INT Summer School on Lattice QCD for Nuclear Physics – August 2012

Exercises for lectures on **Finite density QCD**

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1. Relativistic Bose gas at nonzero chemical potential

Consider a self-interacting complex scalar field in the presence of a chemical potential μ, with the continuum action

$$S = \int d^4x \left[|\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu\left(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi\right) + \lambda |\phi|^4 \right]. \tag{1.1}$$

The euclidean action is complex and satisfies $S^*(\mu) = S(-\mu^*)$. Take $m^2 > 0$, so that at vanishing and small μ the theory is in its symmetric phase.

The lattice action, with lattice spacing $a_{\rm lat} \equiv 1$, is

$$S = \sum_x \left[\left(2d + m^2\right) \phi_x^* \phi_x + \lambda \left(\phi_x^* \phi_x\right)^2 - \sum_{\nu=1}^{4} \left(\phi_x^* e^{-\mu \delta_{\nu,4}} \phi_{x+\hat\nu} + \phi_{x+\hat\nu}^* e^{\mu \delta_{\nu,4}} \phi_x \right) \right], \tag{1.2}$$

where the number of euclidean dimensions is $d = 4$.

i) Show that this action reduces to (1.1) in the continuum limit.

ii) The complex field is written in terms of two real fields ϕ_a $(a = 1, 2)$ as $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. Show that the lattice action then reads

$$S = \sum_x \left[\frac{1}{2}\left(2d + m^2\right) \phi_{a,x}^2 + \frac{\lambda}{4}\left(\phi_{a,x}^2\right)^2 - \sum_{i=1}^{3} \phi_{a,x}\phi_{a,x+\hat i} \right.$$
$$\left. - \cosh\mu\, \phi_{a,x}\phi_{a,x+\hat 4} + i \sinh\mu\, \varepsilon_{ab} \phi_{a,x}\phi_{b,x+\hat 4} \right], \tag{1.3}$$

where ε_{ab} is the antisymmetric tensor, and summation over repeated indices is implied. Note that the '$\sinh\mu$' term is complex.

From now on the self-interaction is ignored and we take $\lambda = 0$. After going to momentum space, the action (1.3) reads

$$S = \sum_p \frac{1}{2} \phi_{a,-p} \left(\delta_{ab} A_p - \varepsilon_{ab} B_p\right) \phi_{b,p} = \sum_p \frac{1}{2} \phi_{a,-p} M_{ab,p} \phi_{b,p}, \tag{1.4}$$

where

$$M_p = \begin{pmatrix} A_p & -B_p \\ B_p & A_p \end{pmatrix}, \tag{1.5}$$

and

$$A_p = m^2 + 4\sum_{i=1}^{3} \sin^2 \frac{p_i}{2} + 2\left(1 - \cosh\mu \cos p_4\right), \qquad B_p = 2\sinh\mu \sin p_4. \tag{1.6}$$

iii) Show that the propagator corresponding to the action (1.4) is

$$G_{ab,p} = \frac{\delta_{ab} A_p + \varepsilon_{ab} B_p}{A_p^2 + B_p^2}. \tag{1.7}$$

iv) Demonstrate that the dispersion relation that follows from the poles of the propagator, taking $p_4 = iE_{\mathbf{p}}$, reads

$$\cosh E_{\mathbf{p}}(\mu) = \cosh\mu \left(1 + \frac{1}{2}\hat{\omega}_{\mathbf{p}}^2\right) \pm \sinh\mu \sqrt{1 + \frac{1}{4}\hat{\omega}_{\mathbf{p}}^2}, \tag{1.8}$$

where

$$\hat{\omega}_{\mathbf{p}}^2 = m^2 + 4\sum_i \sin^2 \frac{p_i}{2}. \tag{1.9}$$

v) Show that this can be written as

$$\cosh E_{\mathbf{p}}(\mu) = \cosh\left[E_{\mathbf{p}}(0) \pm \mu\right], \tag{1.10}$$

such that the (positive energy) solutions are

$$E_{\mathbf{p}}(\mu) = E_{\mathbf{p}}(0) \pm \mu. \tag{1.11}$$

Sketch the spectrum. Note that the critical μ value for onset is $\mu_c = E_{\mathbf{0}}(0)$, so that one mode becomes exactly massless at the transition (Goldstone boson).

