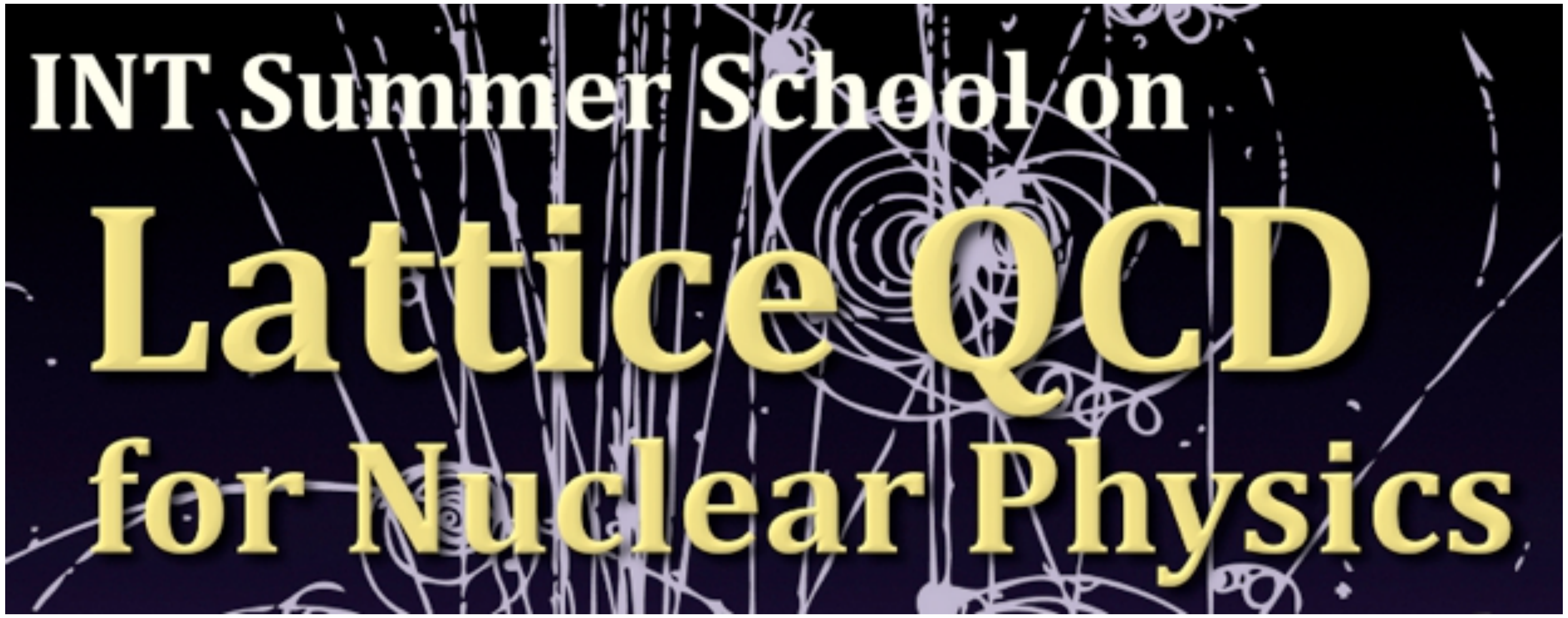


Lectures on Chiral Perturbation Theory

I. Foundations
II. Lattice Applications
III. Baryons
IV. Convergence

Brian Tiburzi

Chiral Perturbation Theory

II. Lattice Applications

Lattice Applications of Chiral Perturbation Theory

- Study quark mass dependence of observables for chiral extrapolations (maybe even interpolations) of lattice QCD data.

- Use the lattice to expose the role chiral symmetry breaking plays in low-energy QCD, hopefully confirm predictions of ChPT.

- **Tailor ChPT to address sources of systematic error in lattice QCD computations of low-energy observables (in conjunction with above).**

Finite volume, partial quenching, discretization

Finite Volume Field Theories

$$Z[\eta] = \int \mathcal{D}\phi \, \exp\left(- \int_V \mathcal{L}(\partial_\mu \phi, \phi, \eta) \right)$$

- Euclidean spacetime volume: hypercubic L
- Boundary conditions: periodic $\phi(x+L) = \phi(x)$

Single valued on hypertorus: no surface terms

Discrete translation symmetry: PBCs not renormalized

$$\phi(x) = \int_k e^{ikx} \phi_k$$

$$\phi(x+L) = \int_k e^{ikx} \phi_k \, e^{ikL}$$

$$k = \frac{2\pi n}{L}$$

That's *it*.

Finite Volume Field Theories

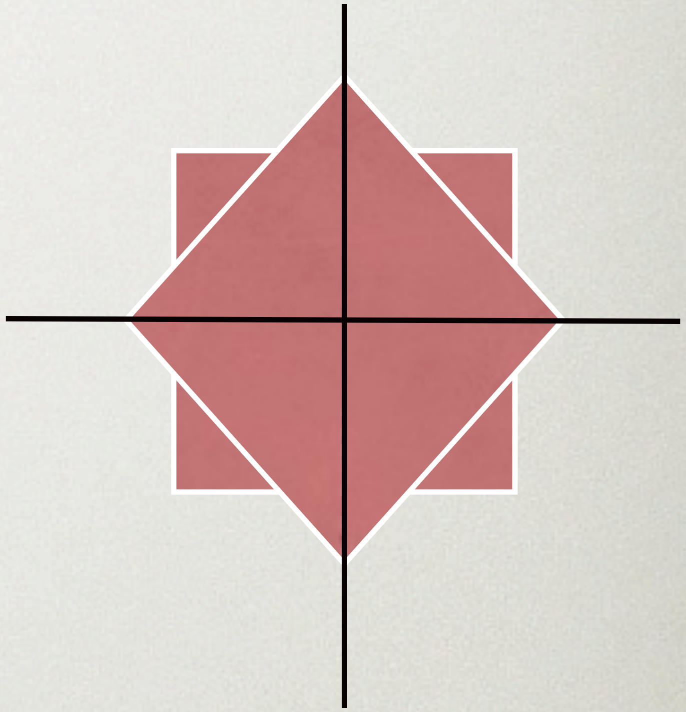
- Euclidean SO(4) invariance reduced to permutation symmetry

