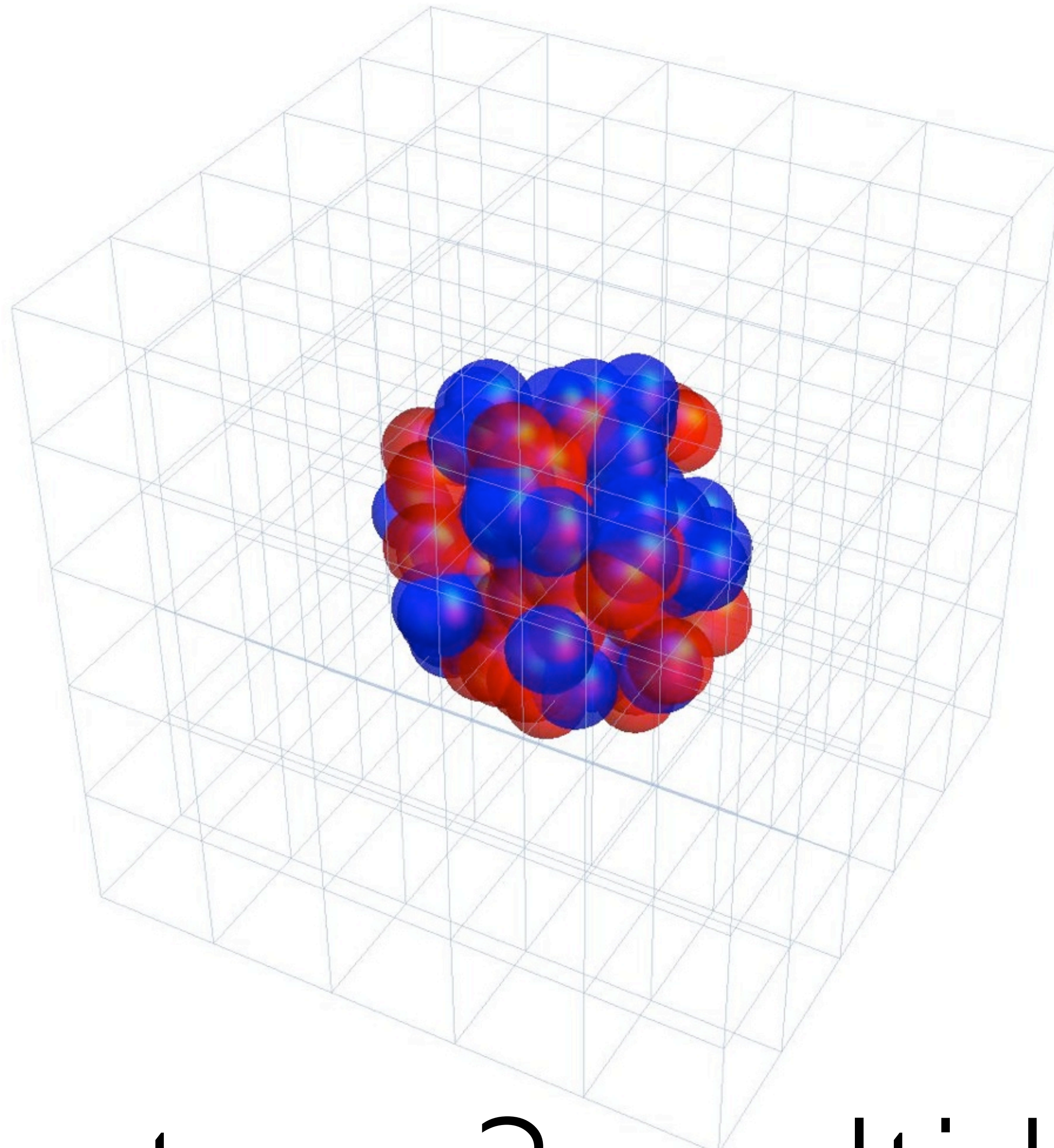

Lecture 3: multi-body

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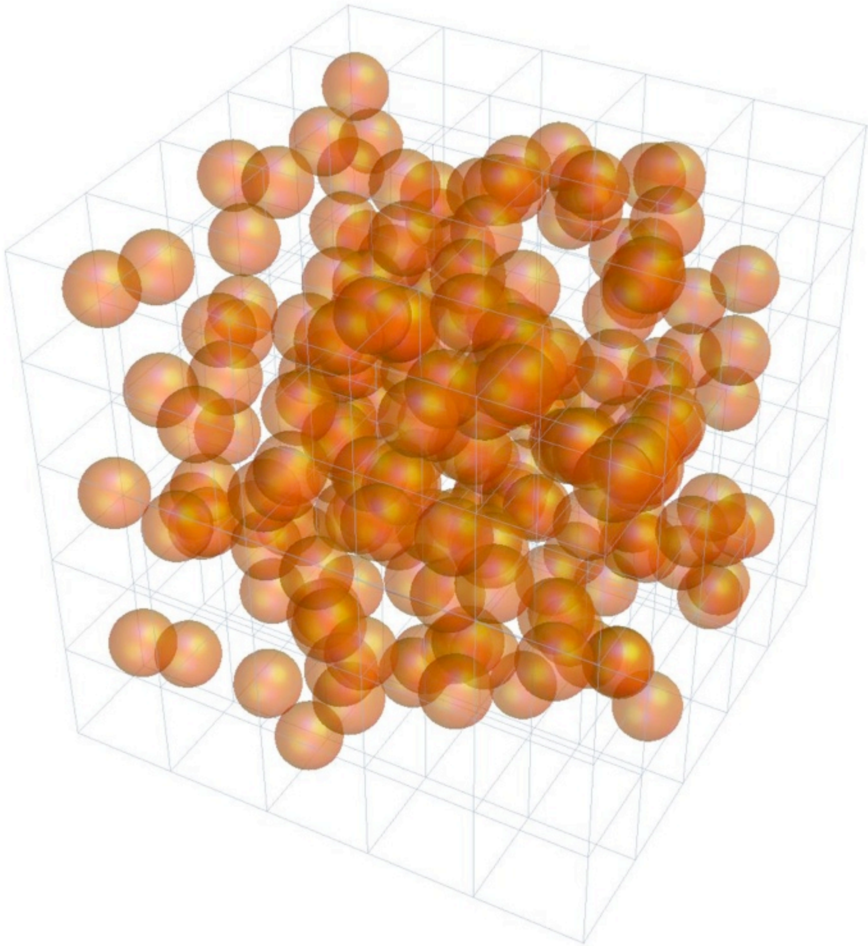
Lecture content

- Multi-hadron interactions: theoretical work
  - Many boson systems
  - Three baryons
  - More: baryons EFT in FV
- Contraction methods
  - Many pions
  - Many nucleons

Theory

Bosons in a box



- Long-standing problem: how do interacting particles confined in a box behave?
    - Uhlenbeck 30's, Bogoliubov 47, Huang&Yang 57
    - Tackled in terms of density expansion
    - For weakly interacting particles, an expansion in a/L might be useful

Bosons in a box

- Hamiltonian formulation = pionless EFT for pions with 2 & 3 body interactions

$$
\begin{aligned}
H \;=\; & \sum_{\mathbf{k}} h_{\mathbf{k}}^{\dagger}\, h_{\mathbf{k}} \left(\frac{|\mathbf{k}|^2}{2M} - \frac{|\mathbf{k}|^4}{8M^3} \right) \\
& + \frac{1}{(2!)^2} \sum_{\mathbf{Q},\mathbf{k},\mathbf{p}} h_{\frac{\mathbf{Q}}{2}+\mathbf{k}}^{\dagger} h_{\frac{\mathbf{Q}}{2}-\mathbf{k}}^{\dagger}\, h_{\frac{\mathbf{Q}}{2}+\mathbf{p}}\, h_{\frac{\mathbf{Q}}{2}-\mathbf{p}} \left(\frac{4\pi a}{M} + \frac{\pi a}{M}\left(a r - \frac{1}{2M^2} \right) \left(|\mathbf{k}|^2 + |\mathbf{p}|^2 \right) \right) \\
& + \frac{\eta_3(\mu)}{(3!)^2} \sum_{\mathbf{Q},\mathbf{k},\mathbf{p},\mathbf{r},\mathbf{s}} h_{\frac{\mathbf{Q}}{3}+\mathbf{k}}^{\dagger} h_{\frac{\mathbf{Q}}{3}+\mathbf{p}}^{\dagger}\, h_{\frac{\mathbf{Q}}{3}-\mathbf{k}-\mathbf{p}}^{\dagger}\, h_{\frac{\mathbf{Q}}{3}+\mathbf{r}} h_{\frac{\mathbf{Q}}{3}+\mathbf{s}}\, h_{\frac{\mathbf{Q}}{3}-\mathbf{r}-\mathbf{s}} \;,
\end{aligned}
$$