vi) The phase-quenched theory corresponds to $\sinh\mu = B_p = 0$. Show that the dispersion relation in the phase-quenched theory is

$$\cosh E_{\mathbf{p}}(\mu) = \frac{1}{\cosh\mu}\left(1 + \frac{1}{2}\hat{\omega}_{\mathbf{p}}^2\right), \tag{1.12}$$

which corresponds to $E_{\mathbf{p}}^2(\mu) = m^2 - \mu^2 + \mathbf{p}^2$ in the continuum limit.

vii) Compare the spectrum of the full and the phase-quenched theory, when $\mu < \mu_c$. At larger μ, it is necessary to include the self-interaction to stabilize the theory. Based on what you know about symmetry breaking, sketch the spectrum in the full and the phase-quenched theory at larger μ as well.

Although the spectrum depends on μ, thermodynamic quantities do not. Up to an irrelevant constant, the logarithm of the partition function is

$$\ln Z = -\frac{1}{2}\sum_p \ln\det M = -\frac{1}{2}\sum_p \ln(A_p^2 + B_p^2), \tag{1.13}$$

and some observables are given by

$$\langle|\phi|^2\rangle = -\frac{1}{\Omega}\frac{\partial \ln Z}{\partial m^2} = \frac{1}{\Omega}\sum_p \frac{A_p}{A_p^2+B_p^2}, \tag{1.14}$$

and

$$\langle n\rangle = \frac{1}{\Omega}\frac{\partial \ln Z}{\partial \mu} = -\frac{1}{\Omega}\sum_p \frac{A_pA_p'+B_pB_p'}{A_p^2+B_p^2}, \tag{1.15}$$

where $\Omega = N_\sigma^3 N_\tau$ and $A_p' = \partial A_p/\partial\mu$, $B_p' = \partial B_p/\partial\mu$.

viii) Evaluate the sums (e.g. numerically) to demonstrate that thermodynamic quantities are independent of μ in the thermodynamic limit at vanishing temperature.

[1] G. Aarts, JHEP **0905** (2009) 052 [arXiv:0902.4686 [hep-lat]].

2. One-dimensional QCD

Consider QCD in one (temporal) dimension, with the staggered fermion action

$$\begin{aligned} S &= \sum \bar\chi(D+m)\chi \\ &= \sum_{x=1}^{n}\left[\frac{1}{2}\bar\chi_x e^{\mu}U_{x,x+1}\chi_{x+1} - \frac{1}{2}\bar\chi_{x+1}e^{-\mu}U^\dagger_{x,x+1}\chi_x + m\bar\chi_x\chi_x\right]. \end{aligned} \tag{2.1}$$

Here n denotes the number of points in the time direction and is taken to be even. The quarks obey anti-periodic boundary conditions. The links $U_{x,x+1}$ are elements of U(N) or SU(N).

Via a unitary transformation, all links but one can be transformed away ("temporal gauge"), i.e. $U_{n,1} \equiv U$, all other U's are unity. The determinant can then be written, up to an overall constant, as [1,2]

$$\det(D+m) = \det{}_C\left(e^{n\mu_c} + e^{-n\mu_c} + e^{n\mu}U + e^{-n\mu}U^\dagger\right). \tag{2.2}$$

The remaining determinant is in colour space and μ_c is related to the mass m as

$$m = \sinh\mu_c. \tag{2.3}$$

The reason for introducing μ_c will become clear below.

i) Show that the determinant has the usual symmetry under complex conjugation.

