INT Summer School on Lattice QCD for Nuclear Physics

Lectures on Chiral Perturbation Theory

I. Foundations
II. Lattice Applications
III. Baryons
IV. Convergence

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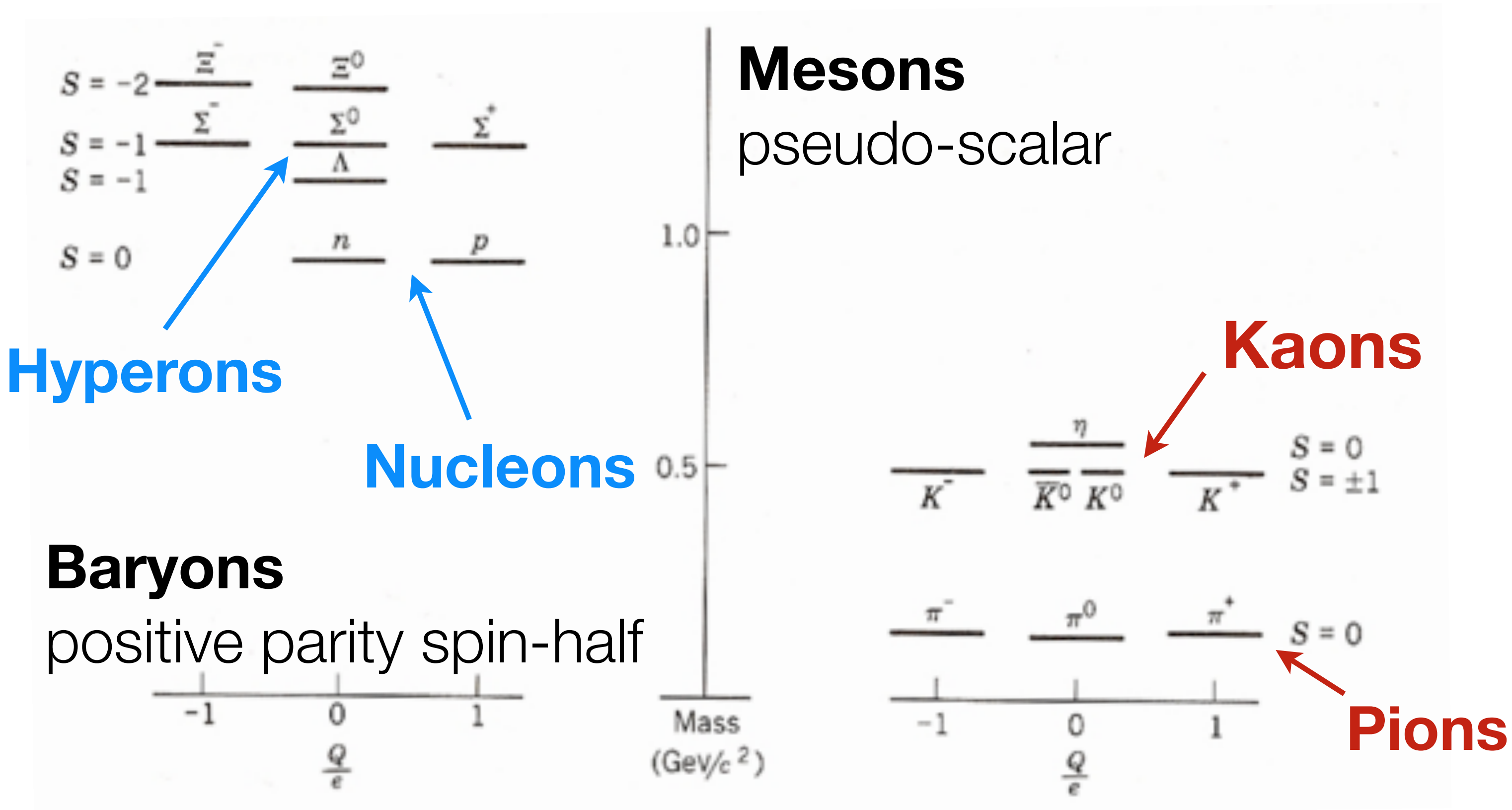
Chiral Perturbation Theory

I. Foundations

Low-energy QCD



Ground state: vacuum

Solutions of QCD (courtesy of nature)

- Spectrum (and properties) of low-lying hadrons indicative of symmetries
  ... *and symmetry breaking*

- ChPT is *the* tool to study such manifestations in low-energy QCD

Massless QCD

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_\psi + \mathcal{L}_{YM}$$

$$\mathcal{L}_\psi = \sum_{i=1}^{N_f} \overline{\psi}_i \not{D} \psi_i$$

- Take two flavors. These will correspond to up and down quarks.
- Massless QCD Lagrange density obviously has global U(2) flavor symmetry but...