kinetic terms

two body interaction

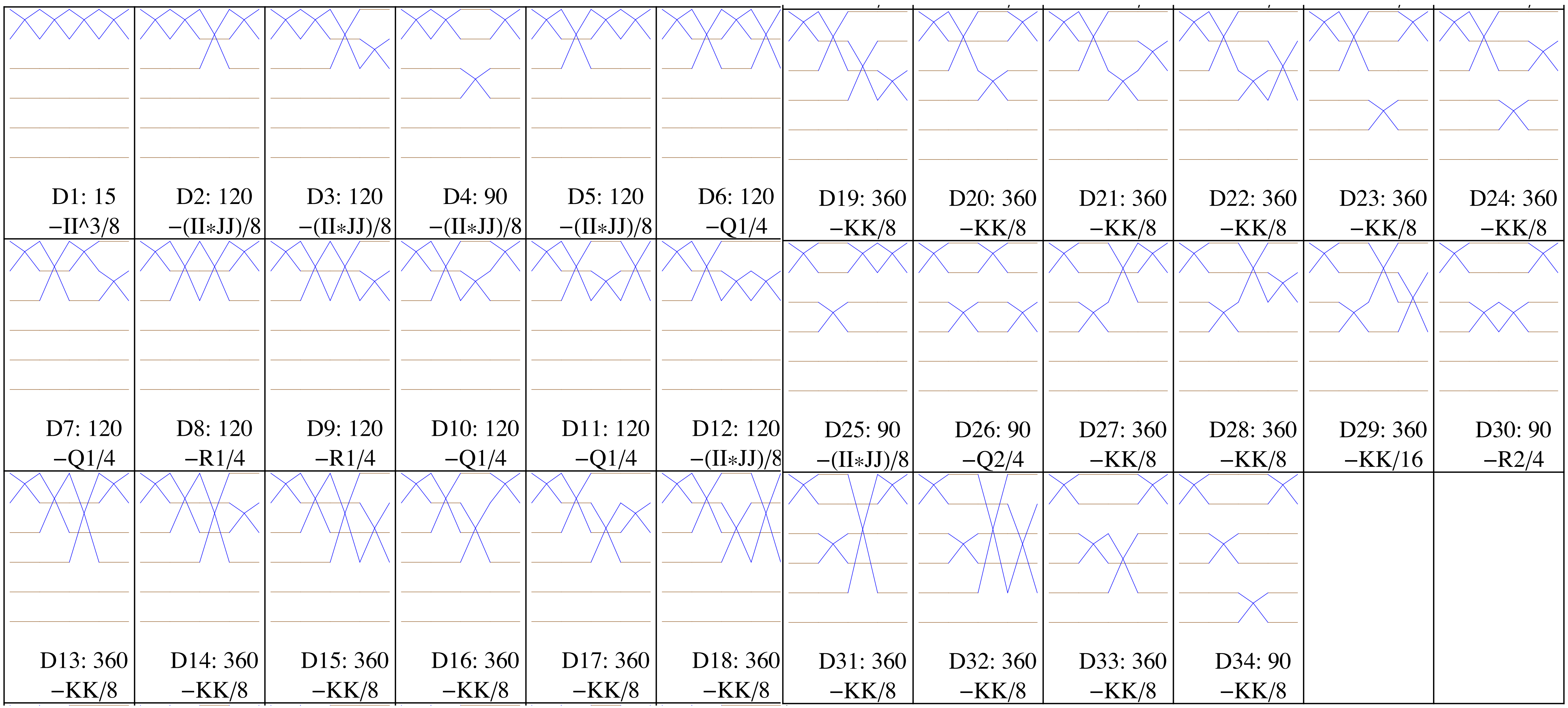
three body interaction

- Time-dependent perturbation theory calculation to determine large volume expansion of *n* particle ground state energies
  - 5th order PT gives energy shift of *n* meson system to $1/L^7$
  - 2 & 3 body interactions (*N* body: $L^{-3(N-1)}$)
  - Relativistic up to particle production threshold
  - Three loop diagrams: 9d integer sums

[Beane, WD & Savage; WD & Savage]

Lots of diagrams!

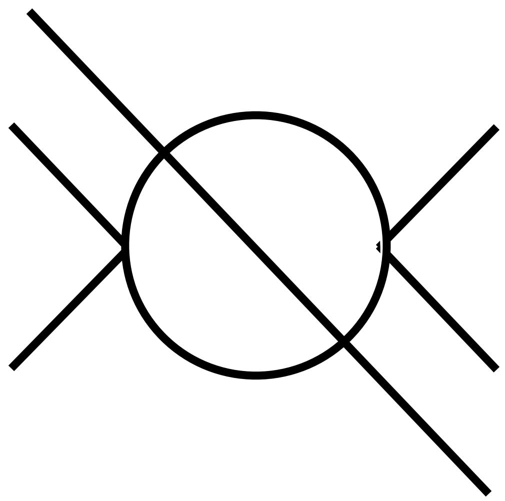
- Eg: non-vanishing $O(V^4)$ contributions for n=6 particles



[WD & Savage]

Three meson interactions

- At 1/L[6], point-like three-boson interaction must occur [Braaten, Nieto '95]
  - IR and UV divergent diagrams appear, needing renormalisation
  - RGI 3BI: $\overline{\eta}_3^{(L)}$ physically meaningful
  - Depends logarithmically on L
- Naive dimensional-analysis $m_\pi f_\pi^4 \overline{\eta}_3^{(L)} \sim 1$
- Combinations of energy shifts isolates the RGI interaction



Multi-boson energies

- Result for shift to $1/L^7$ is

$$\Delta E_0(n,L) \;=\; \frac{4\pi\, a}{M\, L^3}\binom{n}{2}\left\{1-\left(\frac{a}{\pi\, L}\right)\mathcal{I}+\left(\frac{a}{\pi\, L}\right)^2\left[\mathcal{I}^2+(2n-5)\mathcal{J}\right]\right.$$
$$-\left(\frac{a}{\pi\, L}\right)^3\left[\mathcal{I}^3+(2n-7)\mathcal{I}\mathcal{J}+\left(5n^2-41n+63\right)\mathcal{K}\right]$$
$$+\left(\frac{a}{\pi\, L}\right)^4\left[\mathcal{I}^4-6\mathcal{I}^2\mathcal{J}+(4+n-n^2)\mathcal{J}^2+4(27-15n+n^2)\mathcal{I}\,\mathcal{K}\right.$$
$$\left.\left.+(14n^3-227n^2+919n-1043)\mathcal{L}\;\right]\right\}$$
$$+\binom{n}{2}\frac{8\pi^2 a^3 r}{M\, L^6}\left[\,1\;+\;\left(\frac{a}{\pi\, L}\right)3(n-3)\mathcal{I}\;\right]$$
$$+\binom{n}{3}\frac{1}{L^6}\left[\;\eta_3(\mu)\;+\;\frac{64\pi a^4}{M}\left(3\sqrt{3}-4\pi\right)\;\log\left(\mu L\right)\;-\;\frac{96 a^4}{\pi^2 M}\;\mathcal{S}\;\right]\left[1\;-\;6\;\left(\frac{a}{\pi L}\right)\;\mathcal{I}\;\right]$$
$$+\binom{n}{3}\left[\;\frac{192\; a^5}{M\pi^3 L^7}\left(\mathcal{T}_0\;+\;\mathcal{T}_1\; n\right)\;+\;\frac{6\pi a^3}{M^3 L^7}\;(n+3)\;\mathcal{I}\;\right]\;+\;\mathcal{O}\left(L^{-8}\right)\qquad.$$

Geometric coefficients

$$\mathcal{I} = -8.9136329$$
$$\mathcal{J} = 16.532316$$
$$\mathcal{K} = 8.4019240$$
$$\mathcal{L} = 6.9458079$$
$$\mathcal{T}_0 = -4116.2338$$
$$\mathcal{T}_1 = 450.6392$$
$$\mathcal{S}_{\rm MS} = -185.12506$$

- *n*=2: reproduces expansion of Lüscher formula
- Can include higher partial waves, higher body
- Measurement of energies allows extraction of interaction parameters

[WD & M Savage, see also S Tan 07]

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& \qquad + (14n^3 - 227n^2 + 919n - 1043)\mathcal{L}\Big]\Bigg\} \\
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  eg: nucleon-nucleon!
  - Full Lüscher relation is valid, but small a/L, r/L expansions
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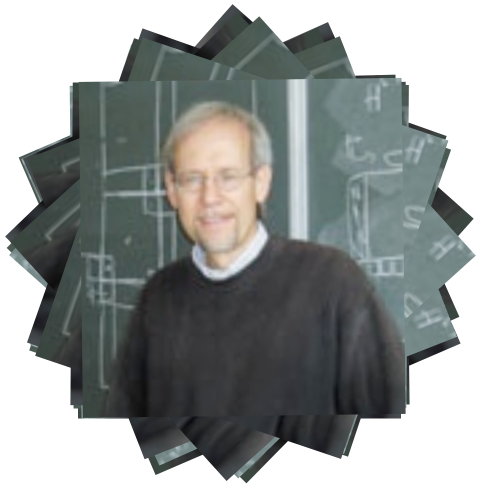

