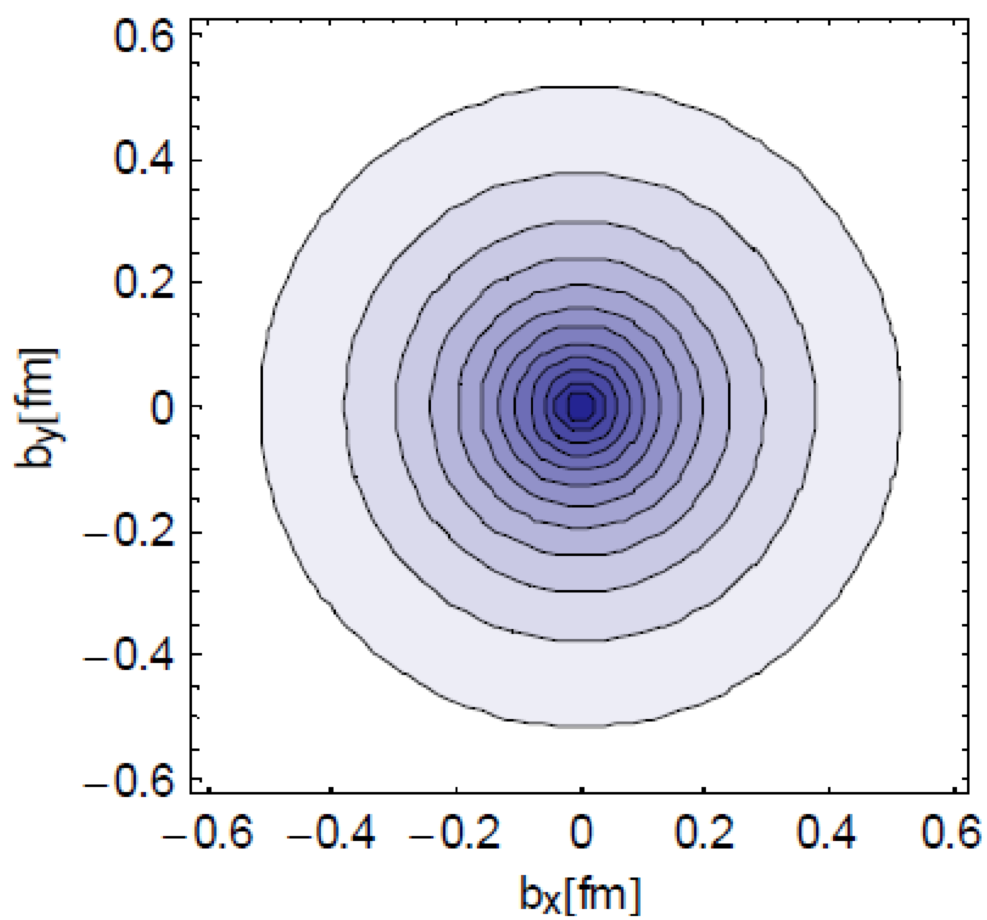
- In terms of charge radii, the d-quark in the proton has a larger charge radius than the u-quark

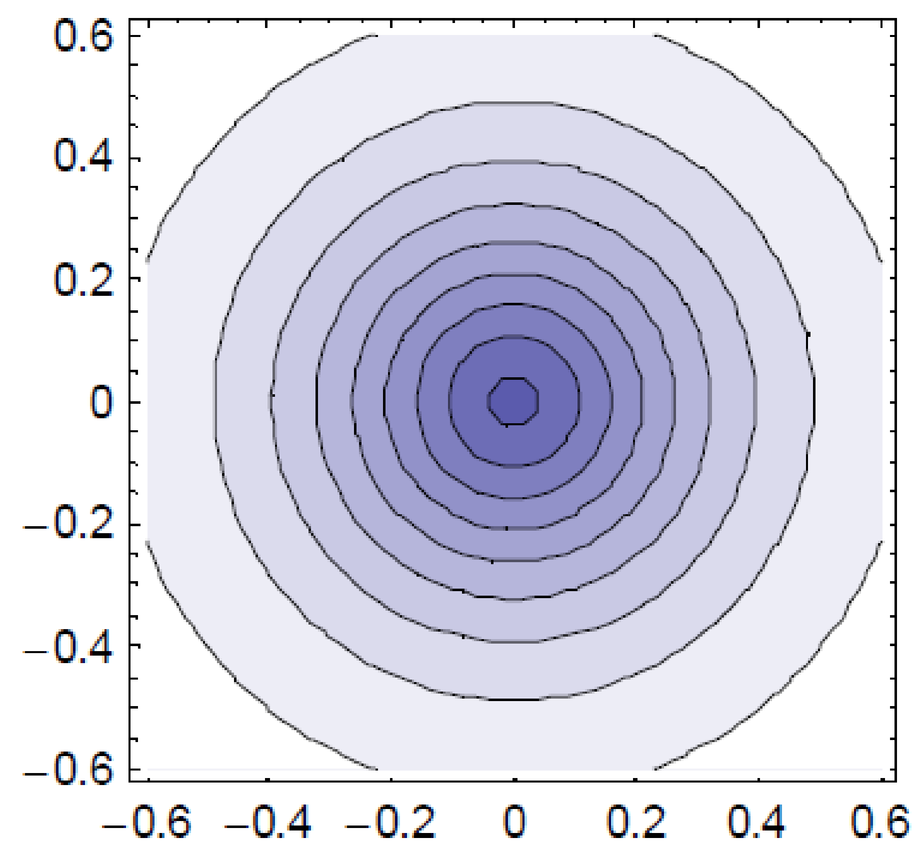


Implications for Transverse Densities

Recall: $$q(b_\perp^2) = \int d^2q_\perp\, e^{-i\vec{b}_\perp \cdot q_\perp} F_1(q^2)$$

up *down*





$$r_{1,2}^d > r_{1,2}^u$$

Ph. Hägler (QCDSF) [PRL 98, 222001 (2007)]