Chiral symmetry

$$\mathcal{P}_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$$ projectors

$$\psi_{L,R} = \mathcal{P}_{L,R}\,\psi$$

$$\overline{\psi}\not{D}\psi = \overline{\psi}_L \not{D} \psi_L + \overline{\psi}_R \not{D} \psi_R$$

- Left- and right-handed fields do not mix: no chirality changing interaction

$$U(2)_L \otimes U(2)_R$$

$$L \qquad\qquad R$$

$$\psi_L \to L\psi_L$$
$$\psi_R \to R\psi_R$$

$$\psi \to (L\mathcal{P}_L + R\,\mathcal{P}_R)\,\psi$$

Massless QCD

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- Left- and right-handed fields do not mix: no chirality changing interaction

Vector subgroup

~~$U(2)_L \otimes U(2)_R$~~ $\quad U(2)_V$

$L = R = V$

$$\psi_L \to V\psi_L$$

$$\psi_R \to V\psi_R$$

$$\psi \to V(\mathcal{P}_L + \mathcal{P}_R)\psi = V\psi$$

Chiral Symmetry of Massless QCD

Action invariant under $U(2)_L \otimes U(2)_R = U(1)_L \otimes U(1)_R \otimes SU(2)_L \otimes SU(2)_R$

$$U(1)_L : \psi_L \to e^{i\theta_L}\psi_L$$
$$U(1)_R : \psi_R \to e^{i\theta_R}\psi_R$$

$$\psi \to \left[\frac{1}{2}(e^{i\theta_R} + e^{i\theta_L}) + \frac{1}{2}(e^{i\theta_R} - e^{i\theta_L})\gamma_5\right]\psi$$

Vector subgroup $\theta_L = \theta_R = \theta$ $\qquad \psi \to e^{i\theta}\psi \qquad U(1)_V$

Axial transformation $-\theta_L = \theta_R = \theta_5$ $\qquad \psi \to [\cos\theta_5 + i\gamma_5 \sin\theta_5]\psi = e^{i\theta_5\gamma_5}\psi$

$U(1)_A$

Exercise:

Consider a non-singlet axial transformation $\psi_i \to [\exp(i\vec{\phi}\cdot\vec{\tau}\gamma_5)]_{ij}\psi_j$

Is there a corresponding symmetry group of the massless QCD action?

Chiral Symmetry of Massless QCD

Action invariant under $U(2)_L \otimes U(2)_R = U(1)_A \otimes U(1)_V \otimes SU(2)_L \otimes SU(2)_R$

- Global symmetries lead to *classically* conserved currents

$$J_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5\psi \qquad J_\mu = \overline{\psi}\gamma_\mu\psi \qquad J^a_{L\mu} = \overline{\psi}_L\gamma_\mu\tau^a\psi_L \qquad J^a_{R\mu} = \overline{\psi}_R\gamma_\mu\tau^a\psi_R$$

(Regulated) Theory not invariant under flavor-singlet axial transformation

$$\partial_\mu J_{5\mu}(x) = \partial_\mu J_{R\mu}(x) - \partial_\mu J_{L\mu}(x) = \begin{cases} -\frac{e}{2\pi} N_f\, \epsilon_{\mu\nu} F_{\mu\nu}(x) & d=2 \\ -\frac{\alpha_s}{4\pi} N_f\, \epsilon_{\mu\nu\rho\sigma} G^A_{\mu\nu}(x) G^A_{\rho\sigma}(x) & d=4 \end{cases}$$

Chiral Anomaly

$U(1)_V \otimes SU(2)_L \otimes SU(2)_R$

The chiral anomaly obstructs chirally invariant lattice regularization of fermions (**see Kaplan's lectures**)

Fate of Symmetries in Low-Energy QCD

$$U(1)_V \otimes SU(2)_L \otimes SU(2)_R$$

- Chiral pairing preferred by vacuum (non-perturbative ground state)

Chiral condensate $\langle \overline{\psi}\psi \rangle = \langle \overline{\psi}_R \psi_L \rangle + \langle \overline{\psi}_L \psi_R \rangle \neq 0$

- Massless quarks *can* change their chirality by scattering off vacuum condensate

$$U(1)_V \otimes SU(2)_L \otimes SU(2)_R \longrightarrow U(1)_V \otimes SU(2)_V$$

- Spontaneously broken symmetries lead to massless bosonic excitations

Nambu-Goldstone Mechanism

Broken generators in coset $SU(2)_L \otimes SU(2)_R / SU(2)_V$

Number of massless particles?