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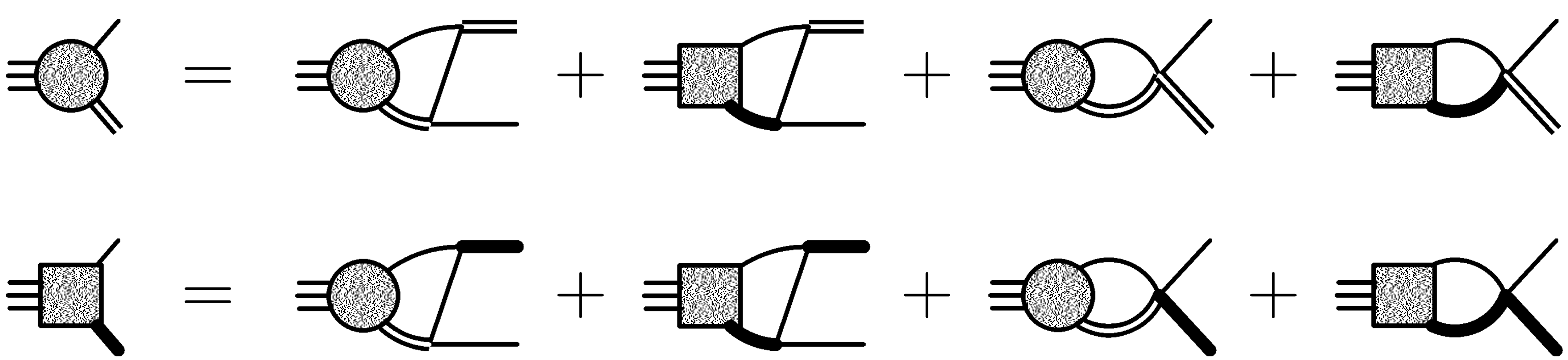

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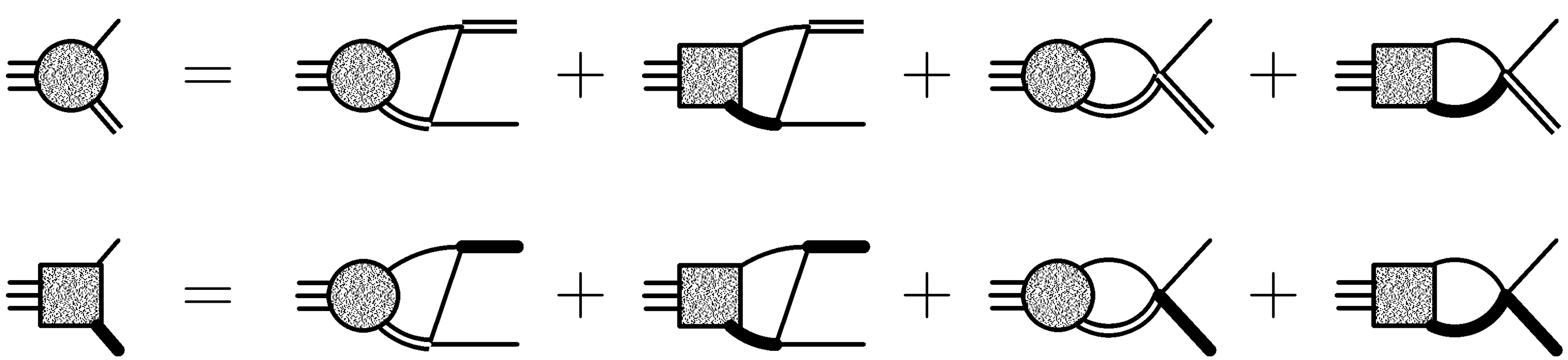


- Need finite volume generalisation of Faddeev

Three nucleon systems

- Kreuzer & Hammer [PLB 694 (2011) 424] studied three-nucleon systems (triton) in pionless EFT (valid at low energies, $p<m_\pi$)
- See also Luu Lattice2008, Polejaeva & Rusetsky [12], Kreuzer & Grießhammer [12]
- Lagrangian involves nucleons (*N*) and dibaryon fields (*s,t*)

J=1 dibaryon field

J=0 dibaryon field

$$\mathcal{L} = N^\dagger \left(i\partial_t + \frac{1}{2}\nabla^2 \right) N + \frac{g_t}{2}\, {t_j}^\dagger t_j + \frac{g_s}{2}\, {s_A}^\dagger s_A$$
$$- \frac{g_t}{2}\, \left[{t_j}^\dagger \left(N^T \tau_2 \sigma_j \sigma_2 N \right) + \text{h.c.} \right] - \frac{g_s}{2}\, \left[{s_A}^\dagger \left(N^T \sigma_2 \tau_A \tau_2 N \right) + \text{h.c.} \right] + \mathcal{L}_3$$

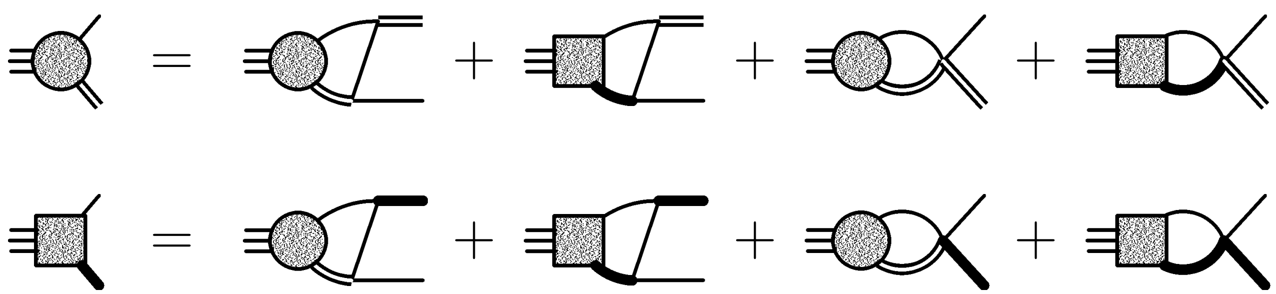
- Three body interaction

three body parameter

$$\mathcal{L}_3 = -\frac{2H(\Lambda)}{\Lambda^2} \Big(g_t^2 N^\dagger (t_j \sigma_j)^\dagger (t_i \sigma_i) N + \frac{g_t g_s}{3} \left[N^\dagger (t_j \sigma_j)^\dagger (s_A \tau_A) N + \text{h.c.} \right]$$
$$+ g_s^2 N^\dagger (s_A \tau_A)^\dagger (s_B \tau_B) N \Big)\,,$$

Three-nucleons in finite volume

- Infinite volume Faddeev eqns correspond to coupled integral equations



- Finite volume: replace loops by momentum sums and dibaryon propagator by periodic version making use of Poisson summation formula

$$\begin{pmatrix} \mathcal{F}_t(\vec{p}) \\ \mathcal{F}_s(\vec{p}) \end{pmatrix} = \frac{1}{\pi^2} \sum_{\vec{n}\in\mathbb{Z}^3} \int_0^\Lambda \mathrm{d}^3 y \, \mathrm{e}^{\imath L \vec{n}\cdot\vec{y}} \left[\mathbf{M}_2(\vec{y}) \, \mathcal{Z}(\vec{p},\vec{y}) + \mathbf{M}_3(\vec{y}) \, \frac{2H(\Lambda)}{\Lambda^2} \right] \begin{pmatrix} \mathcal{F}_t(\vec{y}) \\ \mathcal{F}_s(\vec{y}) \end{pmatrix}$$

$$\mathbf{M}_2(\vec{y}) = \begin{pmatrix} -d_t(\vec{y}) & 3d_s(\vec{y}) \\ 3d_t(\vec{y}) & -d_s(\vec{y}) \end{pmatrix}, \qquad \mathbf{M}_3(\vec{y}) = \begin{pmatrix} -d_t(\vec{y}) & d_s(\vec{y}) \\ d_t(\vec{y}) & -d_s(\vec{y}) \end{pmatrix}, \qquad \mathcal{Z}(\vec{p},\vec{y}) = \left[p^2 + \vec{p}\cdot\vec{y} + y^2 - E_3 \right]^{-1}$$

$$d_{s,t}(\vec{y}) \; = \; (g_{s,t}^2/8\pi) D_{s,t}(E_3 - \vec{y}^{\,2}/2, \vec{y})$$

$$D_{s,t}(p_0,\vec{p}) = \frac{8\pi}{g_{s,t}^2} \left[-\frac{1}{a_{s,t}} + \sqrt{-p_0 + \vec{p}^{\,2}/4 - i\epsilon} - \sum_{\substack{\vec{\jmath}\in\mathbb{Z}^3 \\ \vec{\jmath}\neq\vec{0}}} \frac{1}{|\vec{\jmath}|L} \mathrm{e}^{-|\vec{\jmath}|L\sqrt{-p_0+\vec{p}^{\,2}/4-i\epsilon}} \right]^{-1}$$