Pion Form Factor

$$\langle \pi(p')|J^\mu(\vec{q})|\pi(p)\rangle = P^\mu F_\pi(q^2)$$

$$q^2 = -Q^2 = (p' - p)^2$$

$$P^\mu = p'^\mu + p^\mu$$

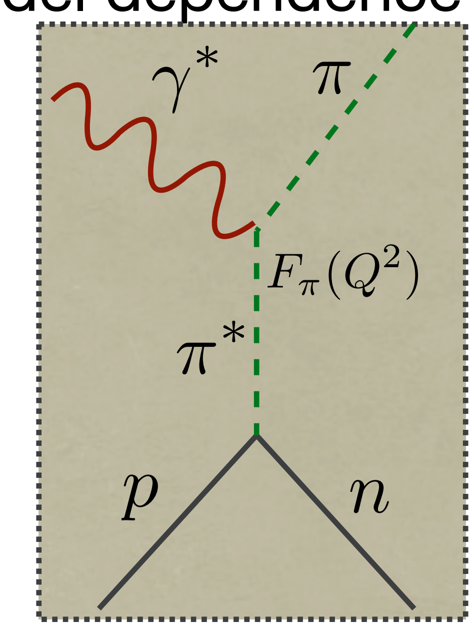
Pion Form Factor

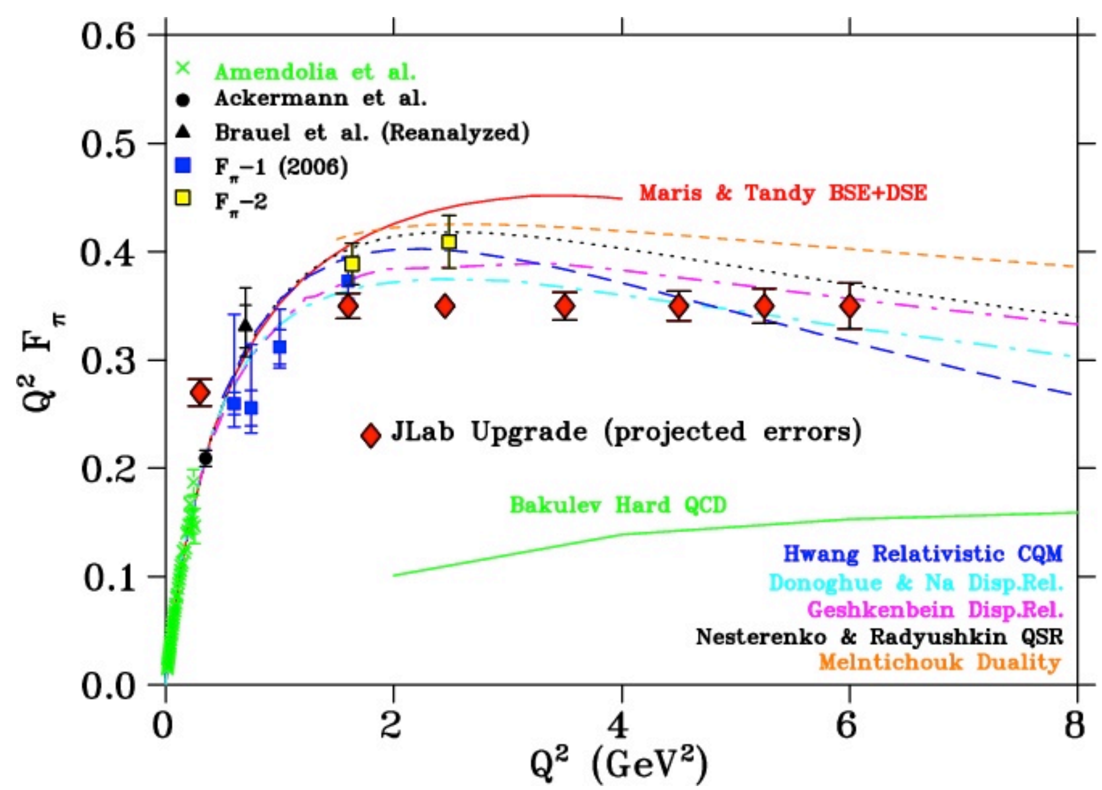
- Asymptotic normalisation known from $\pi \rightarrow \mu + \nu$ decay

$$F_\pi(Q^2 \rightarrow \infty) = \frac{16\pi\alpha_s(Q^2) f_\pi^2}{Q^2}$$

- Allows to study the transition from the soft to hard regimes
- Low Q^2: measured directly by scattering high energy pions from atomic electrons [CERN]
- High Q^2: quasi-elastic scattering off virtual pions [DESY & JLab]

Model dependence





simple_meson_seqsrc_w.cc

```
urceFactory::Instance().registerObject(string("a0-a0"),
                                        mesA0A01SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_x_1"),
                                                                               mesA0RhoX1SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_y_1"),
                                                                               mesA0RhoY1SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-b1_z"),
                                                                               mesA0B1Z1SeqSrc);

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success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-b1_y"),
                                                                               mesA01B1Y1SeqSrc);

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                                                                               mesA01B1X1SeqSrc);

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                                                                               mesA01Pion2SeqSrc);

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                                                                               mesA0A02SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_x_2"),
                                                                               mesA0RhoX2SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_y_2"),
                                                                               mesA0RhoY2SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a1_z"),
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                                                                               mesA0RhoZ2SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a1_y"),
                                                                               mesA0A1Y1SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a1_x"),
                                                                               mesA0A1X1SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-pion_1"),
                                                                               mesA0Pion1SeqSrc);

success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("pion_1-pion_1"),
                                                                               mesPion1Pion1SeqSrc);

// keep for historical purposes
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("pion"),
                                                                               mesPion1Pion1SeqSrc);
```
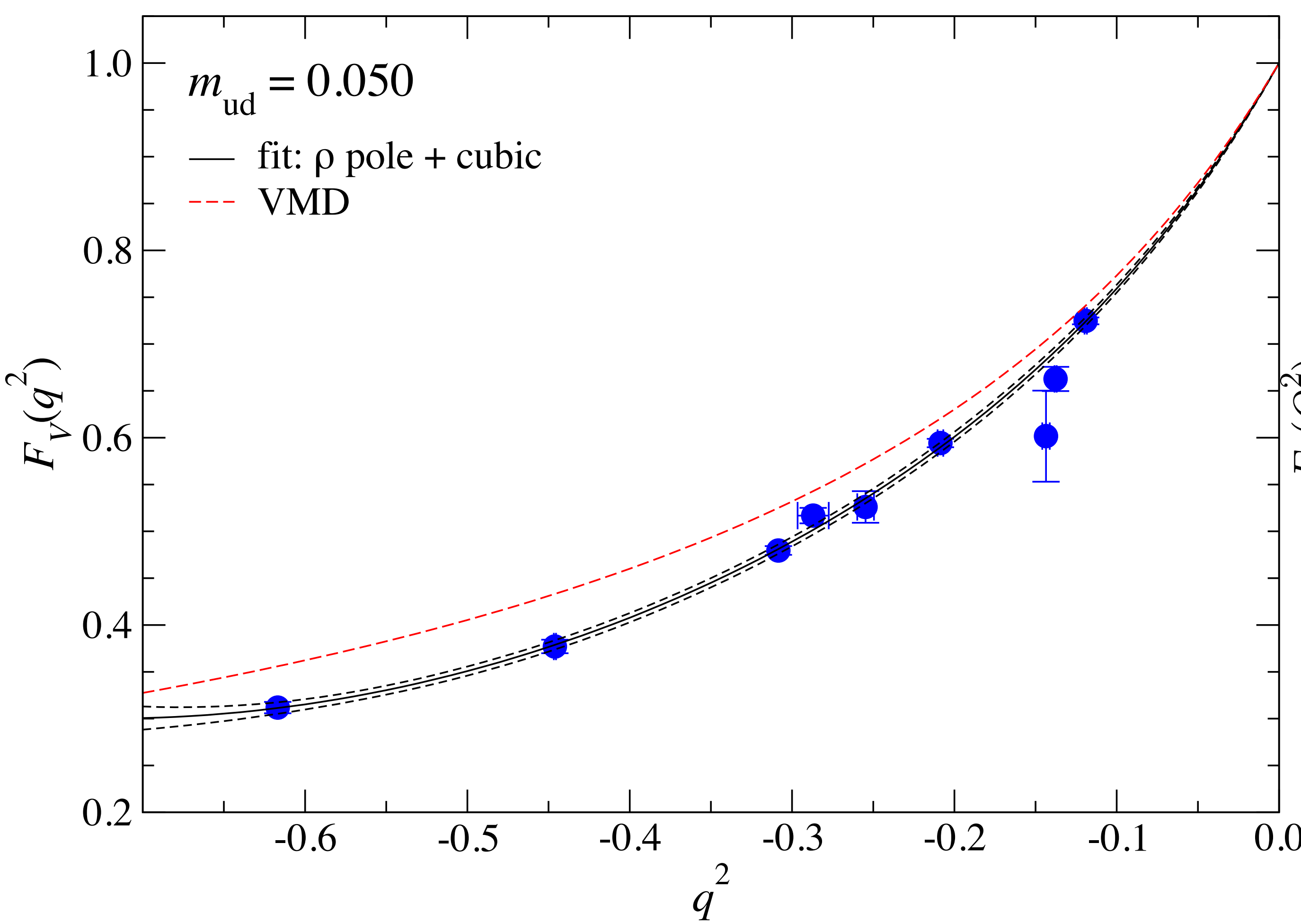
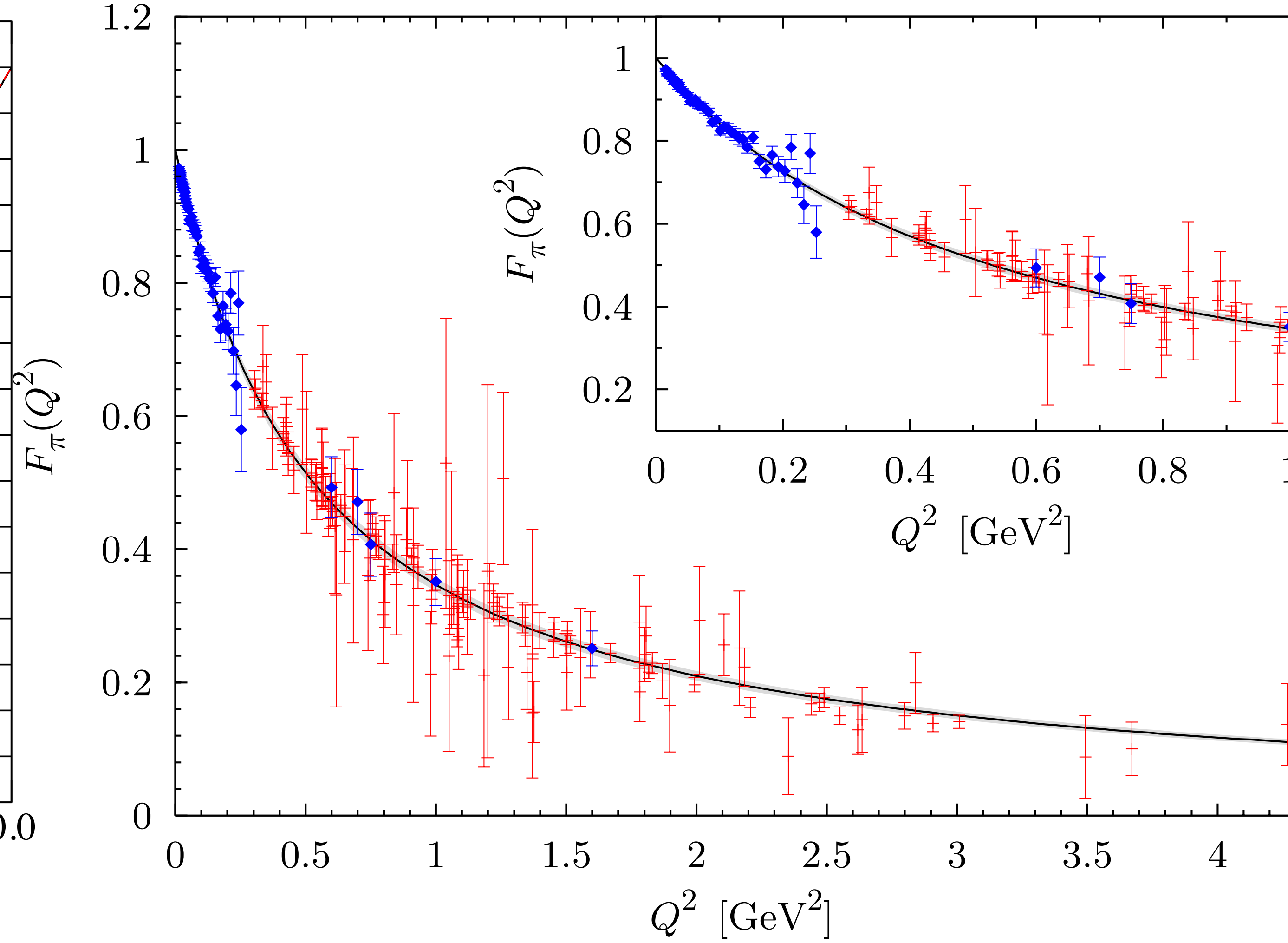