Chiral Condensate

$$\langle \overline{\psi}_{iR} \psi_{jL} \rangle = -\lambda\, \delta_{ji}$$

- Choice for vacuum orientation $\lambda \in \mathbb{R}$ from Vafa-Witten (P)
- After a chiral transformation $\langle \overline{\psi}_{iR} \psi_{jL} \rangle \to L_{jj'} R^{\dagger}_{i'i} \langle \overline{\psi}_{i'R} \psi_{j'L} \rangle = -\lambda (L R^{\dagger})_{ji}$

$$SU(2)_L \otimes SU(2)_R \longrightarrow SU(2)_V$$

- Describe Goldstone fluctuations of vacuum state with fields

$$\delta_{ji} \to \Sigma_{ji}(x) = \delta_{ji} + \ldots \qquad \Sigma \in SU(2)_L \otimes SU(2)_R / SU(2)_V$$

$$[L(x) R^{\dagger}(x)]_{ji} = [e^{i\vec{\theta}_L(x)\cdot\vec{\tau}} e^{-i\vec{\theta}_R(x)\cdot\vec{\tau}}]_{ji} \qquad \vec{\theta}_L = -\vec{\theta}_R$$

$$\Sigma = e^{2i\phi/f} = 1 + \frac{2i\phi}{f} + \ldots$$

Transformation properties

$$\Sigma \to L \Sigma R^{\dagger} \qquad \Sigma \to V \Sigma V^{\dagger} \qquad \phi \to V \phi V^{\dagger}$$

Exercise:

Determine the discrete symmetry properties of the Goldstone modes from the coset's transformation.

Dynamics of Goldstone Bosons: Chiral Lagrangian

$$\Sigma = e^{2i\phi/f} = 1 + \frac{2i\phi}{f} + \ldots \qquad \begin{matrix} \mathrm{Tr}\,\phi = 0 \\ \phi^\dagger = \phi \end{matrix}$$

The Pions

$$\phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}$$

- Build chirally invariant theory of coset field

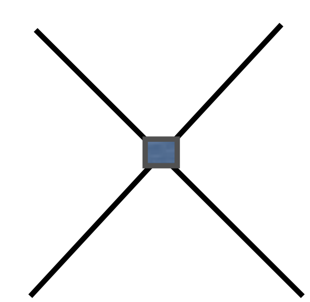
$$\begin{matrix} \Sigma \to L\Sigma R^\dagger \\ \Sigma^\dagger \to R\Sigma^\dagger L \end{matrix} \qquad \Sigma^\dagger\Sigma = 1 \qquad \Longrightarrow \qquad \mathcal{L} = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)$$

- Expand about v.e.v. to uncover Gaußian fluctuations

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,(\partial_\mu\phi\,\partial_\mu\phi) + \mathcal{O}(1/f^2) = \frac{1}{2}\partial_\mu\pi^0\partial_\mu\pi^0 + \partial_\mu\pi^-\partial_\mu\pi^+ + \mathcal{O}(1/f^2)$$

3 massless modes

- Non-linear theory: interactions between multiple pions at “higher orders”



Can treat systematically...

Including Quark Masses

- We began with massless QCD. Quarks have mass, Higgs makes two very light
- Chiral symmetry of action is only approximate: explicit symmetry breaking

$$\Delta\mathcal{L}_\psi = m_q \sum_i \overline{\psi}_i \psi_i = m_q \sum_i \left(\overline{\psi}_{iR}\psi_{iL} + \overline{\psi}_{iL}\psi_{iR}\right)$$

$SU(2)_L \otimes SU(2)_R \longrightarrow SU(2)_V$ $\qquad$ $m_q/\Lambda_{\rm QCD} \ll 1$

- Need to map $\Delta\mathcal{L}_\psi$ onto ChPT operators breaking symmetry in same way

$$\Delta\mathcal{L}_{\rm eff} = -m_q \lambda \, {\rm Tr}\left(\Sigma + \Sigma^\dagger\right)$$