Three-nucleons in finite volume

- Boundary conditions impose cubic symmetry: irreps of SU(2) must be decomposed into irreps of double cover of octahedral group ^{2}O

$$\mathcal{F}(\vec{y}) = \sum_{j=\frac{1}{2},\frac{7}{2},\ldots}^{(G_1^+)} \sum_t F^{(j,t)}(y) \sum_{m_j} \tilde{C}_{jtm_j} |jm_j\rangle \qquad |jm_j\rangle = \sum_{m,s} C^{jm_j}_{\ell(j)m\frac{1}{2}s} |\ell(j)m\rangle \otimes |\tfrac{1}{2}s\rangle$$

subduction coefficients

- Project out partial waves

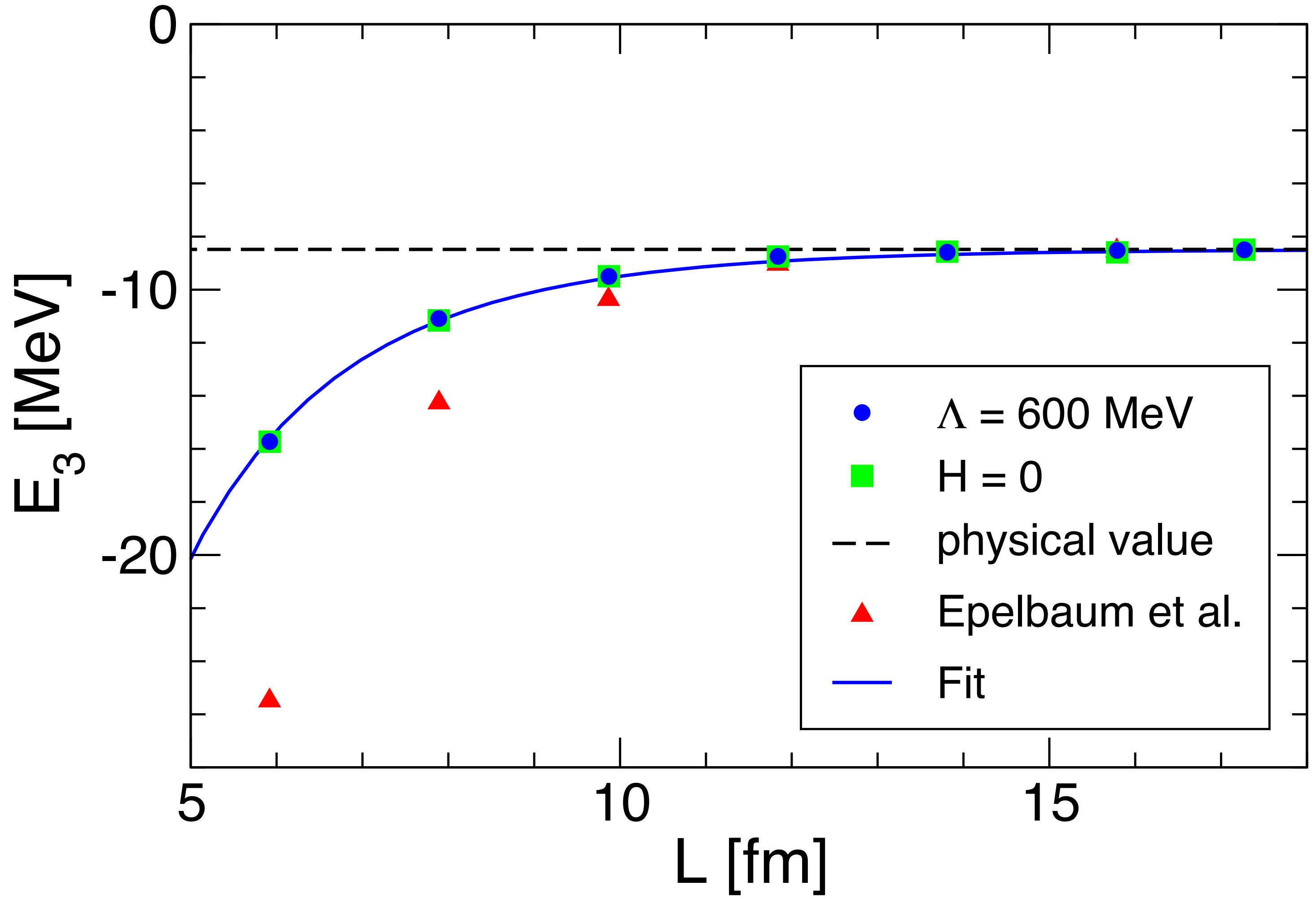
$$\begin{pmatrix} F_t^{(J)}(y) \\ F_s^{(J)}(y) \end{pmatrix} = \frac{4}{\pi} \int_0^\Lambda \frac{\mathrm{d}y\, y^2}{2\ell(J)+1} \sum_j^{(G_1^+)} \left[\mathbf{M}_2(y)\, Z^{(\ell(J))}(p,y) + \mathbf{M}_3(y) \frac{2H(\Lambda)}{\Lambda^2} \delta_{\ell(J),0} \right] \begin{pmatrix} F_t^{(j)}(y) \\ F_s^{(j)}(y) \end{pmatrix}$$
$$\times \left[\delta_{Jj} + \sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ \vec{n}\neq\vec{0}}} \sqrt{4\pi} \sum_{\ell'} i^{\ell'} j_{\ell'}(L|\vec{n}|y) \sqrt{\frac{(2\ell(j)+1)(2\ell'+1)}{2\ell(J)+1}} \right.$$
$$\left. \times \sum_{m(\ell(j)),s(\frac{1}{2})} \frac{\tilde{C}_{j,m+s}}{\tilde{C}_{JM}} Y^*_{\ell'(M-s-m)}(\hat{n}) C^{JM}_{\ell(J)(M-s)\frac{1}{2}s} C^{j,m+s}_{\ell(j)m\frac{1}{2}s} C^{\ell(J)0}_{\ell(j)0\ell'0} C^{\ell(J)(M-s)}_{\ell(j)m\ell'(M-s-m)} \right]$$

where

$$Z^{(\ell)}(p,y) = \frac{2\ell+1}{py} Q_\ell \left(\frac{p^2+y^2-E_3}{py} \right)$$

Triton at finite volume

- Binding energy of triton tuned to physical value at infinite volume

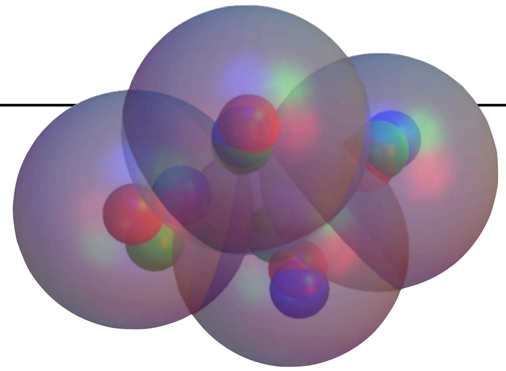


- Fitted by a simple exponential form

$$E_3(L) = E_3(L=\infty)\left[1 + \frac{c}{L}\,\mathrm{e}^{-L/L_0}\right]$$

- Lattice calculations at different volumes would constrain the LEC H(Λ)

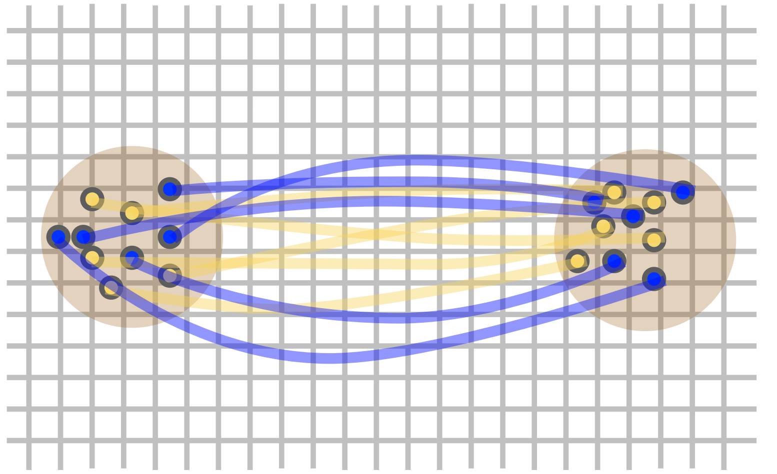
Beyond three baryons



- Philosophical approach is similar to three baryon case
  - Perform lattice calculations and extract eigen-energies of N baryon system
  - Perform EFT calculation in appropriate finite volumes for a range of values of LECs until results match onto lattice calculation
  - Using the determined LECs, perform infinite volume EFT calculation to extract infinite volume binding/scattering information
- Problems
  - Has not really been attempted
  - Four and higher body EFT calculations are computationally demanding
  - Convergence of EFT (for nucleons, pionless EFT is probably not enough) must be carefully investigated