Pion Form Factor

$Q^2 = -q^2$

JLQCD: **arXiv:0810.2590** [hep-lat]

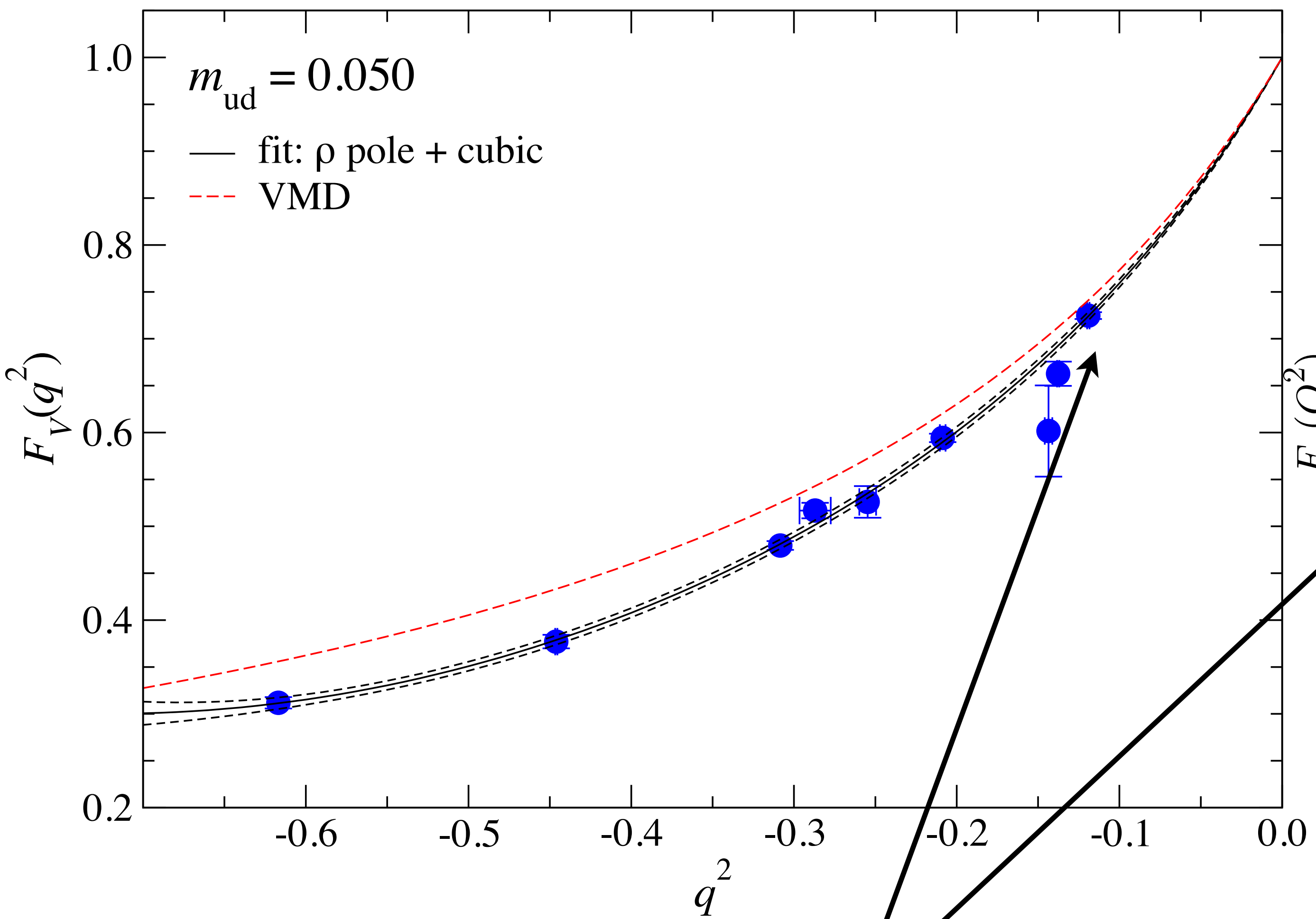


[QCDSF, hep-lat/0608021]