Comments: not chirally invariant $\qquad$ $\Sigma \to L\Sigma R^\dagger$, $\Sigma^\dagger \to R\Sigma^\dagger L^\dagger$

new dimensionful parameter $\qquad$ λ

included only linear quark mass term $\qquad$ m_q^2

Perturbing about chiral limit

Chiral Lagrangian

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - m_q \lambda\, \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)$$

- Expand up to quadratic order $\mathcal{L}_\chi = -4 m_q \lambda + \frac{1}{2}\mathrm{Tr}\left(\partial_\mu \phi\, \partial_\mu \phi\right) + \frac{8 m_q \lambda}{f^2}\frac{1}{2}\mathrm{Tr}\left(\phi\phi\right)$

Pion mass $m_\pi^2 = 8 m_q \lambda / f^2$

- Vacuum energy must be due to chiral condensate (ingredient in our construction)

QCD degrees of freedom

$$Z_{\mathrm{QCD}}[m_q, \ldots] = \int \mathcal{D} \cdots e^{-\int_x (\cdots + m_q \overline{\psi}\psi)}$$

$$-\frac{\partial \log Z_{\mathrm{QCD}}}{\partial m_q} = \langle \overline{\psi}\psi \rangle$$

Low-energy degrees of freedom

Effective field theory

$$Z_{\chi\mathrm{PT}}[m_q, \ldots] = \int \mathcal{D}\Sigma\, e^{-\int_x \mathcal{L}_\chi(\Sigma; m_q)}$$

$$Z_{\mathrm{QCD}}[m_q, \ldots] \equiv Z_{\chi\mathrm{PT}}[m_q, \ldots]$$

Matching

$$\langle \overline{\psi}\psi \rangle = -\frac{\partial \log Z_{\chi\mathrm{PT}}}{\partial m_q} = -\lambda \langle \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right) \rangle = -2 N_f \lambda$$

From before:

$$\langle \overline{\psi}_{iR} \psi_{jL} \rangle = -\lambda\, \delta_{ji}$$

$$\lambda = \lambda$$

Chiral Lagrangian

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)$$

- Expand up to quadratic order $\mathcal{L}_\chi = -4m_q\lambda + \frac{1}{2}\mathrm{Tr}\left(\partial_\mu\phi\,\partial_\mu\phi\right) + \frac{8m_q\lambda}{f^2}\frac{1}{2}\mathrm{Tr}\left(\phi\phi\right)$

Pion mass $m_\pi^2 = 8m_q\lambda/f^2$ $\qquad f^2 m_\pi^2 = 2m_q|\langle\overline{\psi}\psi\rangle|$ (Gell-Mann Oakes Renner)

- Vacuum energy must be due to chiral condensate (ingredient in our construction)

QCD degrees of freedom

$$Z_{\rm QCD}[m_q,\ldots] = \int\mathcal{D}\cdots e^{-\int_x(\cdots+m_q\overline{\psi}\psi)}$$

$$-\frac{\partial\log Z_{\rm QCD}}{\partial m_q} = \langle\overline{\psi}\psi\rangle$$

Low-energy degrees of freedom

$$Z_{\chi\rm PT}[m_q,\ldots] = \int\mathcal{D}\Sigma\,e^{-\int_x\mathcal{L}_\chi(\Sigma;m_q)}$$

Effective field theory

$$Z_{\rm QCD}[m_q,\ldots] \equiv Z_{\chi\rm PT}[m_q,\ldots]$$

Matching

$$\langle\overline{\psi}\psi\rangle = -\frac{\partial\log Z_{\chi\rm PT}}{\partial m_q} = -\lambda\langle\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)\rangle = -2N_f\lambda$$

From before:

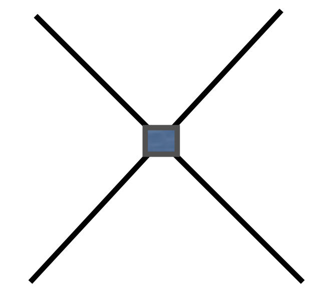
$$\langle\overline{\psi}_{iR}\psi_{jL}\rangle = -\lambda\,\delta_{ji}$$

$$\lambda = \lambda$$

ChPT

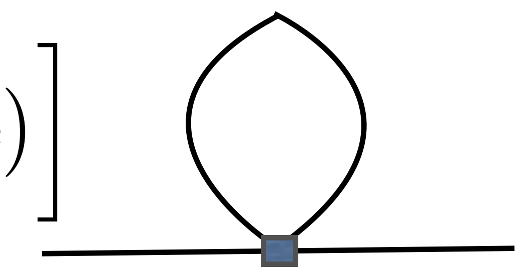
$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - m_q \lambda\, \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)$$