$$\langle \phi(0) | J_\mu | \phi(0) \rangle = Q\, \delta_{\mu,4}$$

Boost

Frame dependence

$$\langle \phi(\vec{v}) | \vec{J}\, | \phi(\vec{v}) \rangle \neq Q\, \vec{v}$$



- Gauge invariance

↕

Ward-Takahashi identity

↕

Ward identity

Finite Volume Field Theories

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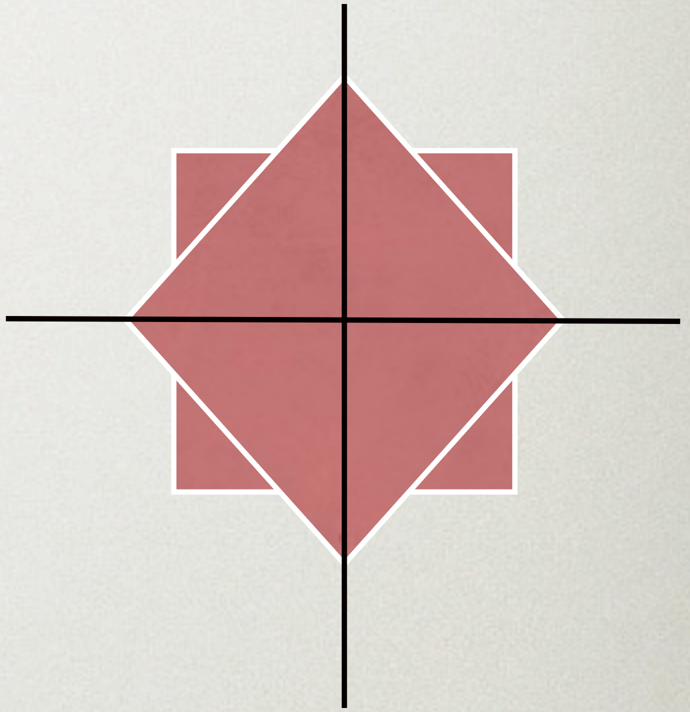

$$\langle \phi(\vec{v}) | \vec{J}\, | \phi(\vec{v}) \rangle \neq Q\, \vec{v}$$

- Gauge invariance

$\updownarrow$

Ward-Takahashi identity

Ward identity

Gauge functions restricted

$\updownarrow$

$$q_\mu J_\mu = Q[G^{-1}(q+p) - G^{-1}(p)]$$

$$J_\mu = Q \frac{\partial G^{-1}}{\partial p_\mu}$$

Long-Range Physics

- Observables: electromagnetic moments, for example

Charge and current distributions are not described by multipole expansion

$$F(q^2) \to F_p(q, L)$$
$$qL = 2\pi n$$

Electric and magnetic fields are not the only gauge invariant quantities

$$e^{i \int_0^L dx_\mu \, A_\mu(x)}$$

- No spontaneous symmetry breaking at finite volume

$$\mathcal{P} \sim e^{-V \int_a^b d\phi \, F(\phi)}$$

Quantum mechanics vs. field theory

Long-range physics modified: Goldstone bosons $k = \frac{2\pi n}{L}$ That's *it*.

Spontaneous Chiral Symmetry Breaking

Chiral symmetry of massless QCD

$SU(2)_L \otimes SU(2)_R$

$$\overline{\psi} \not{D} \psi = \overline{\psi}_L \not{D} \psi_L + \overline{\psi}_R \not{D} \psi_R$$

Vacuum does not respect chiral symmetry

chiral condensate $\langle \overline{\psi}_{iR} \psi_{jL} \rangle = -\lambda \, \delta_{ji}$

$\longrightarrow SU(2)_V$

Chiral perturbation theory $\Sigma \sim \psi_L \overline{\psi}_R$

$$\mathcal{L}_\chi = \frac{f^2}{8} \mathrm{Tr} \left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right) - m_q \lambda \, \mathrm{Tr} \left(\Sigma + \Sigma^\dagger \right)$$

Effective theory for low-energy QCD

$$p^2/(4\pi f)^2 \sim m_\pi^2/(4\pi f)^2 \ll 1$$

Spontaneous Chiral Symmetry Breaking

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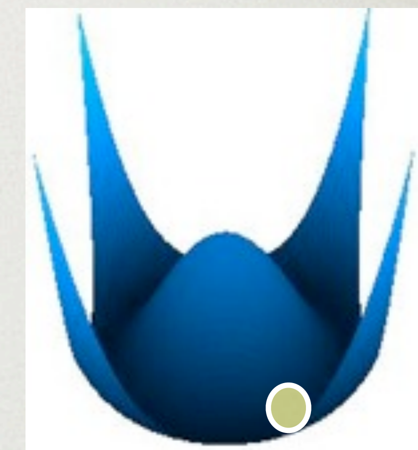

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Effective theory for low-energy QCD

$$p^2/(4\pi f)^2 \sim m_\pi^2/(4\pi f)^2 \ll 1$$

Spontaneous Chiral Symmetry Breaking

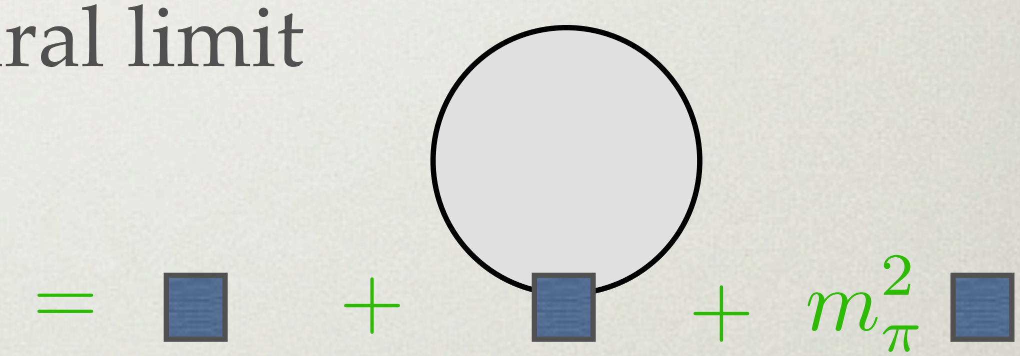
Example: chiral condensate

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(\gamma_\mu D_\mu + m_q)\psi \qquad \mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - m_q \lambda\, \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)$$

$$\langle \overline{\psi}\psi \rangle = -\frac{\partial \log Z_{\mathrm{QCD}}}{\partial m_q} \quad \Rightarrow \quad \langle \overline{\psi}\psi \rangle = -\frac{\partial \log Z_{\chi\mathrm{PT}}}{\partial m_q}$$

