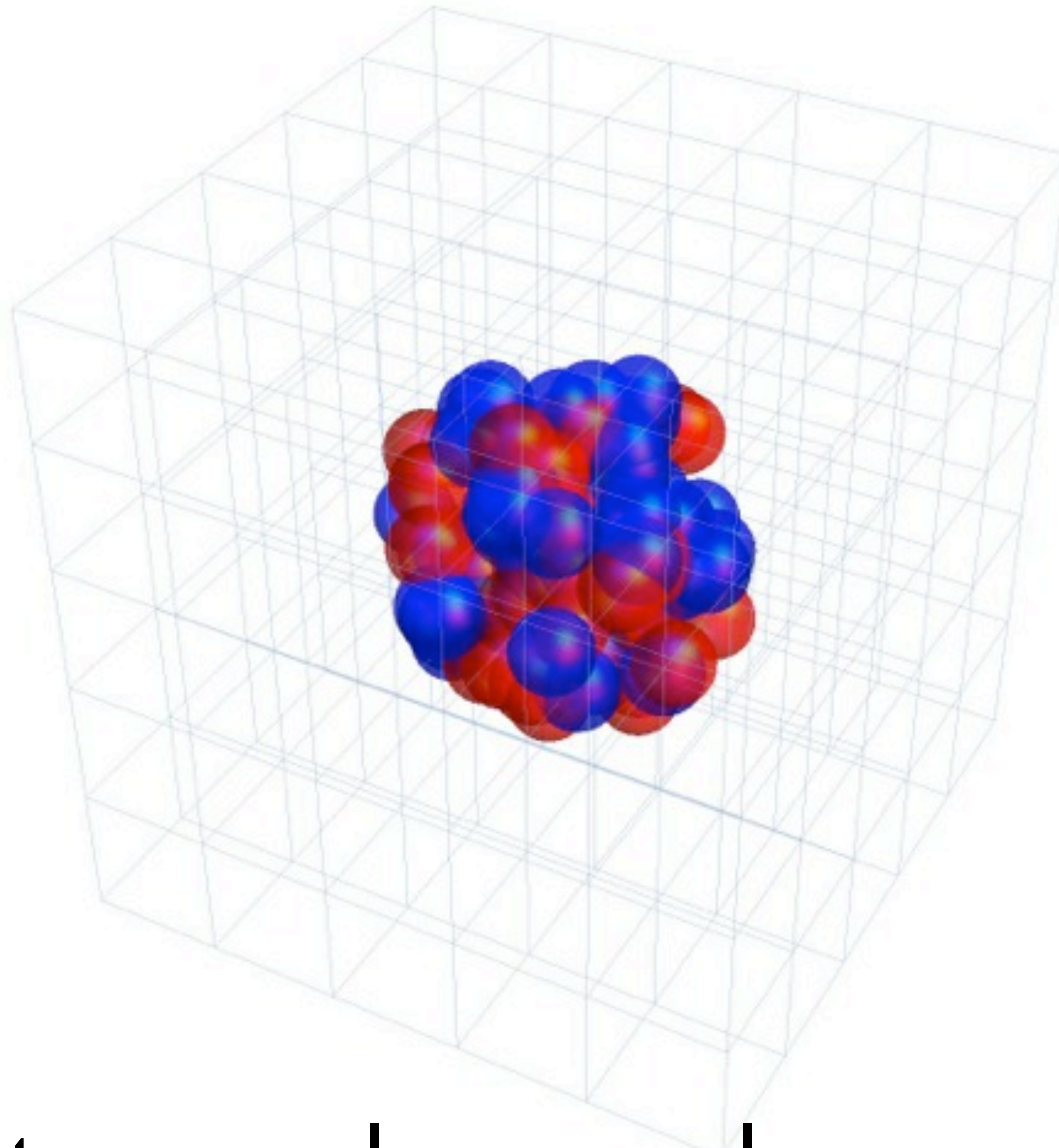

Lecture 1: nuclear physics

William Detmold

Massachusetts Institute of Technology

Lecture content

- Nuclear physics and the role of LQCD
- Scattering theory and bound states
- General forms of potentials
- Potentials for infinitely heavy hadrons

The nuclear landscape

The nuclear landscape

- NP is a vast and somewhat disparate subject

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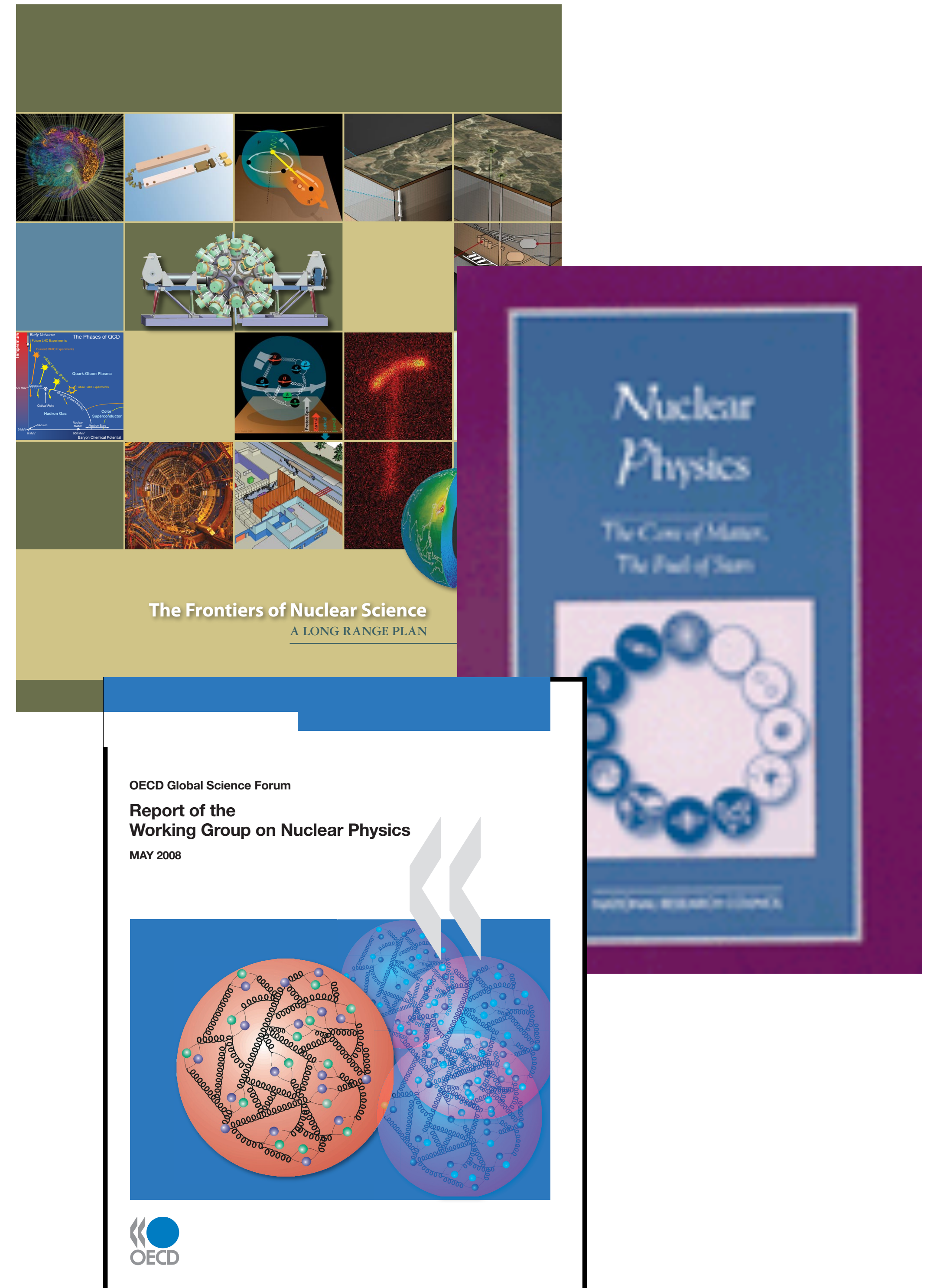

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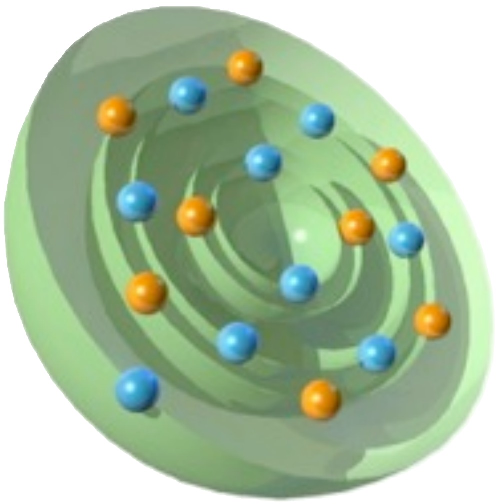

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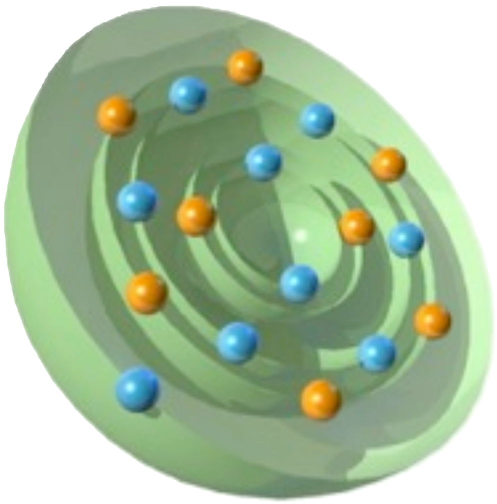

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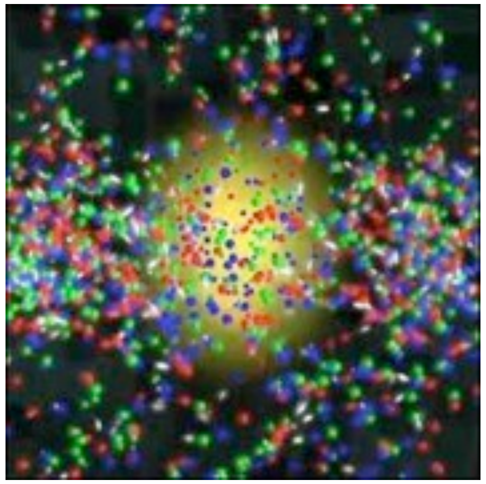

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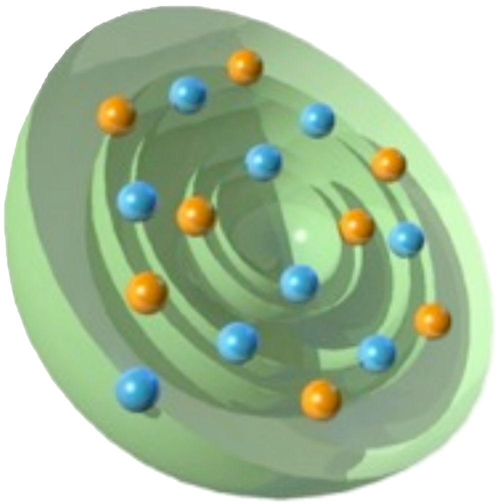

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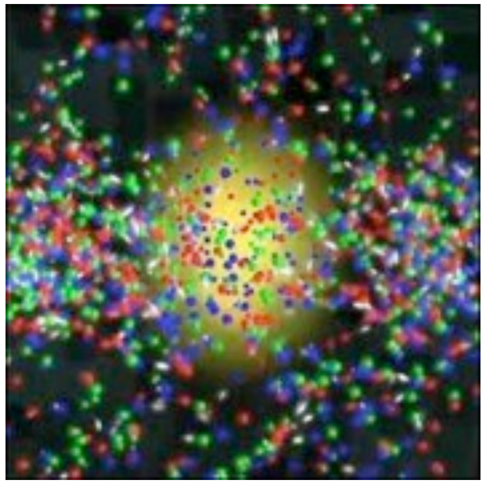
    - Supernovae, neutron stars,...



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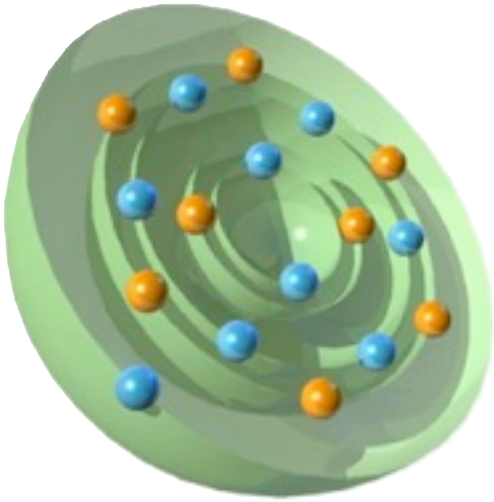

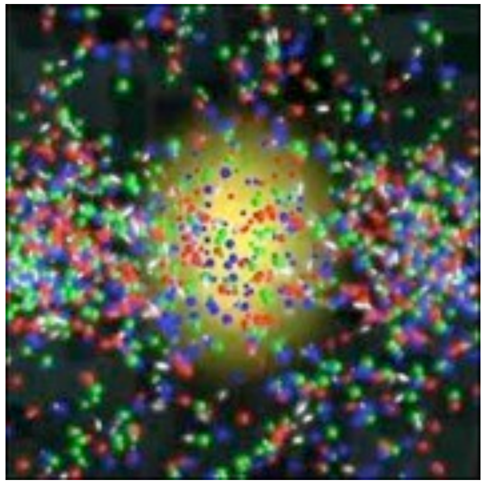

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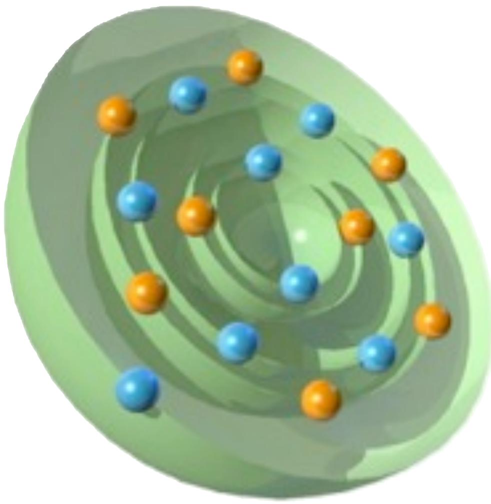

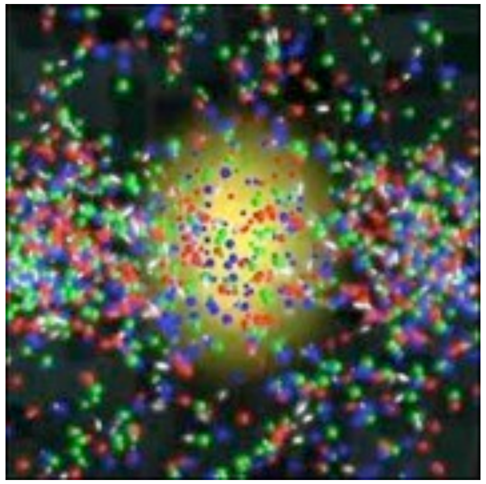

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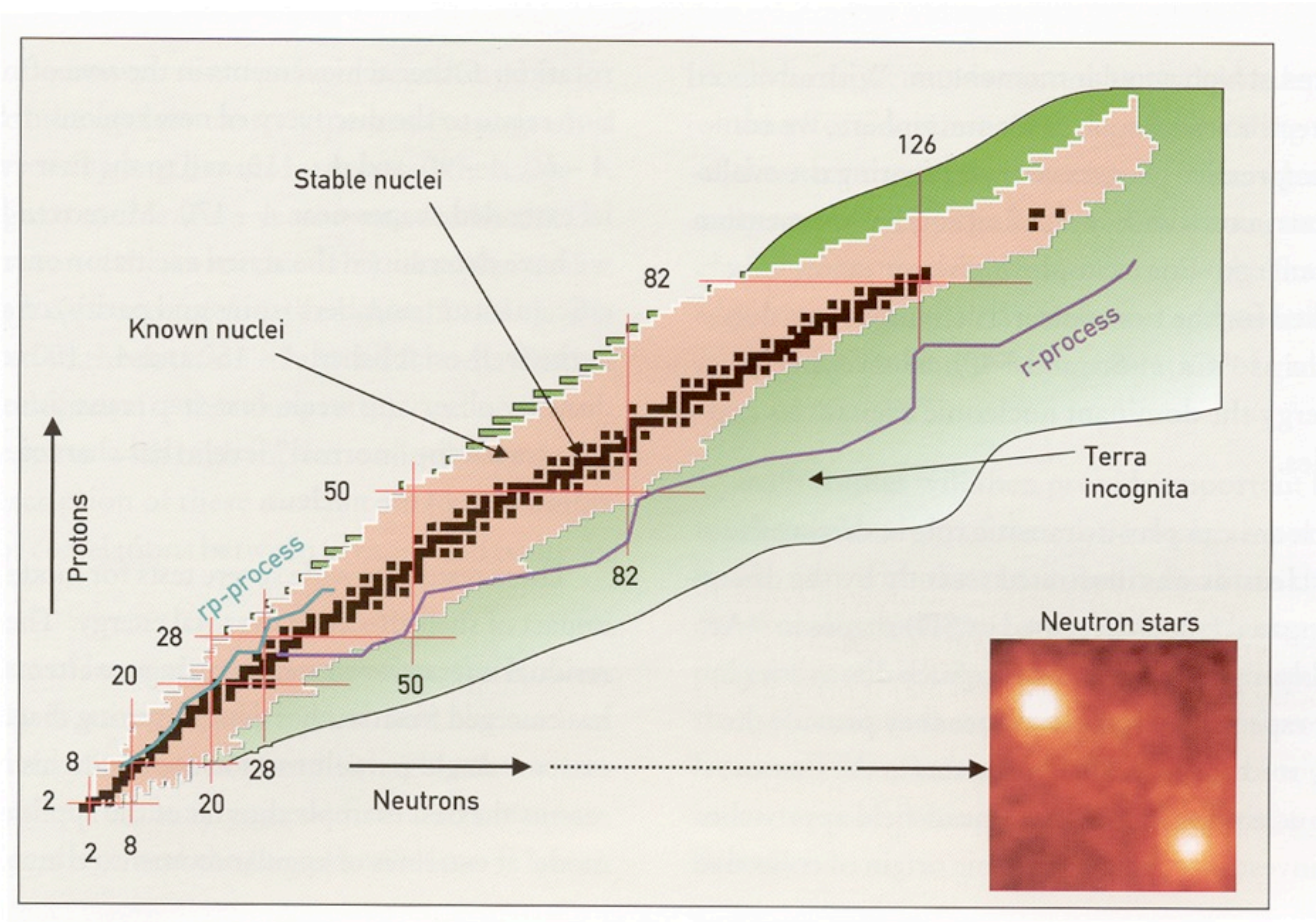
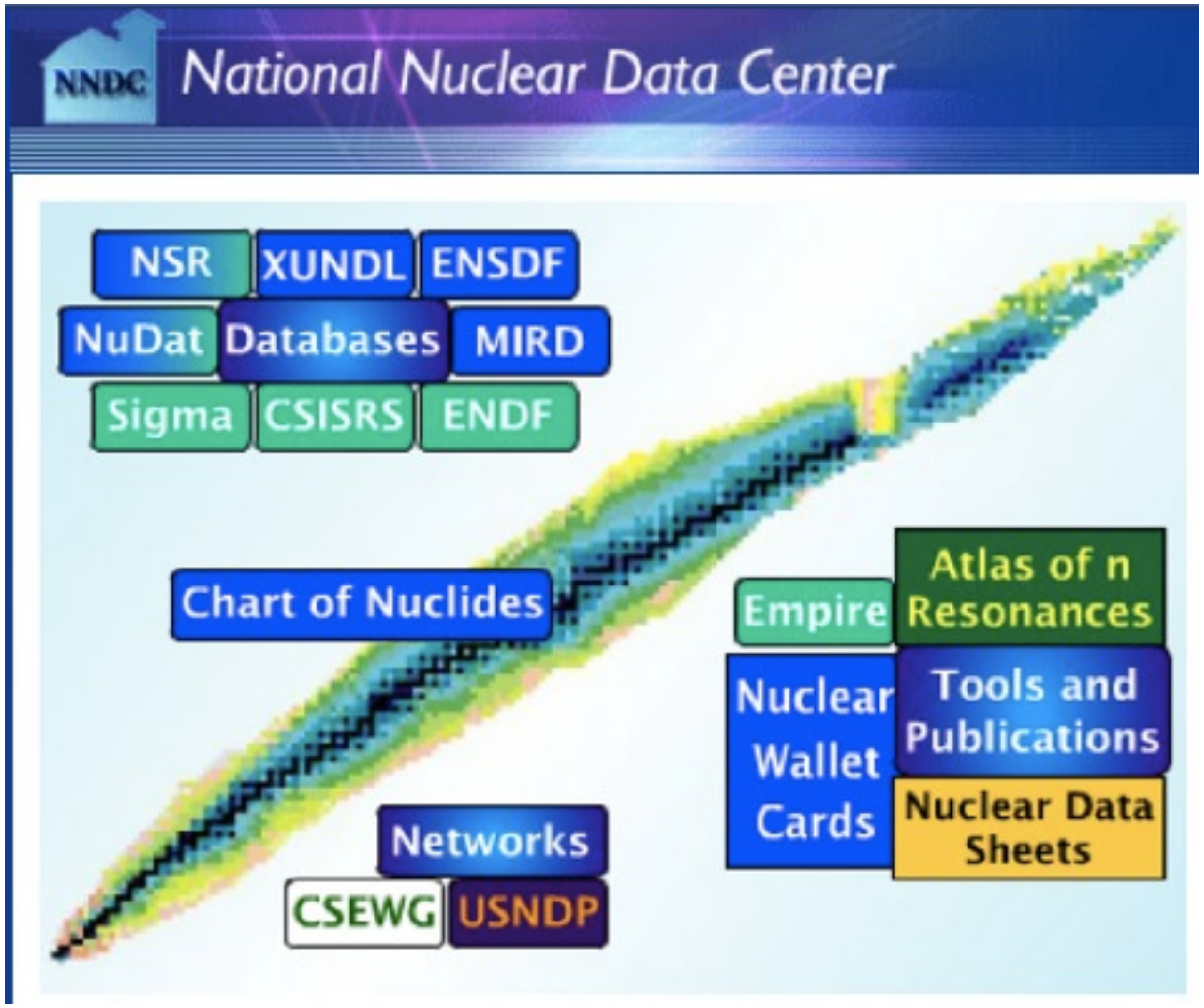


What do we know about nuclei?

- Masses and binding energies
- Sizes and shapes
- Electromagnetic properties; multipole moments
    - Magnetic moments
    - Quadrupole moments
- Excitation spectra
- National Nuclear Data Center
  http://www.nndc.bnl.gov/
  (the PDG of nuclear physics)



Nuclear binding

- Stable isotopes are bound systems of nucleons, so by definition their mass is less than the sum of the masses of the constituents

- The (positive) binding energy is defined as

$$B(Z,N) = Zm_H + Nm_n - M(Z,N)$$

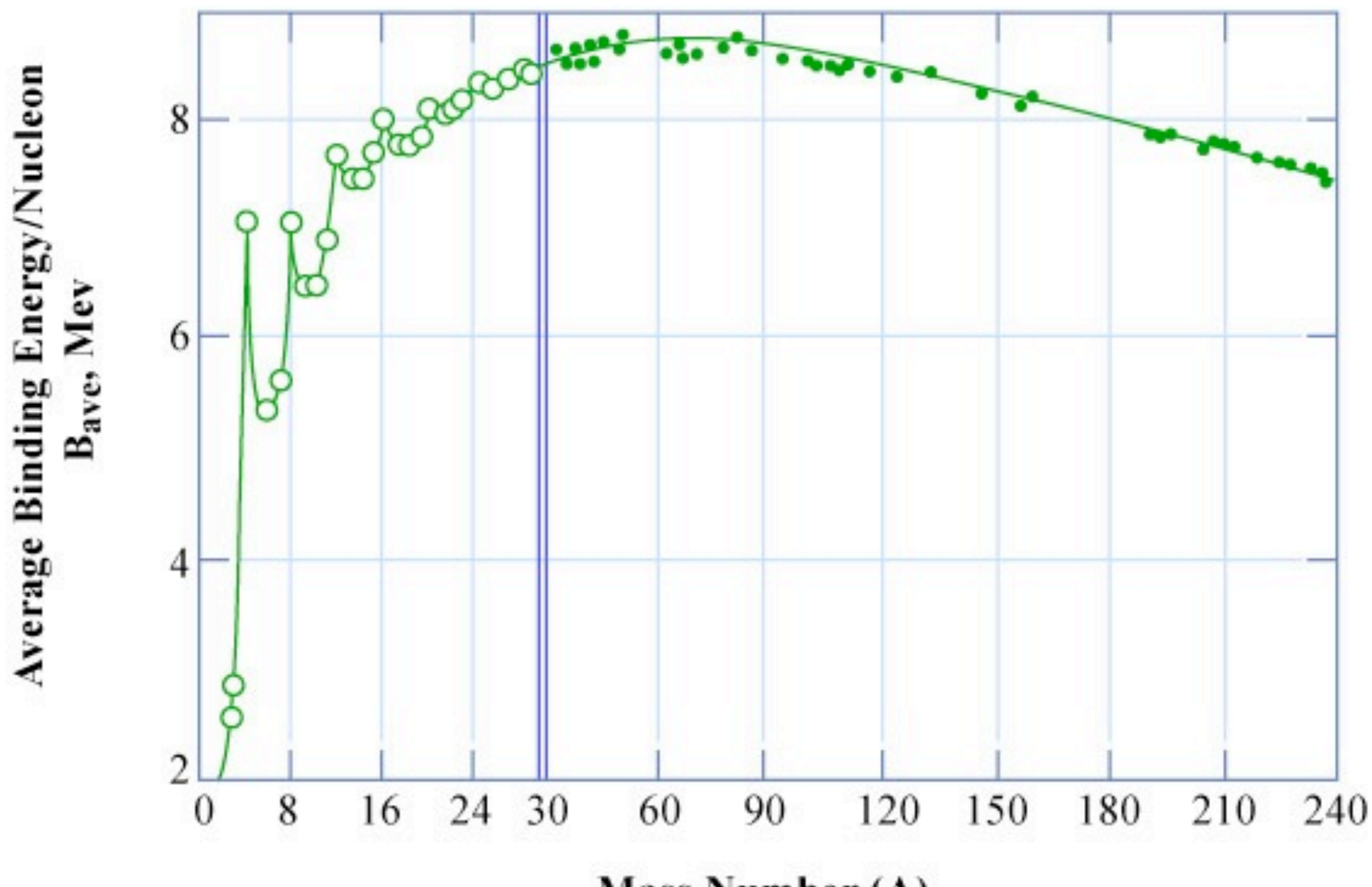
  and the binding energy per nucleon is B/A (shown in figure)