In one dimension, the partition function is simply

$$Z_{N_f} = \int dU\, \det{}^{N_f}(D+m), \tag{2.4}$$

since there is no Yang-Mills action. From now on we take as gauge group U(1): this captures all the essential characteristics in one dimension but also allows one to do the group integral without any effort. We hence write

$$U = e^{i\phi} \qquad\qquad \int dU = \int_0^{2\pi} \frac{d\phi}{2\pi}. \tag{2.5}$$

ii) Show that the partition function for $N_f = 2$ is independent of μ and equal to

$$Z_{N_f=2} = 4 + 2\cosh(2n\mu_c). \tag{2.6}$$

Note that the μ independence is generic in U(N) theories, since μ can be absorbed in the U(1) phase (take μ to be imaginary for this). This is of course not possible in SU(N) theories, where there is no such freedom.

iii) Show that the phase-quenched $N_f = 2$ partition function depends on μ and equals

$$\begin{aligned} Z_{N_f=1+1^*} &= \int dU\ |\det(D+m)|^2 = \int dU\ \det(D(\mu)+m)\det(D(-\mu)+m) \\ &= 2 + 2\cosh(2n\mu_c) + 2\cosh(2n\mu). \end{aligned} \tag{2.7}$$

The chiral condensate and the number density are defined by

$$\Sigma = \frac{1}{n}\frac{\partial \ln Z}{\partial m} \qquad\qquad \langle n_B \rangle = \frac{1}{n}\frac{\partial \ln Z}{\partial \mu}. \tag{2.8}$$

iv) Show that in the full theory one finds

$$\Sigma = \frac{2\sinh(2n\mu_c)}{2+\cosh(2n\mu_c)}\frac{1}{\cosh\mu_c} \to \frac{2\mathrm{sgn}(\mu_c)}{\cosh\mu_c}, \qquad\qquad \langle n_B \rangle = 0. \tag{2.9}$$

The arrow denotes the thermodynamic limit. The μ independence is obvious.

v) Show that in the phase-quenched theory one finds on the other hand

$$\Sigma = \frac{2\sinh(2n\mu_c)}{1+\cosh(2n\mu_c)+\cosh(2n\mu)}\frac{1}{\cosh\mu_c} \to \begin{cases} \frac{2\mathrm{sgn}(\mu_c)}{\cosh\mu_c} & |\mu| < |\mu_c| \\ 0 & |\mu| > |\mu_c| \end{cases}, \tag{2.10}$$

and

$$\langle n_B \rangle = \frac{2\sinh(2n\mu)}{1+\cosh(2n\mu_c)+\cosh(2n\mu)} \to \begin{cases} 0 & |\mu| < |\mu_c| \\ 2\mathrm{sgn}(\mu) & |\mu| > |\mu_c| \end{cases}. \tag{2.11}$$

The full and phase-quenched theories agree when $\mu < \mu_c$ (no μ dependence). The phase-quenched theory undergoes a phase transition at $\mu = \mu_c$, where the density jumps to 2. The interesting region in view of the Silver Blaze problem is therefore this large μ region, where the sign problem is severe and the average phase factor vanishes in the thermodynamic limit:

$$\langle e^{2i\varphi}\rangle_{\rm pq} = \frac{Z_{N_f=2}}{Z_{N_f=1+1^*}} \to 0 \qquad\qquad \det(D+m) = e^{i\varphi}|\det(D+m)|. \tag{2.12}$$

The eigenvalues of D are

$$\lambda_k = \frac{1}{2}e^{i(2\pi(k+\frac{1}{2})+\phi)/n+\mu} - \frac{1}{2}e^{-i(2\pi(k+\frac{1}{2})+\phi)/n-\mu} \qquad\qquad k=1,\ldots,n. \tag{2.13}$$

The $k+\frac{1}{2}$ arises from the antiperiodic boundary conditions and the ϕ/n from uniformly distributing the link U over all links as $U^{1/n}$.

vi) Demonstrate that the eigenvalues lie on an ellipse in the complex plane, determined by

$$\left(\frac{\operatorname{Re}\lambda_k}{\sinh(\mu)}\right)^2 + \left(\frac{\operatorname{Im}\lambda_k}{\cosh(\mu)}\right)^2 = 1. \tag{2.14}$$

The transition in the phase-quenched theory occurs when the quark mass gets inside this ellipse.