- Quadratic fluctuations are the approximate Goldstone bosons of SChSB
- Quartic terms describe interactions $\sim \frac{m_q \lambda}{f^4}\phi^4 \quad \sim \frac{1}{f^2}(\phi \partial_\mu \phi)^2$



- Higher-order interactions renormalize lower-order terms

$$\Delta m_\pi^2 \sim \frac{m_q \lambda}{f^4}\int_k \frac{1}{k^2 + m_\pi^2} \sim \frac{m_q \lambda}{f^2}\left[\Lambda^2 + \frac{m_q \lambda}{f^2}\left(\log \Lambda^2 + \text{finite}\right)\right]$$



power-law divergence

Absorb in renormalized mass,
or just use dimensional regularization

logarithmic divergence

Renormalization requires new
operator in chiral Lagrangian

- ChPT is non-renormalizable (needing infinite local terms to renormalize)

  1). low-energy theory, so who cares?

  2). must be able to order terms in terms of relevance “power counting”

Power Counting

$$\mathcal{L}_\chi = \frac{f^2}{8}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)$$

- Leading-order Lagrangian in expansion in derivatives and quark mass

  $\mathcal{O}(p^2)$ $\partial_\mu \sim p$ $m_q \sim p^2$ Low-energy dynamics of pions

  Propagator $\frac{1}{k^2+m_\pi^2} \sim p^{-2}$ Vertices $\partial_\mu\partial_\mu,\, m_q \sim p^2$ Loop integral $\int_k \sim p^4$

  General Feynman diagram: L loops, I internal lines, V vertices $\sim p^{4L-2I+2V}$

  Euler formula $L = I - V + 1$ $\sim p^{2L+2}$

- Loop expansion: one loop graphs require only $\mathcal{O}(p^4)$ counterterms

  Two loop graphs?

Exercise:
What happens to the power-counting argument in d = 2, 6 dimensions?
Do the results surprise you? Why didn’t I ask about d = 3, 5?

$\mathcal{O}(p^4)$ Chiral Lagrangian

- Construct chirally invariant terms out of coset $\Sigma \to L\Sigma R^\dagger$

$$\Sigma^\dagger \to R\Sigma^\dagger L^\dagger$$

E.g. $\mathcal{O}(p^2)$ Lagrangian $\partial_\mu\Sigma\,\partial_\mu\Sigma^\dagger$

- Construct terms that break chiral symmetry in the same way as mass term

Simplification: add external scalar field to QCD action $\Delta\mathcal{L} = \overline{\psi}_L s\,\psi_R + \overline{\psi}_R s^\dagger \psi_L$

Make the scalar transform to preserve chiral symmetry $s \to L\,s\,R^\dagger$

Giving the scalar a v.e.v. breaks chiral symmetry just as a mass $s = m_q + \cdots$

E.g. $\mathcal{O}(p^2)$ Lagrangian $\Sigma s^\dagger + s\Sigma^\dagger$

$\mathcal{O}(p^4)$ Chiral Lagrangian

Also impose Euclidean invariance, C, P, T

$$\Sigma \to L\Sigma R^\dagger \qquad s \to L\, s\, R^\dagger$$
$$\Sigma^\dagger \to R\Sigma^\dagger L^\dagger \qquad s^\dagger \to R s^\dagger L^\dagger$$

E.g. $[\mathrm{Tr}\left(\Sigma s^\dagger - s\Sigma^\dagger\right)]^2 \to m_q^2[\mathrm{Tr}\left(\Sigma - \Sigma^\dagger\right)]^2 = 0$

Easy to generate terms. Care needed to find *minimal* set.

$$\mathcal{L}_4 = \mathbb{L}_1[\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)]^2 + \mathbb{L}_2\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right)\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right)$$
$$+ \mathbb{L}_3\frac{m_q\lambda}{f^2}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right) + \mathbb{L}_4\frac{(m_q\lambda)^2}{f^4}[\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)]^2$$

$\{\mathbb{L}_j\}$ low-energy constants = Gasser-Leutwyler coefficients, dimensionless

N.B. these are not Gasser-Leutwyler**'s** coefficients

Complete set of counterterms needed to renormalize one-loop ChPT

Additional terms necessary when coupling external fields...