- Sharp rise and spiky for small A

- Maximum at A=58

- Slow decrease above A=60, B/A~8 MeV

- Very important curve for fusion and fission



Semi-empirical mass formula

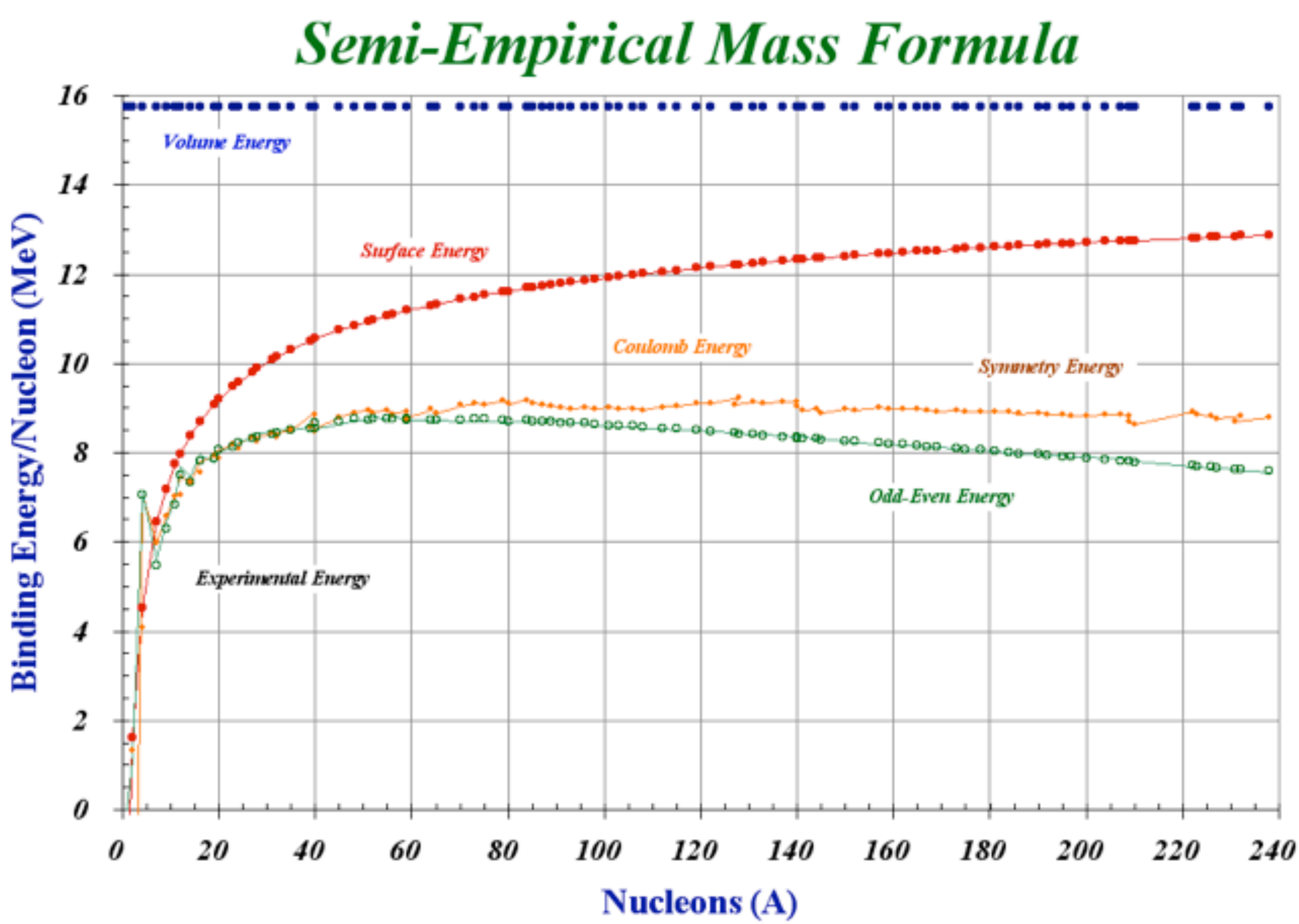
- Weiszacker mass formula:

$$B_{\text{total}} = \alpha A - \beta A^{2/3} - \epsilon \frac{Z(Z-1)}{A^{1/3}} - \frac{(N-Z)^2}{A}\left[\gamma + \frac{\chi}{A^{1/3}}\right] - \frac{\eta}{A^{1/2}}\delta(Z,N)$$

- Parameters are fit to data

$$\alpha = 15.8 \text{ MeV}$$
$$\beta = 17.8 \text{ MeV}$$
$$\gamma = 23.8 \text{ MeV}$$
$$\epsilon = 0.7 \text{ MeV}$$
$$\chi = 33.2 \text{ MeV}$$
$$\eta = 12 \text{ MeV}$$

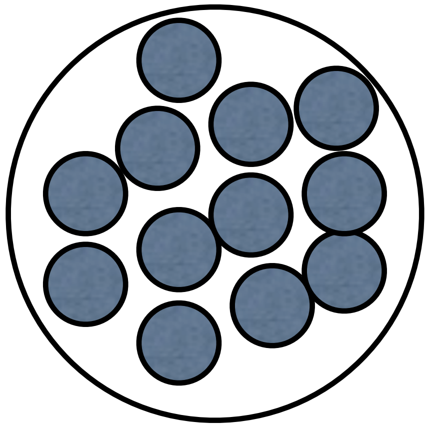
- Gives a good description



Nuclear radii

- Rutherford's experiments in the early 1900s showed that the nucleus was very much smaller than the atom

- Nucleons are about ~0.8 fm in radius, and if we think of them as hard spheres and pack them into a spherical configuration then



$$\frac{4\pi}{3}R_A^3 \equiv V_A = AV_N = \frac{4\pi}{3}A\,r_N^3$$

or equivalently

$$R_A \approx r_0 A^{1/3}$$

(r_0 is a bit larger than proton radius to account for packing fraction)

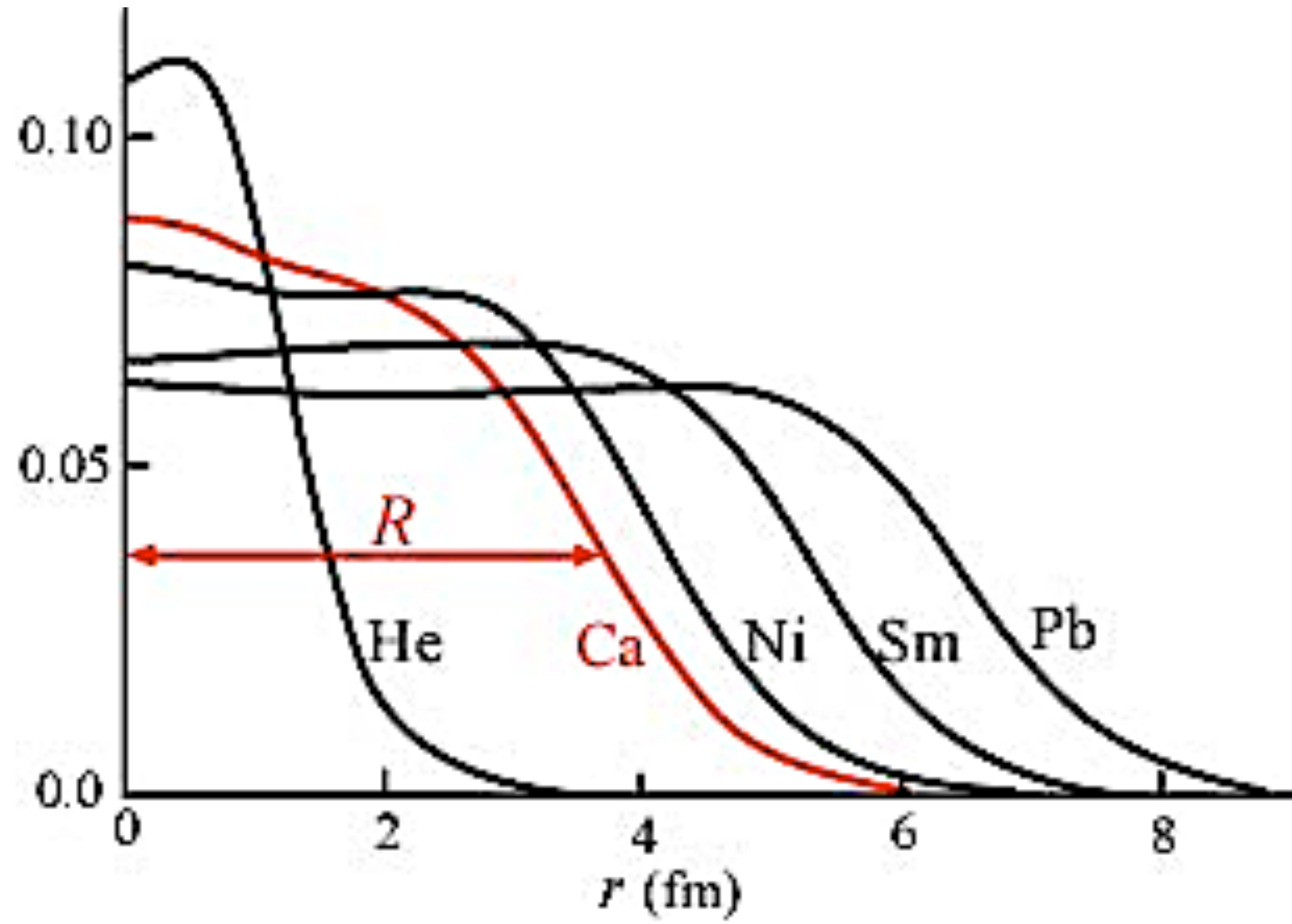
- Charge distribution can be measured in elastic electron scattering just as for proton form factors (also muonic atoms)

- Matter distribution probed by elastic hadron-nucleus scattering

- Charge and matter distributions will agree only if protons and neutrons are arranged similarly

Nuclear charge distributions

- Charge density probed in elastic eA→eA
- Definitive experiments by Hofstadter (1950s)
- For large nuclei, core of constant density inside a diffuse surface with rapidly decreasing density
- For A>20, distribution described by a Fermi distribution



$$\rho(r) = \frac{\rho_0}{1 + e^{(r - R_{1/2})/a}}$$

- a controls the thickness of the surface and $R_{1/2}$ the size of the central region
- Fits give

$$\rho_0 = 0.17\frac{Z}{A}\ \text{fm}^{-1}, \qquad a = 0.54\ \text{fm}, \qquad R_{1/2} = 0.218A^{1/3}\ \text{fm}$$

Neutron skin

- PV electron scattering sees neutrons predominantly over electrons

$$Q_{\rm weak}^{\rm proton} \propto 1 - 4\sin^2\theta_{\rm W} \approx 0.076, \quad Q_{\rm weak}^{\rm neutron} \propto -1$$

  - Provides a theoretically cleaner measurement as EW probe

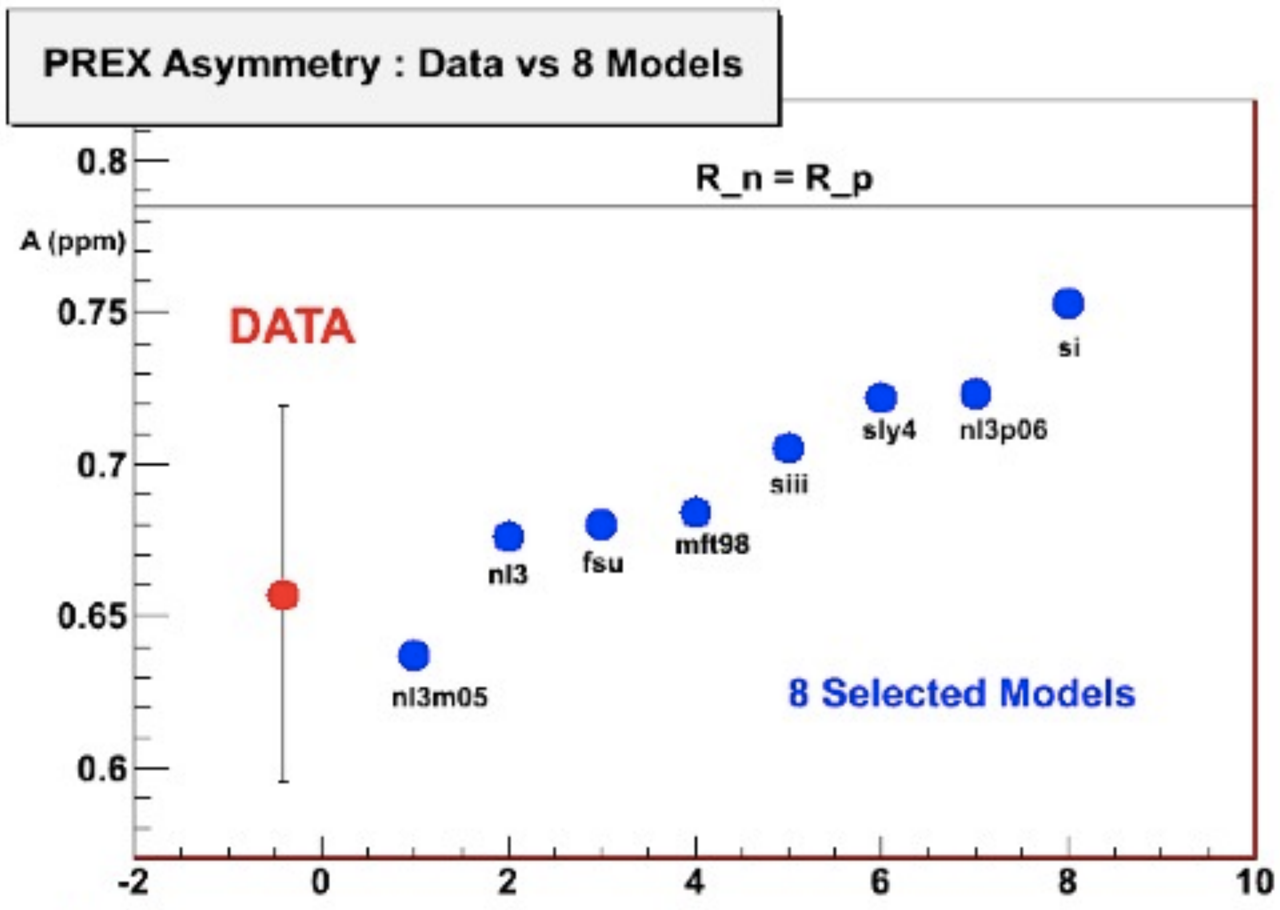
$$A_{\rm PV} = \frac{\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-} = \frac{G_F Q^2}{4\pi\alpha\sqrt{2}}\left[1 - 4\sin^2\theta_W - \frac{F_n(Q^2)}{F_p(Q^2)}\right]$$

- Parity Radius EXperiment (PREX) JLab 2010

- Neutron radius of ^{208}Pb from PREX

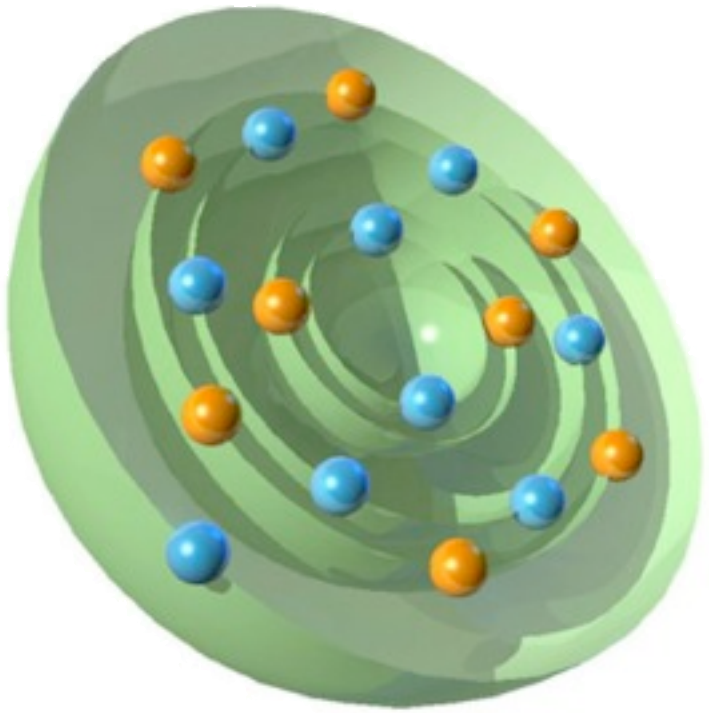
$$R_n - R_p = 0.34 + 0.15 - 0.17 \text{ fm}$$

  - Neutron skin outside protons
  - Constrains Eq of State for neutron stars



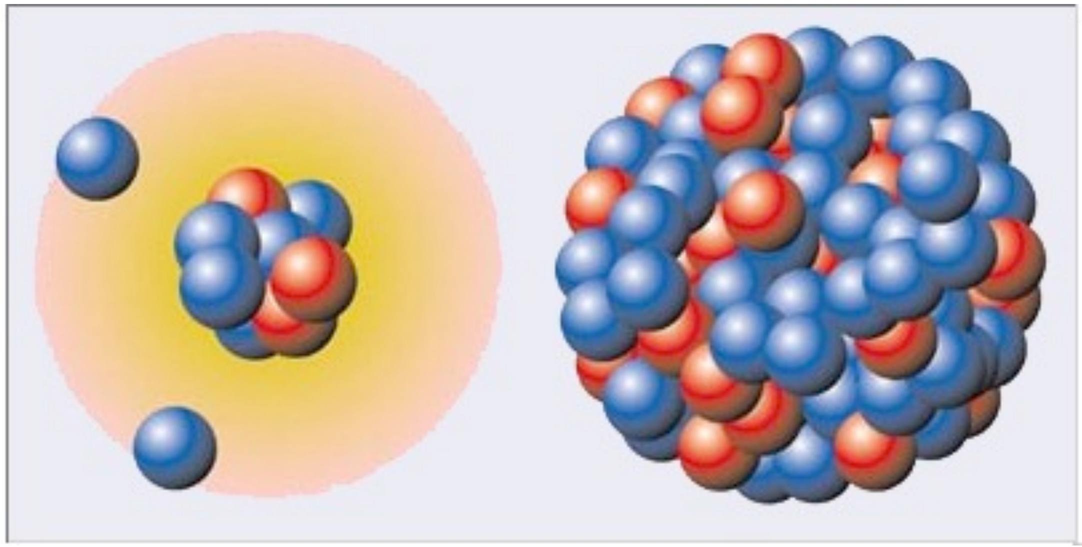
Nuclear theory

- Much disparate phenomenology explored over many years
  - Lots of data
- Proliferation of models cooked up to describe various different aspects
  - Liquid drop model
  - Shell models
  - Vibrational and rotational models
  - "Ab-initio" methods
- To some level of approximation nucleus = a bunch of nucleons with predominantly pairwise interactions

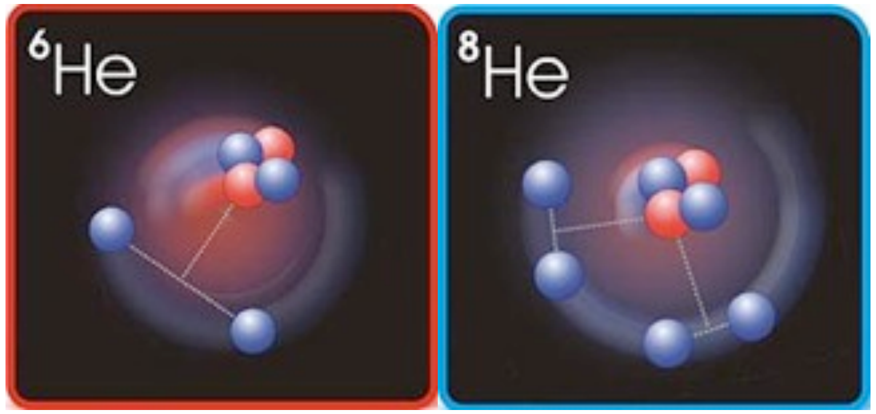


Exotic nuclei

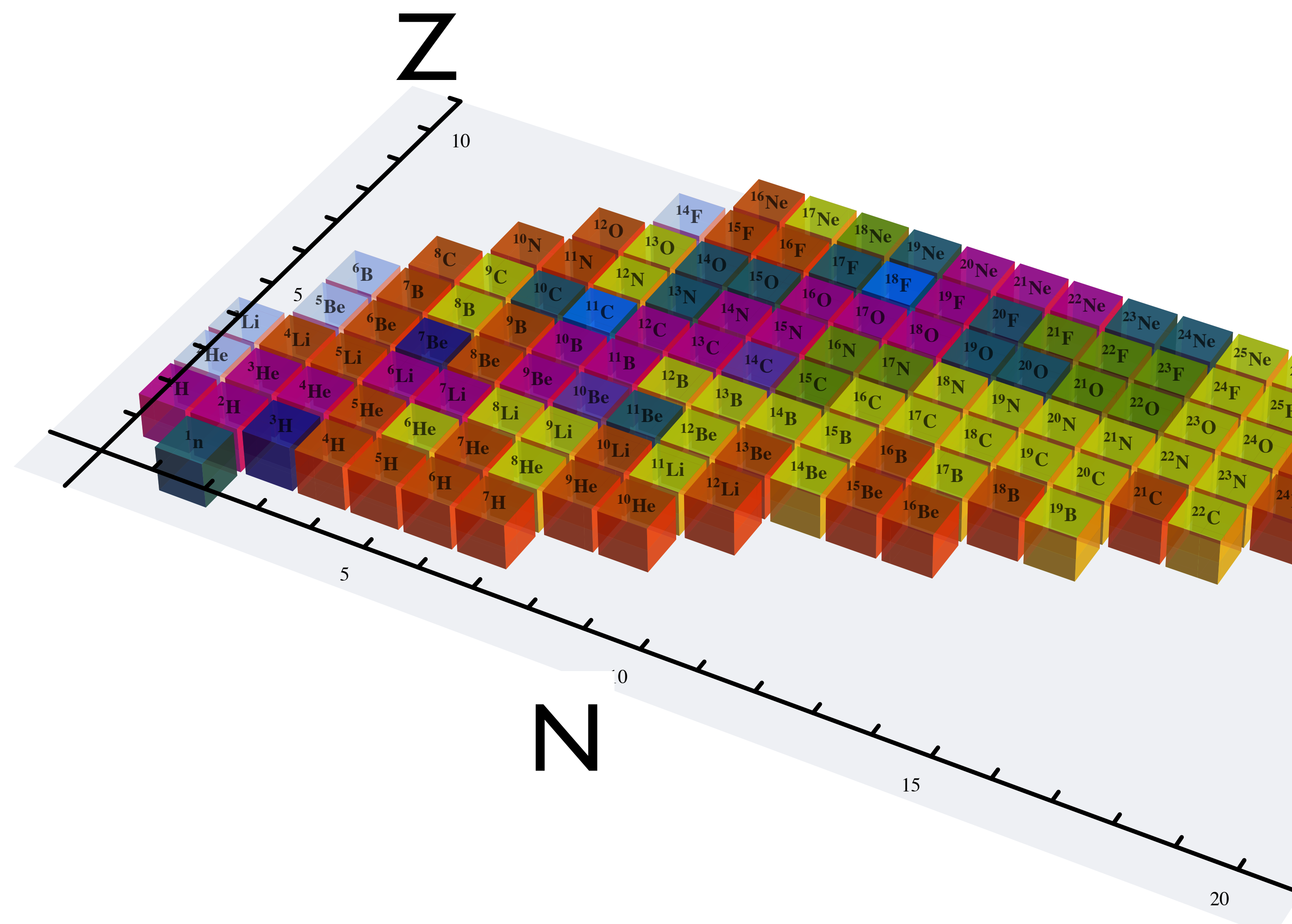
- Not all nuclides are so easily dealt with in these simple models
- One example is halo nuclei in which some nucleons spend significant fractions of time are large distances from the COM
  - Ex ^{11}Li is as large as ^{208}Pb
  - NB: neither of the two subsystems (nn, ^{10}Li) are bound
  - Isotopes of helium are even more strange: four-neutron halo
- Also there are super-deformed nuclei (around N=80, Z=60) that don't fit observed patterns