To compute the eigenvalue density,

$$\rho(z;\mu) = \frac{1}{Z_{N_f}}\int dU\, \det{}^{N_f}(D+m)\sum_k \delta^2(z-\lambda_k), \tag{2.15}$$

we therefore parametrize

$$z = \frac{1}{2}\left(e^{i\alpha+\mu} - e^{-i\alpha-\mu}\right), \tag{2.16}$$

such that

$$\Sigma = \int_0^{2\pi}\frac{d\alpha}{2\pi}\,\frac{\rho(\alpha;\mu)}{z(\alpha)+m}. \tag{2.17}$$

One then finds, for $N_f=2$,

$$\rho(\alpha;\mu) = \frac{4\left[\cosh(n\mu_c)+\cosh(n(\mu+i\alpha))\right]^2}{2+\cosh(2n\mu_c)}. \tag{2.18}$$

vii) Show that in the thermodynamic limit, the eigenvalue density behaves as

$$\rho(\alpha;\mu) = \begin{cases} 2 & |\mu|<|\mu_c| \\ 2e^{2n(|\mu|-|\mu_c|+i\alpha)} & |\mu|>|\mu_c| \end{cases}, \tag{2.19}$$

i.e. it is well-behaved when the full and phase-quenched theories agree, but it is complex and oscillating with a divergent amplitude in the Silver Blaze region.

viii) Show that these oscillations are necessary to find a μ independent chiral condensate by evaluating Eq. (2.17) explicitly. Hint: write $e^{i\alpha} = w$ and use contour integration.

[1] N. Bilic and K. Demeterfi, Phys. Lett. B **212** (1988) 83.
[2] L. Ravagli and J. J. M. Verbaarschot, Phys. Rev. D **76** (2007) 054506 [arXiv:0704.1111 [hep-th]].
[3] G. Aarts and K. Splittorff, JHEP **1008** (2010) 017 [arXiv:1006.0332 [hep-lat]].

3. Strong coupling

The one-link partition function is

$$z(x,y) = \int dU\, e^{\bar{\chi}_x U \chi_y - \bar{\chi}_y U^\dagger \chi_x}. \tag{3.1}$$

The (single flavour) staggered quark field χ_{ix} has a colour index $i = 1, \dots, N$ and $U \in \mathrm{SU}(N)$. This partition function can be written in terms of meson and (anti)-baryon fields,

$$\begin{aligned} M_x &= \bar{\chi}_x \chi_x = \bar{\chi}_{ix}\chi_{ix}, \\ B_x &= \frac{1}{N!}\epsilon_{i_1\dots i_N}\chi_{i_1 x}\cdots\chi_{i_N x}, \\ \bar{B}_x &= \frac{1}{N!}\epsilon_{i_1\dots i_N}\bar{\chi}_{i_N x}\cdots\bar{\chi}_{i_1 x}, \end{aligned} \tag{3.2}$$

as

$$z(x,y) = \sum_{k=0}^{N} \alpha_k \left(M_x M_y\right)^k + \tilde{\alpha}\left(\bar{B}_x B_y + (-1)^N \bar{B}_y B_x\right). \tag{3.3}$$

Here we want to determine the coefficients $\alpha, \tilde{\alpha}$.