Expansion of observables about chiral limit

$$\mathcal{O}(p^2), \quad \mathcal{O}(p^4)$$



Calculable infrared logarithm

$$\langle \overline{\psi}\psi \rangle = -4\lambda \left[1 + \frac{3\, m_\pi^2}{(4\pi f)^2}\left(\log \frac{\mu^2}{m_\pi^2} + 1\right) - \frac{m_\pi^2}{f^2}\mathbb{L}_4(\mu)\right]$$

Chiral Perturbation Theory in Finite Volume

Match finite volume QCD at low energies onto finite volume ChPT

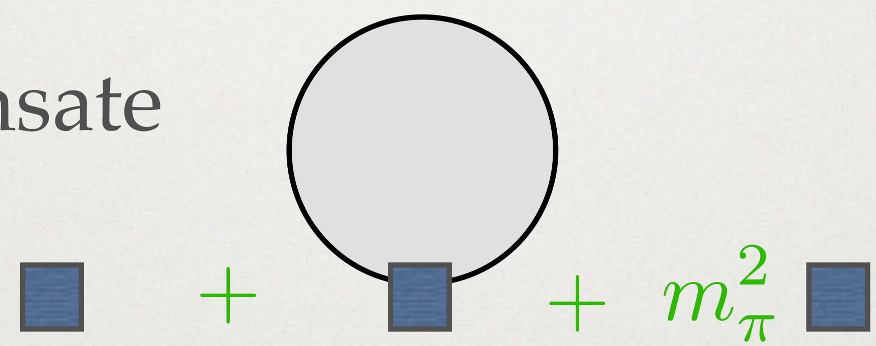
$$Z_{\text{eff}} = \int \mathcal{D}\Sigma \, e^{-\int_V \mathcal{L}(\partial_\mu \Sigma, \Sigma)}$$

+ chiral physics must fit in the box

$$L \gg 1/(4\pi f)$$

Example: chiral condensate

$$\langle \overline{\psi}\psi \rangle = -\frac{\partial \log Z_{\text{eff}}}{\partial m_q} =$$



Infrared logarithm

$$\int_k \frac{1}{k^2 + m_\pi^2} \sim m_\pi^2 \log \frac{m_\pi^2}{\mu^2}$$

$$k = \frac{2\pi n}{L}$$

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \to \frac{1}{L} \sum_{n=-\infty}^{+\infty} \quad \Longrightarrow \quad \frac{1}{L^4} \sum_{n_\mu=-\infty}^{+\infty} \frac{1}{(2\pi\, n_\mu/L)^2 + m_\pi^2}$$

Chiral Perturbation Theory in Finite Volume

Match finite volume QCD at low energies onto finite volume ChPT

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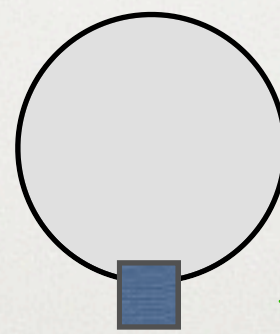
$$\langle \overline{\psi}\psi \rangle = -\frac{\partial \log Z_{\text{eff}}}{\partial m_q}$$ $=$  $+$  $+\; m_\pi^2$ 

Infrared logarithm

$$\int_k \frac{1}{k^2 + m_\pi^2} \sim m_\pi^2 \log \frac{m_\pi^2}{\mu^2}$$

$$k = \frac{2\pi n}{L}$$

$$1 \gg (m_\pi L)^2 \gg (m_\pi / \Lambda_\chi)^2$$

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \to \frac{1}{L} \sum_{n=-\infty}^{+\infty} \quad \Longrightarrow \quad \frac{1}{(fL)^2} \left[\frac{1}{(m_\pi L)^2} + \sum_{n_\mu \neq 0} \frac{1}{4\pi^2 n_\mu^2 + (m_\pi L)^2} \right]$$

Chiral Limit in Finite Volume

Power counting: epsilon regime $\frac{1}{L} \sim \varepsilon, \quad m_\pi \sim \varepsilon^2$

Vertices $\partial_\mu \partial_\mu \sim \varepsilon^2 \quad \lambda m_q \sim \varepsilon^4$

Loop factor $\int_{k_\mu} \to \frac{1}{L^4} \sum_{n_\mu} \sim \varepsilon^4$

Propagator $\frac{1}{m_\pi^2} + \sum_{n_\mu \neq 0} \frac{1}{\left(\frac{2\pi n_\mu}{L}\right)^2 + m_\pi^2}$

Zero mode ε^{-4} Non-zero modes ε^{-2}

Exercise:

Do the leading-order four-pion interactions allow mixing of zero and non-zero modes? Draw all one- and two-loop diagrams for the chiral condensate & count powers of epsilon.

General Feynman diagram

(I internal, V vertices, L loops)

$$L = I - V + 1$$

Only zero mode

$$\sim \varepsilon^{4L+4V-4I} = \varepsilon^4$$

Only non-zero modes, derivative vertices

$$\sim \varepsilon^{4L+2V-2I} = \varepsilon^{2L+2}$$

Zero mode is strongly coupled

$$\Sigma(x) = \Sigma_0 e^{2i\tilde{\phi}(x)/f}$$

Zero Pion Momentum

Chiral Limit in Finite Volume

Zero mode is strongly coupled $\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)$

$\Sigma(x) = \Sigma_0 e^{2i\tilde{\phi}(x)/f}$ but can treat non-perturbatively (Matrix Model)

$$Z_0 = \int \mathcal{D}\Sigma_0 e^{-\int_V \mathcal{L}(\partial_\mu\Sigma_0=0,\Sigma_0)} = \int \mathcal{D}\Sigma_0 e^{2m_q\lambda L^4 \mathfrak{Re}[\mathrm{Tr}(\Sigma_0)]}$$