^{11}Li ^{208}Pb

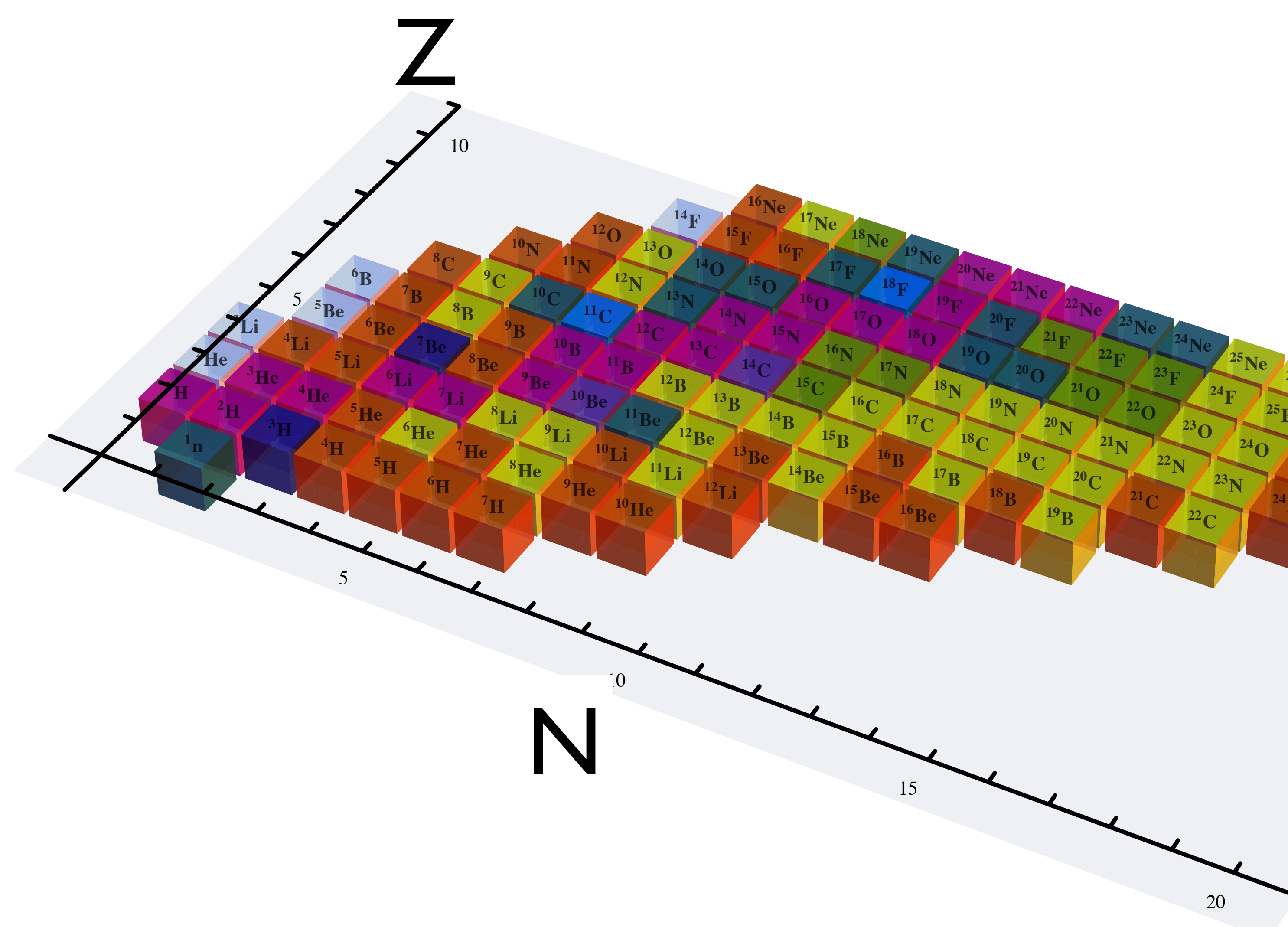


Hypernuclei



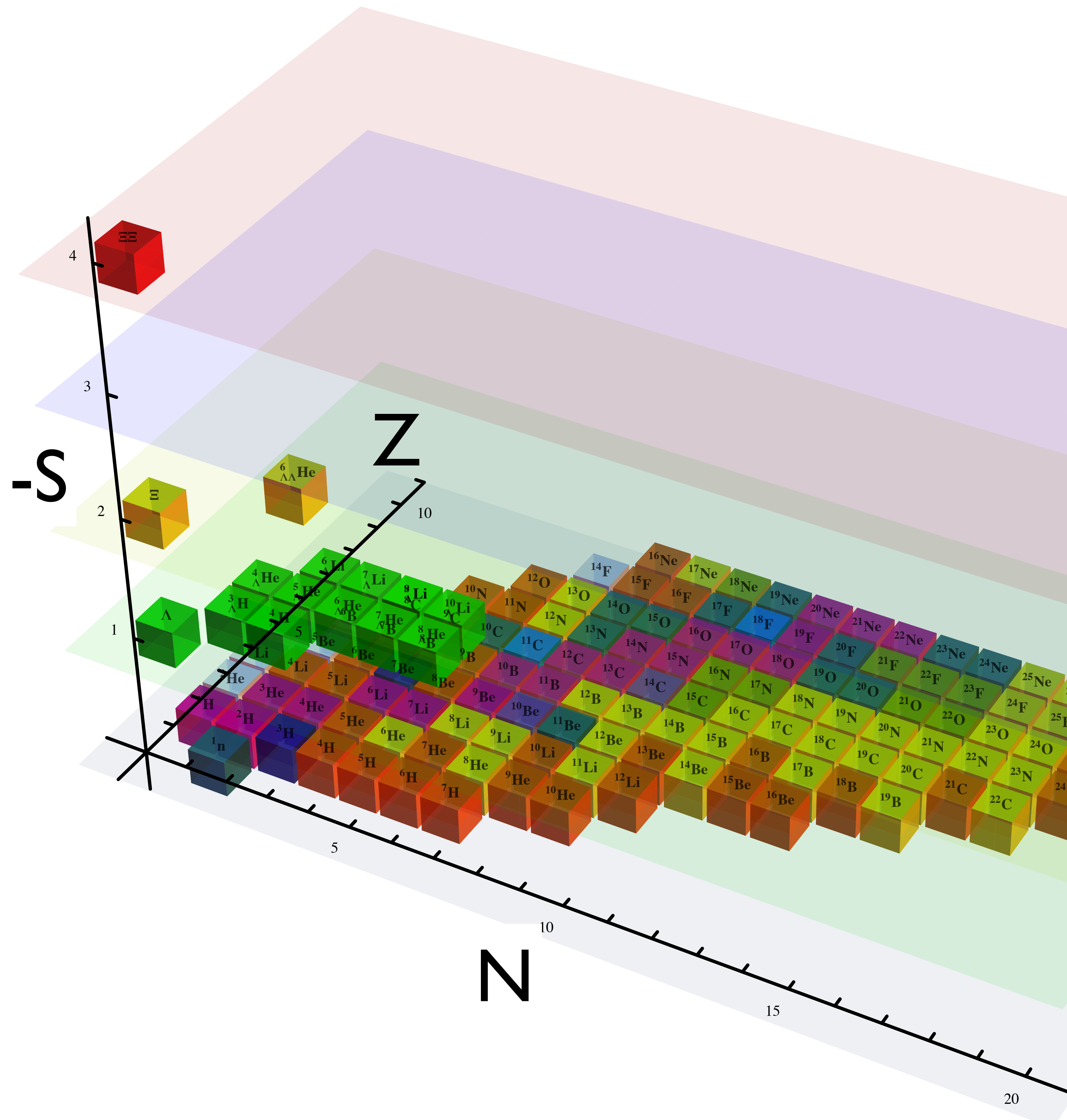
Hypernuclei

- Chart of nuclides



Hypernuclei

- Chart of nuclides
- One plane in chart of hypernuclei



Hypernuclei

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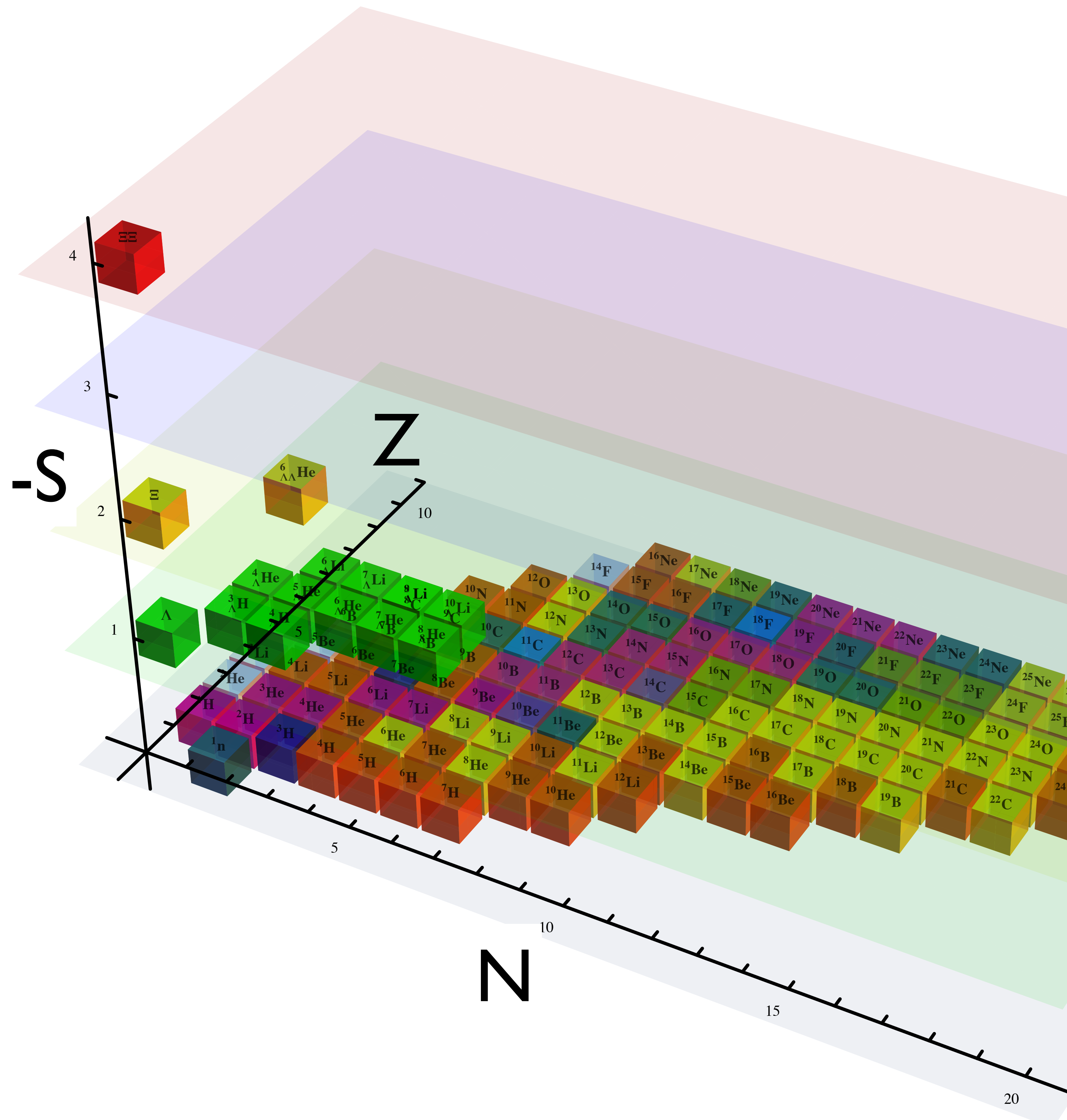
- Hypernuclei = nuclei which contain strangeness



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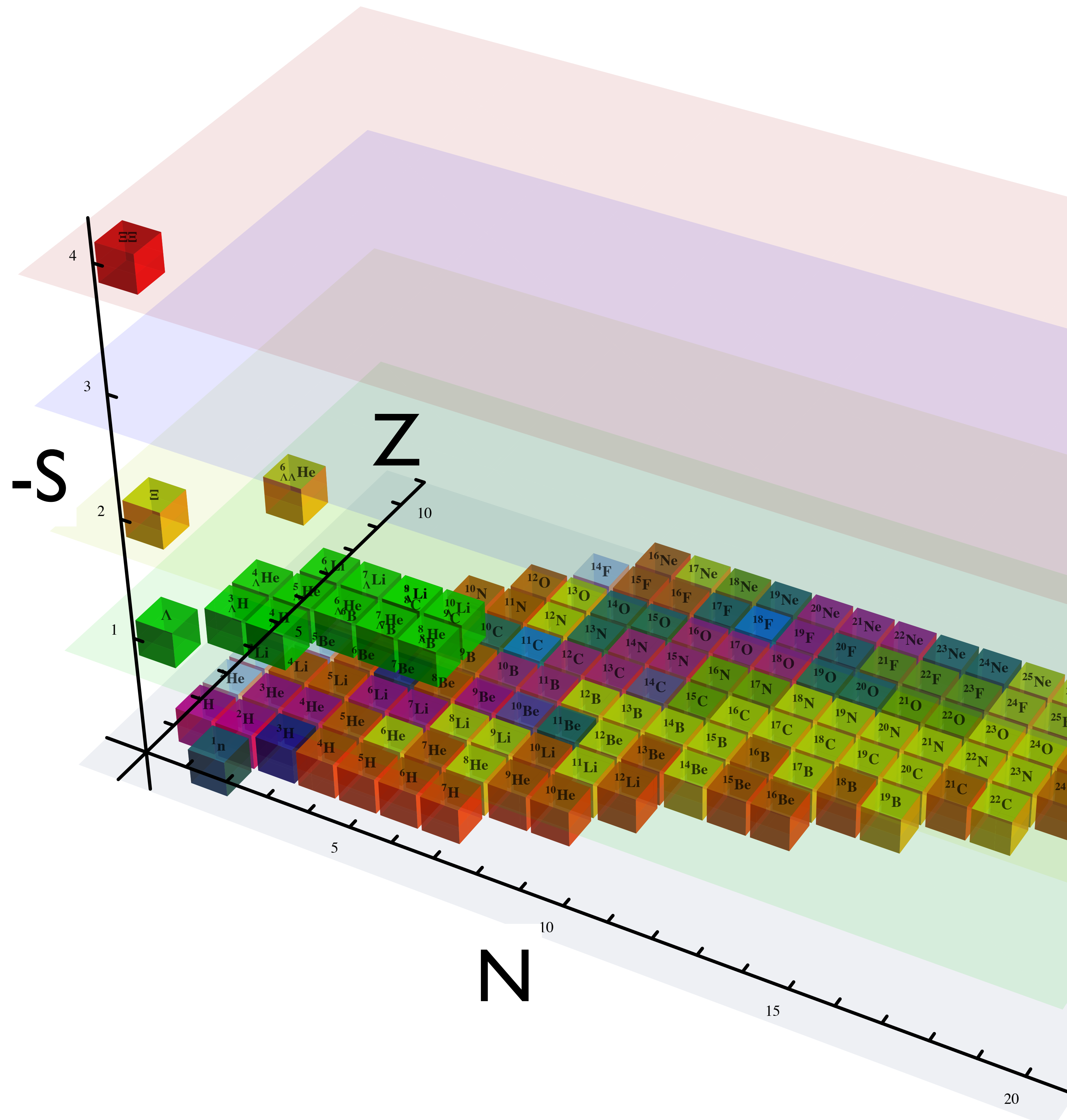
    - Decay weakly but stable under strong interactions



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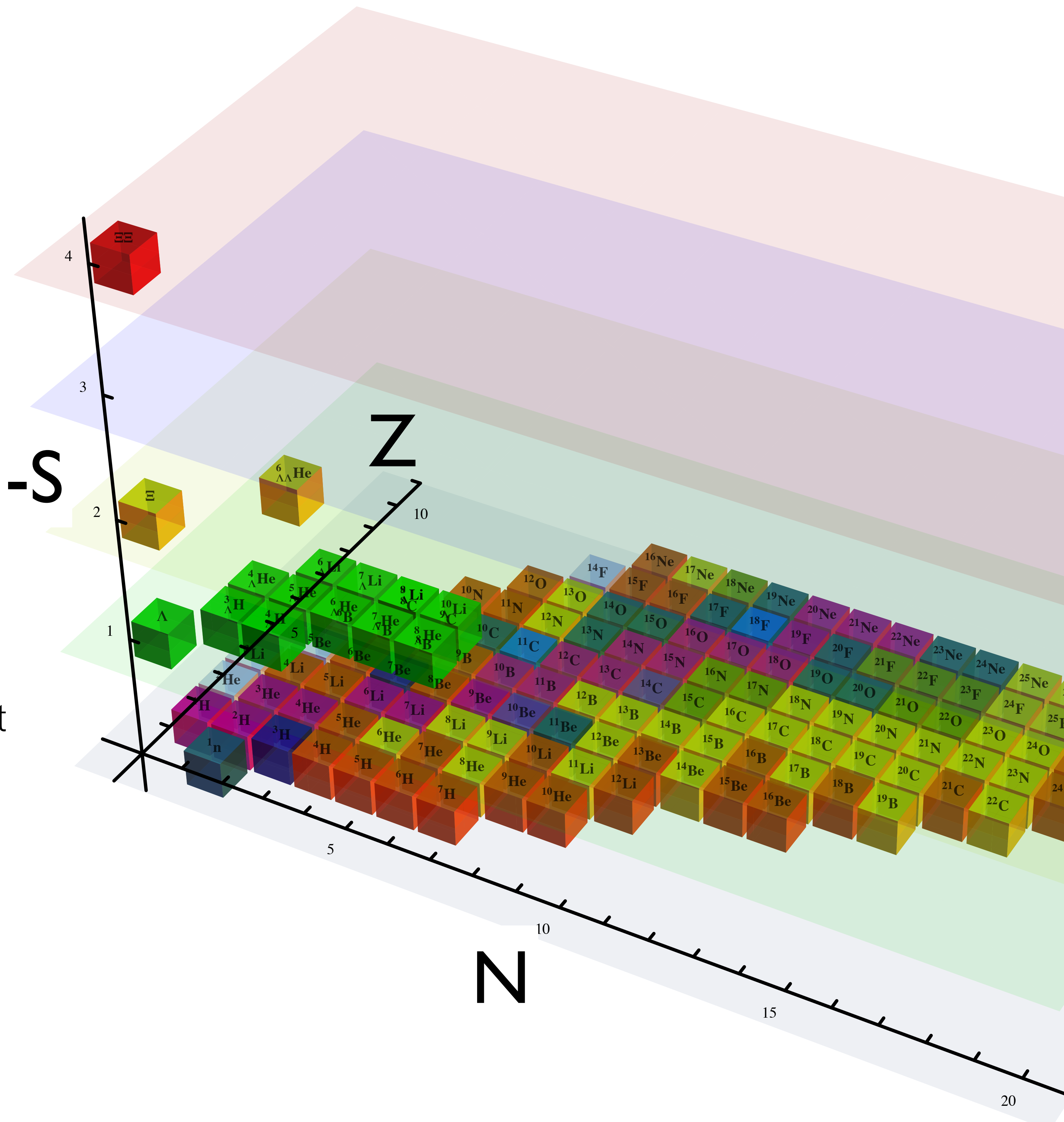

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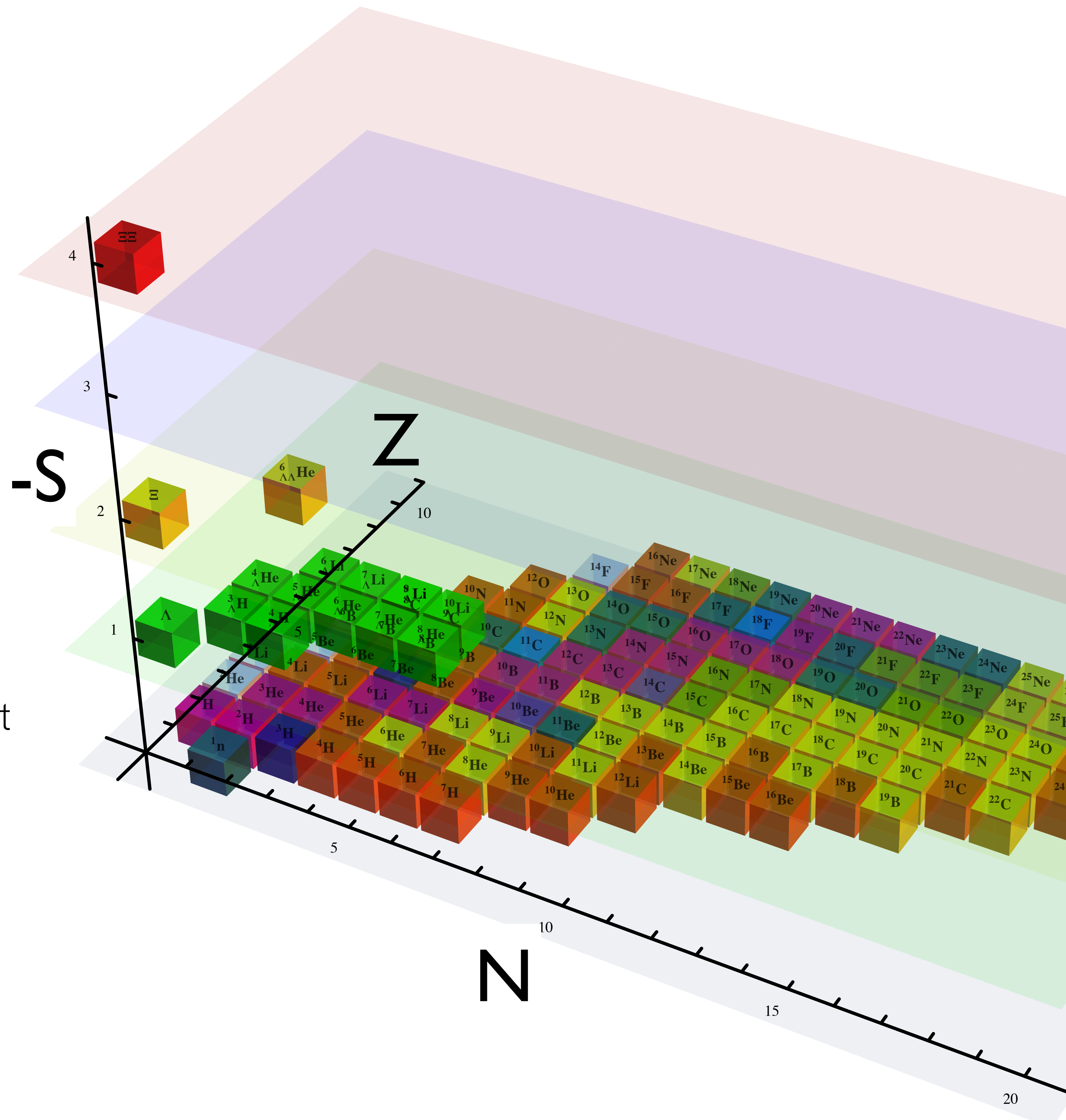
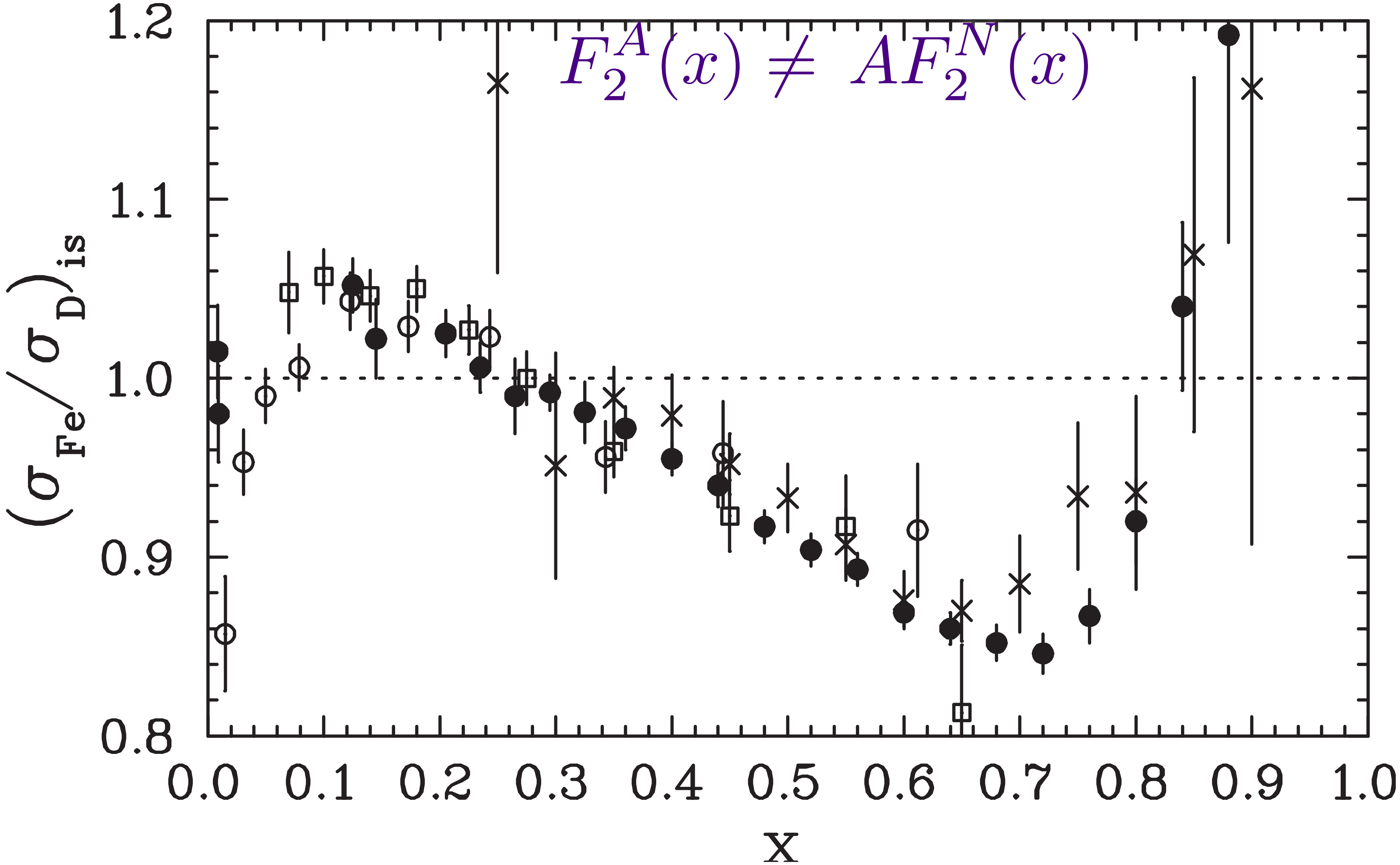
- Much less data to constrain models



The EMC effect

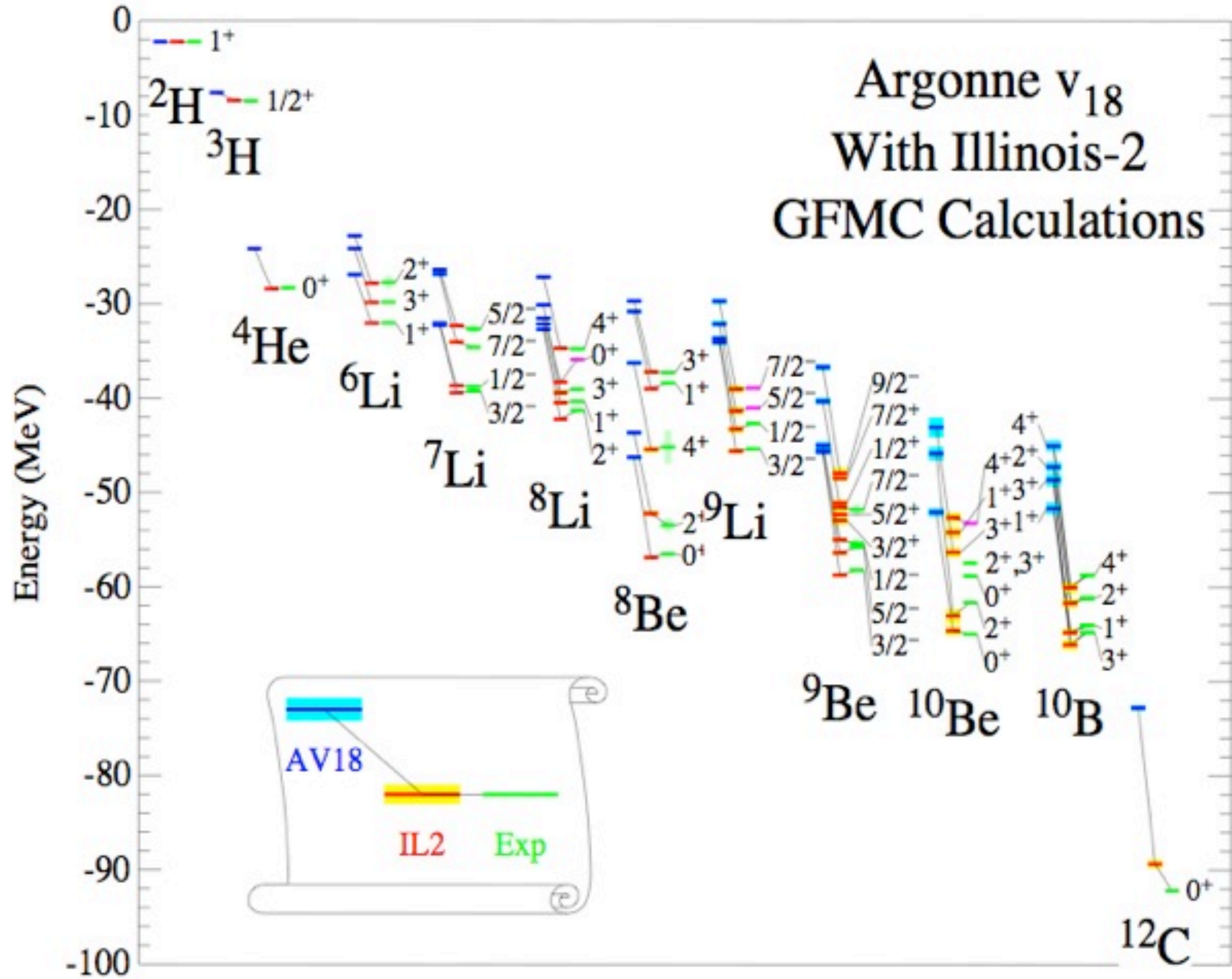
- Nuclei are not just a collection of nucleons
- 1983: deep inelastic scattering on Fe target [EMC]



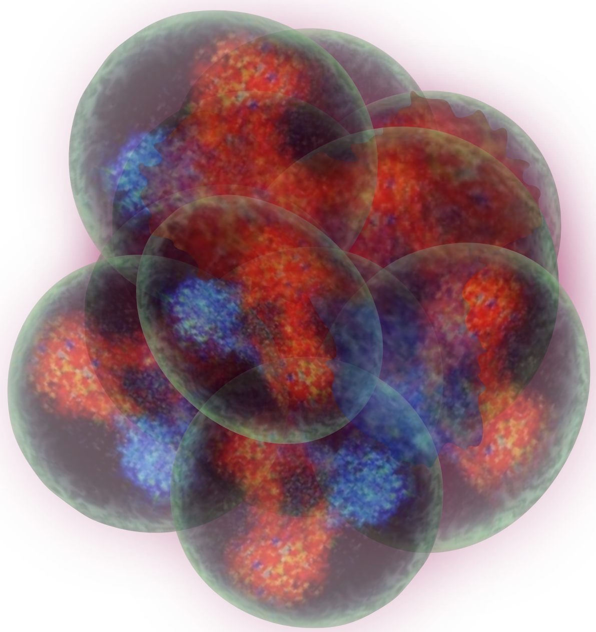
- Proton structure modified in a nuclear environment

Three body physics

- Three body interactions
  - Necessary for accurate description of nuclei in GFMC
  - Unlike 2 body, not much data to constrain 3-body interactions
- Higher body?