i) For the baryon terms quark fields of all colours are needed. Expanding the exponential, show that one finds

$$\begin{aligned} \int dU\, e^{\bar{\chi}_x U \chi_y} &\to \frac{1}{N!}\int dU\, \left(\bar{\chi}_x U \chi_y\right)^N \\ &= \frac{1}{N!}\bar{\chi}_{i_1 x}\chi_{j_1 y}\cdots\bar{\chi}_{i_N x}\chi_{j_N y}\int dU\, U_{i_1 j_1}\cdots U_{i_N j_N}. \end{aligned} \tag{3.4}$$

ii) Using the result for the group integral

$$\int dU\, U_{i_1 j_1}\cdots U_{i_N j_N} = \frac{1}{N!}\epsilon_{i_1\dots i_N}\epsilon_{j_1\dots j_N}, \tag{3.5}$$

show that Eq. (3.4) can be written as $\bar{B}_x B_y$.

iii) Repeat this for the $e^{-\bar{\chi}_y U^\dagger \chi_x}$ term to conclude that $\tilde{\alpha} = 1$.

iv) To determine the coefficients of the meson terms, show first that

$$\int d\chi d\bar{\chi}\, e^{\alpha\bar{\chi}\chi} \left(\bar{\chi}\chi\right)^k = \sum_{n=0}^{N} \frac{\alpha^n}{n!} \int d\chi d\bar{\chi}\, \left(\bar{\chi}\chi\right)^{k+n} = \frac{N!}{(N-k)!}\alpha^{N-k}. \tag{3.6}$$

Note that the only term in the sum that contributes satisfies $k + n = N$.

v) By completing the square, prove the identity

$$\int d\chi_x d\bar{\chi}_x \int dU\, e^{\bar{\chi}_x\chi_x + \bar{\chi}_x U\chi_y - \bar{\chi}_y U^\dagger \chi_x} = e^{\bar{\chi}_y\chi_y}. \tag{3.7}$$

vi) Consider now

$$\int d\chi_x d\bar{\chi}_x\, e^{\bar{\chi}_x\chi_x} z(x,y). \tag{3.8}$$

Substitute Eq. (3.1) as well as

$$z(x,y) = \sum_{k=0}^{N} \alpha_k \left(M_x M_y\right)^k, \tag{3.9}$$

and use the identities derived above to show that this yields

$$\alpha_k = \frac{(N-k)!}{N!k!}. \tag{3.10}$$

[1] I. Montvay and G. Münster, Quantum Fields on a Lattice (1994) CUP.
[2] F. Karsch and K. H. Mütter, Nucl. Phys. B **313** (1989) 541.

4. Fokker-Planck equation

Consider the Langevin process

$$\dot{x}(t) = K(x(t)) + \eta(t), \qquad K = -S'(x), \qquad \langle \eta(t)\eta(t')\rangle_\eta = 2\lambda\delta(t-t'), \tag{4.1}$$

where λ normalizes the noise and the subscript η denotes noise averaging.

We want to derive the associated Fokker-Planck equation

$$\dot{\rho}(x,t) = \partial_x \left(\lambda\partial_x - K\right)\rho(x,t), \tag{4.2}$$

for the distribution $\rho(x,t)$, defined via (the subscript η will be dropped from now on)

$$\langle O(x(t))\rangle = \int dx\, \rho(x,t) O(x). \tag{4.3}$$

We consider the discretized process

$$\delta_n \equiv x_{n+1} - x_n = \epsilon K_n + \sqrt{\epsilon}\eta_n, \qquad \langle \eta_n \eta_{n'} \rangle = 2\lambda \delta_{nn'}. \tag{4.4}$$

ii) Show that

$$\begin{aligned} \langle O(x_{n+1}) \rangle - \langle O(x_n) \rangle &= \langle O'(x_n)\delta_n + \frac{1}{2}O''(x_n)\delta_n^2 + \ldots \rangle \\ &= \epsilon \langle O'(x_n)K_n + \lambda O''(x_n) \rangle + \mathcal{O}(\epsilon^{3/2}). \end{aligned} \tag{4.5}$$

In the $\epsilon \to 0$ limit, this gives

$$\frac{\partial}{\partial t}\langle O(x) \rangle = \langle O'(x)K(x) + \lambda O''(x) \rangle. \tag{4.6}$$

iii) Using Eq. (4.3), demonstrate that this yields the Fokker-Planck equation (4.2) for $\rho(x,t)$. What should λ be to obtain the desired equilibrium distribution?