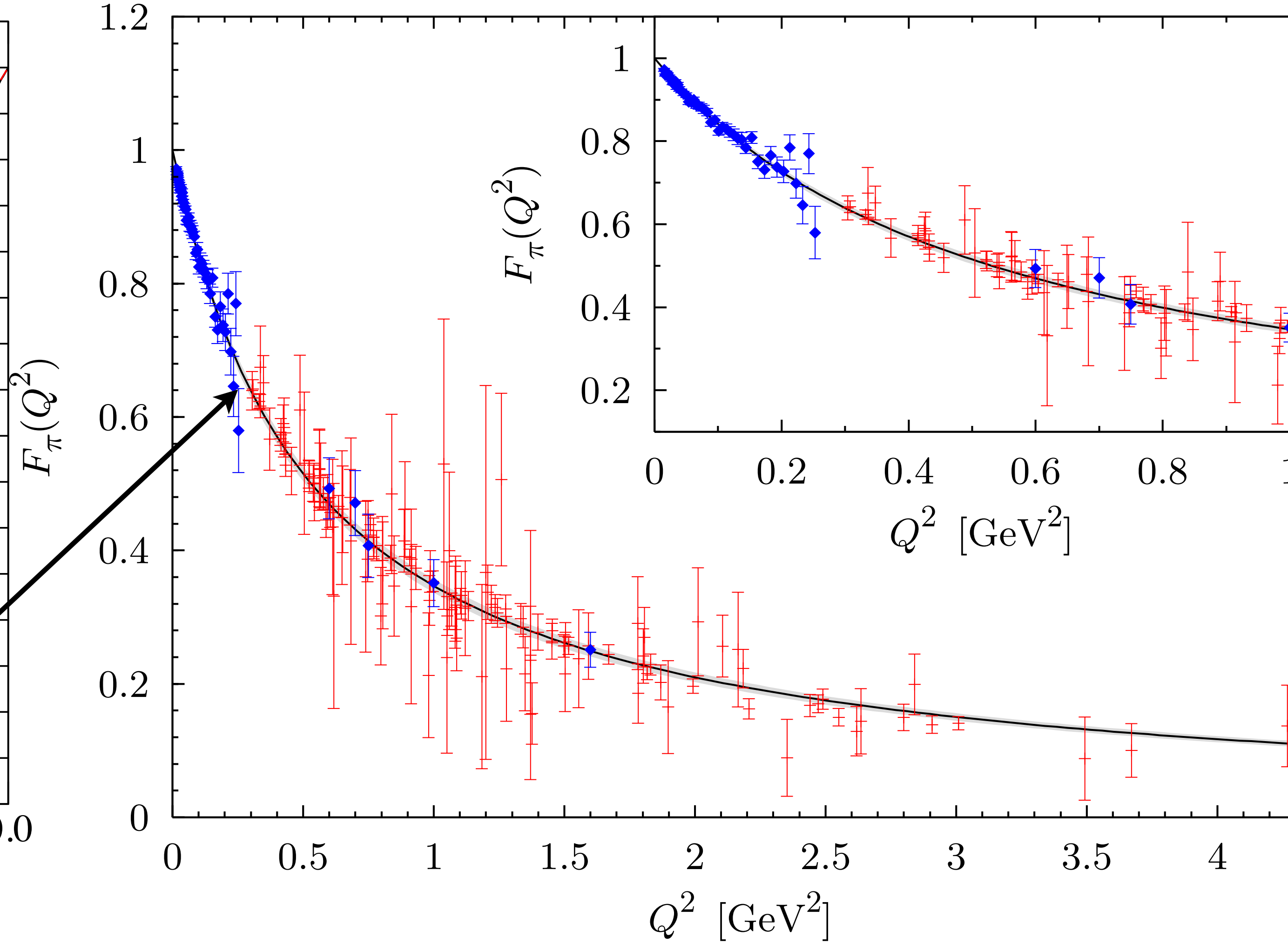


Pion Form Factor

JLQCD: **arXiv:0810.2590** [hep-lat]



[QCDSF, hep-lat/0608021]



Minimum lattice momentum:

$$\vec{p} = \frac{2\pi}{L}\vec{n}$$

affects determination of:

$$\langle r_\pi^2 \rangle$$

Discretised Momentum

- On a periodic lattice with spatial volume L^3, quark fields satisfy

$$\psi(x + \vec{e}_i\, L) = \psi(x), \quad i = 1,\, 2,\, 3$$

$$\int \mathrm{d}^4 p\, e^{-ip(x+\vec{e}_i\, L)} \tilde{\psi}(p) = \int \mathrm{d}^4 p\, e^{-ipx} \tilde{\psi}(p), \quad i = 1,\, 2,\, 3$$

- so we see that momenta are discretised in units of $p_i = \frac{2\pi}{L} n_i, \quad i = 1,\, 2,\, 3$

- For typical lattices, smallest non-zero momentum ~400-500 MeV

- Poor momentum resolution

- Can affect phenomenological observables e.g. form factors

Accessing small momenta:

(partially) twisted boundary conditions

- On a periodic lattice with spatial volume L^3, quark fields satisfy

$$\psi(x + \vec{e}_i\, L) = \psi(x), \quad i = 1,\, 2,\, 3$$

$$\int \mathrm{d}^4 p\, e^{-ip(x+\vec{e}_i\, L)} \tilde{\psi}(p) = \int \mathrm{d}^4 p\, e^{-ipx} \tilde{\psi}(p), \quad i = 1,\, 2,\, 3$$

- so we see that momenta are discretised in units of $p_i = \frac{2\pi}{L} n_i, \quad i = 1,\, 2,\, 3$

- Modify boundary conditions on the valence quarks

$$\psi(x + \vec{e}_i\, L) = e^{i\theta_i} \psi(x), \quad i = 1,\, 2,\, 3$$

- allows to tune the momenta continuously

$$p_i = \frac{2\pi}{L} n_i + \frac{\theta_i}{L}, \quad i = 1,\, 2,\, 3$$

- For a meson with quark flavours (1,2)

$$\vec{p} = \frac{2\pi}{L} \vec{n} + \frac{(\vec{\theta}_1 - \vec{\theta}_2)}{L}$$

Implementation

- Make a unitary Abelian transformation on the fields

$$\psi(x) \longrightarrow \mathcal{U}(\theta, x)\tilde{\psi}(x) = e^{\frac{i\vec{\theta}\cdot\vec{x}}{L}}\tilde{\psi}(x)$$

- Phase factor cancels in all terms of the lattice fermion action except the spatial hopping term

$$\overline{\tilde{\psi}}(x)\left[e^{i\frac{a\theta_i}{L}}U_i(x)(1-\gamma_i)\tilde{\psi}(x+\hat{i}) + e^{-i\frac{a\theta_i}{L}}U_i^\dagger(x-\hat{i})(1+\gamma_i)\tilde{\psi}(x-\hat{i})\right]$$