Contractions

Multi-meson contractions



- An *n* meson correlation function is

$$C_n(t) \propto \langle \left(\sum_{\mathbf{x}} \pi^-(\mathbf{x},t) \right)^n \left(\pi^+(\mathbf{0},0) \right)^n \rangle$$

- For few meson systems can do Wick contractions by hand

- C_n can be written in terms of 12 cpt Grassman valued variables $\boldsymbol{\eta}_i$ and matrix $\boldsymbol{\Pi}$

$$\propto \langle \; (\; \overline{\eta} \Pi \eta \;)^n \; \rangle \qquad\qquad \Pi \; = \; \sum_{\mathbf{x}} \; S(\mathbf{x},t;0,0) \; S^\dagger(\mathbf{x},t;0,0)$$

- Using the 12 cpt Grassman identity

$$\langle \overline{\eta}^{\alpha_1} \overline{\eta}^{\alpha_2} ... \overline{\eta}^{\alpha_n} \eta_{\beta_1} \eta_{\beta_2} ... \eta_{\beta_n} \rangle \; \propto \; \varepsilon^{\alpha_1 \alpha_2 .. \alpha_n \xi_1 .. \xi_{12-n}} \; \varepsilon_{\beta_1 \beta_2 .. \beta_n \xi_1 .. \xi_{12-n}}$$

we can write

$$C_n(t) \; = \; \varepsilon^{\alpha_1 \alpha_2 .. \alpha_n \xi_1 .. \xi_{12-n}} \; \varepsilon_{\beta_1 \beta_2 .. \beta_n \xi_1 .. \xi_{12-n}} \; (\Pi)^{\beta_1}_{\alpha_1} \, (\Pi)^{\beta_2}_{\alpha_2} \, .. \, (\Pi)^{\beta_n}_{\alpha_n}$$

- Appears in the expansion

$$\begin{aligned}
\det(1+\lambda A) \; &= \; \frac{1}{12!} \, \varepsilon^{\alpha_1 \alpha_2 .. \alpha_{12}} \; \varepsilon_{\beta_1 \beta_2 .. \beta_{12}} \; (1+\lambda A)^{\beta_1}_{\alpha_1} \, (1+\lambda A)^{\beta_2}_{\alpha_2} \ldots (1+\lambda A)^{\beta_{12}}_{\alpha_{12}} \\
&= \; \frac{1}{12!} \Big[\, \varepsilon^{\alpha_1 \alpha_2 .. \alpha_{12}} \; \varepsilon_{\alpha_1 \alpha_2 .. \alpha_{12}} \; + \; \lambda \; {}^{12}C_1 \; \varepsilon^{\alpha_1 \alpha_2 .. \alpha_{12}} \; \varepsilon_{\beta_1 \alpha_2 .. \alpha_{12}} \; (\; A \;)^{\beta_1}_{\alpha_1} + \ldots \\
&\qquad + \; \lambda^n \; {}^{12}C_n \; \varepsilon^{\alpha_1 \alpha_2 .. \alpha_n \xi_1 .. \xi_{12-n}} \; \varepsilon_{\beta_1 \beta_2 .. \beta_n \xi_1 .. \xi_{12-n}} \; (\; A \;)^{\beta_1}_{\alpha_1} (\; A \;)^{\beta_2}_{\alpha_2} \ldots (\; A \;)^{\beta_n}_{\alpha_n} \\
&\qquad \ldots \; + \; \lambda^{12} \varepsilon^{\alpha_1 \alpha_2 .. \alpha_{12}} \; \varepsilon_{\beta_1 \beta_2 .. \beta_{12}} \; (\; A \;)^{\beta_1}_{\alpha_1} \ldots (\; A \;)^{\beta_{12}}_{\alpha_{12}} \, \Big]
\end{aligned}$$

Multi-meson contractions

- Can read off form of correlators from

$$\det(1+\lambda A) \;=\; \exp\left(\mathrm{Tr}\left[\log\left[\,1+\lambda A\right]\,\right]\,\right) \;=\; \exp\left(\mathrm{Tr}\left[\sum_{p=1}\frac{(-)^{p-1}}{p}\lambda^p A^p\right]\right)$$

$$= 1 \;+\; \lambda\ \mathrm{Tr}[\ A\ ] \;+\; \frac{\lambda^2}{2}\ \left(\ (\mathrm{Tr}[\ A\ ])^2 \;-\; \mathrm{Tr}\left[\ A^2\ \right]\right)$$

$$+\ \frac{\lambda^3}{6}\ \left(\ 2\mathrm{Tr}\left[\ A^3\ \right]\ -\ 3\mathrm{Tr}[\ A\ ]\,\mathrm{Tr}\left[\ A^2\ \right]\ +\ (\ \mathrm{Tr}[\ A\ ])^3\ \right)\ +\ \ldots$$

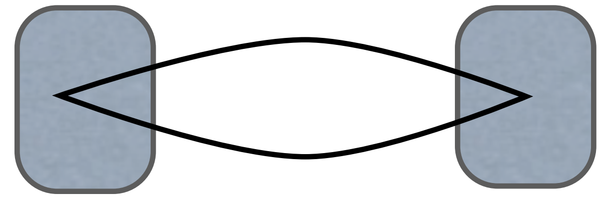
- Eg: $C_3(t) \;\propto\; \mathrm{tr}_{\mathrm{C,S}}\left[\Pi\right]^3 \;-\; 3\ \mathrm{tr}_{\mathrm{C,S}}\left[\Pi^2\right]\mathrm{tr}_{\mathrm{C,S}}\left[\Pi\right] \;+\; 2\ \mathrm{tr}_{\mathrm{C,S}}\left[\Pi^3\right]$
- How do we deal with complexity of contractions?
    - One species: $N_{\mathrm{terms}} \sim e^{\pi\sqrt{2n/3}}/\sqrt{n}$ [Ramanujan & Hardy], two-species is harder, more is not feasible
- How do we go beyond n=12?
    - Need multiple propagator sources but this leads to contraction complexity

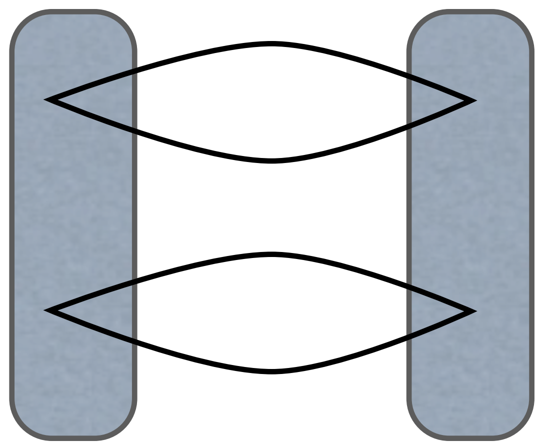
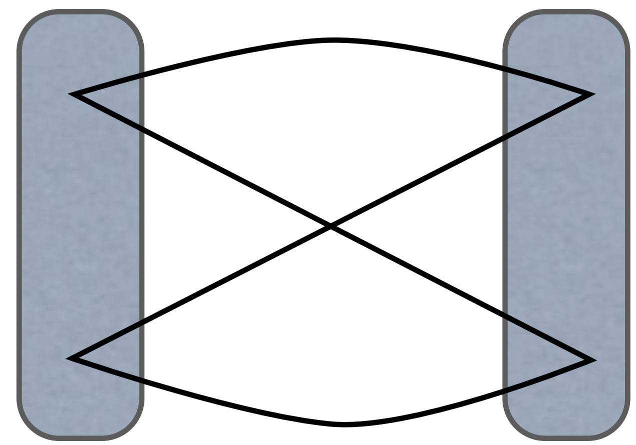
Multi-meson contractions