Exercise:

Determine the effects of strong isospin breaking $m_u \neq m_d$ on the chiral Lagrangian. At what order does the pion isospin multiplet split?

Simplest one-loop computation: Chiral Condensate

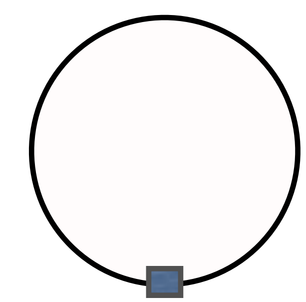
$$\langle\overline{\psi}\psi\rangle = -\frac{\partial \log Z_{\chi\mathrm{PT}}}{\partial m_q} = -\lambda\langle \mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)\rangle = -2N_f\lambda$$

"Tree-Level"

$$\Sigma + \Sigma^\dagger = 2 - \frac{4}{f^2}\phi^2 + \cdots$$

One Loop

$$\Delta\langle\overline{\psi}\psi\rangle = +\frac{4\lambda}{f^2} \times 3\, G_\pi(0) = \frac{12\lambda}{f^2}\int_k \frac{1}{k^2 + m_\pi^2}$$



Dimensionally regulated integral $-\frac{m_\pi^2}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \log 4\pi + \log\frac{\mu^2}{m_\pi^2} + 1\right)$

$$\mathbb{L}_3 \frac{\lambda}{f^2}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right) + 2\mathbb{L}_4\frac{m_q\lambda^2}{f^4}\left[\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)\right]^2$$

$\mathcal{O}(p^4)$

$$= 32\,\mathbb{L}_4\,\frac{m_q\lambda^2}{f^4} = 4\frac{m_\pi^2}{f^2}\,\mathbb{L}_4\,\lambda$$

"Tree-Level"

Final result:

Chiral Logarithm

$$\langle\overline{\psi}\psi\rangle = -4\lambda\left[1 + \frac{3\,m_\pi^2}{(4\pi f)^2}\left(\log\frac{\mu^2}{m_\pi^2} + 1\right) - \frac{m_\pi^2}{f^2}\mathbb{L}_4(\mu)\right]$$

$$\mu^2\frac{d}{d\mu^2}\mathbb{L}_4 = \frac{3}{16\pi^2}$$

Two-Flavor ChPT $\mathcal{L}_2 = \frac{f^2}{8}\left[\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - m_q\lambda\,\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)\right]$

- Leading and next-to-leading order Lagrangian in isospin limit $m_u = m_d$

$$\mathcal{L}_4 = \mathbb{L}_1[\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)]^2 + \mathbb{L}_2\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right)\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right) + \mathbb{L}_3\frac{m_q\lambda}{f^2}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right) + \mathbb{L}_4\frac{(m_q\lambda)^2}{f^4}[\mathrm{Tr}\left(\Sigma+\Sigma^\dagger\right)]^2$$

- Compute quark mass dependence of chiral condensate, pion mass, pion-pion scattering, ..., in terms of a few low-energy constants

$$\langle\overline{\psi}\psi\rangle = A_0\left[1 + B_0\,m_q\left(\log m_q + C_0\right)\right] \qquad \checkmark$$

$$m_\pi^2 = A_1\,m_q\left[1 + B_1\,m_q\left(\log m_q + C_1\right)\right]$$

$$a_{\pi\pi}^{I=2} = A_2\sqrt{m_q}\left[1 + B_2\,m_q\left(\log m_q + C_2\right)\right]$$

- Further applications: electroweak properties of pions require external fields

Incorporating External Fields in ChPT

- **Start by incorporating external gauge fields in QCD**

$$\mathcal{L}_\psi = \overline{\psi}_L \not{D}_L \psi_L + \overline{\psi}_R \not{D}_R \psi_R$$

$$(D_L)_\mu = \partial_\mu + ig\, G_\mu + iL_\mu$$

$$(D_R)_\mu = \partial_\mu + ig\, G_\mu + iR_\mu$$

E.g. external vector field $L_\mu = R_\mu = Qe\mathcal{A}_\mu$

Local invariance $[SU(2)_L] \otimes [SU(2)_R]$

$$\psi_L \longrightarrow L(x)\psi_L$$
$$\psi_R \longrightarrow R(x)\psi_R$$

and

$$L_\mu \longrightarrow L(x)L_\mu L^\dagger(x) + i[\partial_\mu L(x)]L^\dagger(x)$$
$$R_\mu \longrightarrow R(x)R_\mu R^\dagger(x) + i[\partial_\mu R(x)]R^\dagger(x)$$