[1] P. H. Damgaard and H. Hüffel, Phys. Rept. **152** (1987) 227.

5. Gaussian model

Consider the complex integral

$$Z = \int_{-\infty}^{\infty} dx\, \rho(x), \qquad \rho(x) = e^{-S}, \qquad S = \frac{1}{2}\sigma x^2, \qquad \sigma = a + ib. \tag{5.1}$$

i) Show that the corresponding complex Langevin equations are given by

$$\dot{x} = K_x + \eta, \qquad K_x = -ax + by, \tag{5.2}$$

$$\dot{y} = K_y, \qquad K_y = -ay - bx, \tag{5.3}$$

where $\langle \eta(t)\eta(t') \rangle = 2\delta(t - t')$.

ii) Demonstrate that these Langevin equations are solved by

$$x(t) = e^{-at}\left[\cos(bt)x(0) + \sin(bt)y(0)\right] + \int_0^t ds\, e^{-a(t-s)}\cos[b(t-s))]\eta(s), \tag{5.4}$$

$$y(t) = e^{-at}\left[\cos(bt)y(0) - \sin(bt)x(0)\right] - \int_0^t ds\, e^{-a(t-s)}\sin[b(t-s)]\eta(s). \tag{5.5}$$

iii) Show that the expectation values in the infinite time limit are given by

$$\langle x^2 \rangle = \frac{1}{2a}\frac{2a^2 + b^2}{a^2 + b^2}, \qquad \langle y^2 \rangle = \frac{1}{2a}\frac{b^2}{a^2 + b^2}, \qquad \langle xy \rangle = -\frac{1}{2}\frac{b}{a^2 + b^2}. \tag{5.6}$$

iv) Demonstrate this yields the desired result

$$\langle x^2 \rangle \to \langle (x+iy)^2 \rangle = \frac{a-ib}{a^2+b^2} = \frac{1}{a+ib} = \frac{1}{\sigma}. \tag{5.7}$$

The Fokker-Planck equation for the (real and positive) weight $P(x,y;t)$, defined via

$$\langle O(x(t)+iy(t)) \rangle = \int dxdy\, P(x,y;t) O(x+iy), \tag{5.8}$$

is given by

$$\dot{P}(x,y;t) = \left[\partial_x \left(\partial_x - K_x\right) - \partial_y K_y\right] P(x,y;t) \tag{5.9}$$

Since the original integral is Gaussian, the equilibrium distribution $P(x,y)$ is also Gaussian and can be written as

$$P(x,y) = N \exp\left[-\alpha x^2 - \beta y^2 - 2\gamma xy\right], \tag{5.10}$$

where N is a normalization constant.

v) Using the Fokker-Planck equation, show that the coefficients are given by

$$\alpha = a, \qquad \beta = a\left(1 + \frac{2a^2}{b^2}\right), \qquad \gamma = \frac{a^2}{b}, \tag{5.11}$$

and demonstrate that this gives the previously computed expectation values

$$\langle x^2 \rangle = \frac{\int dxdy\, P(x,y) x^2}{\int dxdy\, P(x,y)}, \tag{5.12}$$

etc.

vi) From the equivalence

$$\int dx\, \rho(x) O(x) = \int dxdy\, P(x,y) O(x+iy), \tag{5.13}$$

it follows that the real distribution is related to the original complex one via

$$\rho(x) = \int dy\, P(x-iy,y). \tag{5.14}$$

Verify this explicitly (up to the undetermined normalization).