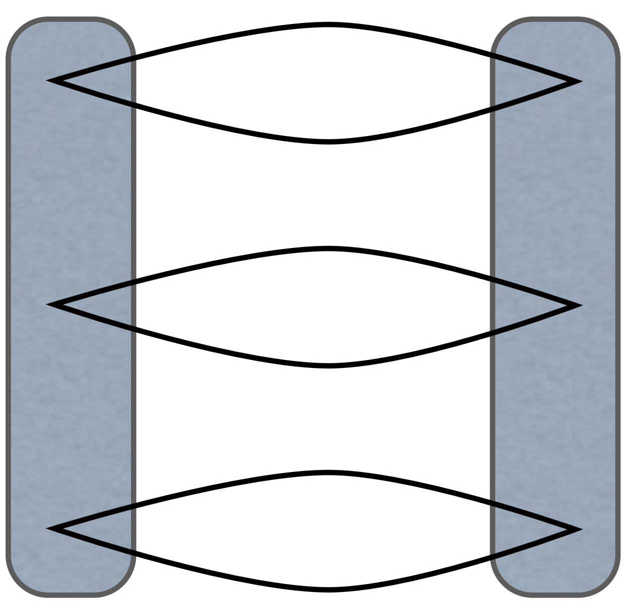
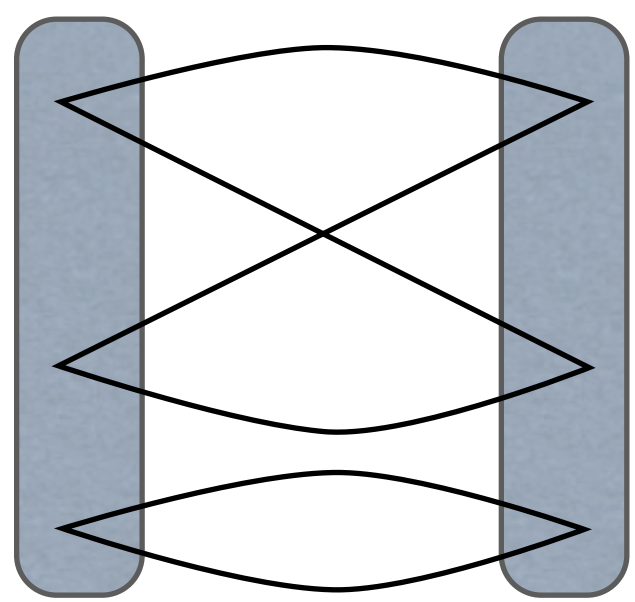
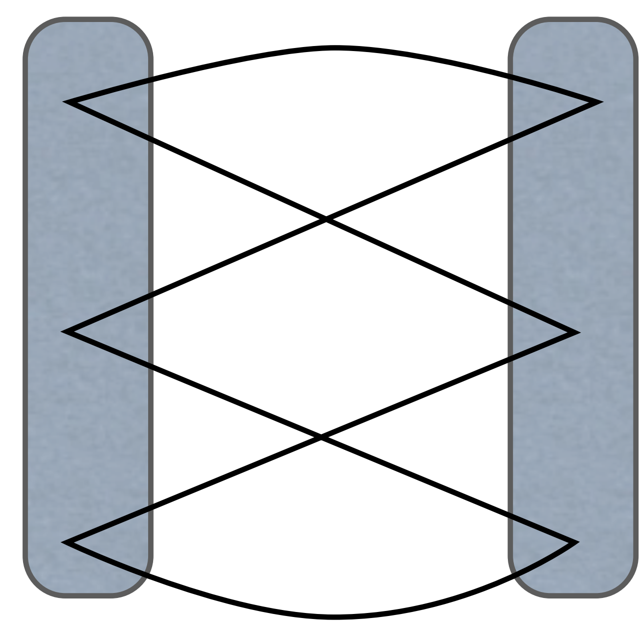
$$
\begin{aligned}
C_{13}(t) \;=\; & T_1^{13} - 78T_2T_1^{11} + 572T_3T_1^{10} + 2145T_2^2T_1^9 - 4290T_4T_1^9 - 25740T_2T_3T_1^8 + 30888T_5T_1^8 \\
& -25740T_2^3T_1^7 + 68640T_3^2T_1^7 + 154440T_2T_4T_1^7 - 205920T_6T_1^7 + 360360T_2^2T_3T_1^6 \\
& -720720T_3T_4T_1^6 - 864864T_2T_5T_1^6 + 1235520T_7T_1^6 + 135135T_2^4T_1^5 - 1441440T_2T_3^2T_1^5 \\
& +1621620T_4^2T_1^5 - 1621620T_2^2T_4T_1^5 + 3459456T_3T_5T_1^5 + 4324320T_2T_6T_1^5 - 6486480T_8T_1^5 \\
& +1601600T_3^3T_1^4 - 1801800T_2^3T_3T_1^4 + 10810800T_2T_3T_4T_1^4 + 6486480T_2^2T_5T_1^4 - 12972960T_4T_5T_1^4 \\
& -14414400T_3T_6T_1^4 - 18532800T_2T_7T_1^4 + 28828800T_9T_1^4 - 270270T_2^5T_1^3 + 7207200T_2^2T_3^2T_1^3 \\
& -16216200T_2T_4^2T_1^3 + 20756736T_5^2T_1^3 + 5405400T_2^3T_4T_1^3 - 14414400T_3^2T_4T_1^3 - 34594560T_2T_3T_5T_1^3 \\
& -21621600T_2^2T_6T_1^3 + 43243200T_4T_6T_1^3 + 49420800T_3T_7T_1^3 + 64864800T_2T_8T_1^3 - 103783680T_{10}T_1^3 \\
& -9609600T_2T_3^3T_1^2 + 32432400T_3T_4^2T_1^2 + 2702700T_2^4T_3T_1^2 - 32432400T_2^2T_3T_4T_1^2 \\
& -12972960T_2^3T_5T_1^2 + 34594560T_3^2T_5T_1^2 + 77837760T_2T_4T_5T_1^2 + 86486400T_2T_3T_6T_1^2 \\
& -103783680T_5T_6T_1^2 + 55598400T_2^2T_7T_1^2 - 111196800T_4T_7T_1^2 - 129729600T_3T_8T_1^2 \\
& -172972800T_2T_9T_1^2 + 283046400T_{11}T_1^2 + 135135T_2^6T_1 + 3203200T_3^4T_1 - 16216200T_4^3T_1 \\
& -7207200T_2^3T_3^2T_1 + 24324300T_2^2T_4^2T_1 - 62270208T_2T_5^2T_1 + 86486400T_6^2T_1 \\
& -4054050T_2^4T_4T_1 + 43243200T_2T_3^2T_4T_1 + 51891840T_2^2T_3T_5T_1 - 103783680T_3T_4T_5T_1 \\
& +21621600T_2^3T_6T_1 - 57657600T_3^2T_6T_1 - 129729600T_2T_4T_6T_1 - 148262400T_2T_3T_7T_1 \\
& +177914880T_5T_7T_1 - 97297200T_2^2T_8T_1 + 194594400T_4T_8T_1 + 230630400T_3T_9T_1 \\
& +311351040T_2T_{10}T_1 - 518918400T_{12}T_1 + 4804800T_2^2T_3^3 - 32432400T_2T_3T_4^2 \\
& +41513472T_3T_5^2 - 540540T_2^5T_3 - 9609600T_3^3T_4 + 10810800T_2^3T_3T_4 \\
& +3243240T_2^4T_5 - 34594560T_2T_3^2T_5 + 38918880T_4^2T_5 - 38918880T_2^2T_4T_5 \\
& -43243200T_2^2T_3T_6 + 86486400T_3T_4T_6 + 103783680T_2T_5T_6 - 18532800T_2^3T_7 \\
& +49420800T_3^2T_7 + 111196800T_2T_4T_7 - 148262400T_6T_7 + 129729600T_2T_3T_8 \\
& -155675520T_5T_8 + 86486400T_2^2T_9 - 172972800T_4T_9 - 207567360T_3T_{10} \\
& -283046400T_2T_{11} + 479001600T_{13}
\end{aligned}
$$

$$T_i^j = \mathrm{tr}\left[X^i\right]^j$$

Few pion contractions

$C_{1\pi}(t) =$ 

$C_{2\pi}(t) =$  $-$ 

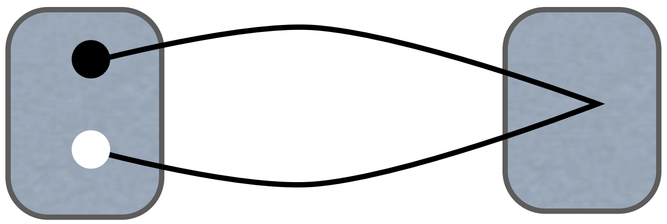
$C_{3\pi}(t) =$  -3  -2 

Meson blocks

- Define a partly contracted pion correlator

$$\Pi \equiv R_1 = \sum_{\mathbf{x}} S_u(\mathbf{x},t;x_0)\gamma_5 S_d(x_0;\mathbf{x},t)\gamma_5 = \sum_{\mathbf{x}} S_u(\mathbf{x},t;x_0) S_d^{\dagger}(\mathbf{x},t;x_0)$$

  - Time-dependent 12x12 matrix (spin-colour indices)

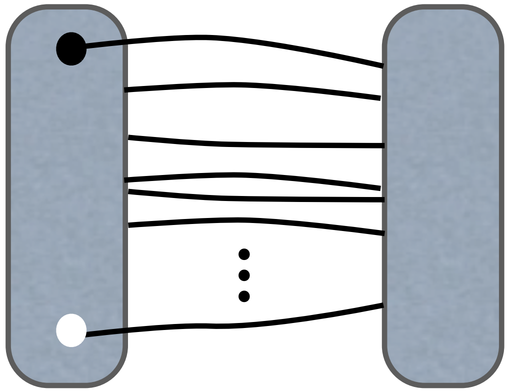


- Correlators (⟨...⟩ indicates color-spin trace)

$$C_1(t) = \langle \Pi \rangle, \quad C_2(t) = \langle \Pi \rangle^2 - \langle \Pi^2 \rangle, \ldots$$