- **Then incorporate external gauge fields in ChPT**

$$\Sigma \rightarrow L(x)\Sigma R^\dagger(x)$$

Need covariant derivative $D_\mu\Sigma \rightarrow L(x)[D_\mu\Sigma]R^\dagger(x)$

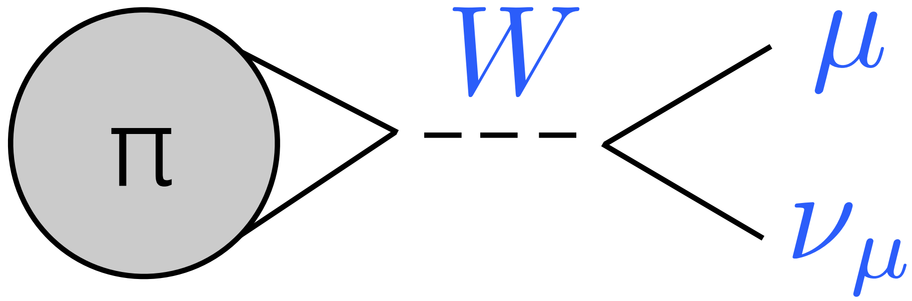
$$D_\mu\Sigma = \partial_\mu\Sigma + iL_\mu\Sigma - i\Sigma R_\mu^\dagger$$

Leading-order chiral Lagrangian [with external fields counted as $\mathcal{O}(p)$]

$$\mathcal{L}_2 = \frac{f^2}{8}\left[\mathrm{Tr}\left(D_\mu\Sigma D_\mu\Sigma^\dagger\right) - m_q\lambda\,\mathrm{Tr}\left(\Sigma + \Sigma^\dagger\right)\right]$$

*Additional operators at higher orders

What is f?



$$\pi \to \mu + \nu_\mu$$

Pion weak decay $\Delta\mathcal{L} = W_\mu^- J_{\mu L}^+$ $\qquad J_{\mu L}^+ = \overline{u}_L \gamma_\mu d_L$

Strong part factorizes into QCD matrix element
(the rest you learned how to compute in QFT)

$$\langle\, 0\,|J_{\mu L}^+|\,\pi(\vec{p})\,\rangle = i p_\mu\, f_\pi$$

$$\Gamma_{\pi\to\mu+\nu_\mu} = \frac{G_F^2}{8\pi} f_\pi^2 m_\mu^2 m_\pi |V_{ud}|^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

pion decay constant

$$f_\pi = 132\,\mathtt{MeV}$$

ChPT current matches the QCD current

$$\tau^+ = \frac{1}{2}(\tau^1 + i\tau^2)$$

$$J_{\mu L}^a = \left.\frac{\partial \mathcal{L}_\chi}{\partial L_\mu^a}\right|_{L_\mu=0} = \frac{f^2}{4}\mathrm{Tr}\left(i\tau^a \Sigma \partial_\mu \Sigma^\dagger\right) + \cdots = \frac{f}{2}\mathrm{Tr}\left(\tau^a \partial_\mu \phi\right) + \cdots$$

$$\langle\, 0\,|J_{\mu L}^+|\,\pi(\vec{p})\,\rangle = i p_\mu\,(f + \cdots)$$

$$f_\pi = f\,[1 + B m_q\,(\log m_q + C)]$$

Dimensionless power counting $\quad p^2/\Lambda_\chi^2$

$$\Lambda_\chi = 2\sqrt{2}\pi f \sim 1.2\,\mathtt{GeV} \qquad m_\pi^2/\Lambda_\chi^2$$

Exercise:

The masses of hadrons are modified by electromagnetism.

Construct all leading-order electromagnetic mass operators by promoting the electric charge matrix to fields transforming under the chiral group. (Notice that no photon fields will appear in the electromagnetic mass operators because there are no *external* photon lines.) Which pion masses are affected by the leading-order operators? Finally give an example of a next-to-leading order operator, or find them all.