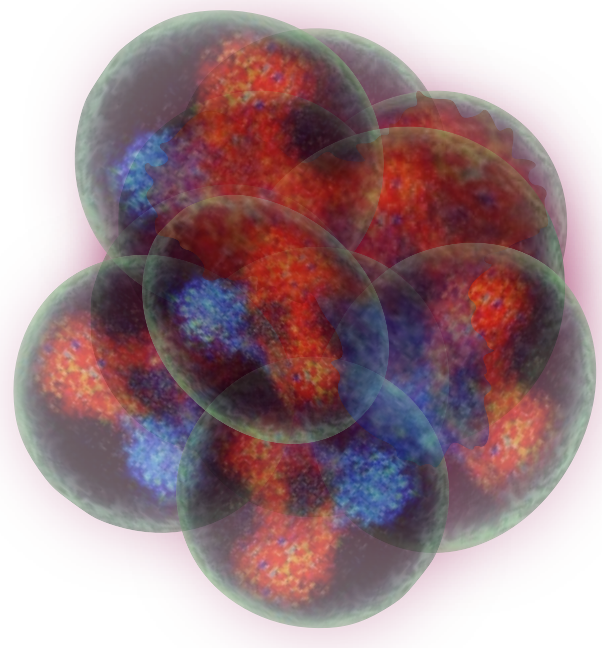


LQCD in nuclear physics



LQCD in nuclear physics

- Nuclear physics: an emergent phenomenon of the Standard Model



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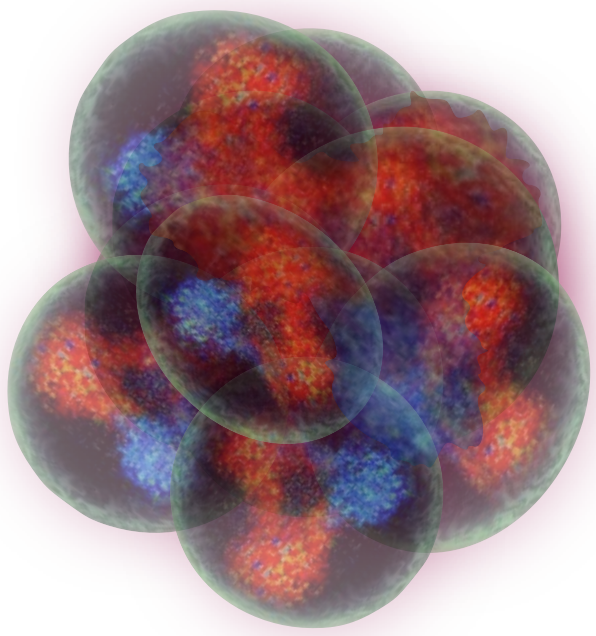

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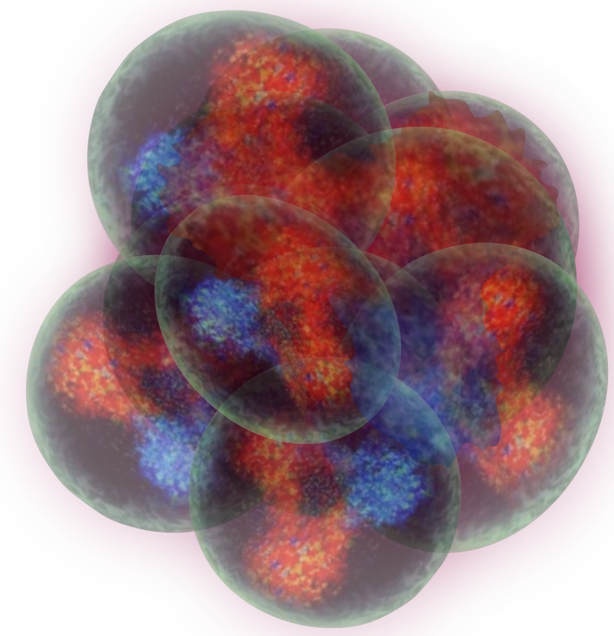

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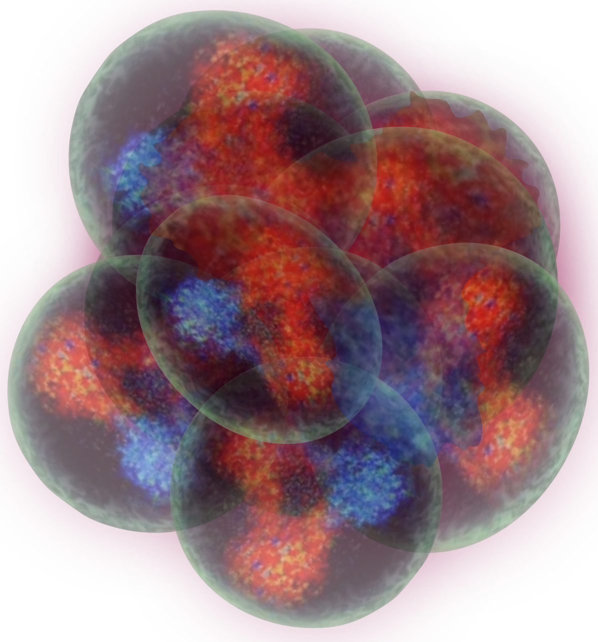

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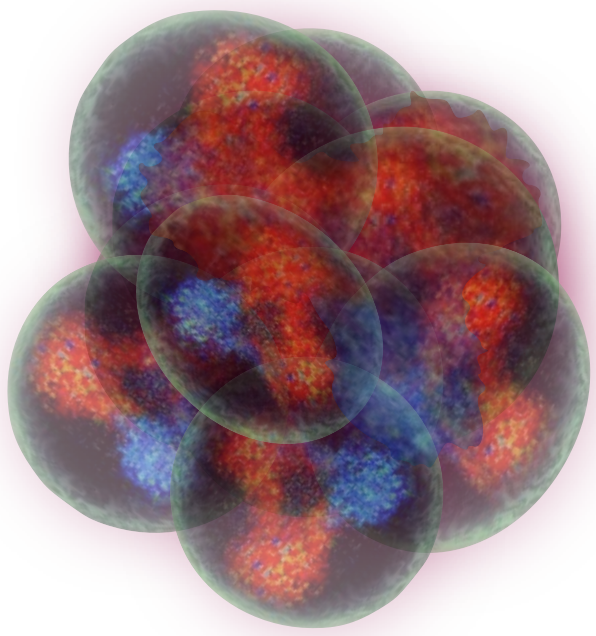

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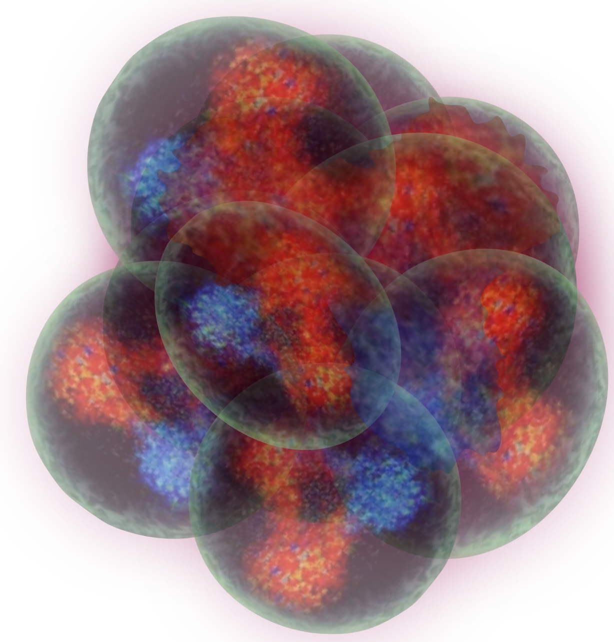

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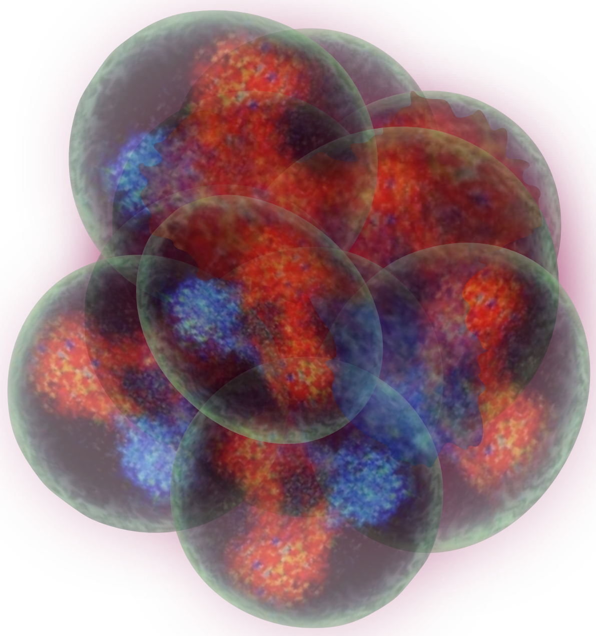
  - Lots of challenges



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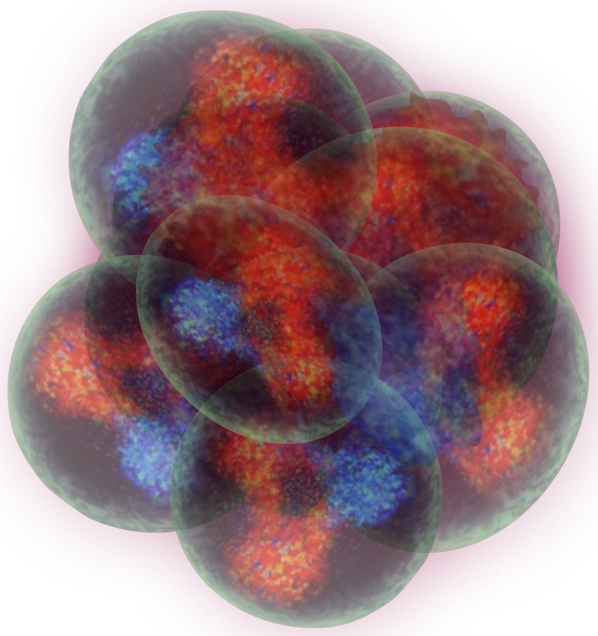
  - Enabled by growth in HPC
  - Lots of challenges
  - Lots of recent progress



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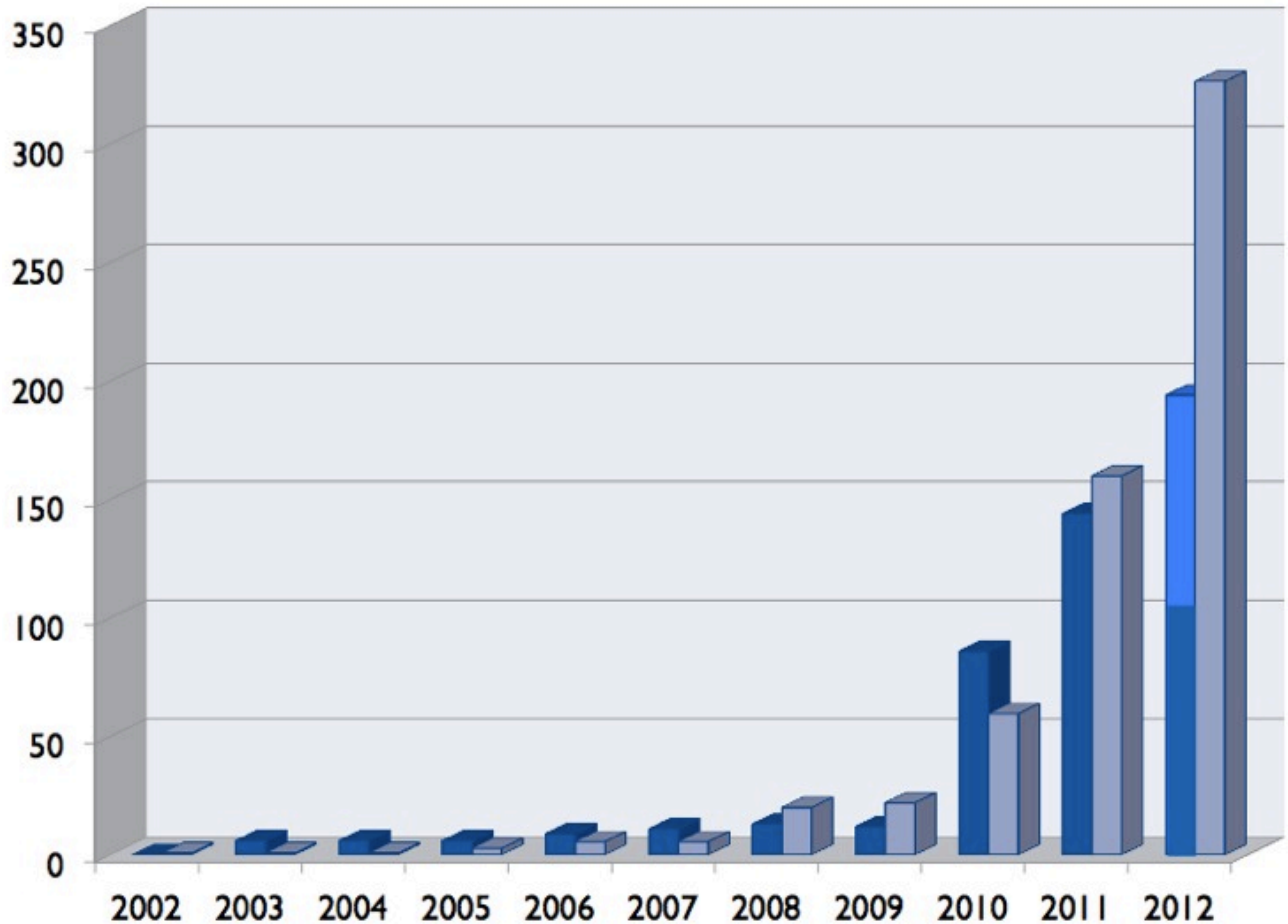

[compiled by Tom Luu, LLNL]

Nuclear physics from LQCD

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- Can we compute the mass of 208Pb in QCD?

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- Yes, consider

$$\langle 0 | T q_1(t) \ldots q_{624}(t) \overline{q}_1(0) \ldots \overline{q}_{624}(0) | 0 \rangle$$

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- Long time behaviour gives ground state energy up to EW effects

$$\stackrel{t\to\infty}{\longrightarrow} \# \exp(-M_{Pb} t)$$

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- But...

An (exponentially hard)^2 problem?

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- Complexity: number of
  Wick contractions = (A+Z)!(2A-Z)!

$$a_i^\dagger(t_1)a_j^\dagger(t_1)a_j(t_1)a_i(t_1)a_i^\dagger(t_2)a_j^\dagger(t_2)a_j(t_2)a_i(t_2)$$

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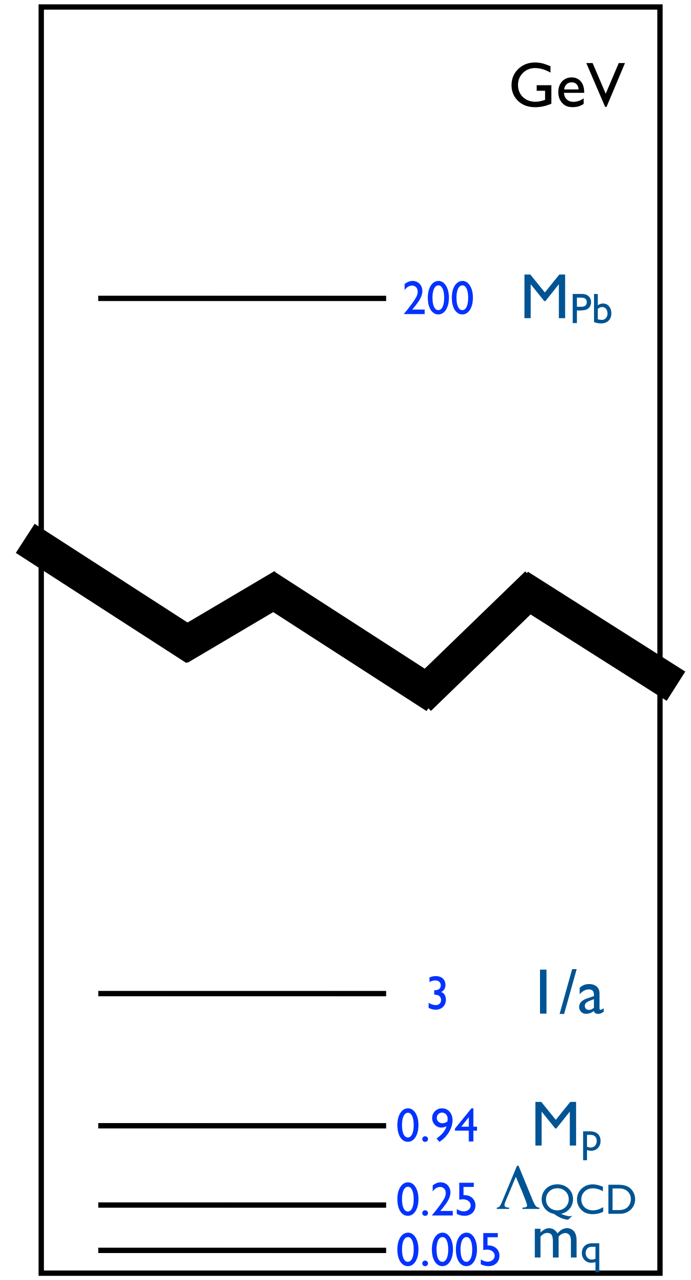
- Dynamical range of scales (numerical precision)



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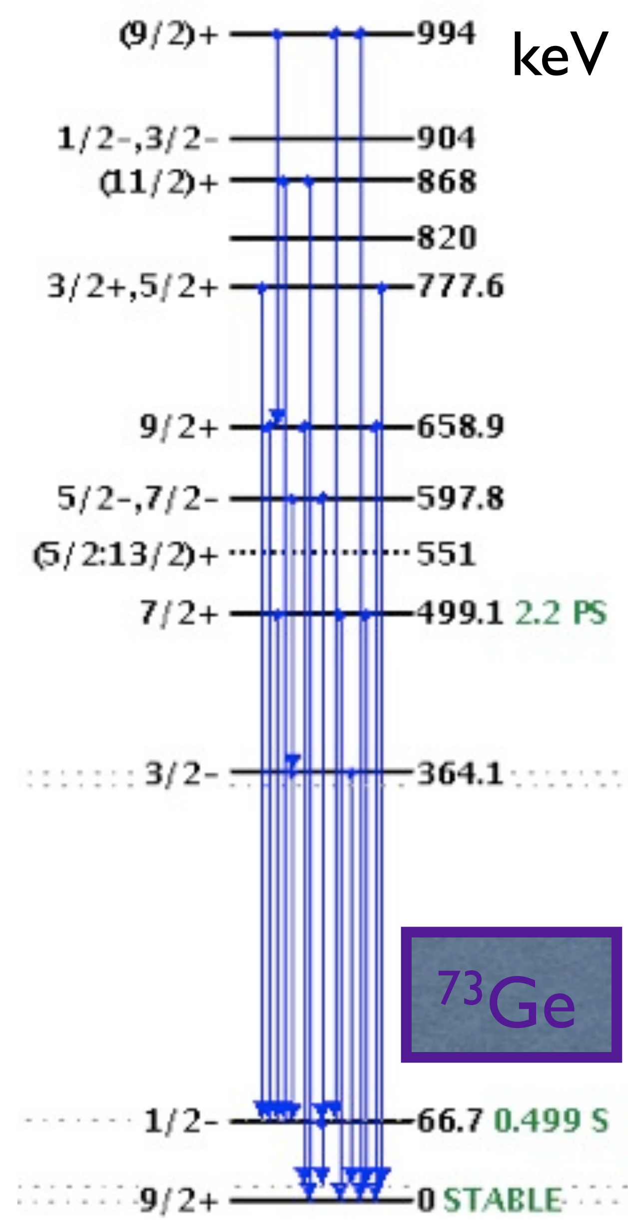
- Dynamical range of scales (numerical precision)

- Small energy splittings



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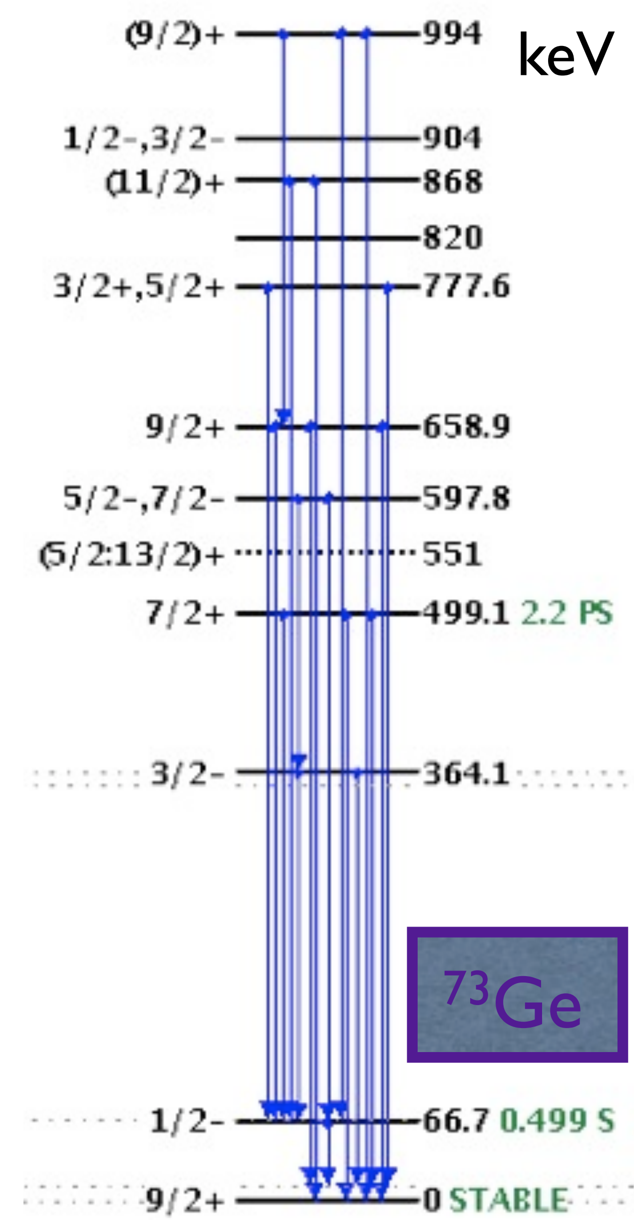
- Dynamical range of scales
  (numerical precision)

- Small energy splittings

- Importance sampling: statistical
  noise exponentially increases with A



The trouble with baryons

- Importance sampling of QCD functional integrals
  ➤ correlators determined stochastically

- Variance in single nucleon correlator (C) determined by

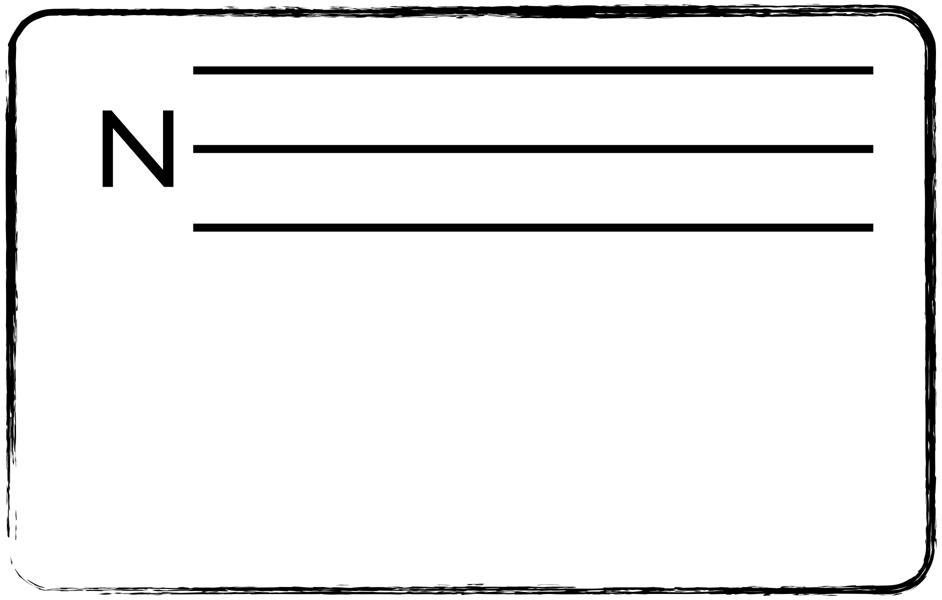
$$\sigma^2(C) = \langle C C^\dagger \rangle - |\langle C \rangle|^2$$

- For nucleon:

$$\frac{\text{signal}}{\text{noise}} \sim \exp\left[-(M_N - 3/2 m_\pi)t\right]$$

- For nucleus A:

$$\frac{\text{signal}}{\text{noise}} \sim \exp\left[-A(M_N - 3/2 m_\pi)t\right]$$



[Lepage '89]

The trouble with baryons

- Importance sampling of QCD functional integrals
  ➤ correlators determined stochastically

- Variance in single nucleon correlator (C) determined by

$$\sigma^2(C) = \langle C C^\dagger \rangle - |\langle C \rangle|^2$$

- For nucleon:

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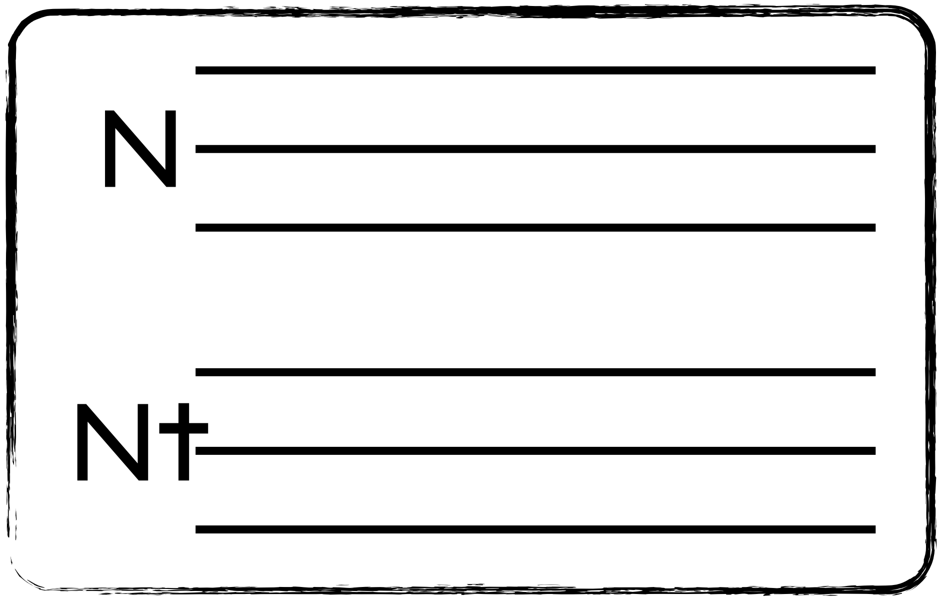

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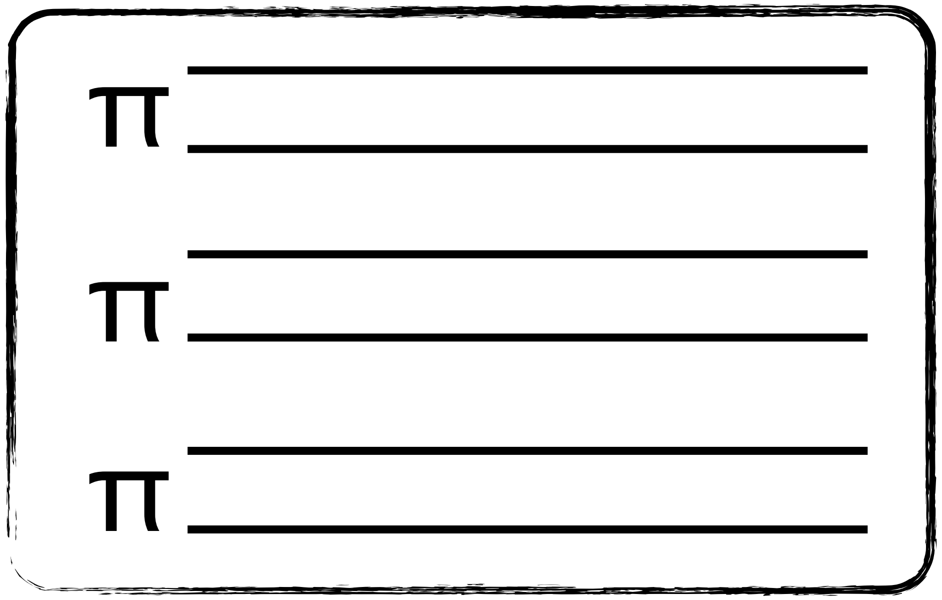