- In practice, compute quark propagator with gauge links

$$\{U_i(x)\} \longrightarrow \{e^{i\frac{a\theta_i}{L}}U_i(x)\}$$

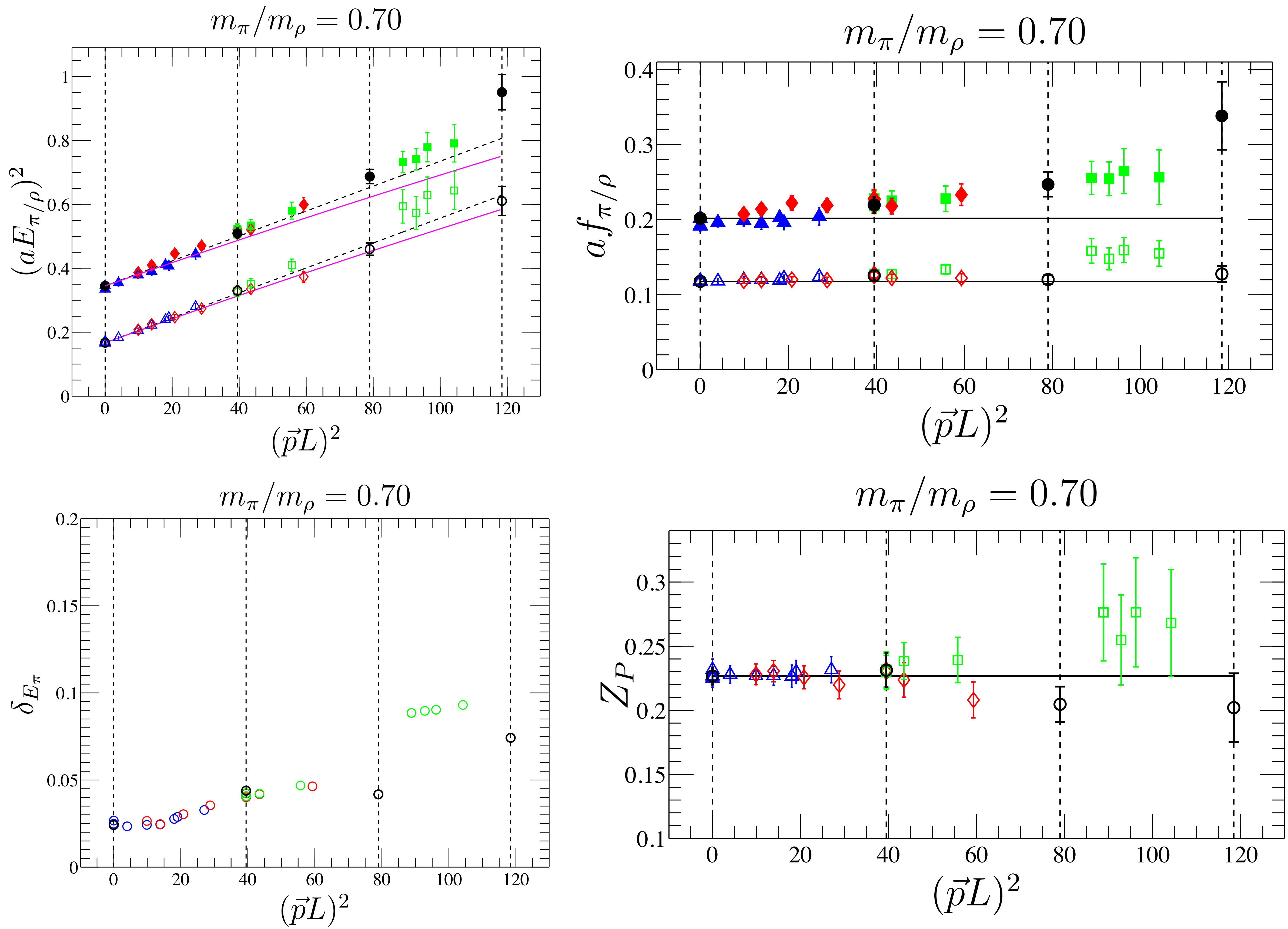
- Twisted boundary conditions for sea quarks requires generating new set of gauge fields for each twist

  - only twist valence quarks ⇨ *partially twisted boundary conditions*

  - Introduces an additional finite size effect that is, however, exponentially suppressed

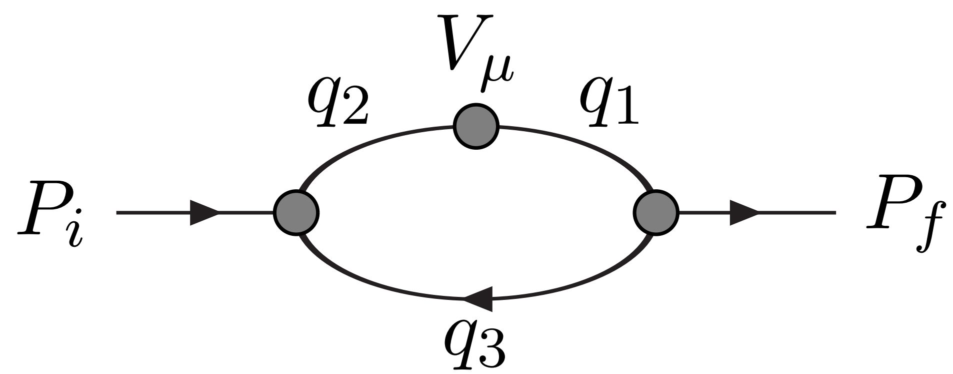
Additional Finite Volume Effects (dispersion relation) [hep-lat/0506016]

$$E_{\pi,\rho}^2 = m_{\pi,\rho}^2 + \left(\vec{p}_{\rm lat} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L}\right)^2$$



Implementation

[RBC/UKQCD, hep-lat/0705005]