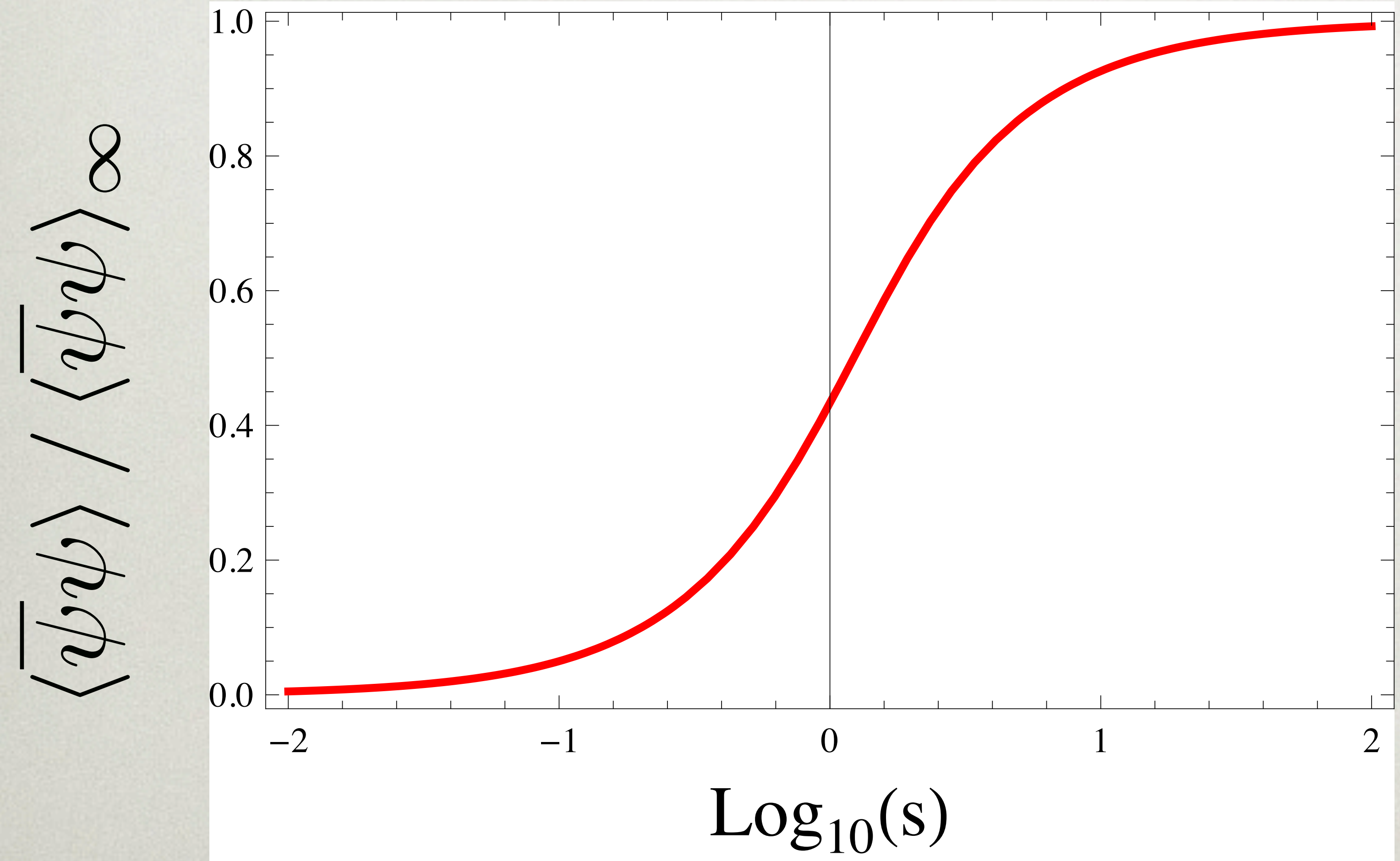
SU(2) simple to evaluate $Z_0 = I_1(2s)/s, \quad s = 2m_q\lambda L^4 = \frac{1}{4}f^2 m_\pi^2 L^4$

$$\langle\overline{\psi}\psi\rangle = -\frac{\partial \log Z_{\mathrm{eff}}}{\partial m_q} \qquad \langle\overline{\psi}\psi\rangle / \langle\overline{\psi}\psi\rangle_\infty = \frac{\partial}{\partial s}\log I_1(2s)/s$$

Chiral Limit in Finite Volume

$$k = \frac{2\pi n}{L}$$

That's *it*.



Scaling variable

$$s = \frac{1}{4} V f^2 m_\pi^2$$

Chiral Limit in Finite Volume

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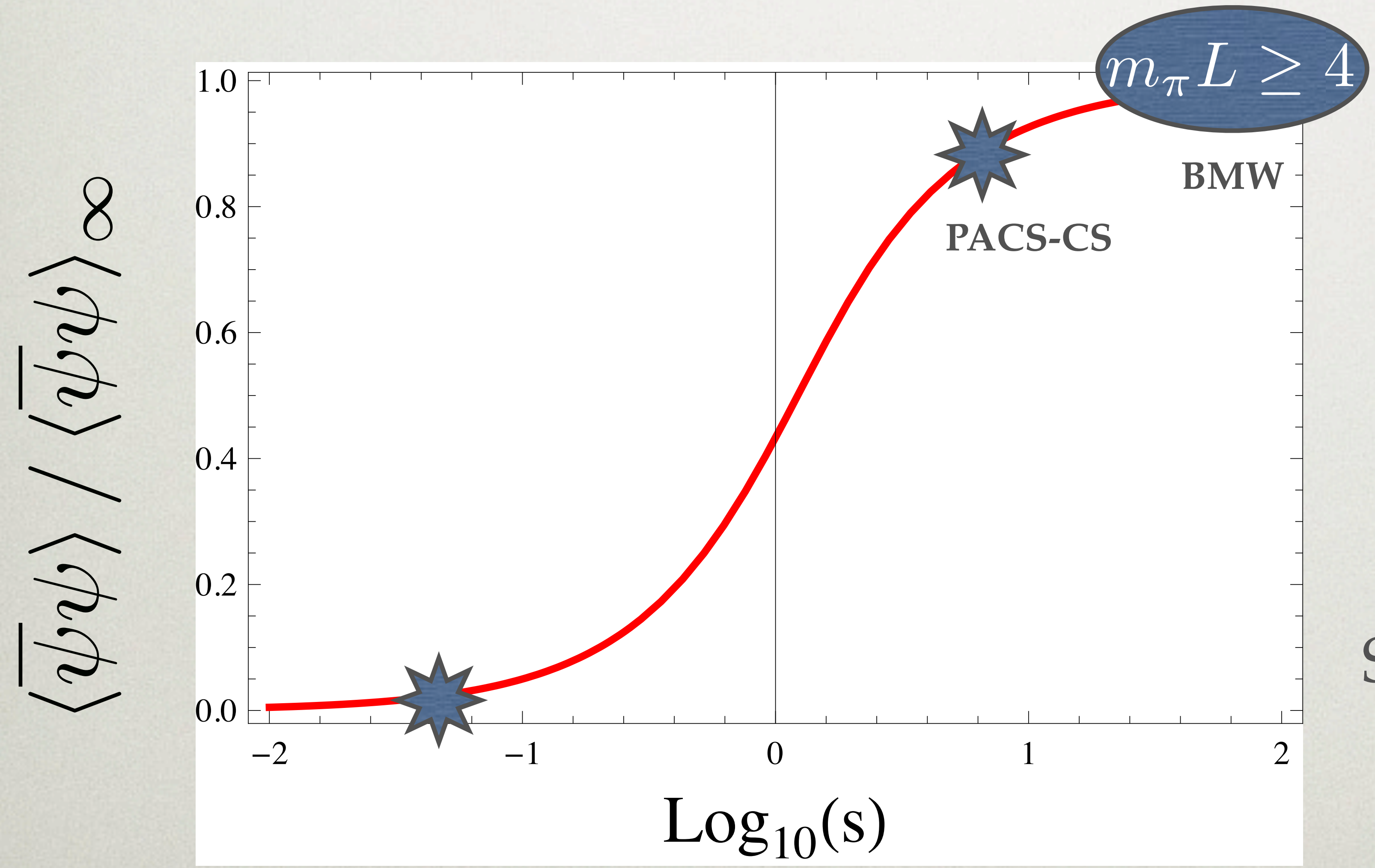