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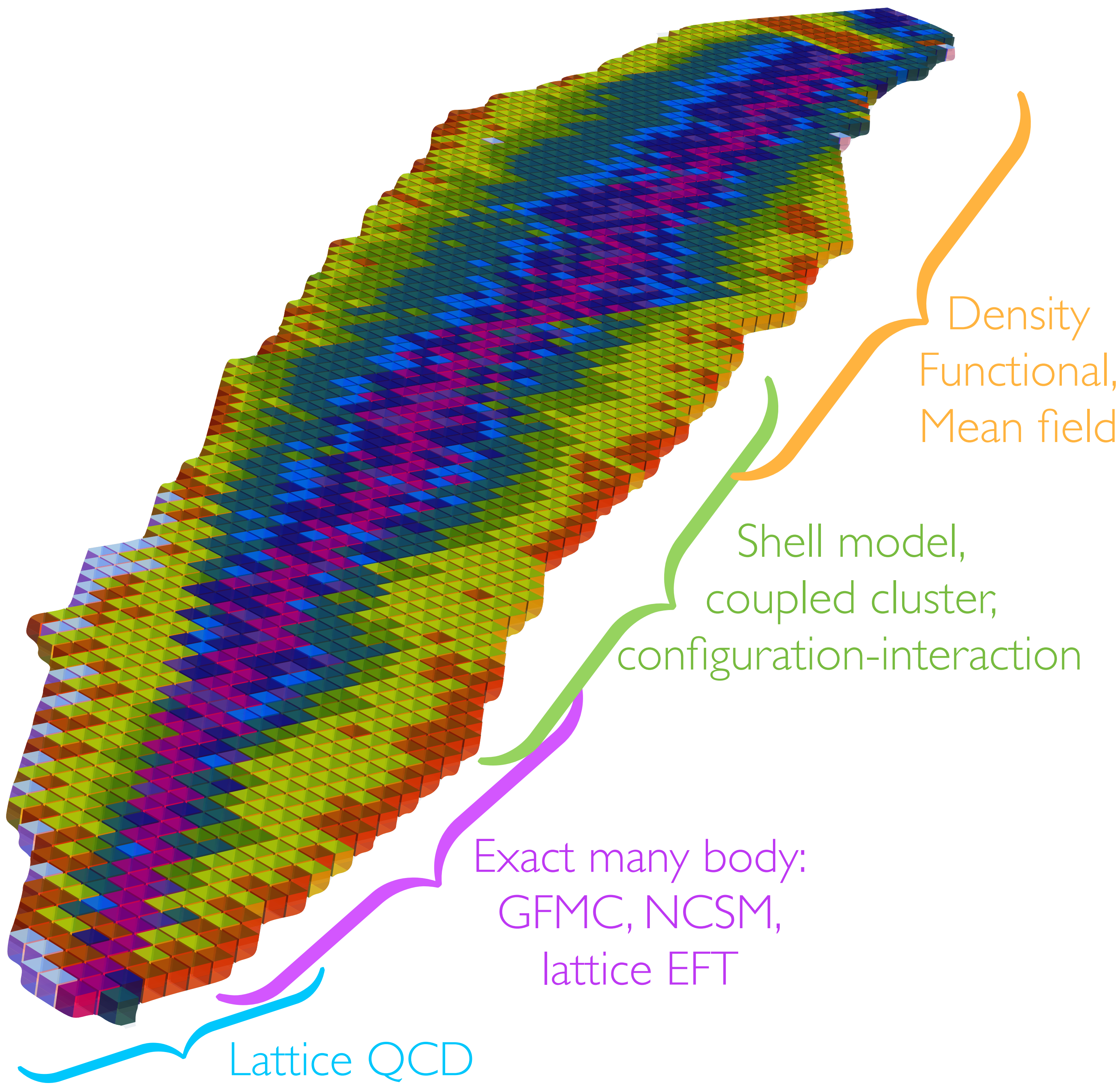
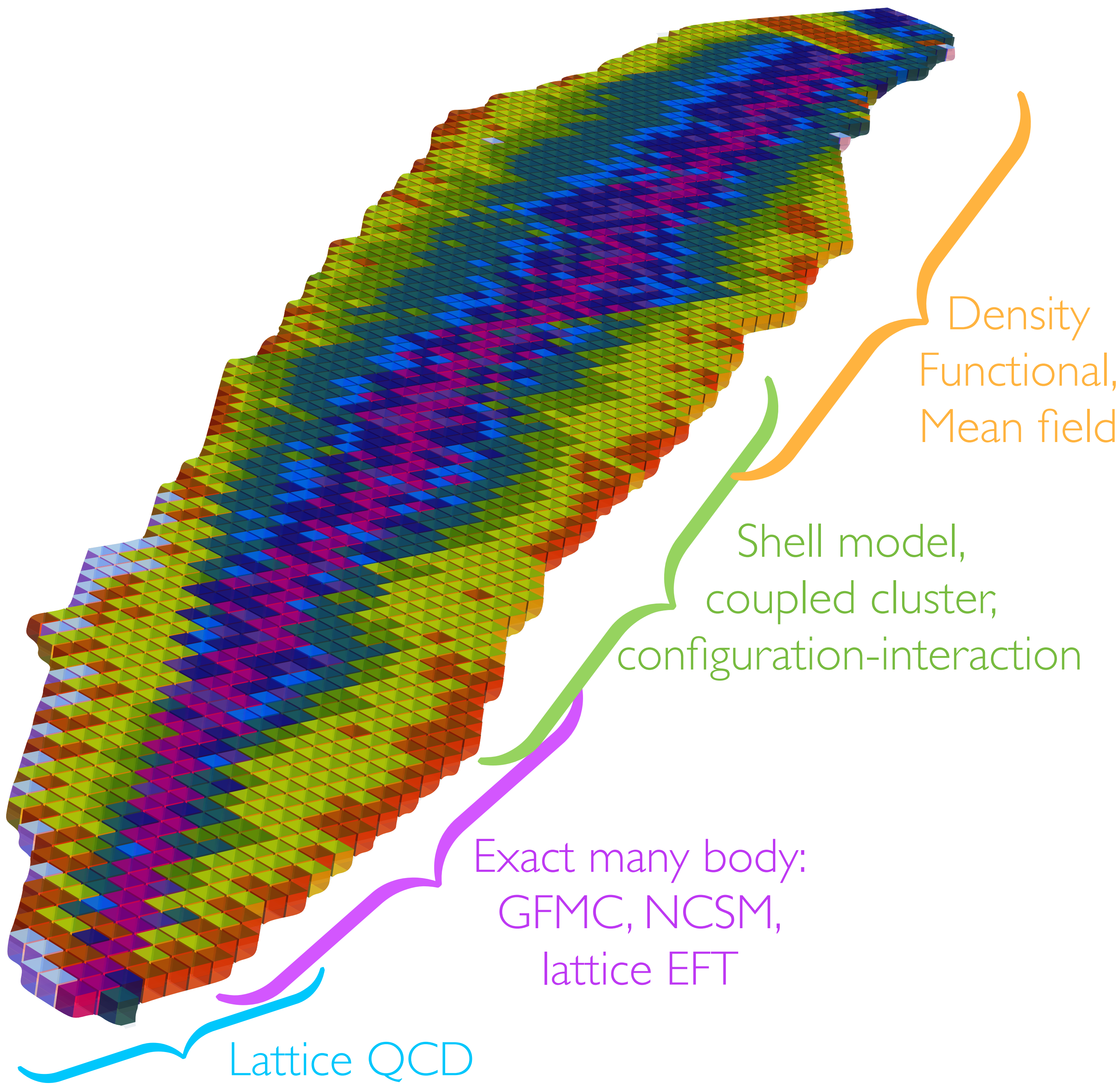
LQCD in nuclear physics



LQCD in nuclear physics

- Very difficult to explore all of NP from QCD



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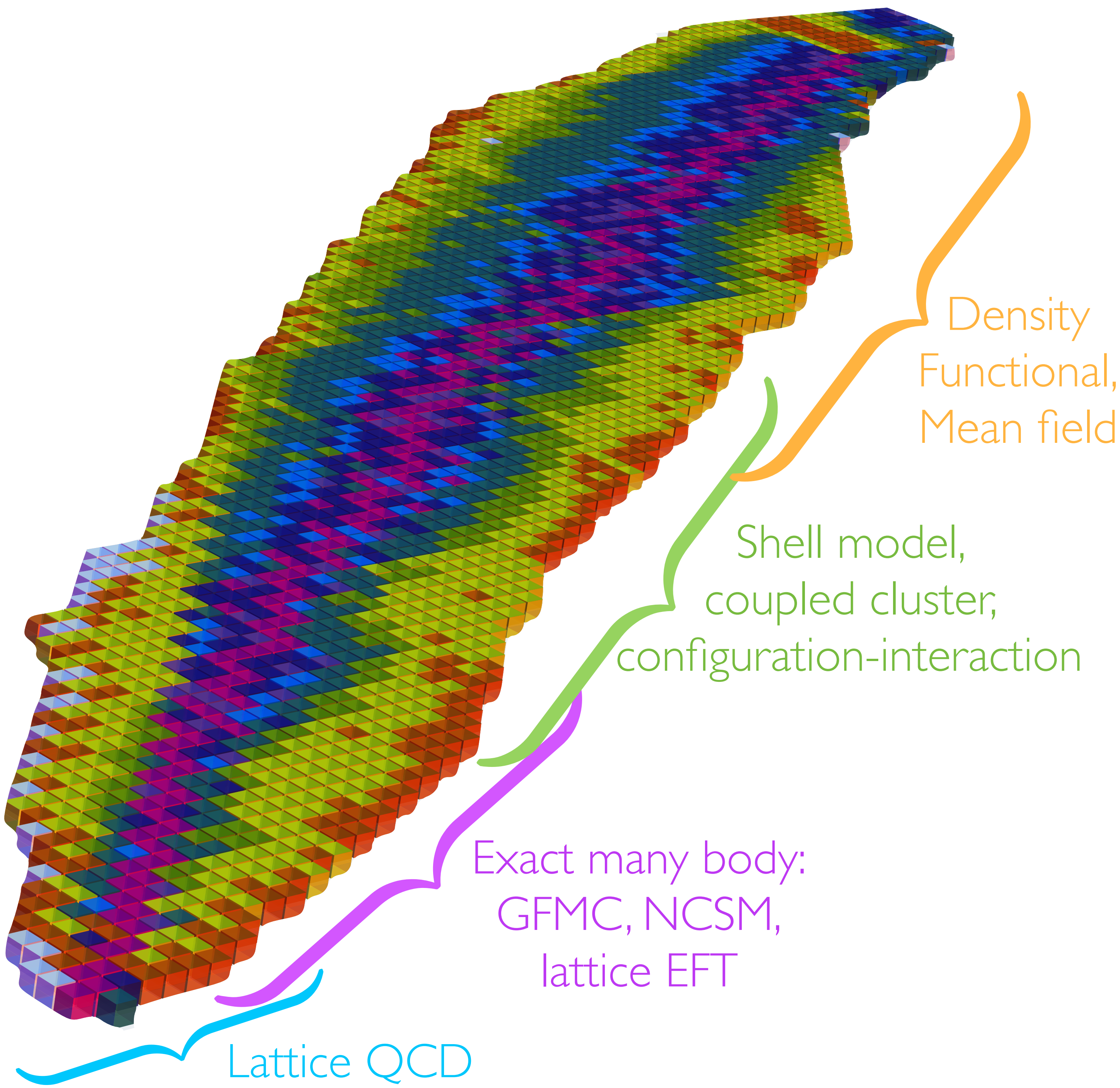

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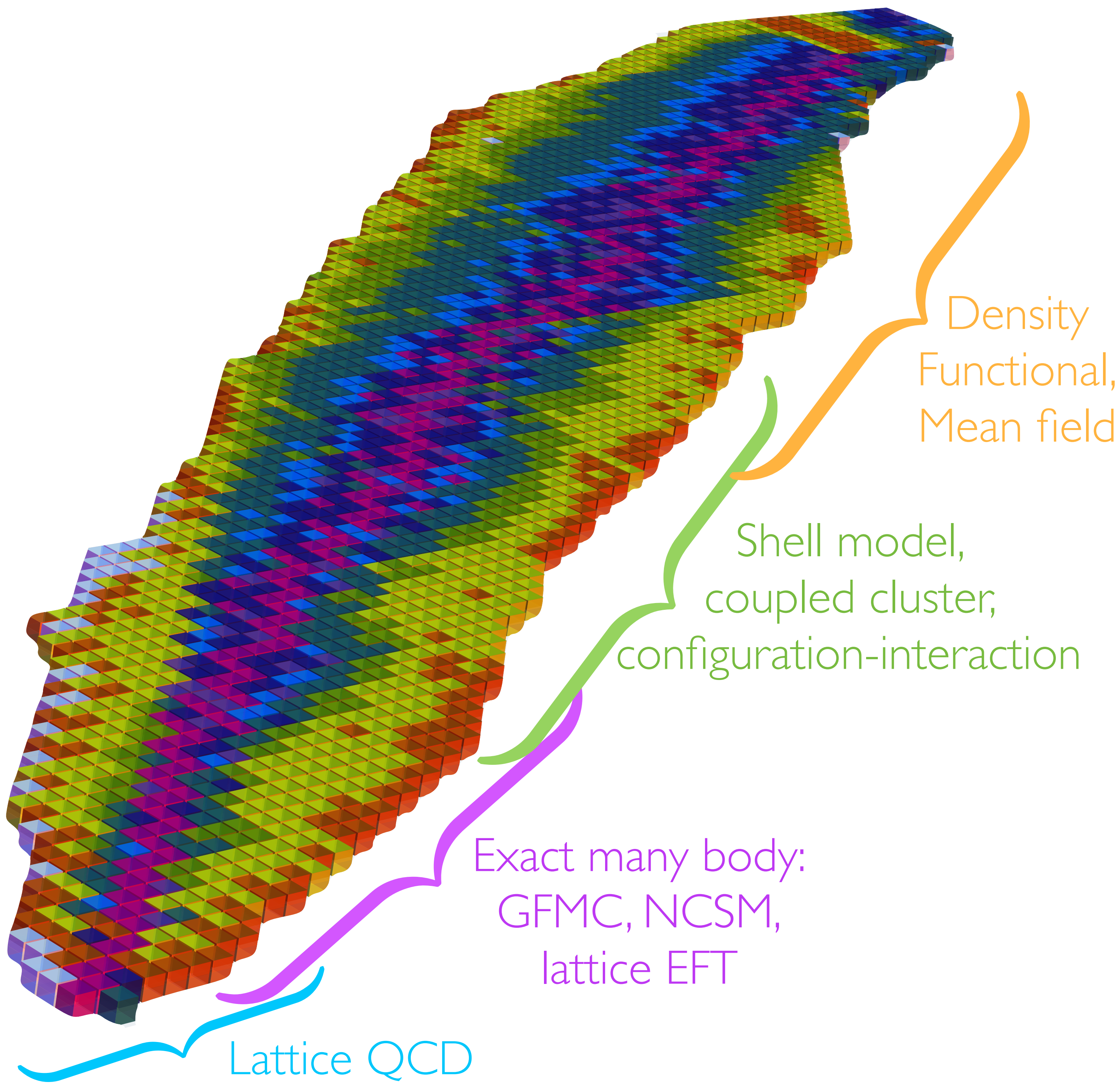

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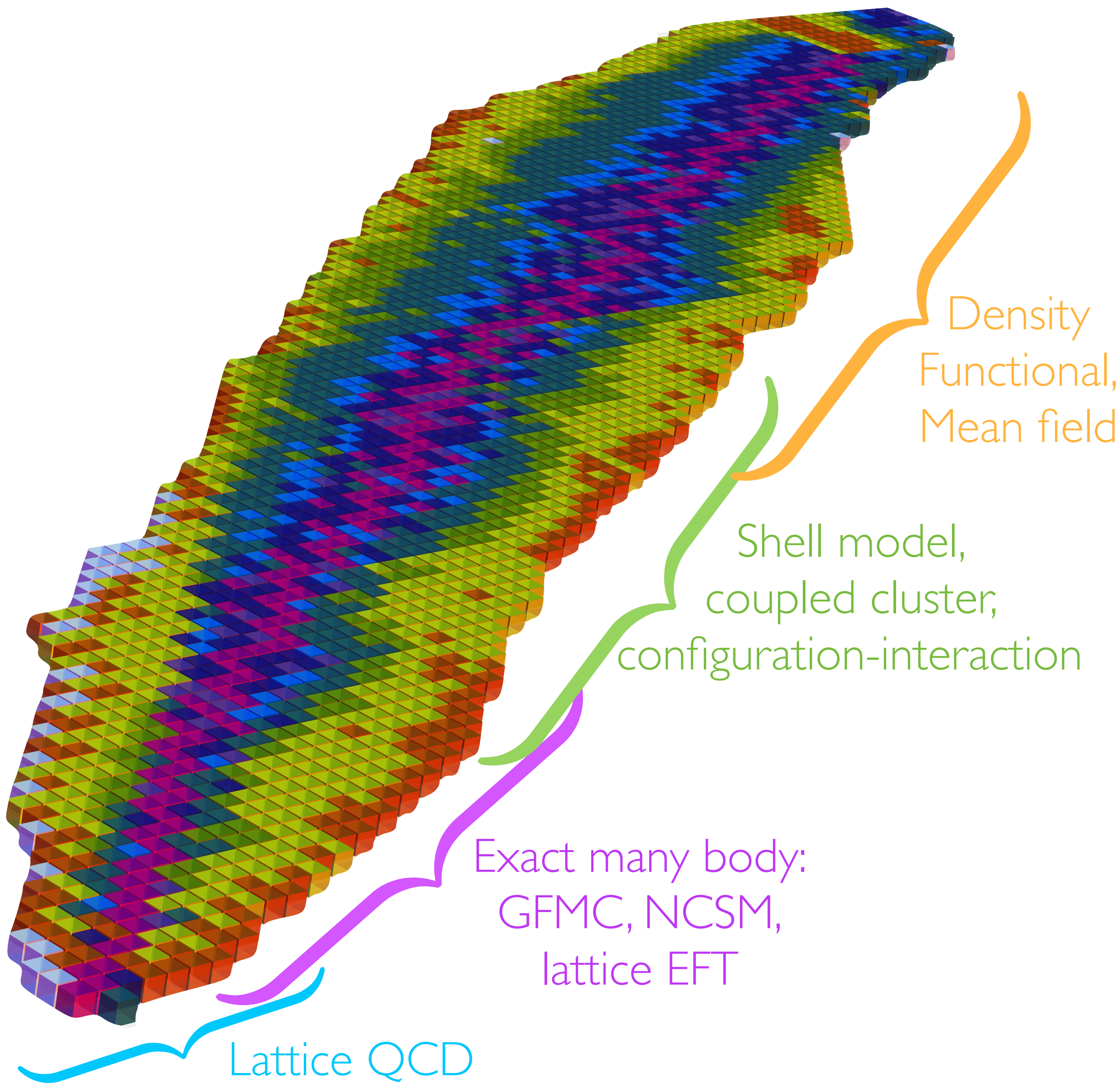

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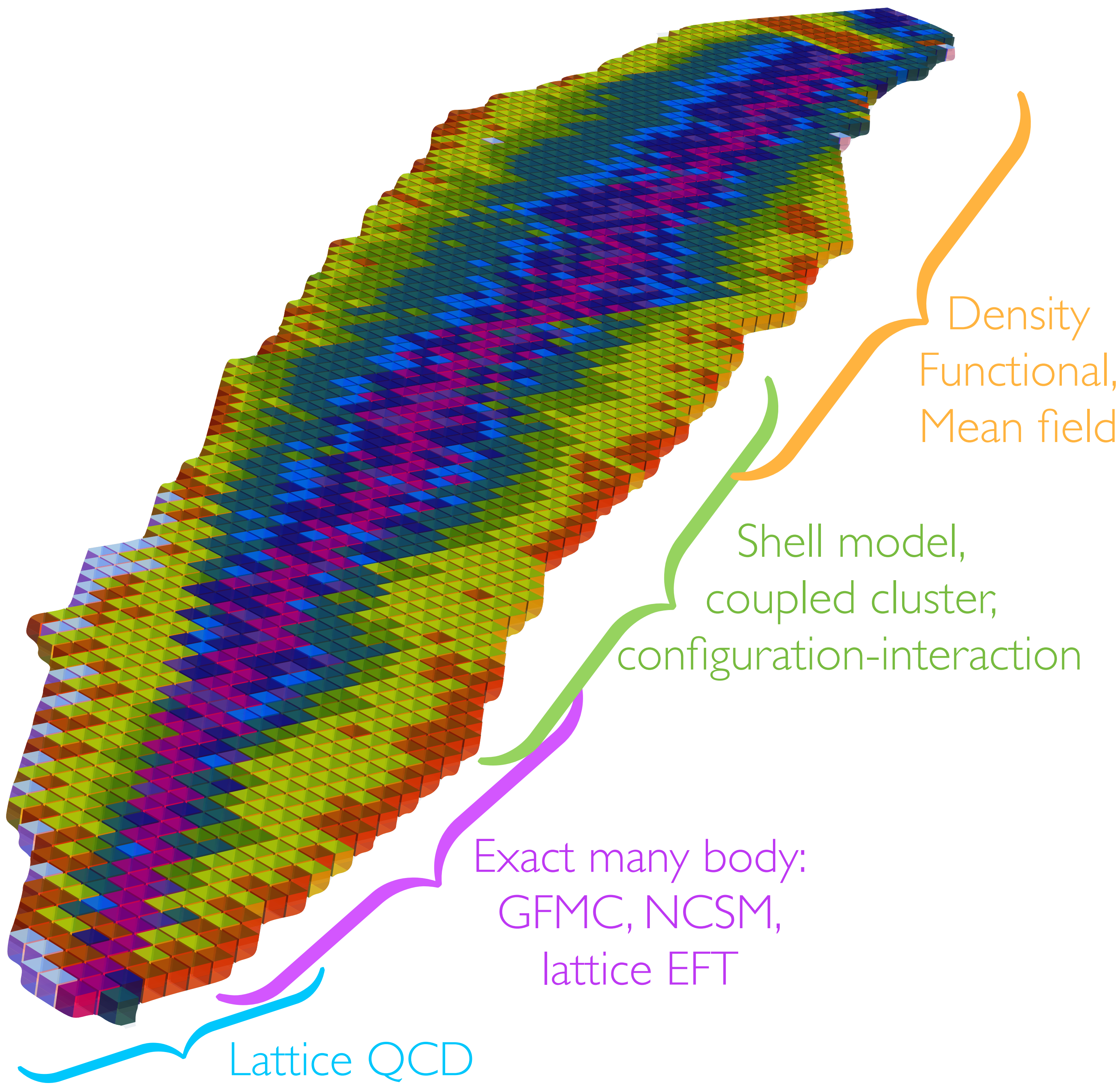

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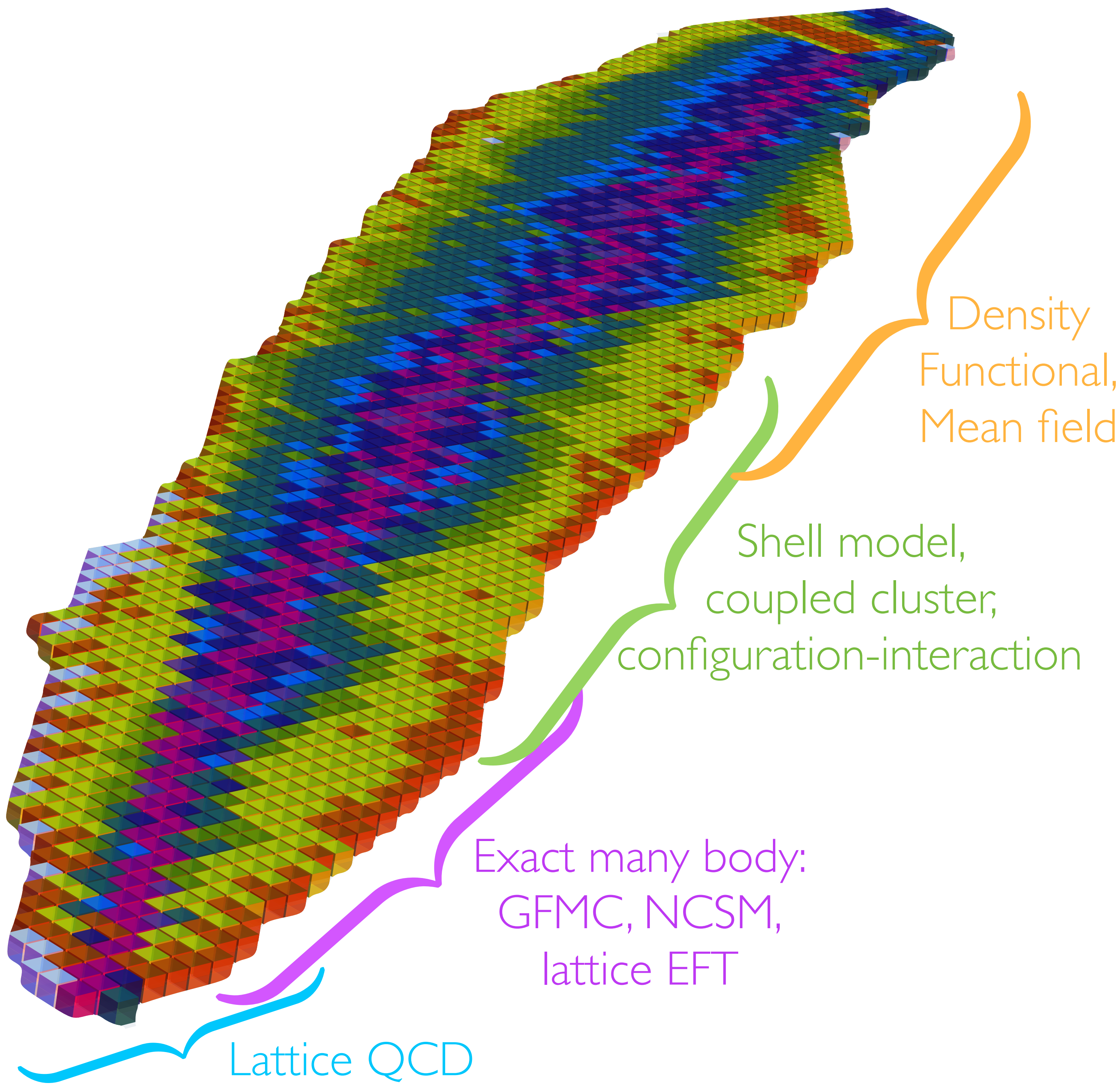

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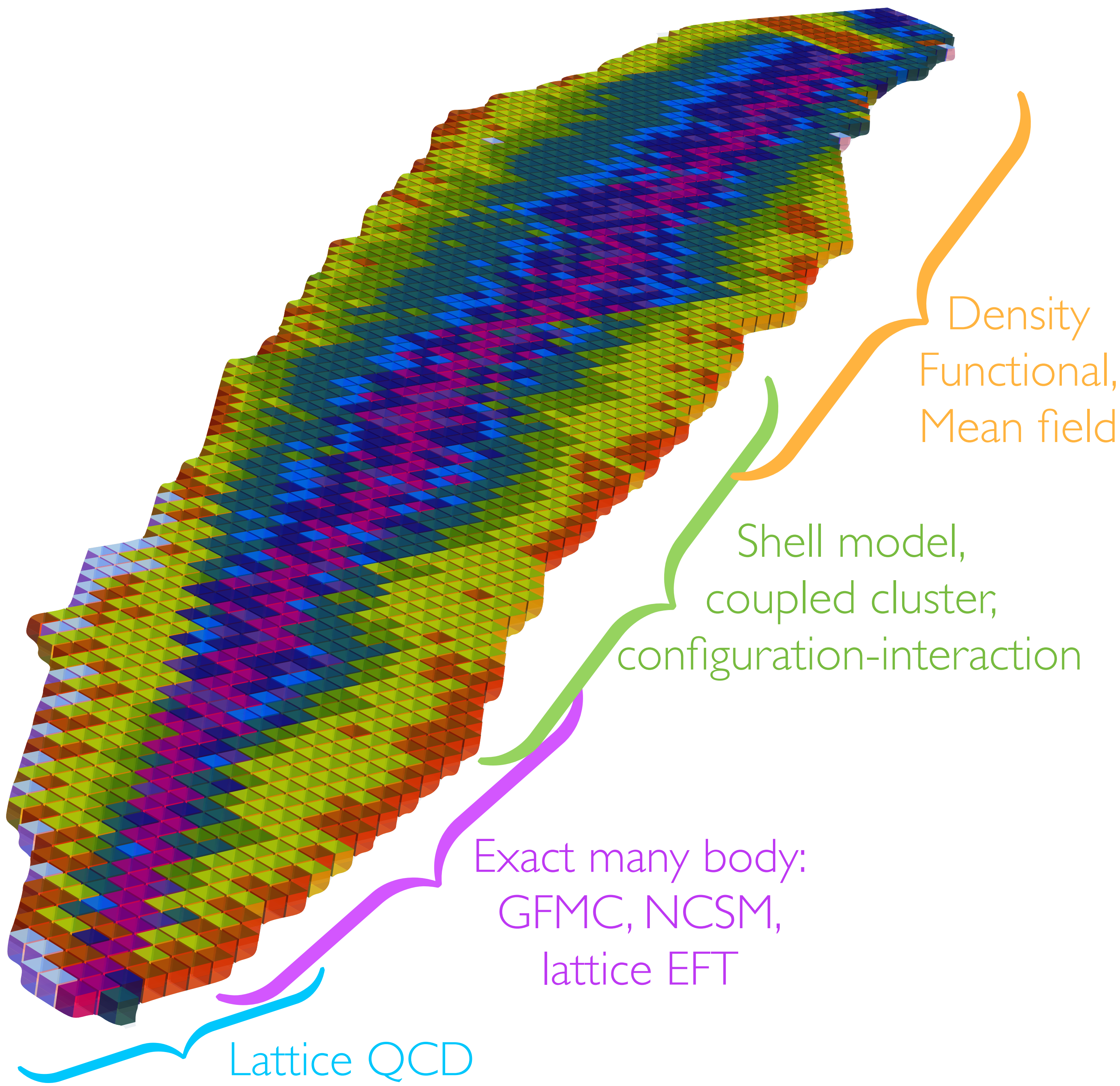
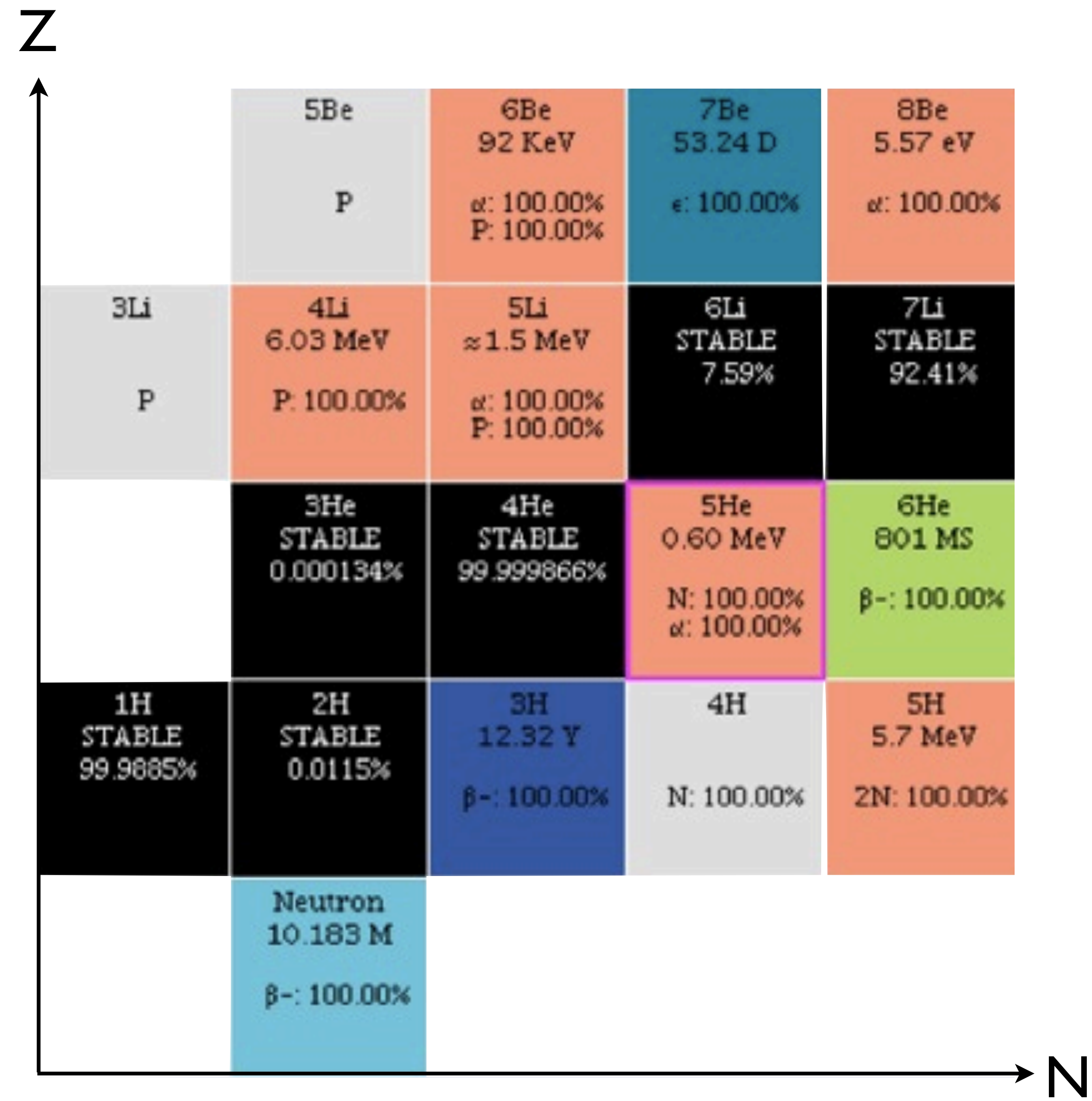
  - ... for now ...