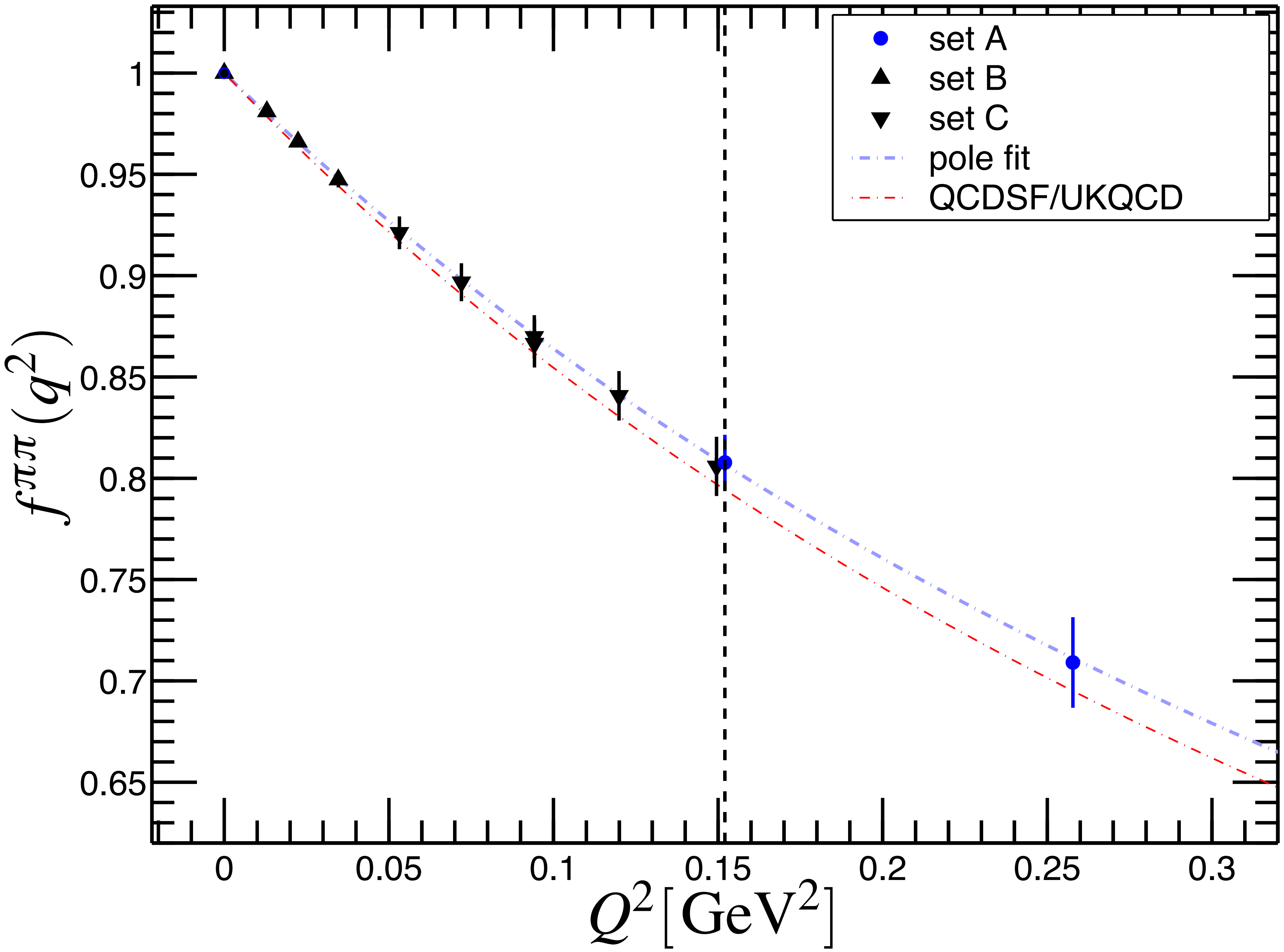
- Use different (twisted) boundary conditions when computing the propagators either side of the current

- E.g. One possibility would be

$$\vec{\theta}_{q_2} = \vec{\theta}_{q_3} = \vec{0}, \quad \vec{\theta}_{q_1} \neq \vec{0}$$

Pion charge radius

[RBC/UKQCD, arXiv:0804.3971]



$m_\pi \approx 330\mathrm{MeV}$

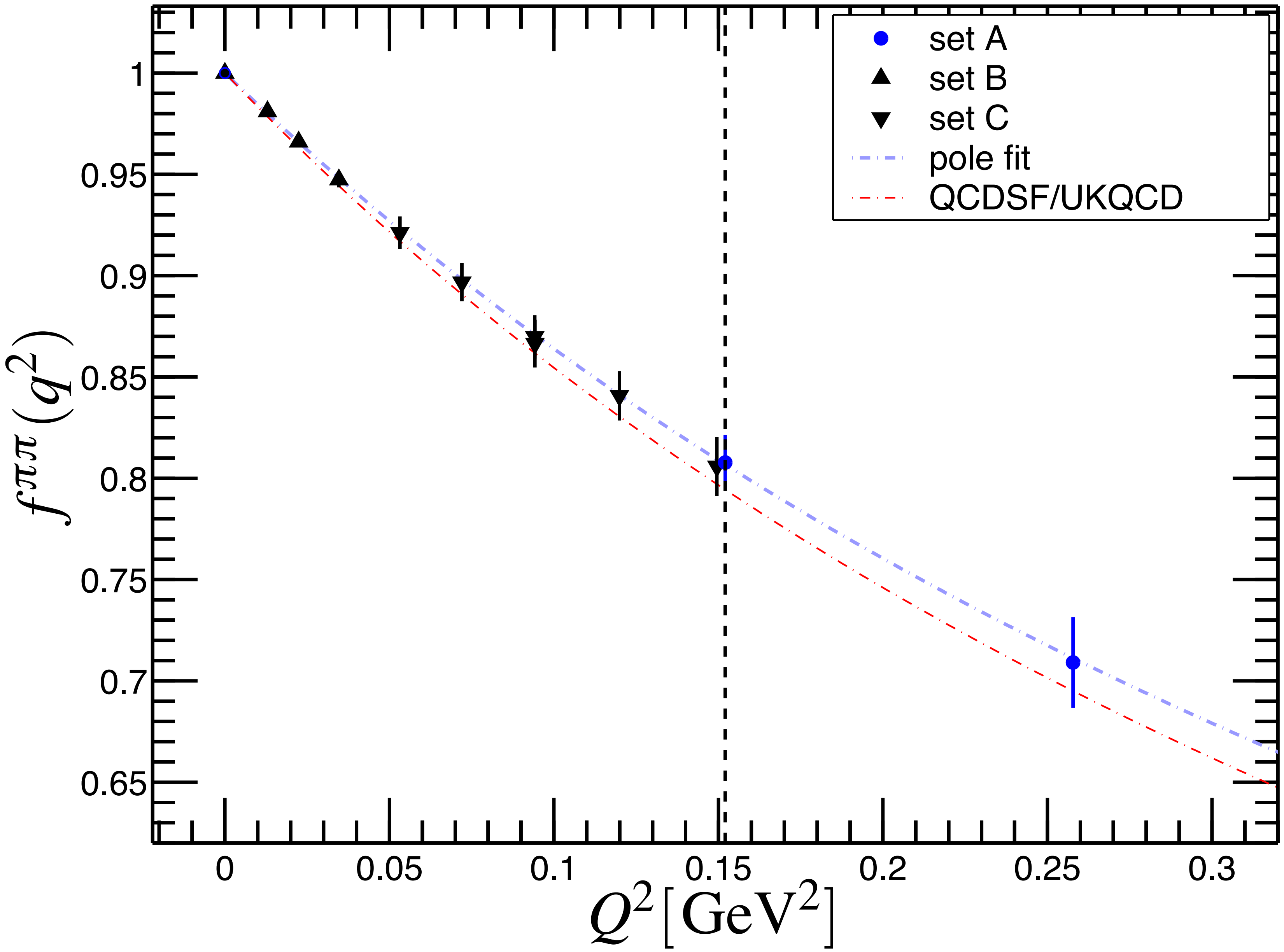
$24^3 \times 64$

$a \approx 0.114$ fm

maximum Q^2	linear	quadratic	cubic	pole
0.013 GeV^2	0.354(28)(11)	–	–	0.361(29)(12)
0.022 GeV^2	0.354(26)(11)	0.353(35)(11)	–	0.364(27)(12)
0.035 GeV^2	0.353(25)(11)	0.355(32)(11)	0.351(41)(11)	0.366(27)(12)
0.150 GeV^2	0.332(28)(11)	0.387(44)(13)	0.406(56)(13)	0.382(37)(12)

Pion charge radius

[RBC/UKQCD, 0804.3971]