- Functional definition

$$\Pi_{ij} = \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_1(t)$$



- Generalises to

$$(R_n)_{ij} \equiv \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_n(t)$$

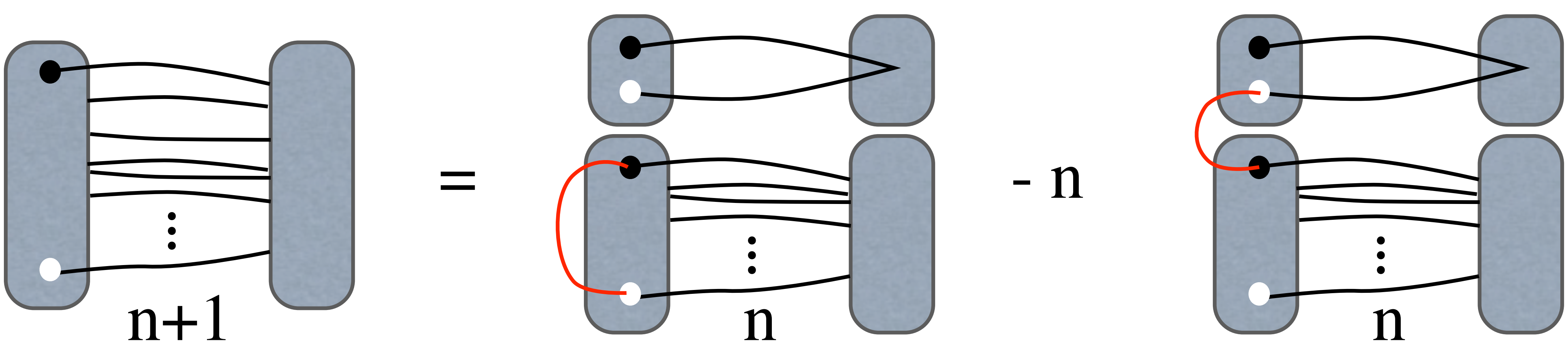
Recursion relation

[WD, M Savage, PRD 82 (2010) 014511]

- Contractions are not simply related
- Block objects *<u>are</u>* simply related
- Recursion relation

$$R_{n+1} = \langle R_n \rangle \; R_1 - n \; R_n \; R_1$$

- Initial condition is that $R_1 = \Pi, \qquad R_j = 0, \; \forall j < 1$
- Can also construct a descending recursion as we know that R_{13}=0

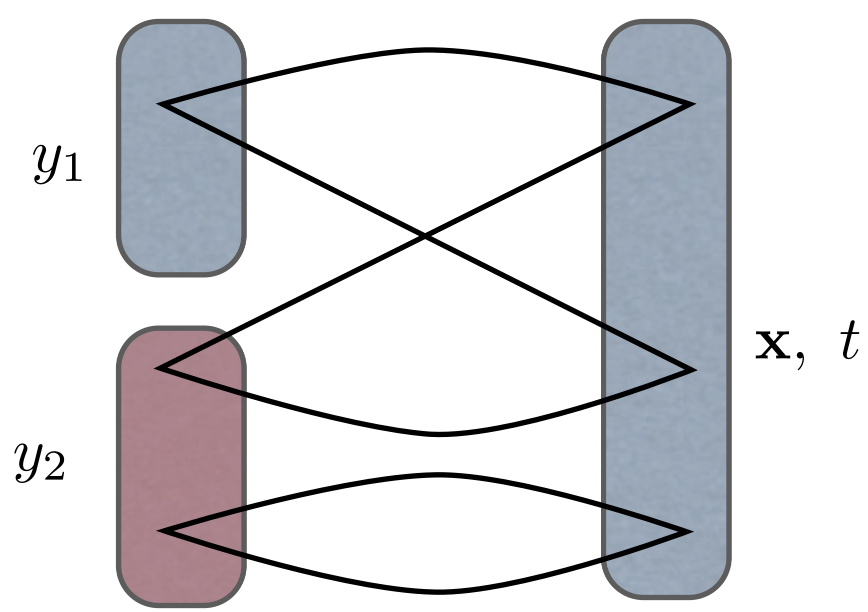


Multi-source systems

- To get beyond n=12, need to consider multi-source systems
- Consider two sources first

$$C_{(n_1\pi_1^+\ ,\ n_2\pi_2^+)}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x},t) \right)^{n_1+n_2} \left(\pi^-(\mathbf{y_1},0)\right)^{n_1} \left(\pi^-(\mathbf{y_2},0)\right)^{n_2} \right\rangle$$

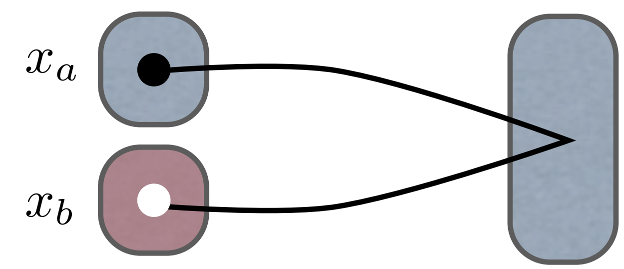
- $C_{(2,1)}(t)$ contains contractions like (sink position summed over timeslice, so no exclusion problem until n=12L[3])



Multi-source systems

- Multiple types of blocks needed

$$A_{ab} = \sum_{\mathbf{x}} S_u(\mathbf{x}, t; x_a) S_d^\dagger(\mathbf{x}, t; x_b)$$



- Two species case has a simple recursion relation:
  First define

$$P_1 = \left(\begin{array}{c|c} A_{11}(t) & A_{12}(t) \\ \hline 0 & 0 \end{array} \right) \quad , \quad P_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline A_{21}(t) & A_{22}(t) \end{array} \right)$$

Then $Q_{(n1,n2)}$ (generalisations of the R_n) satisfy

$$Q_{(n_1+1,n_2)} = \langle\, Q_{(n_1,n_2)} \,\rangle\, P_1 \; - \; (n_1+n_2)\, Q_{(n_1,n_2)} \; P_1$$
$$+ \langle\, Q_{(n_1+1,n_2-1)} \,\rangle\, P_2 \; - \; (n_1+n_2)\, Q_{(n_1+1,n_2-1)} \; P_2$$

Extensions

- Recursions also constructed for
  - *m*-source systems
  - *k*-species systems: π's, K's, D's, B's, ...
  - *m*-source, *k*-species systems

$$T_{\mathbf{n}+\mathbf{1}_{rs}} = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle\, T_{\mathbf{n}+\mathbf{1}_{rs}-\mathbf{1}_{ij}} \,\rangle\, P_{ij} \; - \; \overline{\mathcal{N}}\, T_{\mathbf{n}+\mathbf{1}_{rs}-\mathbf{1}_{ij}} \;\; P_{ij}$$

  where subscripts are matrices

- Implemented as matrix multiplications - computationally tractable
- Each iteration involves essentially two-body contractions
- Without tracking which source a given pion came from, cost is $\sim n^3$

Improved meson contraction methods

- Enlarge matrix $\mathbf{\Pi}$ to 12Nx12N using N source locations

$$A = P_1 + P_2 + \ldots + P_N = \left(\begin{array}{c|c|c|c} P_{1,1} & P_{1,2} & \ldots & P_{1,N} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline P_{k,1} & P_{k,2} & \ldots & P_{k,N} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline P_{N,1} & P_{N,2} & \ldots & P_{N,N} \end{array}\right) \quad \text{where} \quad P_{k,i}(t) = \sum_{\mathbf{x}} S(\mathbf{x}, t; \mathbf{y}_i, 0) S^\dagger(\mathbf{x}, t; \mathbf{y}_k, 0),$$

- New approaches based on determinantal nature: ~n^3 scaling

$$\det[1 + \lambda A] = 1 + \lambda C_{1\pi} + \lambda^2 C_{2\pi} + \ldots + \lambda^{12N} C_{12N\pi}$$

  - Vandermonde system

$$\begin{pmatrix} \frac{\det[1+\lambda_1 A]-1}{\lambda_1} \\ \frac{\det[1+\lambda_2 A]-1}{\lambda_2} \\ \vdots \\ \frac{\det[1+\lambda_{12N} A]-1}{\lambda_{12N}} \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{12N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{12N-1} \\ \vdots & & & & \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{12N-1} \end{pmatrix} \cdot \begin{pmatrix} C_{1\pi} \\ C_{2\pi} \\ \vdots \\ C_{12N\pi} \end{pmatrix}$$

  - Fourier analysis
  - Combination method

- Implement contractions in momentum space

[WD, K Orginos, Z Shi, 1205.4224]