Few nucleon systems

- Zoom in on few nucleon systems
- Nucleons are J=1/2, I=1/2 state
- Two body: I,J =0,1
  - isosinglet, spin triplet: deuteron
  - isotriplet, spin singlet: nn, np, pp unbound
- Three body: ^{3}He stable, triton (^{3}H) almost stable
- Four body: alpha particle (^{4}He)
- No stable 5,8 body systems



HW: what would happen to the above chart if we turned off EM and weak interactions?

LQCD in nuclear physics

- *What is to be gained from ab initio nuclear physics (ie from QCD)?*
    - QCD on the same footing as QED: known physics, just calculate!
    - Nuclear physics: lots of puzzles to be solved
        - Many things that current nuclear theory gets wrong or can't constrain: eg A_y, YN scattering, high density matter
    - Standard Model predictions
        - Comparison to experiment: in principle test QCD (or expt!)
        - Predictions without experiment: reliably calculate quantities where experiments are unavailable ortoo expensive
    - Beyond the Standard Model
        - Better constrain test of the SM: eg. V_{ud} from superallowed nuclear β-decays
        - Exploration of alternate universes where quark masses, charges etc differ

Basic scattering theory

- Two particles scattering described by one-body potential in relative coordinate (consider simplest case where V depends only on positions but not gradients)

$$L = \frac{1}{2}m_A\dot{\mathbf{r}}_A^2 + \frac{1}{2}m_B\dot{\mathbf{r}}_B^2 - V(|\mathbf{r}_A - \mathbf{r}_B|)$$

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \underbrace{\frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(|\mathbf{r}|)}$$

$$H_{\text{int}} = \frac{\mathbf{p}^2}{2\mu} - V(|\mathbf{r}|)$$

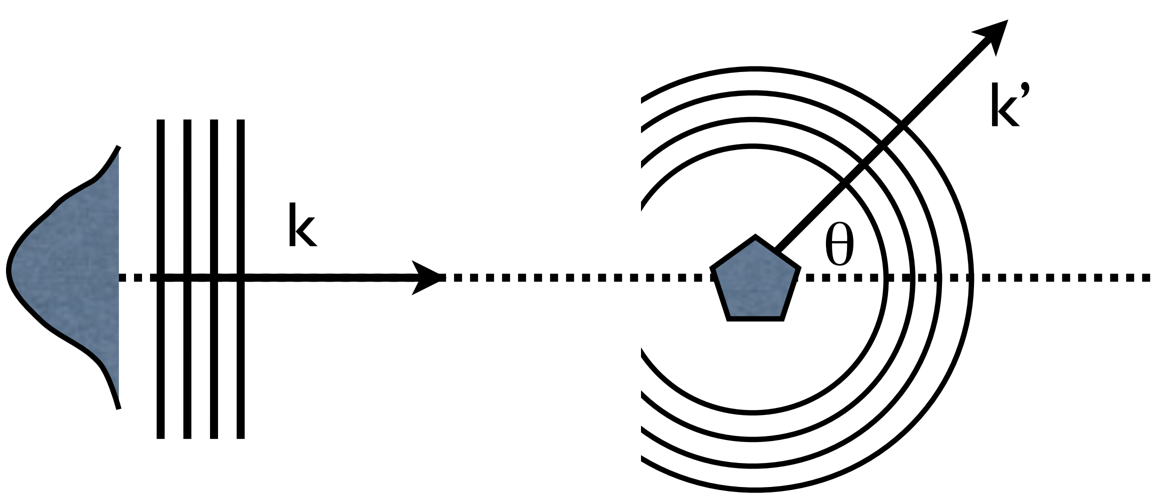
$$\mathbf{R} = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B}{m_A + m_B}$$

$$\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$$

$$M = m_A + m_B$$

$$\mu = \frac{m_A m_B}{m_A + m_B}$$

- Consider a wave packet localised at **x**=-∞ at t=-∞ incident on a scattering centre at **x**=0
- Aim to determine what outgoing state is at t→∞ (probability that the wave packet ends up traveling at some angle w.r.t. incoming direction
- Simplest approach is to decompose the wave packet into plane-waves, study plane-wave scattering and then convolve with wave-packet amplitudes at the end (if needed)
- Hopefully a review



Scattering in a central potential

- Incident wave function is then

$$\psi_k(r) = e^{ikz} = e^{ikr\cos\theta}$$

- Decompose in angular momentum modes

$$\psi_k(r) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

- Spherical Bessel functions j_l and Legendre polynomials P_l
- Satisfies the non-relativistic Schrödinger equation with no potential
- Now consider a central potential (no angular dependence) V(r)
- Schrödinger equation for a given E

$$\hat{H}\psi = E\psi$$

$$\left[-\frac{1}{2\mu}\nabla^2 + V(r)\right]\psi(\mathbf{r}) = -E\psi(\mathbf{r})$$

- NB: wf depends on E

Scattering in a central potential

- Since potential is central, we can separate the radial and angular dependence of the wf (sum over l)

$$\psi(\mathbf{r}) = \frac{u_l(r)}{r} Y_l^m(\Omega)$$

- Now ∇^2 can be written in spherical coords as (we knew this already to write $Y_l^m(\boldsymbol{\Omega})$ above)

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

- The radial function satisfies

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + 2\mu V(r) - k^2\right] u_l(r) = 0$$

  where $k^2 = 2\mu E$

- General solution in terms of spherical Bessel (j_l) and Neumann (n_l) functions

- Finite solution everywhere requires $u_l(r \to 0) \to 0$

$$u_l(r) = r\left[A_l j_l(kr) + B_l n_l(kr)\right]$$

Scattering in a central potential

- General solution

$$u_l(r) = r\left[A_l j_l(kr) + B_l n_l(kr)\right]$$

- Boundary condition at the origin implies that B_ℓ=0 if V=0
- For V≠0, we can define $\tan\delta_\ell = -\frac{B_\ell}{A_\ell}$

$$u_\ell(r) = rA_l\left[j_\ell(kr) - \tan\delta_l\ n_l(kr)\right]$$

- Normalising to unit coefficient, the asymptotic form is

$$u_\ell(r \gg \ell) \sim \frac{1}{k}\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)$$

- So the outgoing scattered solution is a linear combinations of the solutions

$$\psi^+ = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta)\frac{u_\ell(r)}{r} \quad \xrightarrow{r\to\infty} \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta)\frac{\sin\left(kr - \ell\pi/2 + \delta_\ell\right)}{kr}$$

for unknown coefficients a_ℓ and phases (phase shifts) δ_ℓ

Scattering in a central potential

- We match this onto the expectation of incoming plane waves and outgoing scattered plane waves at large distance

$$\psi^+ = e^{ikr\cos\theta} + f(\theta)\frac{e^{ikr}}{r} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell \frac{\sin(kr - l\pi/2)}{kr} P_\ell(\cos\theta) + f(\theta)\frac{e^{ikr}}{r}$$

- This leads to

$$a_\ell = (2\ell+1)i^l e^{i\delta_\ell}$$

- and hence

$$\psi^+ = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell e^{i\delta_\ell} P_\ell(\cos\theta)\frac{\sin(kr - l\pi/2 + \delta_\ell)}{kr}$$

- Finally matching the outgoing plane wave components gives the angular dependence

$$f(\cos\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta_\ell} - 1)P_\ell(\cos\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$$

Scattering flux and cross section

- Probability current density (QM)

$$\mathbf{j} = \frac{1}{\mu}\mathrm{Im}\left(\psi^* \nabla \psi\right)$$

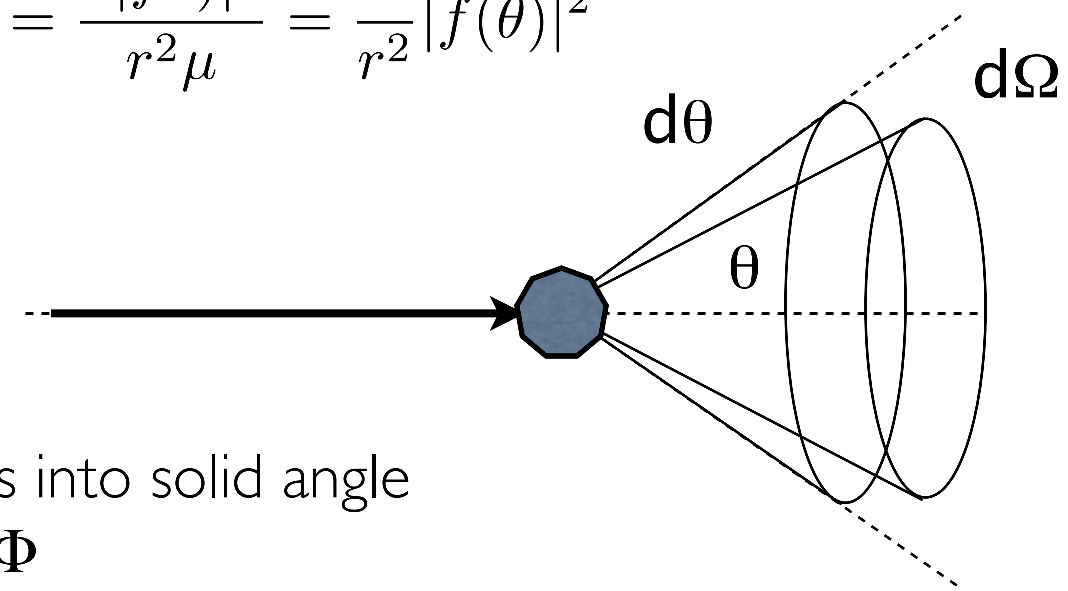
- Ingoing and outgoing components

$$\mathbf{j}_{\mathrm{in}} = \frac{1}{\mu}\mathrm{Im}\left(e^{-ikz}\frac{d}{dz}e^{ikz}\right) = \frac{k}{\mu} = v$$

$$\mathbf{j}_{\mathrm{out}} = \frac{1}{\mu}\mathrm{Im}\left(f^*(\theta)\frac{e^{-ikr}}{r}\frac{\partial}{\partial r}\left[f(\theta)\frac{e^{ikr}}{r}\right]\right) = \frac{k|f\theta)|^2}{r^2\mu} = \frac{v}{r^2}|f(\theta)|^2$$

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{n\Phi}\frac{dN}{d\Omega}$$



where *dN* is the number of observed scatterings into solid angle *d*$\mathbf{\Omega}$ from *n* scattering centres with incident flux $\mathbf{\Phi}$

- Thus this is given in terms of f = the scattering amplitude

$$\frac{d\sigma}{d\Omega} = \frac{\mathbf{j}_{\mathrm{out}} r^2}{\mathbf{j}_{\mathrm{in}}} = |f(\theta)|^2$$

Scattering cross section

- Differential cross section

$$\frac{d\sigma}{d\Omega} = |f(\cos\theta)|^2$$
$$= \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)\right|^2$$

- Legendre polynomials are orthogonal

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

- Thus the total cross section is

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell = \sum_{\ell=0}^{\infty}\sigma_\ell$$

- Partial waves bounded above

$$\sigma_\ell \leq \frac{4\pi}{k^2}(2\ell+1)$$

Low energy s-wave scattering

- At very low energy, can focus on s-wave (**ℓ**=0)

$$\sigma = \sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \qquad \left[V_{\rm eff}^{(\ell)}(r) = V(r) + \frac{\ell(\ell+1)}{2\mu r^2}\right]$$

- For finite range potential, cross section must be bounded for all k so the phase shift must vanish at least as fast as $\delta_0 \to -k\ a$ (where a is a constant)

- So for small energy,

$$\sigma \stackrel{k\to 0}{\longrightarrow} 4\pi a^2$$

where a is called the scattering length

- In this limit, the radial wave function behaves as

$$u_0(r) = \frac{\sin(kr+\delta_0)}{k} \xrightarrow{k\to 0} r - a \qquad \left[\ \frac{d^2 u_0(r)}{dr} = 0 \quad \text{for } r > R_V\ \right]$$

- So the scattering length is interpreted as the radius at which the solution vanishes for low energy

- The only information learnt from low energy scattering is the scattering length

Effective range expansion

- If the momentum of the scattering plane wave is small compared to the inverse radius on the potential, we can preform a Taylor expansion of the phase shift
  - Coefficients will encode information about the potential/wave-function
- The scattering amplitude

$$f(\theta) = \frac{1}{k} e^{i\delta} \sin\delta = \frac{1}{k} \frac{\sin\delta}{\cos\delta - i\sin\delta} = \frac{1}{k\ \cot\delta - i\ k}$$

- Taylor expand phase shift

$$k\ \cot\delta(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + r_1 k^4 + \ldots$$

where r_0 is the effective range and r_1 is called the first shape parameter

- Radius of convergence set by the inverse range of the potential
- Reproduces the definition of the scattering length for vanishing k

General two-body potentials

- The most general form of a potential is non-local: the action of the interaction at a point *r* depends on all other points

$$\langle \vec{r}|\widehat{V}|\psi\rangle = \int d^3r' \langle \vec{r}|\widehat{V}|\vec{r}'\rangle\langle \vec{r}'|\psi\rangle = \int d^3r \; V(\vec{r},\vec{r}')\psi(\vec{r}')$$

- Separable potentials satisfy: $V(\vec{r},\vec{r}') = f^*(\vec{r})f(\vec{r}')$

- Local potentials depend only a single point

$$V(\vec{r},\vec{r}') = V(\vec{r})\delta^{(3)}(\vec{r}-\vec{r}')$$

- What can the general structure of a two-body potential be?

  - Built from available degrees of freedom: positions, momenta, spins and isospins of the two particles: r_i, p_i, $\boldsymbol{\sigma}_i$, $\boldsymbol{\tau}_i$ for i=1,2

$$V = V(\vec{r}_i, \vec{p}_i, \sigma_i, \tau_i; i=1,2)$$

  - Constrained by appropriate symmetries

  - Should be a scalar (i.e. no free indices)

General two-body potentials

- Work in terms of relative and centre of mass coordinates

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad \vec{R} = \tfrac{1}{2}(\vec{r}_1 - \vec{r}_2), \quad \vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2), \quad \vec{P} = \vec{p}_1 + \vec{p}_2.$$

$$V = V(\vec{r}, \vec{p}, \vec{R}, \vec{P}, \sigma_i, \tau_i; i = 1, 2)$$

- Translations and boost

$$\vec{r}_i \to \vec{r}_i + \vec{a} \Longrightarrow \vec{r} \to \vec{r}, \quad \vec{R} \to \vec{R} + \vec{a}$$
$$\vec{p}_i \to \vec{p}_i + \vec{b} \Longrightarrow \vec{p} \to \vec{p}, \quad \vec{P} \to \vec{P} + \vec{2}b$$

invariance requires

$$V(\ldots, \vec{R} + \vec{a}, \vec{P} + 2\vec{b}, \ldots) = V(\ldots, \vec{R}, \vec{P}, \ldots) \quad \forall\ \vec{a}, \vec{b}$$

so

$$V = V(\vec{r}, \vec{p}, \sigma_i, \tau_i; i = 1, 2)$$

- Should be invariant under parity, time-reversal and particle interchange 1↔2
    - Greatly restricts allowed structures

General two-nucleon potentials

- The most general form of the potential between two spin 1/2, isospin 1/2 fermions is (each V is a function of r^2,p^2,L^2)

$$
\begin{aligned}
V = {} & V_1 + V_\tau \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + V_\sigma \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + V_{\sigma\tau} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \\
& + V_{LS} \mathbf{L} \cdot \mathbf{S} + V_{LS\tau} \mathbf{L} \cdot \mathbf{S} \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \\
& + V_T S_{12} + V_{T\tau} S_{12} \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \\
& + V_Q (\mathbf{L} \cdot \mathbf{S})^2 + V_{Q\tau} (\mathbf{L} \cdot \mathbf{S})^2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \\
& + V_{pT} S_{12}^p + V_{pT\tau} S_{12}^p \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2
\end{aligned}
$$

  where $S_{12} = 3\boldsymbol{\sigma}_1 \cdot \hat{r} \boldsymbol{\sigma}_2 \cdot \hat{r} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ and $S_{12}^p = S_{12}(\hat{r} \rightarrow \hat{p})$

- Five rows correspond to central, spin-orbit, tensor, quadratic LS, momentum-tensor (usually dropped as not accessible in elastic scattering) potentials

  - spin-orbit, quadratic LS, and momentum-tensor potentials are non-local

  - NB: spin-orbit term only connects states of the same L

Example: Reid soft core potential

- A simple phenomenological successful potential is the Reid soft-core potential
- Up to a few minor details, for each total spin, isospin, L<3, take

$$V = V_C(\mu r) + V_{12}(\mu r) S_{12} + V_{LS}(\mu r) \mathbf{L} \cdot \mathbf{S}$$

where

$$V_C(x) = \sum_{n=1}^{n_{max}} a_n \frac{e^{-nx}}{x}, \qquad V_{LS}(x) = \sum_{n=1}^{n_{max}} c_n \frac{e^{-nx}}{x},$$

$$V_{12}(x) = \frac{b_1}{x} \left[\left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) e^{-x} - \left(\frac{b_0}{x} + \frac{1}{x^2} \right) e^{-b_0 x} \right] + \sum_{n=2}^{n_{max}} b_n \frac{e^{-nx}}{x},$$

- Contains many parameters that can be determined by performing fits to the NN scattering data (about 4000 data points)

Example: AV$_{18}$ potential

Probably the "best" available potential - but takes 4 journal pages and 40 parameters to define!!

PHYSICAL REVIEW C VOLUME 51, NUMBER 1 JANUARY 1995

Accurate nucleon-nucleon potential with charge-independence breaking

R. B. Wiringa
Physics Division, Argonne National Laboratory, Argonne, Illinois 60439

V. G. J. Stoks
School of Physical Sciences, The Flinders University of South Australia, Bedford Park, South Australia 5042, Australia

R. Schiavilla
CEBAF Theory Group, Newport News, Virginia 23606
and Department of Physics, Old Dominion University, Norfolk, Virginia 23529

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We present a new high-quality nucleon-nucleon potential with explicit charge dependence and charge asymmetry, which we designate Argonne v_{18}. The model has a charge-independent part with 14 operator components that is an updated version of the Argonne v_{14} potential. Three additional charge-dependent and one charge-asymmetric operators are added, along with a complete electromagnetic interaction. The potential has been fit directly to the Nijmegen pp and np scattering database, low-energy nn scattering parameters, and deuteron binding energy. With 40 adjustable parameters it gives a χ^2 per datum of 1.09 for 4301 pp and np data in the range 0–350 MeV.

PACS number(s): 13.75.Cs, 12.39.Pn, 21.30.+y

I. INTRODUCTION

Traditionally, nucleon-nucleon (NN) potentials are constructed by fitting np data for $T = 0$ states and either np or pp data for $T = 1$ states. Examples of potentials fit to np data in all states are the Argonne v_{14} [1], Urbana v_{14} [2], and most of the Bonn potentials [3,4]. In contrast, the Reid [5], Nijmegen [6], and Paris [7] potentials were fit to pp data for $T = 1$ channels. Unfortunately, potential models which have been fit only to the np data often give a poor description of the pp data [8], even after applying the necessary corrections for the Coulomb interaction. By the same token, potentials fit to pp data in $T = 1$ states give only a mediocre description of np data. Fundamentally, this problem is due to charge-independence breaking in the strong interaction.

In the present work we construct an updated version of the Argonne potential that fits both pp and np data, as well as low-energy nn scattering parameters and deuteron properties. The strong interaction potential is written in an operator format that depends on the values of S, T, and T_z of the NN pair. We then project the potential into a charge-independent (CI) part that has 14 operator components (as in the older Argonne v_{14} model) and a charge-independence breaking (CIB) part that has three charge-dependent (CD) and one charge-asymmetric (CA) operators. We also include a complete electromagnetic potential, containing Coulomb, Darwin-Foldy, vacuum polarization, and magnetic moment terms with finite-size effects. We designate the new model Argonne v_{18}.

In a number of applications it is important for a NN potential to reproduce correct np and pp scattering parameters. For example, in thermal neutron radiative capture on the proton, $^1H(n,\gamma)^2H$, it is crucial to have the correct singlet np scattering length in the initial state to get the cross section. However, in low-energy proton weak capture, $^1H(p,e^+\nu_e)^2H$, it is equally important that the correct pp scattering length be provided by the interaction. Clearly, a complete potential model should meet both requirements.

Another important application is in the formulation of three-nucleon (NNN) potentials. In general, nuclei are underbound using only NN potentials fit to the scattering data. Nontrivial many-nucleon interactions are expected to make up a portion of the missing binding energy. Phenomenologically we may choose to construct a many-body Hamiltonian, such as

$$H = \sum_i \frac{-\hbar^2}{2M_i}\nabla_i^2 + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} , \qquad (1)$$

and constrain the strength parameters of the NNN potential by requiring that H gives the correct trinucleon binding energy. Similar considerations apply if we choose a relativistic formulation. Clearly, such constraints are ambiguous or even meaningless if the NN potential used in the calculations does not adequately describe the two-nucleon data. For ^3He (^3H), in which the NN interaction underbinds by $\sim$1 MeV, there are two np pairs and one pp (nn) pair. To a good approximation, the two np pairs will be in the $S = 1$, $T = 0$ state 75% of the time, and in the $S = 0$, $T = 1$ state 25% of the time, while the pp (nn) pair will be pure $S = 0$, $T = 1$. If the chosen NN potential fits only the more repulsive pp (nn) data in the $T = 1$ state, we would get a smaller NN contribution to the binding energy and thus overestimate the NNN potential strength required. By the same token, a model fit to np data in the $T = 1$ state would be too attractive and we would underestimate the NNN potential. The difference can be as much as 0.4 MeV, leading to variations in

the NNN potential strength of order ±20%. This would have significant effects in larger many-body systems.

Because we include a complete electromagnetic potential and fit low-energy nn scattering parameters, the present model also can be used to study charge-symmetry breaking, as in the ^3H-^3He mass difference [9], or more generally the Nolen-Schiffer anomaly [10]. The electromagnetic potential is in principle well known and is the longest-range part of the interaction. Potential models commonly fit the deuteron energy to better than 1 keV accuracy. Since we find that the electromagnetic terms give a non-negligible 18 keV repulsion in the deuteron and moderate shifts in the np and nn scattering lengths, we deem it desirable to include these terms explicitly.

The major goal of the present work is to construct a nonrelativistic potential that can be used easily in nuclear many-body calculations and that accurately fits both pp and np data. We adopt the local operator structure of the older Argonne v_{14} and Urbana v_{14} potentials, which have been used extensively in calculations of finite nuclei, nuclear matter, and neutron stars [11–13]. The assumption of an underlying operator structure relates all partial waves in a simple manner, without imposing a one-boson-exchange (OBE) form which might be too restrictive at short distances. Recently, the Nijmegen group has shown [14] that it is feasible to construct potential models which fit the NN data with the almost perfect χ^2 per datum of 1. However, these models differ in each partial wave and thus implicitly introduce nonlocalities from one partial wave to the next that may be difficult to characterize and treat accurately in many-body calculations. When they limit the potential to an OBE form, which has a local operator structure (save for a nonlocal part in the central potential) describing all partial waves simultaneously, the χ^2 per datum increases to 1.87, albeit with a much smaller number of parameters. The present model is a compromise between these two approaches, adopting a phenomenological form (unrestricted by an OBE picture) at short distances, but maintaining a local operator structure. The potential was directly fit to the Nijmegen NN scattering database [15,16], which contains 1787 pp and 2514 np data in the range 0–350 MeV, and has an excellent χ^2 per datum of 1.09.

In Sec. II we present the analytical form of the potential in the various spin and isospin states. Special attention is given to the electromagnetic part of the interaction. The free parameters are fit to the NN scattering data and deuteron binding energy in Sec. III, where we also present the phase shifts. Section IV discusses the projection of the potential into operator format. Static deuteron properties and electromagnetic form factors, with relativisitic and exchange current contributions, are presented in Sec. V. Conclusions and an outlook are given in Sec. VI.

II. FORM OF THE POTENTIAL IN S,T,T_z STATES

The NN potential is written as a sum of an electromagnetic (EM) part, a one-pion-exchange (OPE) part, and an intermediate- and short-range phenomenological part:

$$v(NN) = v^{EM}(NN) + v^{\pi}(NN) + v^{R}(NN) . \qquad (2)$$

The EM interaction is the same as that used in the Nijmegen partial-wave analysis, with the addition of short-range terms and finite-size effects [17–19]. (Values for the masses and other physical constants used in the following formulae are given in Table I.) For pp scattering we include one- and two-photon Coulomb terms, the Darwin-Foldy term, vacuum polarization, and the magnetic moment interaction, each with an appropriate form factor:

$$v^{EM}(pp) = V_{C1}(pp) + V_{C2} + V_{DF} + V_{VP} + V_{MM}(pp) . \qquad (3)$$

Here

$$V_{C1}(pp) = \alpha' \frac{F_C(r)}{r} , \qquad (4)$$

$$V_{C2} = -\frac{\alpha}{2M_p^2}\left[(\nabla^2 + k^2)\frac{F_C(r)}{r} + \frac{F_C(r)}{r}(\nabla^2 + k^2)\right] \approx -\frac{\alpha\alpha'}{M_p}\left[\frac{F_C(r)}{r}\right]^2 , \qquad (5)$$

$$V_{DF} = -\frac{\alpha}{4M_p^2} F_\delta(r) , \qquad (6)$$

$$V_{VP} = \frac{2\alpha\alpha'}{3\pi}\frac{F_C(r)}{r}\int_1^\infty dx\, e^{-2m_e r x}\left[1 + \frac{1}{2x^2}\right]\frac{(x^2-1)^{1/2}}{x^2} , \qquad (7)$$

$$V_{MM}(pp) = -\frac{\alpha}{4M_p^2}\mu_p^2\left[\frac{2}{3}F_\delta(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{F_t(r)}{r^3}S_{ij}\right] - \frac{\alpha}{2M_p^2}(4\mu_p - 1)\frac{F_{ls}(r)}{r^3}\mathbf{L}\cdot\mathbf{S} . \qquad (8)$$

Example: AV$_{18}$ potential

Probably the "best" available potential - but takes 4 journal pages and 40 parameters to define!!