Scaling variable

$$s = \frac{1}{4} V f^2 m_\pi^2$$

Epsilon regime in principle allows clean access to low-energy constants ...

Zero Pion Winding

Finite Volume ChPT

Power counting: ***p-regime***

$$\frac{1}{L} \sim p \qquad m_\pi \sim p$$

Vertices $\partial_\mu \partial_\mu, \ \lambda m_q \sim p^2$

General Feynman diagram

(I internal, V vertices, L loops)

Loop factor $\int_{k_\mu} \rightarrow \frac{1}{L^4} \sum_{n_\mu} \sim p^4$

$$L = I - V + 1$$

$$\sim p^{4L-2I+2V} = p^{2L+2}$$

Propagator $\frac{1}{m_\pi^2} + \sum_{n_\mu \neq 0} \frac{1}{\left(\frac{2\pi n_\mu}{L}\right)^2 + m_\pi^2}$

All modes $\sim p^{-2}$

Mimics infinite volume power counting

$$(m_\pi L)^2 \gtrsim 1 \gg m_\pi^2 / \Lambda_\chi^2$$

Infinite volume limit $m_\pi L \rightarrow \infty$

Poisson Formula

Mode sums required in ***p-regime***. . .
Goal: trade in momentum sums for momentum integrals

$$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) = \int_{-\infty}^{\infty} dx\, e^{-ixk}\,\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{2\pi i x n/L}$$

Reminder: mode representation

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{2\pi i(x-y)n/L} \overset{\text{FV}}{=} \delta(x-y)$$

$$x, y \in (-L/2, L/2\,)$$

Poisson Formula

Mode sums required in ***p-regime***. . .
Goal: trade in momentum sums for momentum integrals

$$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) \quad = \quad \sum_{\nu=-\infty}^{\infty}\int_{\nu L-L/2}^{\nu L+L/2} dx\, e^{-ixk}\,\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{2\pi i x n/L}$$

Reminder: mode representation

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Reminder: mode representation

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Finite Volume Propagator

Poisson Formula $$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) = \sum_{\nu=-\infty}^{\infty} e^{i\nu Lk}$$

Finite volume propagator

$$D^{\mathrm{FV}}(x,0) = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{2\pi i n x/L}\, G(2\pi n/L\,)$$

$$G(k) = \frac{1}{k^2+m^2}$$

$$k = \frac{2\pi n}{L}$$

That's *it*.

Finite Volume Propagator

Poisson Formula $$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) = \sum_{\nu=-\infty}^{\infty} e^{i\nu L k}$$

Finite volume propagator

$$D^{\mathrm{FV}}(x,0) = \int_{-\infty}^{\infty} dk\, e^{ikx} G(k)\, \frac{1}{L}\sum_{n=-\infty}^{\infty}\delta(k-2\pi n/L\,)$$

Finite Volume Propagator

Poisson Formula $$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) \quad = \quad \sum_{\nu=-\infty}^{\infty} e^{i\nu L k}$$

Finite volume propagator

$$D^{\mathrm{FV}}(x,0) \quad = \quad \sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, e^{ik(x+\nu L)}G(k)$$

Finite Volume Propagator

Poisson Formula $$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) = \sum_{\nu=-\infty}^{\infty} e^{i\nu L k}$$

Finite volume propagator

$$D^{\mathrm{FV}}(x,0) = \sum_{\nu=-\infty}^{\infty} D^{\infty}(x+\nu L,0)$$

Finite Volume Propagator

Poisson Formula $$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) \;=\; \sum_{\nu=-\infty}^{\infty} e^{i\nu Lk}$$

Finite volume propagator

$$D^{\mathrm{FV}}(x,0) \;=\; \sum_{\nu=-\infty}^{\infty} D^{\infty}(x+\nu L,0)$$

Infinite volume limit: $\nu = 0$

Volume corrections: $\nu \neq 0$

Finite Volume Propagator

Poisson Formula $$\frac{1}{L}\sum_{n=-\infty}^{\infty}\delta\left(k-\frac{2\pi n}{L}\right) = \sum_{\nu=-\infty}^{\infty} e^{i\nu L k}$$