Many baryon systems

- Many baryon correlator construction is messier
- Interpolating fields – minimal expression as weighted sums

color/spin/flavour/spatial indices

$$\bar{\mathcal{N}}^h = \sum_{k=1}^{N_w} \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

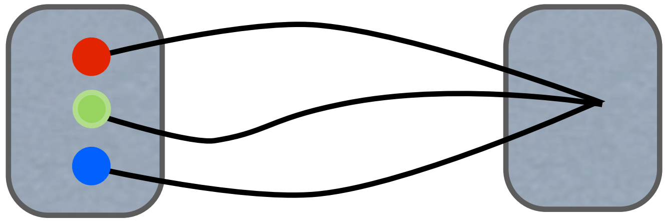
  - Generation of weights can be automated (symbolic code) for given quantum numbers
    - Specify final quantum numbers (spin, isospin, strangeness etc)
    - Build up from states of smaller quantum numbers just by using rules of eg angular momentum addition
  - Similar ideas by Doi and Endres [1205.0585]
  - Contraction just reads in weights and can be implemented independent of the particular process being considered

[WD, K Orginos, 1207.1452]

Many baryon systems

- Given a complex many baryon system to perform contractions for, always possible to group colour singlets at one end (sink)
- Contractions can be written in terms of baryon blocks (objects that are contracted at sink)
- A particular set of quantum numbers b for the block is select by a weighted sum of components of quark propagators

$$\mathcal{B}_b^{a_1,a_2,a_3}(\mathbf{p},t;x_0) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \sum_{k=1}^{N_{B(b)}} \tilde{w}_b^{(c_1,c_2,c_3),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,i_3}$$
$$\times S(c_{i_1},x;a_1,x_0)S(c_{i_2},x;a_2,x_0)S(c_{i_3},x;a_3,x_0)$$



- Can be generalised to multi-baryon blocks if desired although storage requirements rapidly increase

Many baryon systems

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h'^{(a'_1,a'_2\cdots a'_{n_q}),k'}\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

Many baryon systems

- Contractions

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N_w'}\sum_{k=1}^{N_w} \tilde{w}_h'^{(a_1',a_2'\cdots a_{n_q}'),k'}\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

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$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

$$= \; e^{-\mathcal{S}_{eff}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} S(a'_{j_1};a_{i_1})S(a'_{j_2};a_{i_2})\cdots S(a'_{j_{n_q}};a_{i_{n_q}})$$

Many baryon systems

- Contractions

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$
$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$
$$= \; e^{-\mathcal{S}_{eff}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$
$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} S(a'_{j_1};a_{i_1})S(a'_{j_2};a_{i_2})\cdots S(a'_{j_{n_q}};a_{i_{n_q}})$$

- Make a particular choice of correlation function (momentum projection at sink) and express in terms of blocks (quark-hadron level contraction)

Many baryon systems

- Contractions

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h'^{(a'_1,a'_2\cdots a'_{n_q}),k'}\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$
$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$
$$= e^{-\mathcal{S}_{eff}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h'^{(a'_1,a'_2\cdots a'_{n_q}),k'}\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$
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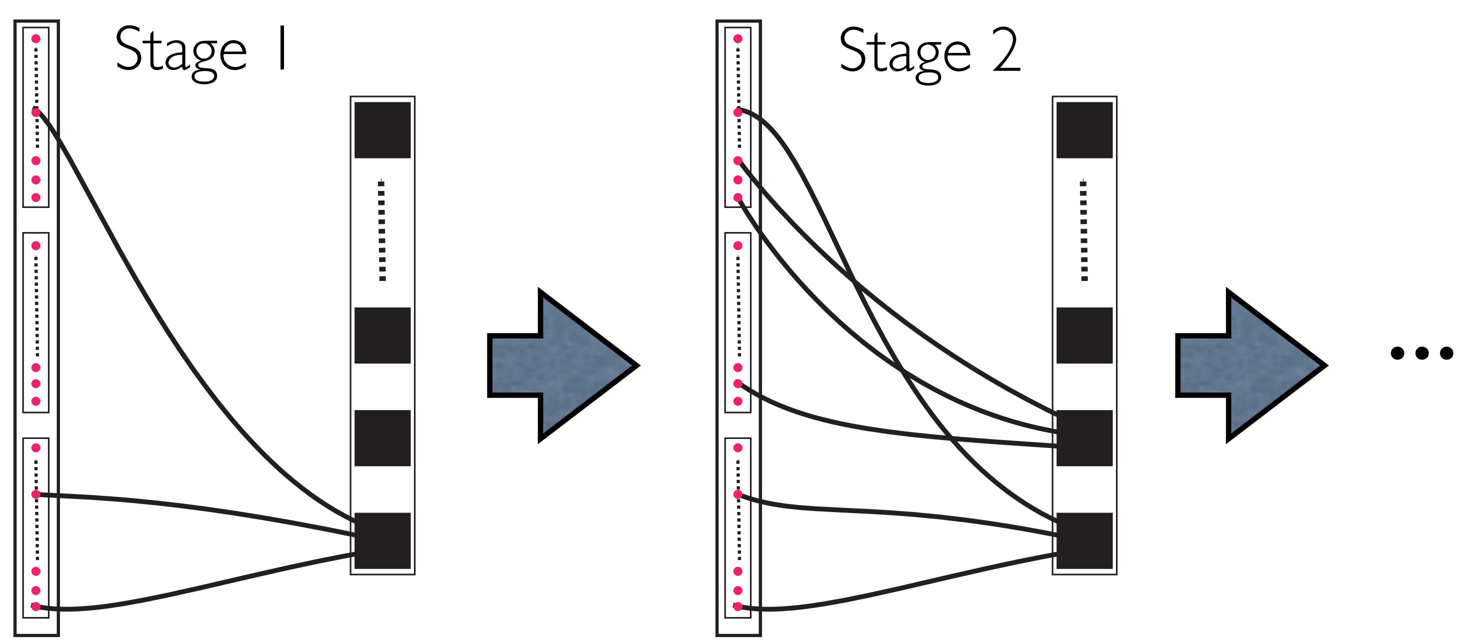

Many baryon systems

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \;\sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

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$$= \; e^{-\mathcal{S}_{eff}[U]} \;\sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} S(a'_{j_1};a_{i_1})S(a'_{j_2};a_{i_2})\cdots S(a'_{j_{n_q}};a_{i_{n_q}})$$

Many baryon systems

- Contractions

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \;\sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

$$= \; e^{-\mathcal{S}_{eff}[U]} \;\sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\; \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} S(a'_{j_1};a_{i_1})S(a'_{j_2};a_{i_2})\cdots S(a'_{j_{n_q}};a_{i_{n_q}})$$

Many baryon systems

- Contractions

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k}\times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1})\times\bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

$$= e^{-\mathcal{S}_{eff}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k}\times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} S(a'_{j_1};a_{i_1})S(a'_{j_2};a_{i_2})\cdots S(a'_{j_{n_q}};a_{i_{n_q}})$$

- Or write as determinant (quark-quark level contraction)

$$\langle\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\mathcal{U}\; e^{-\mathcal{S}_{eff}} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k}\times\det G(\mathbf{a}';\mathbf{a})$$

where

$$G(\mathbf{a}';\mathbf{a})_{j,i} = \begin{cases} S(a'_j;a_i) & a'_j\in\mathbf{a}' \text{ and } a_i\in\mathbf{a} \\ \delta_{a'_j,a_i} & \text{otherwise}\end{cases}$$

Many baryon systems

- Contractions

$$\left[\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\right]_U = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S_{QCD}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k}\times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} q(a'_{j_{n_q}})\cdots q(a'_{j_2})q(a'_{j_1}) \times \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

$$= e^{-\mathcal{S}_{eff}[U]} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k}\times$$

$$\sum_{\mathbf{j}}\sum_{\mathbf{i}} \epsilon^{j_1,j_2,\cdots,j_{n_q}}\epsilon^{i_1,i_2,\cdots,i_{n_q}} S(a'_{j_1};a_{i_1})S(a'_{j_2};a_{i_2})\cdots S(a'_{j_{n_q}};a_{i_{n_q}})$$

- Or write as determinant (quark-quark level contraction)

$$\langle\mathcal{N}_1^h(t)\bar{\mathcal{N}}_2^h(0)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\mathcal{U}\; e^{-\mathcal{S}_{eff}} \sum_{k'=1}^{N'_w}\sum_{k=1}^{N_w} \tilde{w}_h^{\prime(a'_1,a'_2\cdots a'_{n_q}),k'}\;\tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k}\times \det G(\mathbf{a}';\mathbf{a})$$

where

$$G(\mathbf{a}';\mathbf{a})_{j,i} = \begin{cases} S(a'_j;a_i) & a'_j\in\mathbf{a}' \text{ and } a_i\in\mathbf{a} \\ \delta_{a'_j,a_i} & \text{otherwise} \end{cases}$$

- Determinant can be evaluated in polynomial number of operations