TABLE I. Values of fundamental constants adopted in this work.

$\hbar c$	197.32705	MeV fm
m_{π^0}	134.9739	MeV/c^2
$m_{\pi^\pm}$	139.5675	MeV/c^2
M_p	938.27231	MeV/c^2
M_n	939.56563	MeV/c^2
α^{-1}	137.03599	
μ_p	2.79285	μ_0
μ_n	−1.91304	μ_0

The Coulomb interaction includes an energy dependence through the $\alpha' \equiv 2k\alpha/(M_p v_{\rm lab})$ [20], which is significantly different from α at even moderate energies ($\sim 20\%$ difference at $T_{\rm lab} = 250$ MeV). The vacuum polarization and two-photon Coulomb interaction are important for fitting the high-precision low-energy scattering data. The F_C, F_δ, F_t, and F_{ls} are short-range functions that represent the finite size of the nucleon charge distributions. They have been obtained under the assumption that the nucleon form factors are well represented by a dipole form

$$G_E^p = \frac{G_M^p}{\mu_p} = \frac{G_M^n}{\mu_n} = G_D = \left(1+\frac{q^2}{b^2}\right)^{-2}, \qquad (9)$$

where $b = 4.27$ fm^{-1}. The functions are given by

$$F_C(r) = 1 - \left(1 + \frac{11}{16}x + \frac{3}{16}x^2 + \frac{1}{48}x^3\right)e^{-x},$$
$$F_\delta(r) = b^3\left(\frac{1}{16} + \frac{1}{16}x + \frac{1}{48}x^2\right)e^{-x}, \qquad (10)$$
$$F_t(r) = 1 - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{144}x^5\right)e^{-x},$$
$$F_{ls}(r) = 1 - \left(1 + x + \frac{1}{2}x^2 + \frac{7}{48}x^3 + \frac{1}{48}x^4\right)e^{-x},$$

with $x = br$. The derivation of F_C is given in [21], while the others are related by $F_\delta = -\nabla^2(F_C/r)$, $F_t = (F_C/r)'' - (F_C/r)'/r$, and $F_{ls} = (F_C/r)'/r$. In the limit of point nucleons, $F_C = F_t = F_{ls} = 1$ and $F_\delta = 4\pi\delta^3(\mathbf{r})$. These form factors are illustrated in Fig. 1. The use of F_C in V_{VP} is an approximate method of removing the $1/r$ singularity (the logarithmic singularity remains) which is justified by its short range and the overall smallness of the term. Similarly, the use of F_C^2 in V_{C2} is an approximate method of removing the $1/r^2$ singularity. We note that because we use the Sachs nucleon form factors, there are no additional magnetic Darwin-Foldy terms [22].

For the np system we include a Coulomb term attributable to the neutron charge distribution in addition to the interaction between magnetic moments,

$$v^{\rm EM}(np) = V_{C1}(np) + V_{MM}(np). \qquad (11)$$

Here

$$V_{C1}(np) = \alpha\beta_n\frac{F_{np}(r)}{r}, \qquad (12)$$

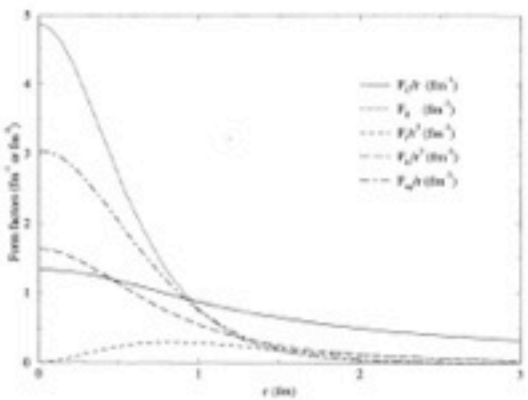


FIG. 1. Form factors in the electromagnetic interaction.

where the function F_{np} is obtained assuming the neutron electric form factor [22]

$$G_E^n = \beta_n q^2\left(1+\frac{q^2}{b^2}\right)^{-3}. \qquad (13)$$

Here $\beta_n \equiv [dG_E^n/dq^2]_{q=0} = 0.0189$ fm^2, the experimentally measured slope [23]. We have checked this form factor in a self-consistent calculation of the deuteron structure function $A(q^2)$ used to extract G_E^n [24] and find it gives a fairly good fit to the data. This simple form leads to

$$F_{np}(r) = b^2\left(15x + 15x^2 + 6x^3 + x^4\right)\frac{e^{-x}}{384}. \qquad (14)$$

The F_{np} is also shown in Fig. 1. The magnetic moment interaction is given by

$$V_{MM}(np) = -\frac{\alpha}{4M_nM_p}\mu_n\mu_p\left[\frac{2}{3}F_\delta(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{F_t(r)}{r^3}S_{ij}\right] - \frac{\alpha}{2M_nM_r}\mu_n\frac{F_{ls}(r)}{r^3}(\mathbf{L}\cdot\mathbf{S} + \mathbf{L}\cdot\mathbf{A}), \qquad (15)$$

where M_r is the nucleon reduced mass. The term proportional to $\mathbf{A} = \frac{1}{2}(\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j)$ is a "class IV" charge-asymmetric force [25], which mixes spin-singlet and spin-triplet states. Its contribution is very small, and we only include it when we construct the magnetic moment scattering amplitude [19].

Finally, for nn scattering, we neglect the Coulomb interaction between the neutron form factors, so there is only a magnetic moment term

$$v^{\rm EM}(nn) = V_{MM}(nn) = -\frac{\alpha}{4M_n^2}\mu_n^2\left[\frac{2}{3}F_\delta(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{F_t(r)}{r^3}S_{ij}\right]. \qquad (16)$$

The charge-dependent structure of the OPE potential is the same as that used in the Nijmegen partial-wave analysis and reads

$$v^\pi(pp) = f_{pp}^2 v_\pi(m_{\pi^0}),$$
$$v^\pi(np) = f_{pp}f_{nn}v_\pi(m_{\pi^0}) + (-)^{T+1}2f_c^2 v_\pi(m_{\pi^\pm}), \qquad (17)$$
$$v^\pi(nn) = f_{nn}^2 v_\pi(m_{\pi^0}),$$

where T is the isospin and

$$v_\pi(m) = \left(\frac{m}{m_s}\right)^2 \tfrac{1}{3}mc^2\left[Y_\mu(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + T_\mu(r)S_{ij}\right]. \qquad (18)$$

(Strictly speaking, the neutron-proton mass difference gives rise to an OPE "class IV" force as well, which again we only explicitly include when we construct the OPE scattering amplitude [19].) Here $Y_\mu(r)$ and $T_\mu(r)$ are the usual Yukawa and tensor functions with the exponential cutoff of the Urbana and Argonne v_{14} models

$$Y_\mu(r) = \frac{e^{-\mu r}}{\mu r}\left(1 - e^{-cr^2}\right),$$
$$T_\mu(r) = \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right)\frac{e^{-\mu r}}{\mu r}\left(1 - e^{-cr^2}\right)^2, \qquad (19)$$

where $\mu = mc/\hbar$. The scaling mass m_s, introduced in Eq. (18) to make the coupling constant dimensionless, is taken to be the charged-pion mass, $m_{\pi^\pm}$. The Nijmegen partial-wave analysis of NN scattering data below 350 MeV finds very little difference between the coupling constants [26], so we choose them to be charge independent, i.e., $f_{pp} = -f_{nn} = f_c \equiv f$, with the recommended value $f^2 = 0.075$. Thus all charge dependence in Eqs. (17) is due simply to the difference in the charged- and neutral-pion masses.

The remaining intermediate- and short-range phenomenological part of the potential is expressed, as in the Argonne v_{14} model, as a sum of central, L^2, tensor, spin-orbit, and quadratic spin-orbit terms (abbreviated as $c, l2, t, ls, ls2$, respectively) in different S, T, and T_z states:

$$v_{ST}^R(NN) = v_{ST,NN}^c(r) + v_{ST,NN}^{l2}(r)L^2 + v_{ST,NN}^t(r)S_{12} + v_{ST,NN}^{ls}(r)\mathbf{L}\cdot\mathbf{S} + v_{ST,NN}^{ls2}(r)(\mathbf{L}\cdot\mathbf{S})^2. \qquad (20)$$

Each of these terms is given the general form

$$v_{ST,NN}^i(r) = I_{ST,NN}^i T_\mu^2(r) + \left[P_{ST,NN}^i + \mu r\, Q_{ST,NN}^i + (\mu r)^2 R_{ST,NN}^i\right]W(r), \qquad (21)$$

where $\mu = \frac{1}{3}(m_{\pi^0} + 2m_{\pi^\pm})c/\hbar$ is the average of the pion masses and $T_\mu(r)$ is given by Eq. (19). Thus the $T_\mu^2(r)$ term has the range of a two-pion-exchange force. The $W(r)$ is a Woods-Saxon function which provides the short-range core:

$$W(r) = \left[1 + e^{(r-r_0)/a}\right]^{-1}. \qquad (22)$$

The four sets of constants $I_{ST,NN}^i$, $P_{ST,NN}^i$, $Q_{ST,NN}^i$, and $R_{ST,NN}^i$ are parameters to be fit to data. However, we also impose a regularization condition at the origin which reduces the number of free parameters by one for each $v_{ST,NN}^i$. We require that

$$v_{ST,NN}^t(r=0) = 0,$$
$$\left.\frac{\partial v_{ST,NN}^{i\neq t}}{\partial r}\right|_{r=0} = 0. \qquad (23)$$

Since the tensor part of the OPE potential already vanishes at $r = 0$, the first condition is satisfied by setting $P_{ST,NN}^t = 0$. The second condition is equivalent to fixing, for $i \neq t$,

$$Q_{ST,NN}^i = -\frac{1}{\mu W(0)}\left[P_{ST,NN}^i\frac{\partial W}{\partial r} + \delta_{ic}\frac{\partial v_{ST}^\pi}{\partial r}\right]_{r=0}, \qquad (24)$$

where we only have to evaluate the derivative of the spin-spin part of the OPE potential.

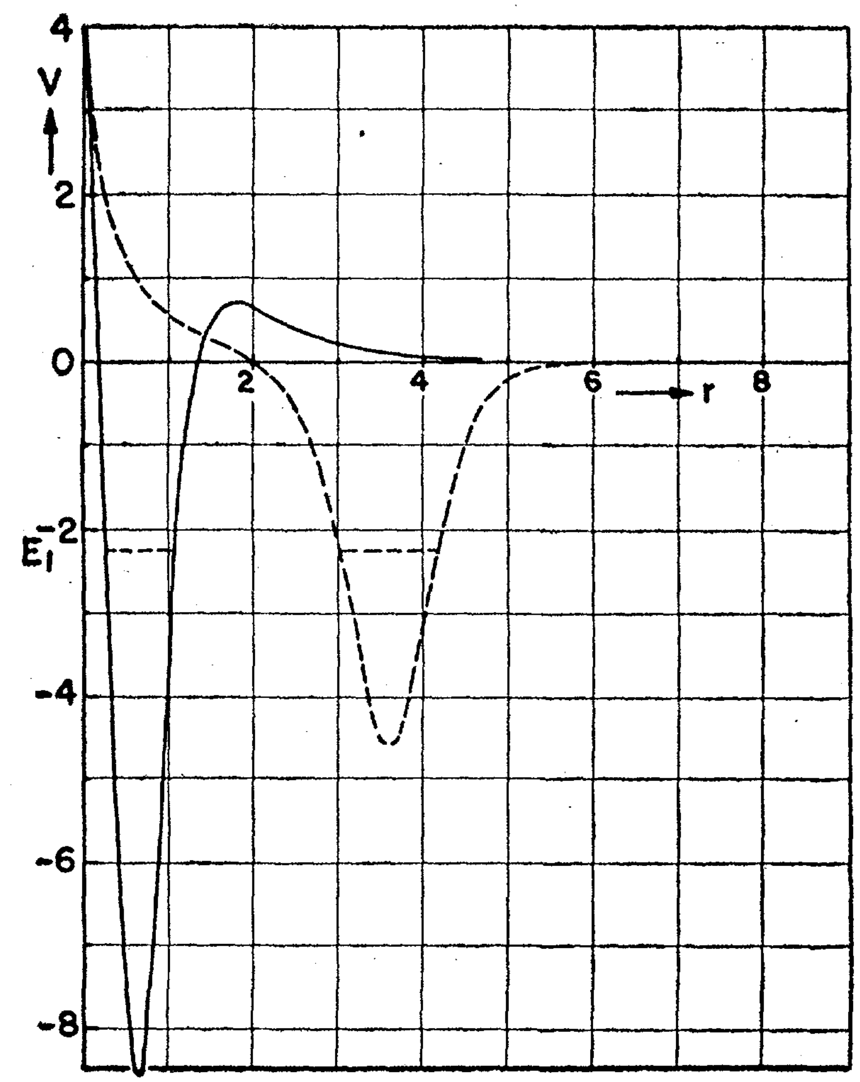
III. DATA FITTING

An initial survey of possible potential forms was made by fitting to the χ^2 hypersurface of the Nijmegen partial-wave analysis of pp and np data [16]. These studies helped select the final form of the potential ($\sim$10 variations were tried) and the values of the function shape parameters c, r_0, and a. Eventually, the cutoff parameter in the OPE functions $Y_\mu(r)$ and $T_\mu(r)$ was set at $c = 2.1$ fm^{-2}, while the parameters in the short-range Woods-Saxon $W(r)$ were set at $r_0 = 0.5$ fm and $a = 0.2$ fm. This value of c is slightly different from the 2.0 fm^{-2} used in the Urbana and Argonne v_{14} models, while r_0 and a are the same. Attempts to make a softer-core model led to a poorer fit. Sensitivity to the OPE coupling constant was also checked before the recommended value [26], $f^2 = 0.075$, was adopted as satisfactory.

Once these four parameters were set, a preliminary fit of the remaining parameters $I_{ST,NN}^i$, $P_{ST,NN}^{i\neq t}$, $Q_{ST,NN}^t$, and $R_{ST,NN}^i$ to the phase shifts was made. The final values were obtained by a direct fit to the Nijmegen pp and np scattering data base and the deuteron binding energy. We use nonrelativistic kinematics, i.e., the deuteron binding energy is taken as $E_d = \kappa^2/2M_r$. In practice, we found no benefit to including an $R_{ST,NN}^i$ in spin-singlet states, so these values were set to zero. Also, we found no indication of a need for charge dependence in the phenomenological part of spin-triplet states. In the final fit

Hadron potentials

- Encode of information about scattering amplitude
  - Useful phenomenologically
- Not uniquely defined: many phase equivalent potentials
- Defined up to unitary transformations
- Local potentials are energy dependent: only guaranteed to reproduce phase shift at a given energy
- Presence of inelastic channels (all QFT contexts)automatically renders a potential non-local [Feshbach non-locality – see eg Balentikin et al nucl-th/9709007] or equivalently local and energy dependent



[V Bargmann, RMP 21, (1949) 488]
Phase equivalent potentials

Static *BB* potentials

- Static limit for heavy quark ($m_b \to \infty$): *B* meson (*bd*) mass is infinite

$$M_b = m_b + \overline{\Lambda} + \mathcal{O}(1/m_b)$$

  - Momentum excitations are degenerate

- Heavy quark spin decouples: *B* and *B** degenerate

- Two static hadrons: defined, observable potential

$$V_{I,s_l}(\mathbf{r}, L) = E_{I,s_l}(\mathbf{r}, L) - 2E_B(L)$$

  - Potential is local (momentum excitations degenerate)

- LDOF quantum numbers ⊃ *NN* (has central and tensor interactions)

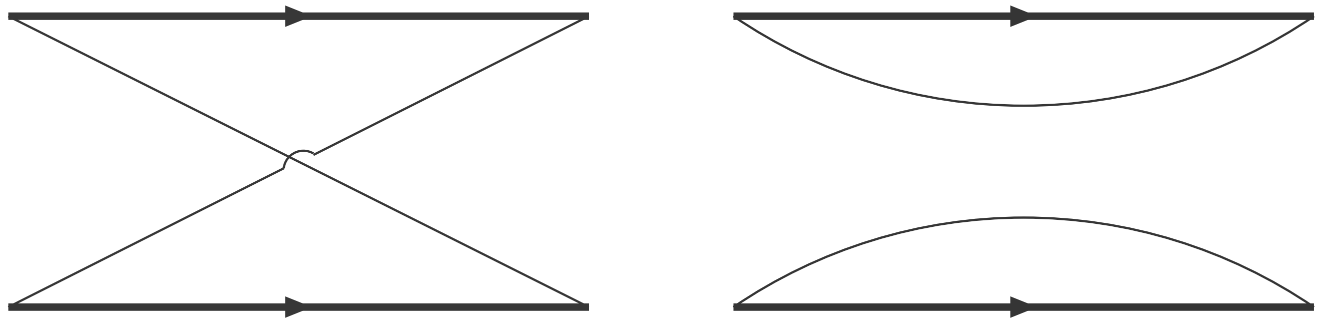
- EFT: potential has same form as NN for large distance

$$V_{BB}(|\mathbf{r}| \gg \Lambda_\chi) \longrightarrow \# \frac{g^2_{BB^*\pi}}{f^2} \frac{e^{-m_\pi |\mathbf{r}|}}{|\mathbf{r}|} \qquad V_{NN}(|\mathbf{r}| \gg \Lambda_\chi) \longrightarrow \# \frac{g^2_{A}}{f^2} \frac{e^{-m_\pi |\mathbf{r}|}}{|\mathbf{r}|}$$

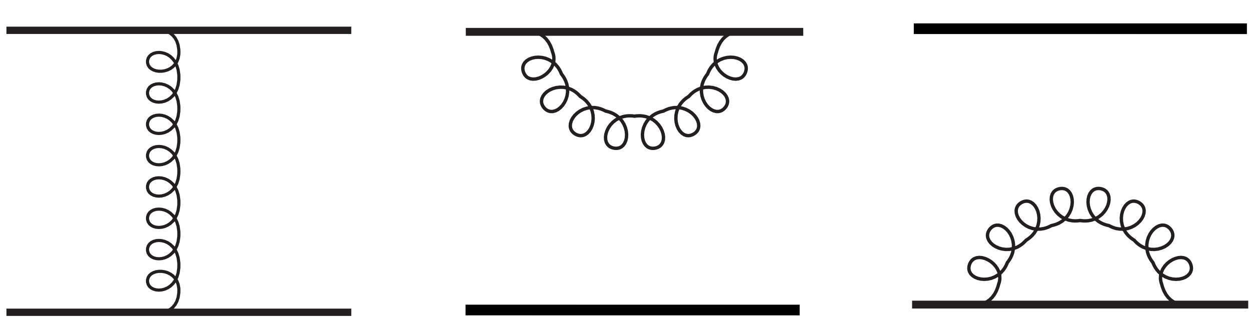
- Short range very different: $1/|\mathbf{r}|$ Coulomb

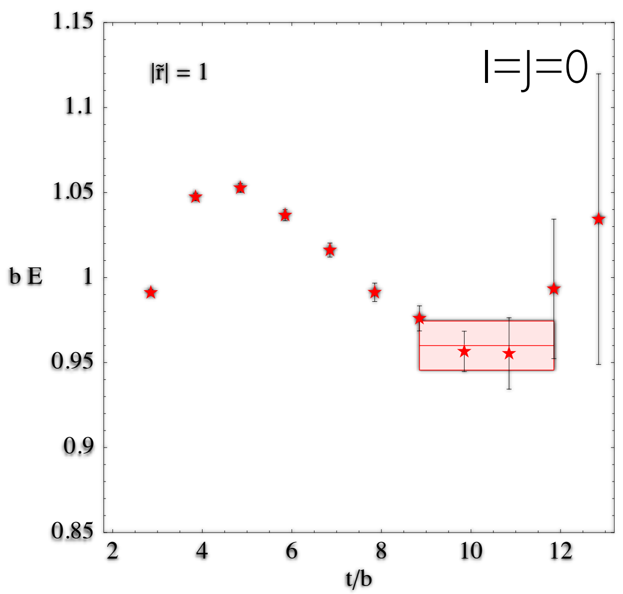
Static *BB* potentials

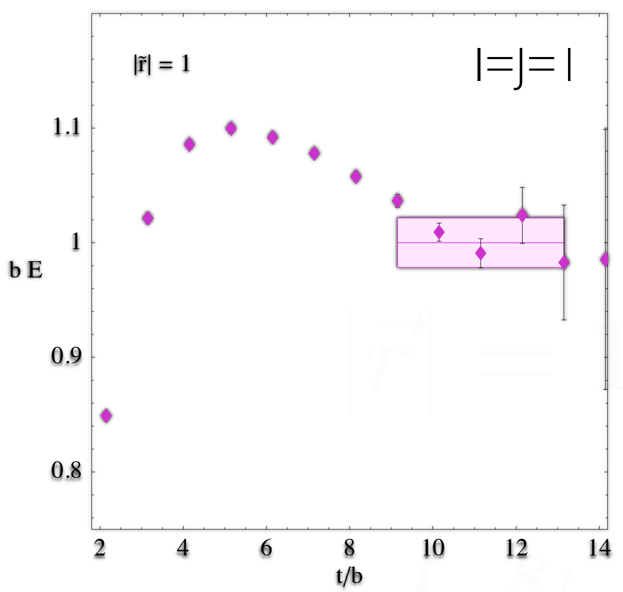
- Two types of contractions



- Compute correlators for given I,J and for each **r**
    - Extract lattice potential from correlator ratios
- Perturbatively correct for lattice distortion of Coulomb interaction (one gluon exchange)
    - Subtract lattice PT and add back in continuum PT

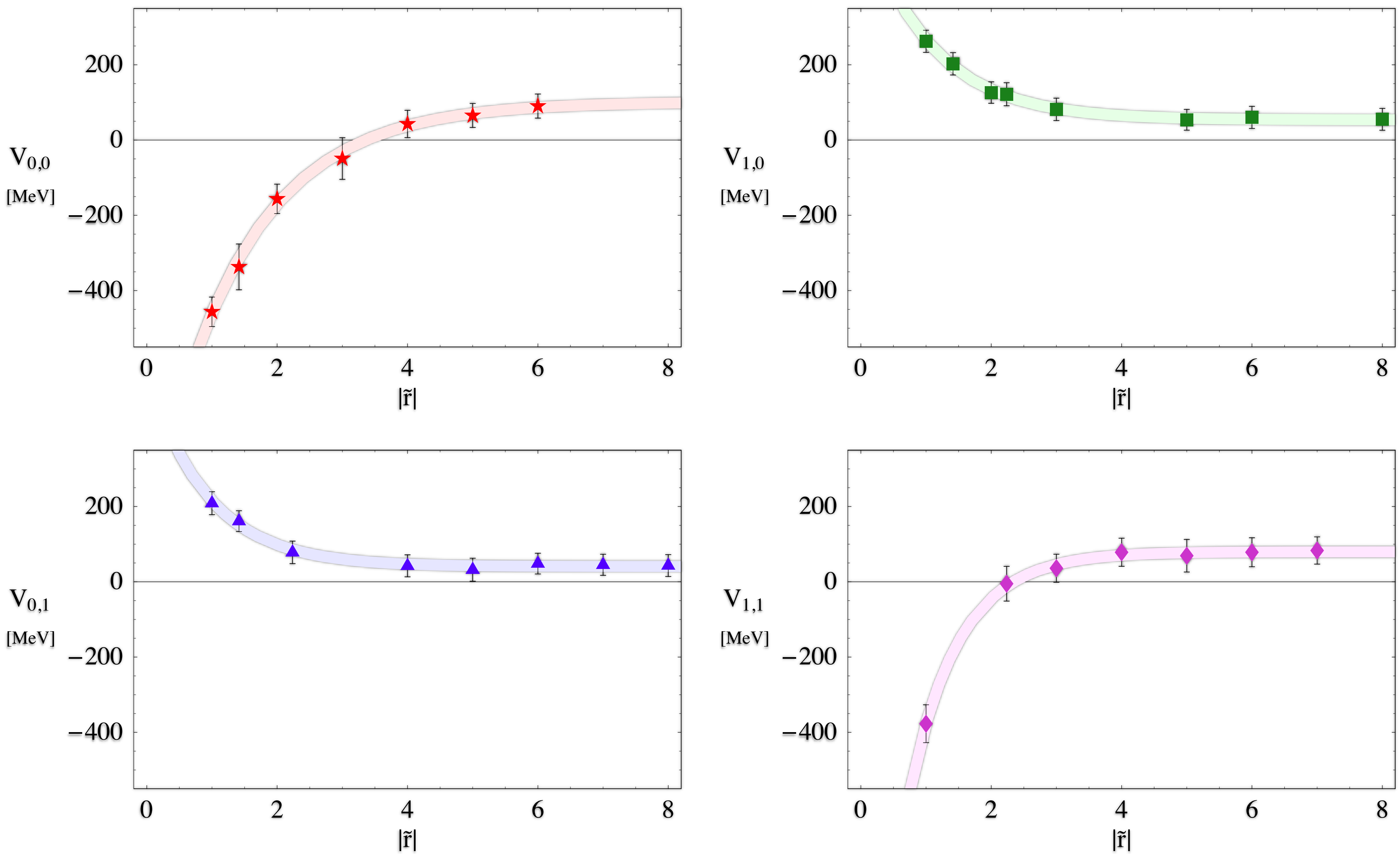






[WD, K Orginos and M Savage, PRD.76.114503]

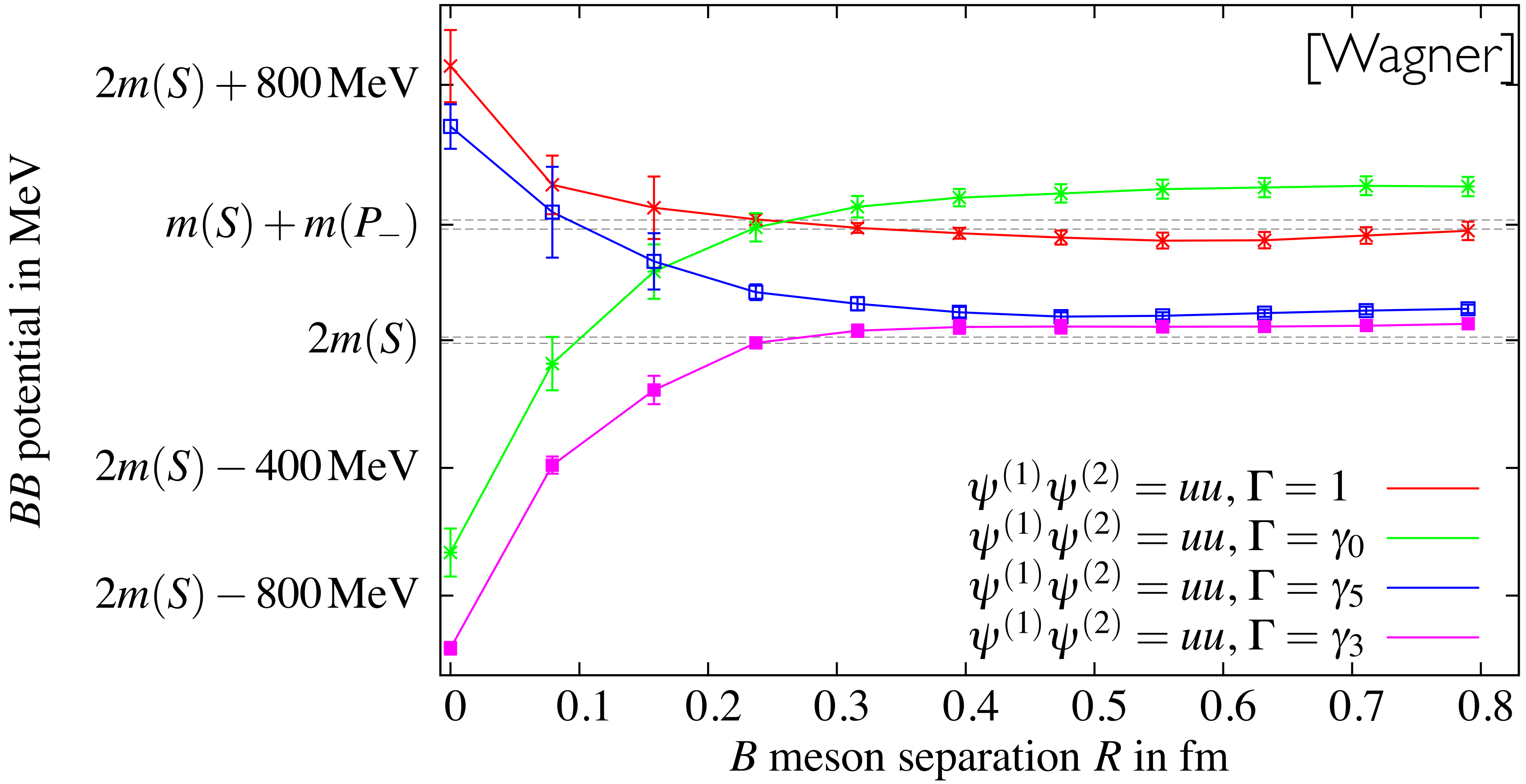
"Continuum" potentials



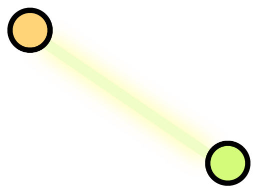
[WD, K Orginos and M Savage, PRD.76.114503]