Finite volume propagator

$$D^{\mathrm{FV}}(x,0) = \sum_{\nu=-\infty}^{\infty} D^{\infty}(x+\nu L,0)$$

Infinite volume limit: $\nu = 0$

Volume corrections: $\nu \neq 0$

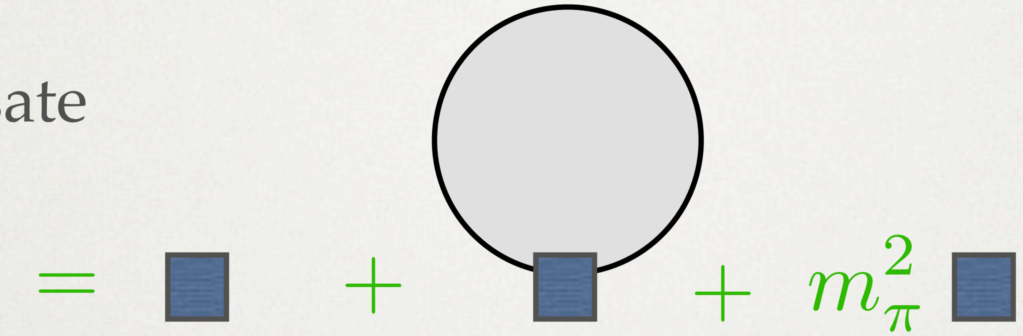
pion wraps around the world

Chiral Perturbation Theory in Finite Volume

Example: chiral condensate

One-loop contribution $\propto D^{\mathrm{FV}}(0,0)$

$$\langle\overline{\psi}\psi\rangle = -\frac{\partial \log Z_{\mathrm{eff}}}{\partial m_q}$$



Winding number expansion

$$D^{\mathrm{FV}}(0,0) = D^{\infty}(0,0) + d\left[D^{\infty}(L,0) + D^{\infty}(-L,0)\right] + \ldots$$

$\nu = 0$ $\qquad \nu = 1$ $\qquad \nu = -1$

Asymptotically large volume

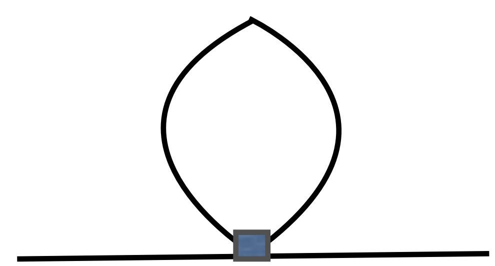
$$D^{\infty}(x,0) = \frac{m}{4\pi^2\sqrt{x^2}} K_1\left(m\sqrt{x^2}\right) = \frac{m^2}{2(2\pi m\sqrt{x^2})^{3/2}} e^{-m\sqrt{x^2}} + \ldots$$

Result with leading finite volume correction

$$\langle\overline{\psi}\psi\rangle = \langle\overline{\psi}\psi\rangle^{\infty}_{m_q=0}\left[1 + \frac{3\,m_\pi^2}{(4\pi f)^2}\left(\log\frac{\mu^2}{m_\pi^2} + 1 - 8\sqrt{2\pi}\frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}}\right) - \frac{m_\pi^2}{f^2}\mathbb{L}_4(\mu)\right]$$

Exercise:

In addressing finite volume corrections, one typically considers lattices with finite spatial volume and infinite temporal extent.



Why is this done? How are the above results modified? How does the pion mass scale with volume for asymptotically large volumes?

Lattice Applications of Chiral Perturbation Theory

- Study quark mass dependence of observables for chiral extrapolations (maybe even interpolations) of lattice QCD data.

- Use the lattice to expose the role chiral symmetry breaking plays in low-energy QCD, hopefully confirm predictions of ChPT.

- **Tailor ChPT to address sources of systematic error in lattice QCD computations of low-energy observables (in conjunction with above).**

Finite volume, partial quenching, discretization

Partial Quenching

QCD

$$\langle H' \,|\, \mathcal{O} \,|\, H \rangle = \int \mathcal{D}A_\mu \, \mathrm{Det}(\not{D} + m_q) \, e^{-S[A]} \frac{1}{\not{D} + m_q} \cdots \frac{1}{\not{D} + m_q}$$

Quenched QCD

$$\langle H' \,|\, \mathcal{O} \,|\, H \rangle = \int \mathcal{D}A_\mu \qquad 1 \qquad e^{-S[A]} \frac{1}{\not{D} + m_q} \cdots \frac{1}{\not{D} + m_q}$$

Partially Quenched QCD

$$\langle H' \,|\, \mathcal{O} \,|\, H \rangle = \int \mathcal{D}A_\mu \, \mathrm{Det}(\not{D} + m_{\mathrm{sea}}) \, e^{-S[A]} \frac{1}{\not{D} + m_{\mathrm{val}}} \cdots \frac{1}{\not{D} + m_{\mathrm{val}}}$$

Paradigm also useful for mixed actions, QCD + QQED, disconnected diagrams

$$\not{D} \to \not{D}_{\mathrm{sea}}$$

Quenching and Bosonic Quarks

$$\langle H' \,|\, \mathcal{O} \,|\, H \rangle = \int \mathcal{D}A_\mu \quad 1 \quad e^{-S[A]} \frac{1}{\not{D} + m_q} \cdots \frac{1}{\not{D} + m_q}$$

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} u \\ d \\ \tilde{u} \\ \tilde{d} \end{pmatrix}$$

$$\mathcal{L}_{\mathrm{QQCD}} = \overline{\psi}\,(\not{D} + m_q)\;\psi + \overline{\tilde{\psi}}\,(\not{D} + m_q)\,\tilde{\psi} = \overline{\Psi}\,(\not{D} + m_q)\,\Psi$$

graded vector

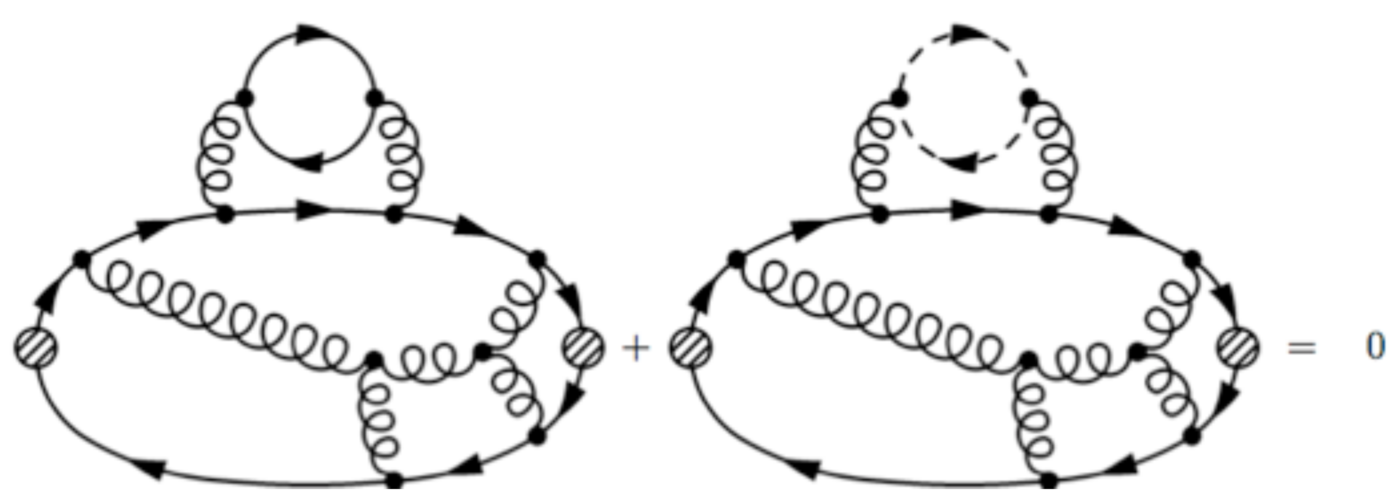
ψ fermionic quarks (Grassmann anticommuting)