$m_\pi \approx 330\mathrm{MeV}$

$24^3 \times 64$

$a \approx 0.114$ fm

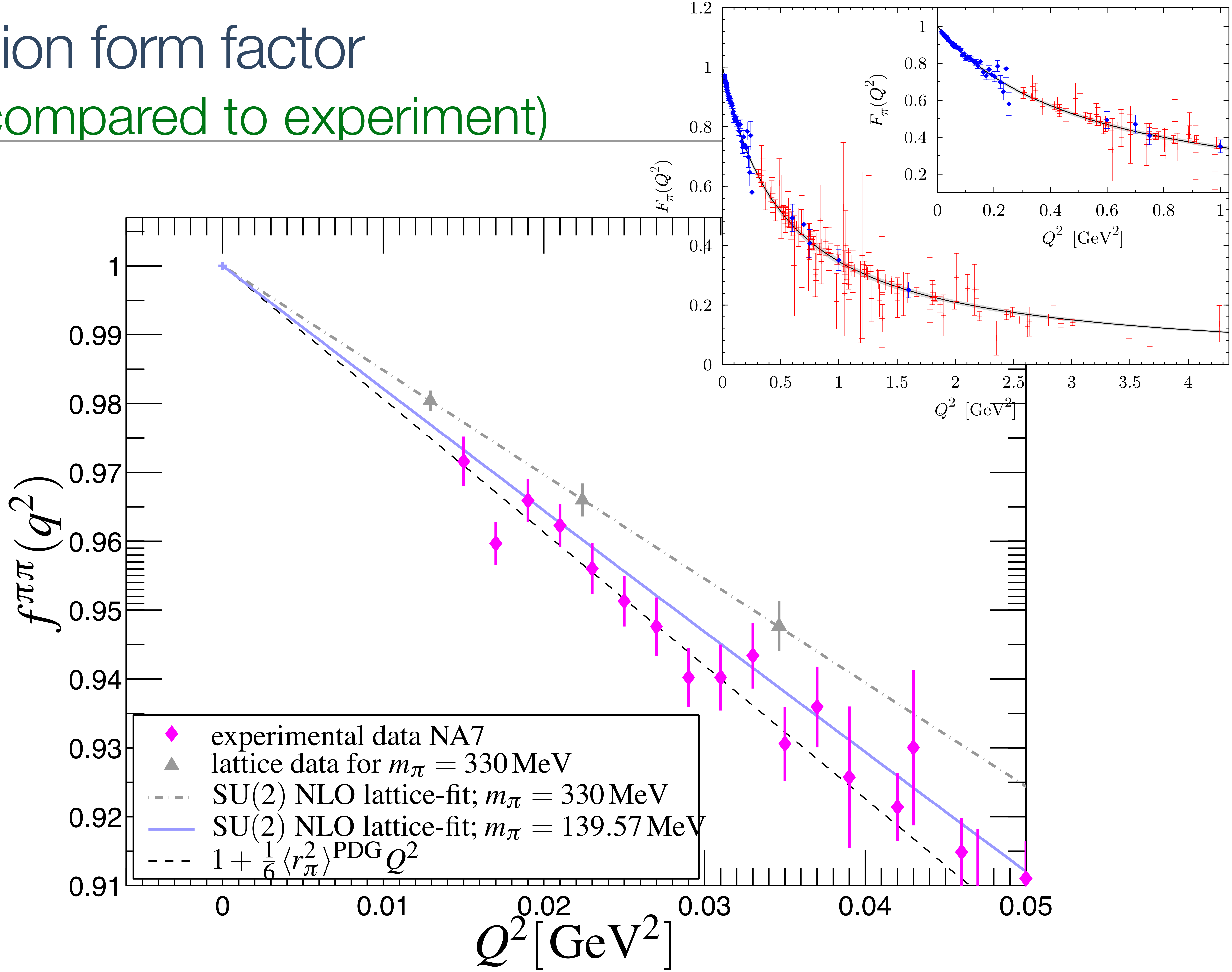
$Q^2_{\mathrm{max}} = 0.035\,[\mathrm{GeV}]^2$

$$\langle r_\pi^2 \rangle_{(2),\mathrm{NLO}} = -\frac{12 l_6^r}{f^2} - \frac{1}{8\pi^2 f^2}\left(\log \frac{m_\pi^2}{\mu^2} + 1\right)$$

[0.418(28)]