Unquenched

- Previous quenched study: Michael & Pennanen 99
- Recent unquenched studies
  - ETMC: [Wagner, 1008.1538]
  - QCDSF [Bali & Hetzenegger 1111.2222]
  - Zac Brown [forthcoming]
- Qualitatively similar



Periodic lattice



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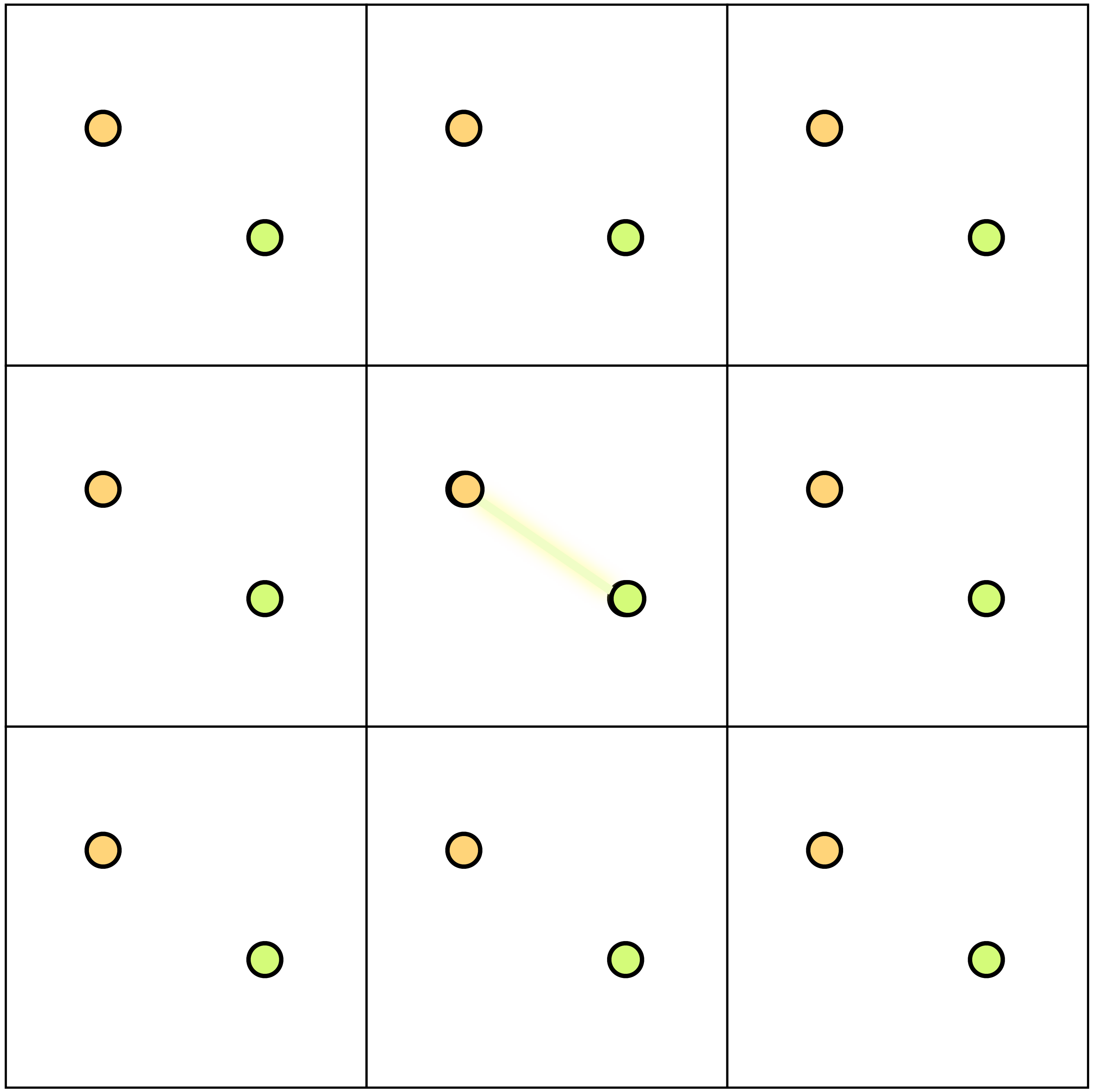

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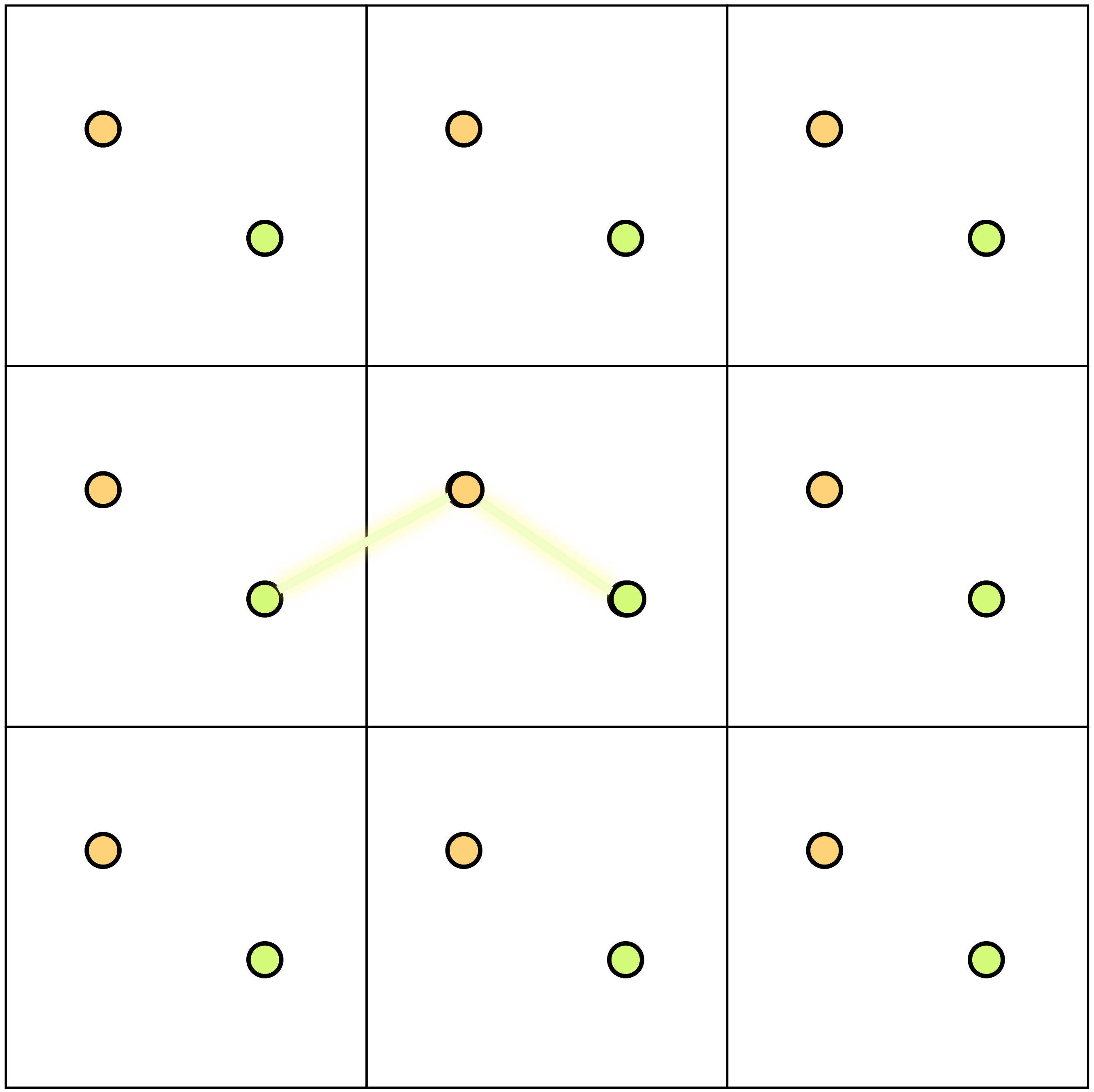

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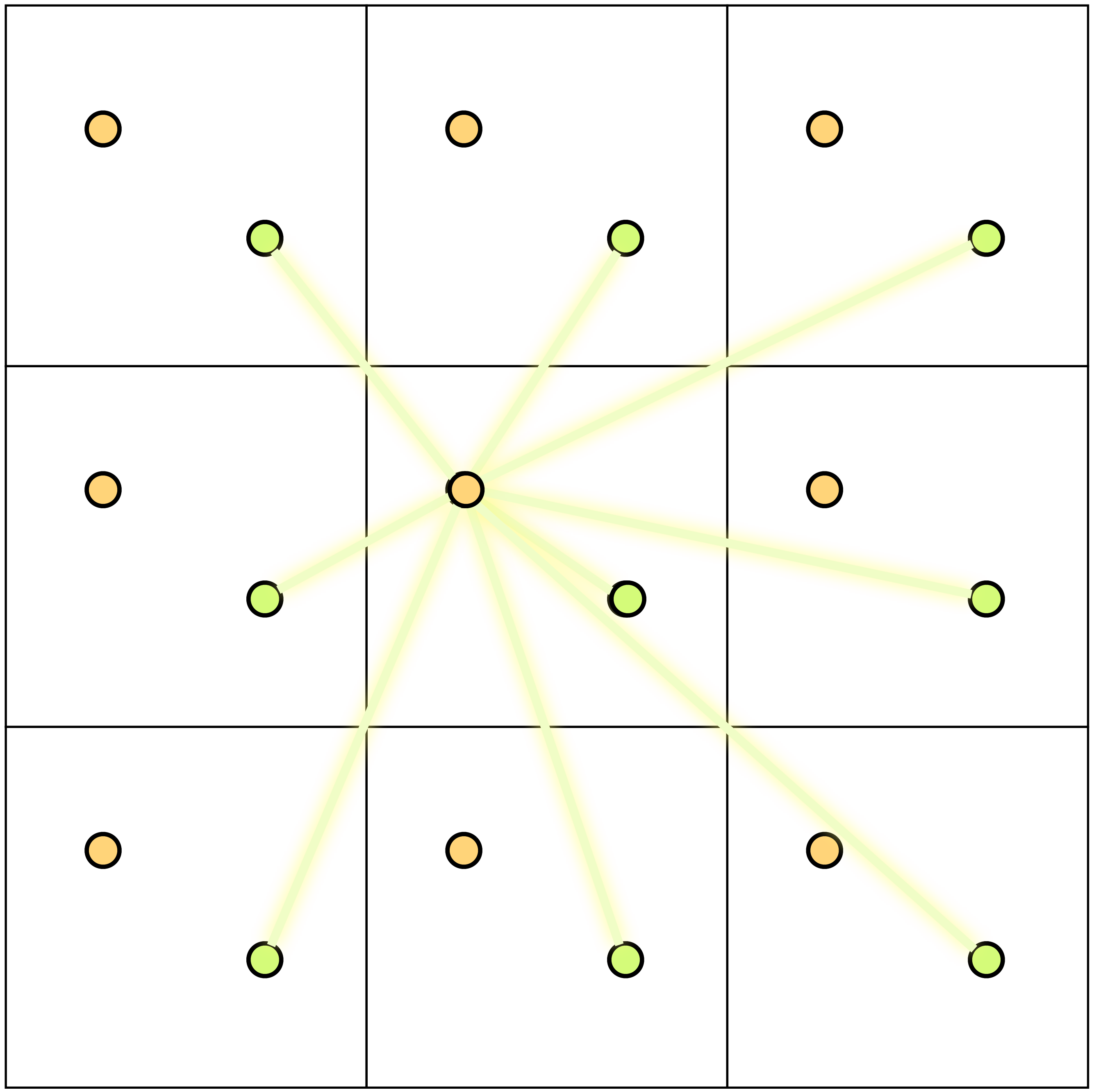


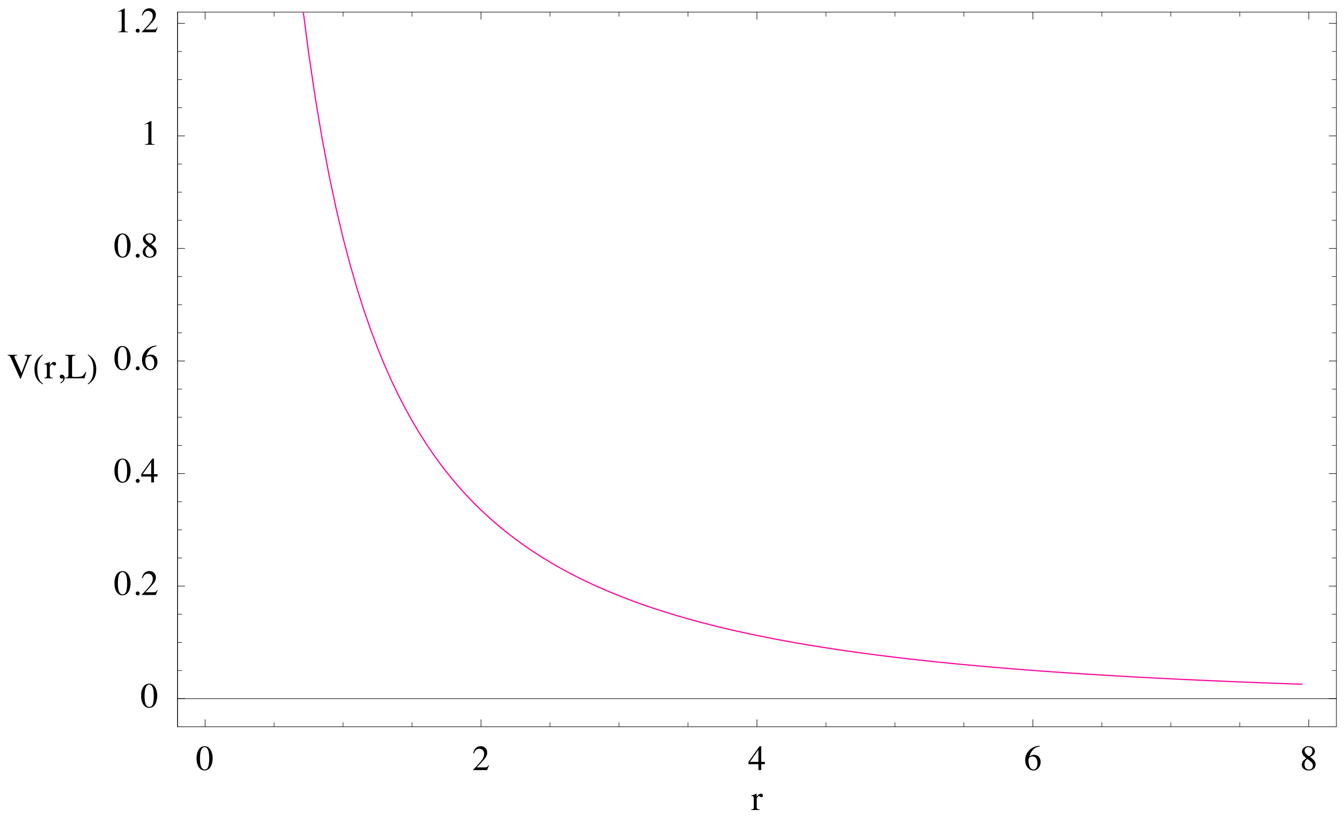
Finite volume effects

- Periodicity of lattice modifies potentials:

$$V(\mathbf{r}, L) = V(\mathbf{r}) + \sum_{\mathbf{n}\neq 0} V(\mathbf{r} + \mathbf{n}\, L)$$

assuming single particle exchange

- Strictly: V from $V^{(L)}$ impossible
  - Long range potential from EFT
  - Short range: FV effects smallest



HW: assuming a simple Yukawa interaction with the physical pion mass

$$V(|\mathbf{r}|) = V_0 \frac{e^{-m_\pi |\mathbf{r}|}}{|\mathbf{r}|}$$

and a 3 fm side length of the lattice volume, calculate the ratio of the lattice potential to the infinite volume potential at a separation of 1 fm.

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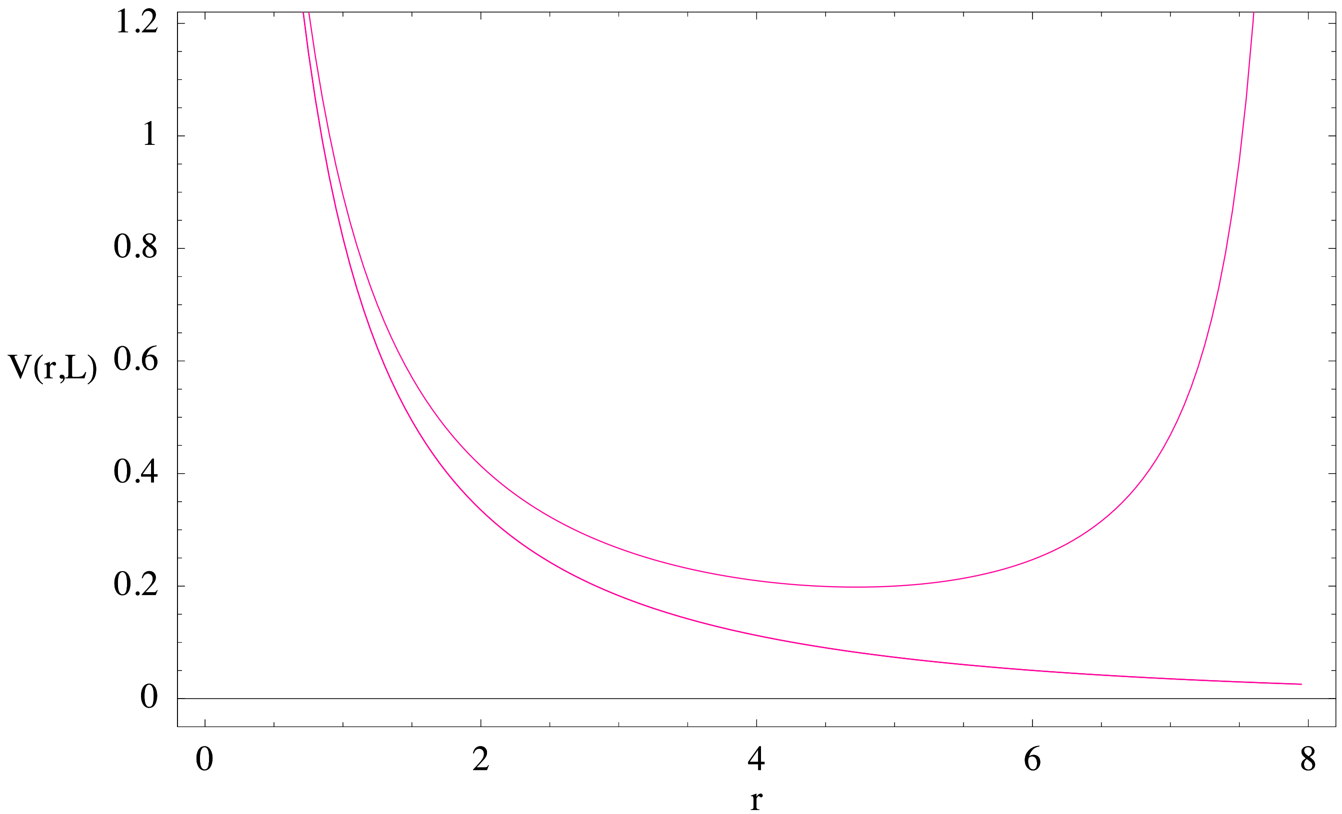

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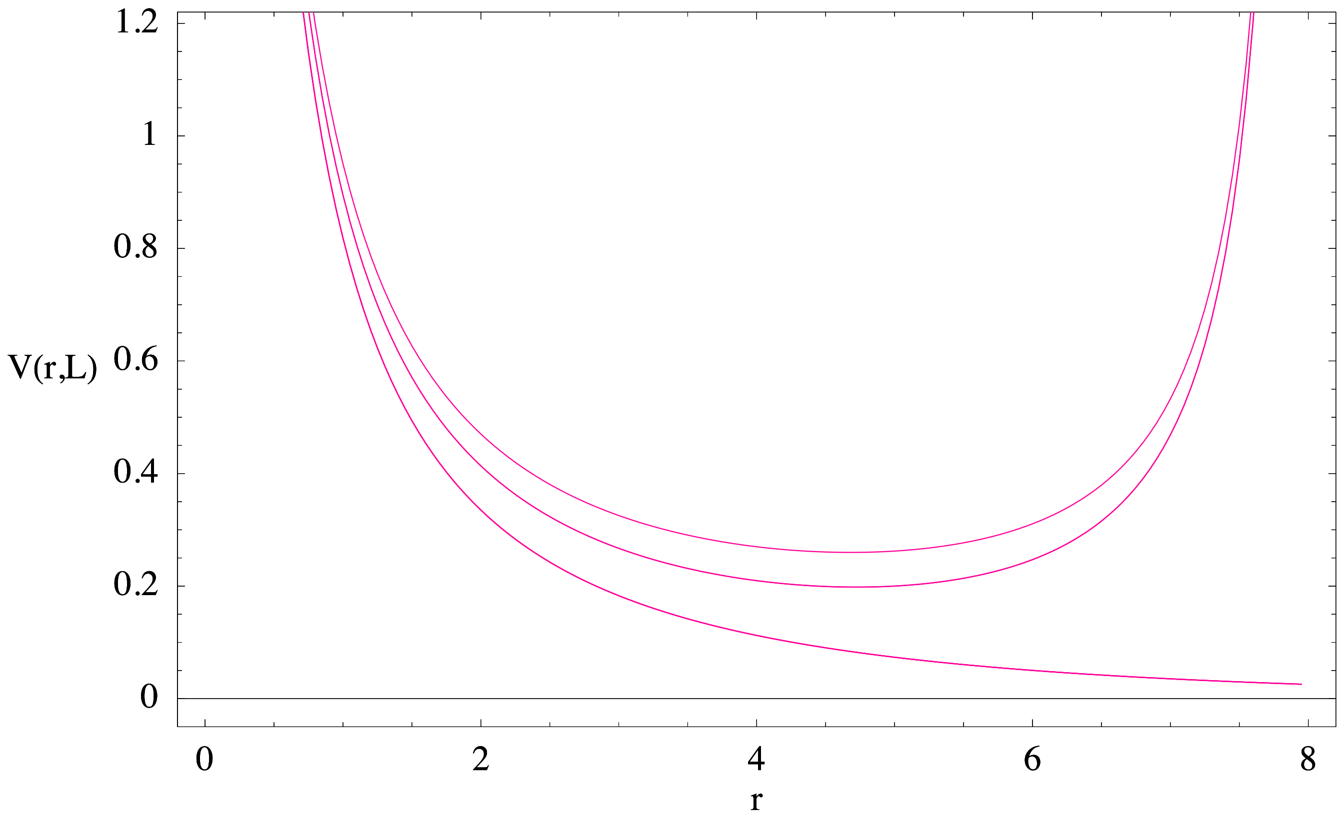

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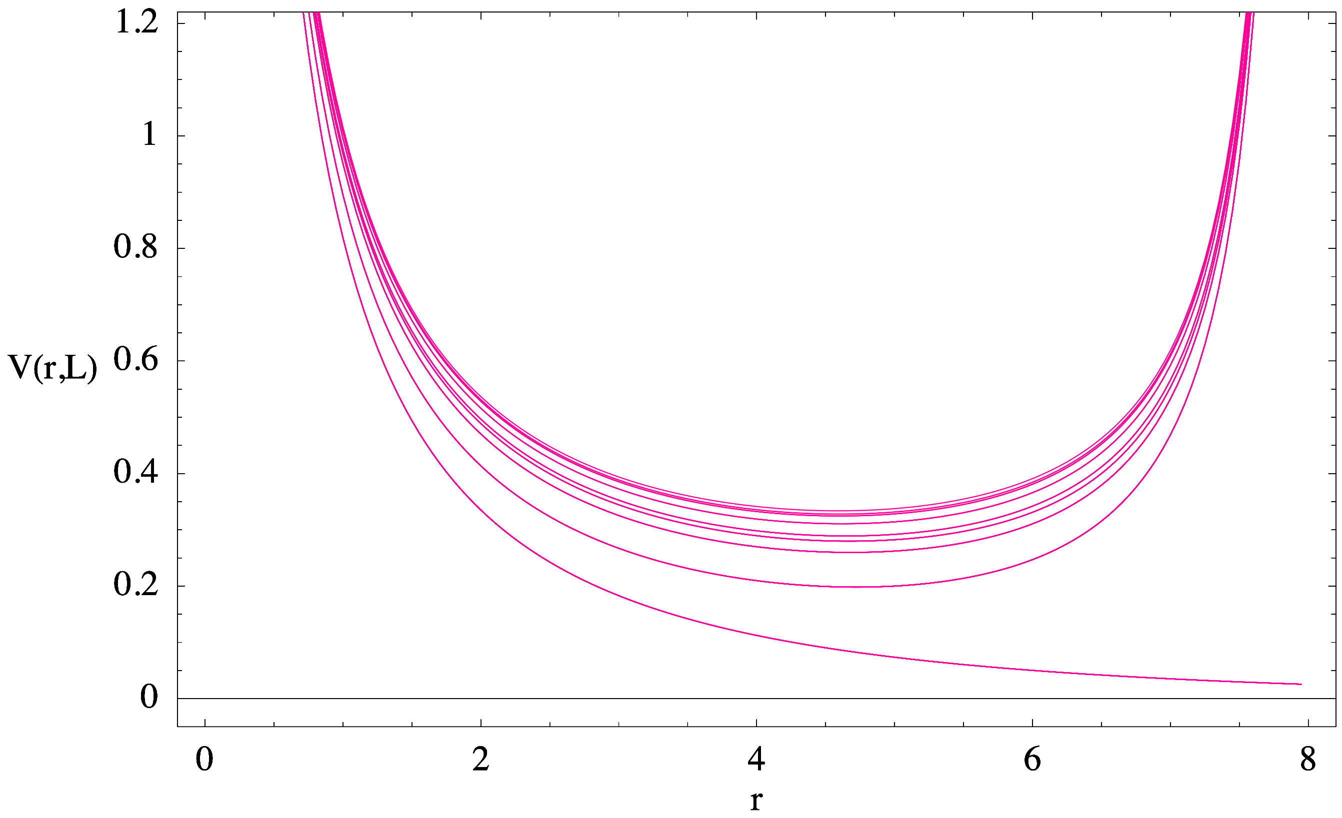

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Away from the static limit: the derivative expansion

- The general non-local potential U(r,r') can formally be written in terms of a local, energy (momentum) dependent potential

$$U(\mathbf{r},\mathbf{r}') = V(\mathbf{r},\nabla)\delta^{(3)}(\mathbf{r}-\mathbf{r}')$$

  - Energy dependent potential can be expanded (can have linear terms if has spin)

$$V(\mathbf{r},\nabla) = V(\mathbf{r}) + \mathcal{O}(\nabla^2)$$

  - Expansion parameter is ∇/M essentially a velocity expansion $|\mathbf{p}|/M_{\bar{Q}Q} \sim v$
    - Interesting study of quarkonium potential using LQCD [Ikeda&Iidal Kawanai&Sasaki]
    - $v \sim 0.1,\ 0.3$ for bottomonium, charmonium

- For nucleon-nucleon expansion parameter is $\Lambda_{\mathrm{QCD}}/M_N$ so expansion is marginal