$\tilde{\psi}$ bosonic quarks (commuting)

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S[\overline{\psi},\psi]} = \mathrm{Det}(\not{D} + m_q)$$

$$\int \mathcal{D}\overline{\tilde{\psi}}\mathcal{D}\tilde{\psi} e^{-S[\overline{\tilde{\psi}},\tilde{\psi}]} = \frac{1}{\mathrm{Det}(\not{D} + m_q)}$$

Fermionic quarks couple to external sources



Partial Quenching and Bosonic Quarks

$$\langle H' \,|\, \mathcal{O} \,|\, H \rangle = \int \mathcal{D}A_\mu \, \mathrm{Det}(\not{D} + m_{\mathrm{sea}})\, e^{-S[A]} \frac{1}{\not{D} + m_{\mathrm{val}}} \cdots \frac{1}{\not{D} + m_{\mathrm{val}}}$$

$$\Psi = \begin{pmatrix} \psi \\ \psi' \\ \tilde{\psi} \end{pmatrix}$$

graded vector

$$\mathcal{L}_{\mathrm{PQQCD}} = \overline{\psi}\,(\not{D} + m_{\mathrm{val}})\,\psi + \overline{\tilde{\psi}}\,(\not{D} + m_{\mathrm{val}})\,\tilde{\psi} + \overline{\psi}'\,(\not{D} + m_{\mathrm{sea}})\,\psi' = \overline{\Psi}\,(\not{D} + M)\,\Psi$$

ψ, ψ' fermionic quarks (Grassmann anticommuting)

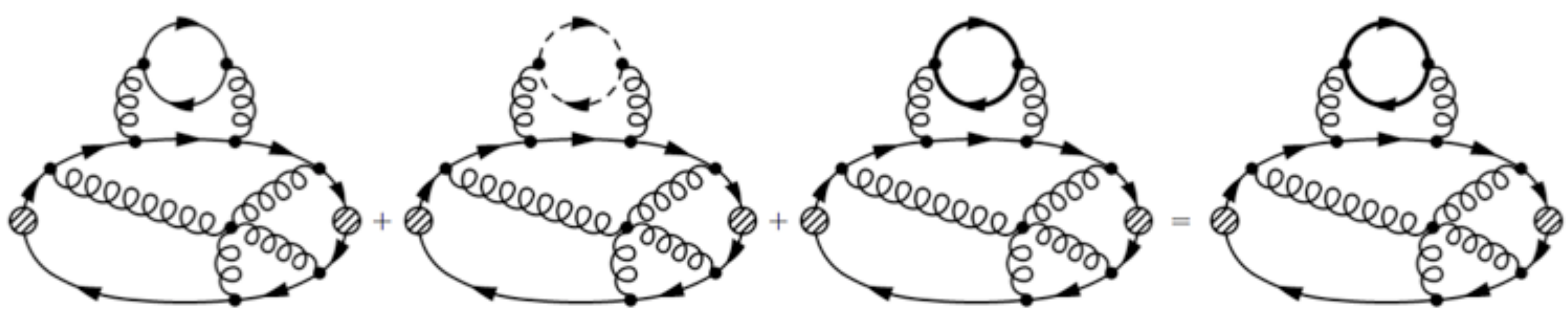
$\tilde{\psi}$ bosonic quarks (commuting)

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S[\overline{\psi},\psi]} = \mathrm{Det}(\not{D} + m_{\mathrm{val}})$$

$$\int \mathcal{D}\overline{\psi}'\mathcal{D}\psi' e^{-S[\overline{\psi}',\psi']} = \mathrm{Det}(\not{D} + m_{\mathrm{sea}})$$

$$\int \mathcal{D}\overline{\tilde{\psi}}\mathcal{D}\tilde{\psi} e^{-S[\overline{\tilde{\psi}},\tilde{\psi}]} = \frac{1}{\mathrm{Det}(\not{D} + m_{\mathrm{val}})}$$

Valence fermionic quarks couple to external sources



Graded Symmetries

$U(4|2)_V$

$$\mathcal{L}_{\mathrm{PQQCD}}(m_{\mathrm{val}} = m_{\mathrm{sea}} = 0) = \overline{\Psi} \not{D} \Psi$$

$$\Psi_A = \begin{pmatrix} \psi_a \\ \phi_\alpha \end{pmatrix}$$

$$\Psi_A \xrightarrow{U(4|2)_V} \mathcal{U}_{AB} \Psi_B$$

$$\mathcal{U}_{AB} = \begin{pmatrix} \mathcal{A}_{4\times 4} & \mathcal{B}_{4\times 2} \\ \mathcal{C}_{2\times 4} & \mathcal{D}_{2\times 2} \end{pmatrix}_{AB}$$

$\mathcal{A}, \mathcal{D}$ bosonic

$\mathcal{B}, \mathcal{C}$ fermionic

$$\mathcal{M}_{AB} \to [\mathcal{U}\mathcal{M}\mathcal{U}^\dagger]_{AB}$$

$$\mathrm{Str}\mathcal{M} = \sum_A (-)^{g(A)} \mathcal{M}_{AA} = \sum_a \mathcal{M}_{aa} - \sum_\alpha \mathcal{M}_{\alpha\alpha}$$

Grading $g(A) = \begin{cases} 1, & A = a \\ 0, & A = \alpha \end{cases}$

Exercise:

Show that the graded trace is invariant under graded unitary transformations.