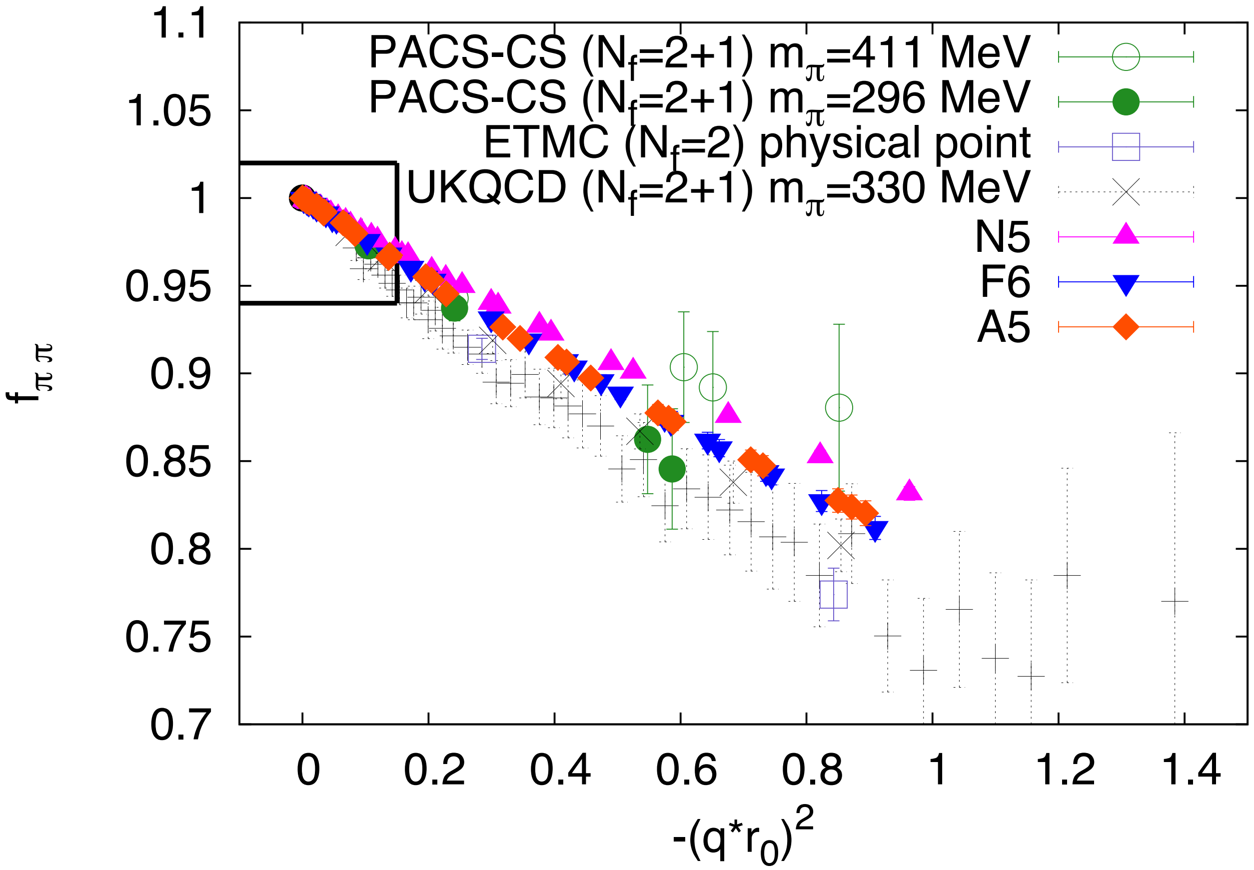
[0.452(11)]$_{exp}$

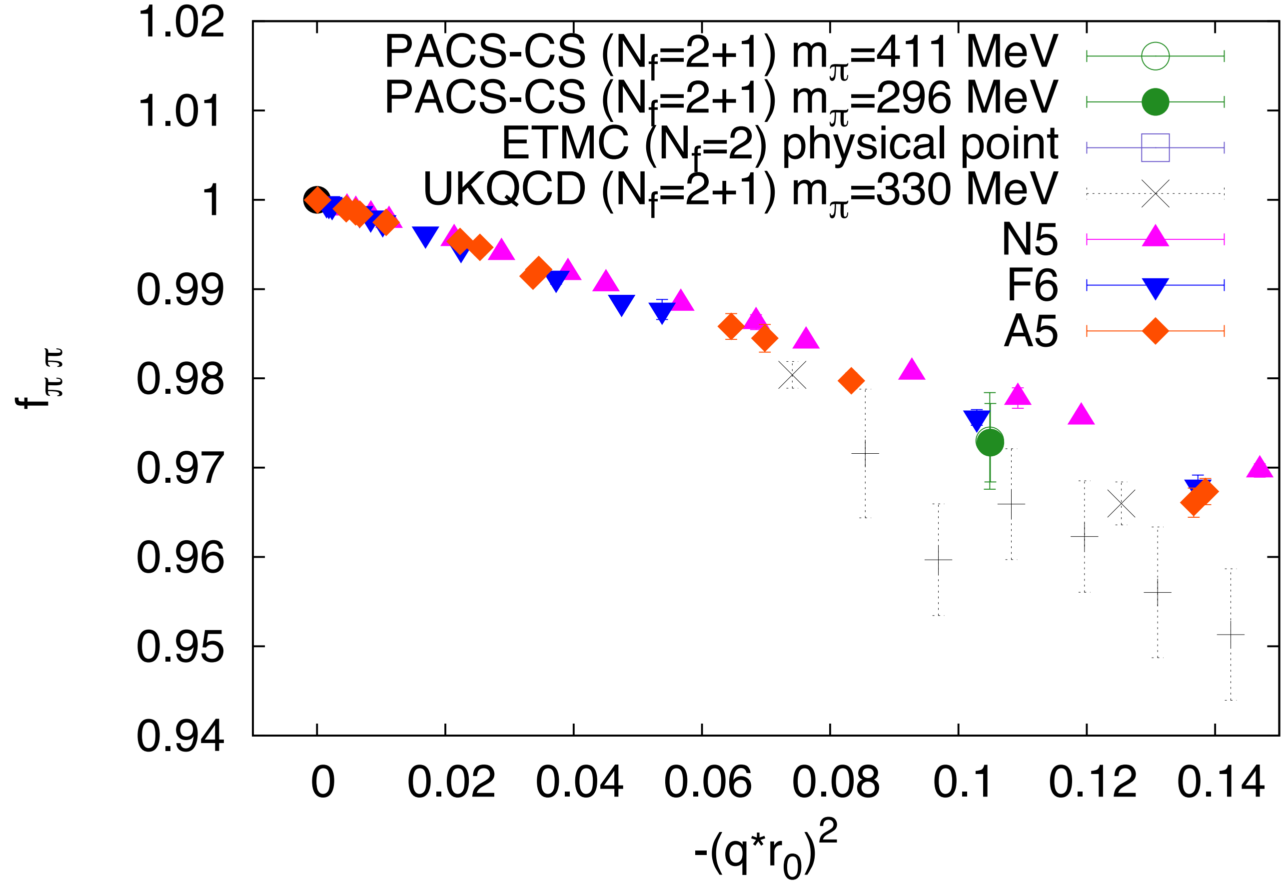
Pion form factor
(compared to experiment)

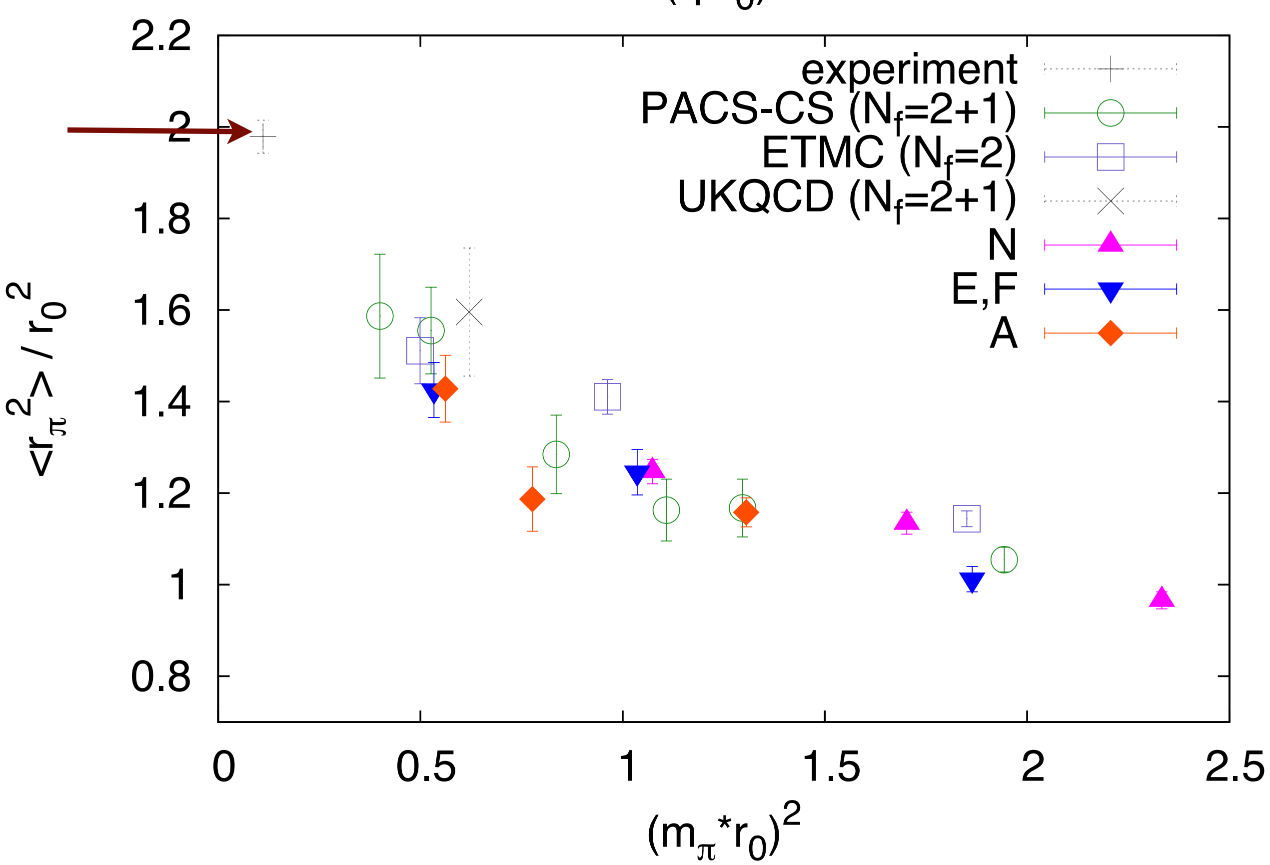


Pion Form Factor

[Mainz: 1109.0196]







- Many choices of twist angles giving access to extremely small Q^2
- Radii results increasing towards the experimental point at smaller quark masses

Other Hadron Form Factors

- Pion and nucleon form factors have received the most attention
- Small amount of work on form factors of other hadrons, e.g.
  - Hyperons — CSSM: hep-lat/0604022, QCDSF: 1101.2806, H-W.Lin et al.: 0812.4456
  - Delta — CSSM: 0902.4046, Alexandrou et al.: 0810.3976
  - Rho — CSSM: hep-lat/0703014, QCDSF: PoS LAT2008, 051
  - $\gamma N \to \Delta$ transition — Alexandrou et al.: 1011.3233
- Non-zero quadrupole moment ⇒ hadron deformation

Review: [Alexandrou et al.: 1201.4511]