Partially Quenched Chiral Perturbation Theory

- Many details omitted! $\mathcal{L}_{\rm PQQCD}(m_{\rm val} = m_{\rm sea} = 0) = \overline{\Psi}_L \not{D} \Psi_L + \overline{\Psi}_R \not{D} \Psi_R$

  Spontaneous chiral symmetry breaking $S(4|2)_L \otimes SU(4|2)_R \longrightarrow SU(4|2)_V$

$$\Sigma = e^{2i\Phi/f} \qquad \Phi = \left(\begin{array}{cc|c} \Phi_{\overline{\psi}\psi} & \Phi_{\overline{\psi}\psi'} & \Phi_{\overline{\psi}\tilde{\psi}} \\ \Phi_{\overline{\psi}'\psi} & \Phi_{\overline{\psi}'\psi'} & \Phi_{\overline{\psi}'\tilde{\psi}} \\ \hline \Phi_{\overline{\tilde{\psi}}\psi} & \Phi_{\overline{\tilde{\psi}}\psi'} & \Phi_{\overline{\tilde{\psi}}\tilde{\psi}} \end{array}\right)$$

Explicit symmetry breaking
$\overline{\Psi}_L M \Psi_R + \overline{\Psi}_R M \Psi_L$

- PQChPT $\mathcal{L}_2 = \frac{f^2}{8}\left[\mathrm{Str}\left(D_\mu \Sigma D_\mu \Sigma^\dagger\right) - \lambda\,\mathrm{Str}\left(M\Sigma + \Sigma^\dagger M\right)\right] + \frac{1}{2}\mu_0^2\left[\mathrm{Str}(\Phi)\right]^2$

Can show that f, λ are the same as in ChPT

Matching $m_{\rm sea} = m_{\rm val}$

$U(4|2)_L \otimes U(4|2)_R \longrightarrow U(1)_A \otimes U(1)_V \otimes SU(4|2)_V$

$\mu_0 \to \infty \qquad \mathrm{Str}\,\Phi = \eta'_{\rm val} + \eta'_{\rm sea} - \tilde{\eta}'$

Flavor *neutral* meson propagators have double poles

Exercise:

Find the tree-level masses of all *charged* mesons in PQChPT.

Lattice Applications of Chiral Perturbation Theory

- Study quark mass dependence of observables for chiral extrapolations (maybe even interpolations) of lattice QCD data.

- Use the lattice to expose the role chiral symmetry breaking plays in low-energy QCD, hopefully confirm predictions of ChPT.

- **Tailor ChPT to address sources of systematic error in lattice QCD computations of low-energy observables (in conjunction with above).**

Finite volume, partial quenching, discretization

Discretization Effects in Chiral Perturbation Theory

- Continuum limit of lattice data necessary to connect with reality
- ChPT is a low-energy effective theory. Why should short range physics show up?

Discretization Effects in Chiral Perturbation Theory

- Continuum limit of lattice data necessary to connect with reality

- ChPT is a low-energy effective theory. Why should short range physics show up?

- Most solutions of fermion doubling problem break chiral symmetry at zero quark mass

propagation of infrared modes
determined by nature of discretization

Symanzik's Effective Field Theory

$a \ll \Lambda_{\mathrm{QCD}}^{-1}$

- Near continuum, lattice QCD can be described by an effective theory built from continuum operators respecting the underlying lattice theory (different for each lattice action)

$$S_{\mathrm{ymanzik}} = S_0 + aS_1 + a^2 S_2 + \ldots$$

gauge invariance, C, P, T, hypercubic invariance, ...

$$S_i = \sum_j c_j^{(i)} \mathcal{O}_j^{(i)}$$

- Leading order is just QCD action (perhaps after fine tuning) $S_0 = \overline{\psi}\,(\not{D} + m_q)\,\psi$

Euclidean invariance recovered as an accidental symmetry

$$\mathcal{O} = \sum_\mu \overline{\psi} D_\mu D_\mu D_\mu \gamma_\mu \psi$$

*Logarithms buried in coefficients

$\log(1/a)$ $\qquad c_j^{(i)}(g^2)$

ChPT for the Wilson Action

- Wilson solves fermion doubling by breaking chiral symmetry...
  Accordingly not imposed on the Symanzik theory for Wilson action.

  (Quark mass renormalization $\frac{1}{a}\overline{\psi}\psi$ requires fine tuning to have light quarks)

- Chiral symmetry breaking operator

$$S_1 = c_{\rm sw}\overline{\psi}\sigma_{\mu\nu}G_{\mu\nu}\psi = c_{\rm sw}\left[\overline{\psi}_L\sigma_{\mu\nu}G_{\mu\nu}\psi_R + \overline{\psi}_R\sigma_{\mu\nu}G_{\mu\nu}\psi_L\right]$$

- Incorporate into chiral perturbation theory

$$\mathcal{L}_\chi = \frac{f^2}{8}{\rm Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - m_q\lambda\,{\rm Tr}\left(\Sigma+\Sigma^\dagger\right) - a\,c_{\rm sw}\lambda_a{\rm Tr}\left(\Sigma+\Sigma^\dagger\right)$$

$$m_\pi^2 = \frac{8}{f^2}\left(m_q\lambda + a\,c_{\rm sw}\lambda_a\right)$$

Chiral logarithms $\log[m_\pi^2(m_q,a)]$

Action improvement will diminish size of coefficient $c_{\rm sw}$

ChPT for Mixed Lattice Actions

- For computational economy, can use different fermion actions for valence/sea

  overlap valence / domain wall sea   domain wall valence / staggered sea

- Symanzik action is expressed as a partially quenched theory

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$$

$$\Psi_{\rm sea} = \psi'$$

Continuum limit symmetry $SU(4|2)_L \otimes SU(4|2)_R$

Finite lattice spacing $SU(2|2)_L \otimes SU(2|2)_R \otimes SU(2)_L \otimes SU(2)_R$

- Explicit breaking, e.g. $\mathcal{O}_{\rm mix} = \sum_{\mu} \left(\overline{\Psi}\gamma_\mu \Psi\right)\left(\overline{\Psi}_{\rm sea}\gamma_\mu \Psi_{\rm sea}\right)$

A consequence: mixed meson masses are not protected from additive renormalization at finite lattice spacing

$\Phi_{\overline{\psi}\psi'}, \Phi_{\overline{\psi}'\psi}$ $\Delta(m_\pi^2)_{\rm val,sea} = a^2 C_{\rm mix}$

Exercise:

Write down all dimension-6 four-quark operators in the Symanzik action for a general mixed-action theory. Classify the operators according to symmetry. Which ones are absent in a theory describing Wilson valence quarks and overlap sea quarks?