QCD at nonzero chemical potential and the sign problem

INT lectures 2012

V: complex Langevin dynamics

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Where are we?

complex weight:

- straightforward importance sampling not possible
- overlap problem

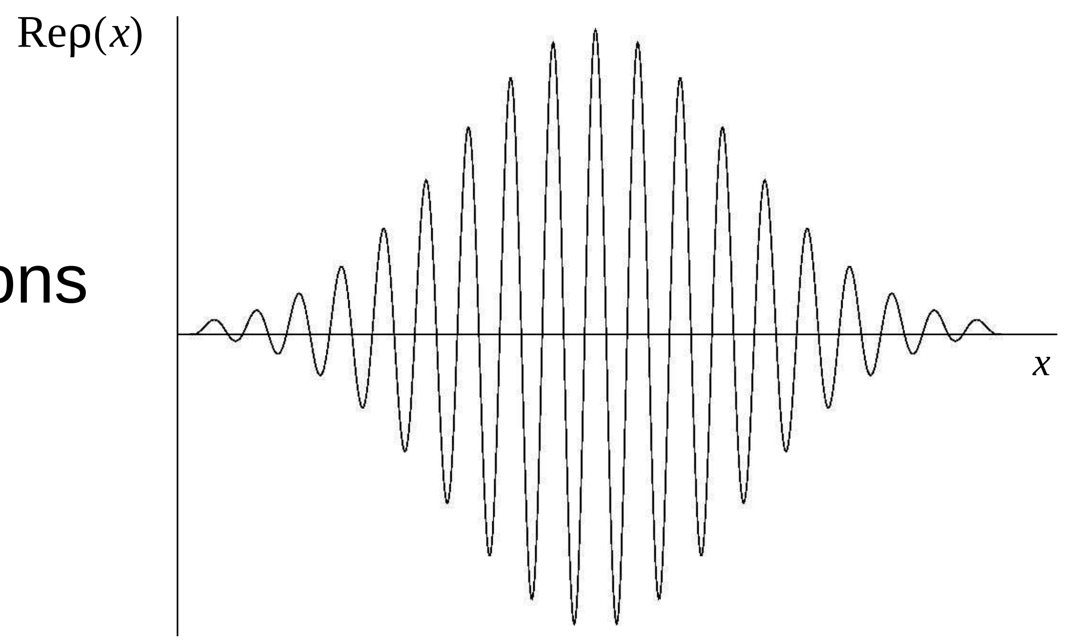
various possibilities:

- preserve overlap as best as possible
- use approximate methods at small μ
- do something radical:
  - rewrite partition function in other dof
  - explore field space in a different way
  - . . .

Overlap problem

- configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
- cancelation between configurations with ‘positive’ and ‘negative’ weight

dominant configurations in the path integral?



Complex integrals

- consider simple integral

$$Z(a,b) = \int_{-\infty}^{\infty} dx\, e^{-S(x)} \qquad\qquad S(x) = ax^2 + ibx$$

- complete the square/saddle point approximation:

  into complex plane

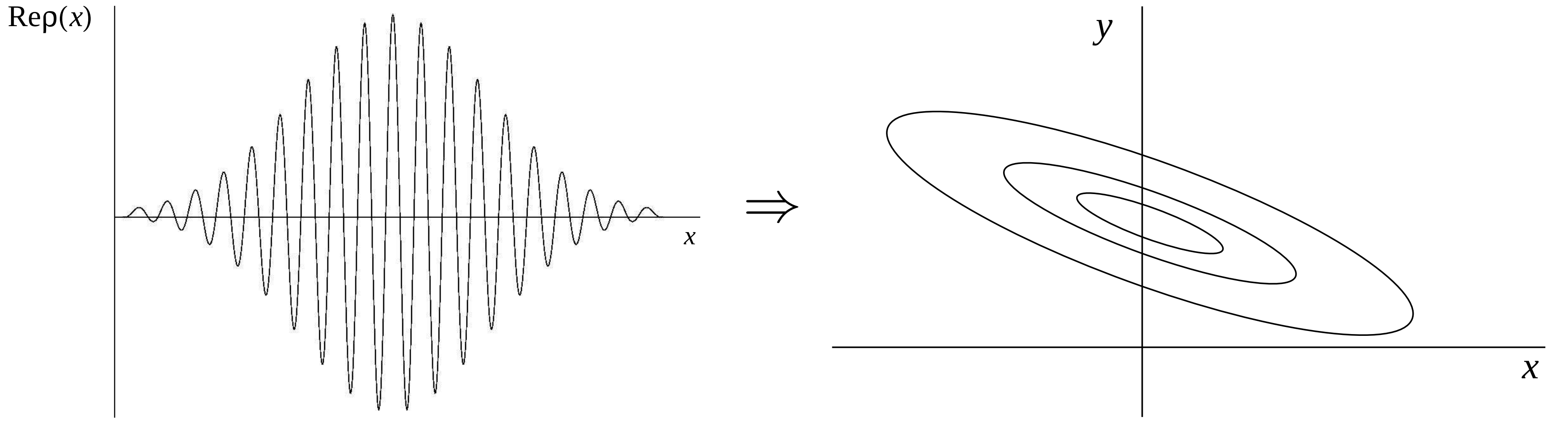
- lesson: don't be real(istic), be more imaginative

radically different approach:

- complexify all degrees of freedom $x \to z = x + iy$
- enlarged complexified space
- new directions to explore

Complexified field space

dominant configurations in the path integral?



real and positive distribution $P(x, y)$: how to obtain it?

$\Rightarrow$ solution of stochastic process

complex Langevin dynamics

Parisi 83, Klauder 83

Gaussian integral

- consider complex Gaussian integral

$$Z(a,b) = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 - ibx} \qquad \left(= \sqrt{\frac{2\pi}{a}} e^{-\frac{1}{2}b^2/a} \right)$$

  complex action $S^*(b) = S(-b^*)$ [assume $a > 0$ and real]

- phase quenched theory

$$Z_{\rm pq} = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = Z(a,0) = \sqrt{\frac{2\pi}{a}}$$

- sign problem: average phase factor

$$\langle e^{-ibx} \rangle_{\rm pq} = \frac{Z(a,b)}{Z(a,0)} = e^{-\frac{1}{2}b^2/a}$$

Gaussian integral

- average phase factor: one degree of freedom only

$$\langle e^{-ibx} \rangle_{\rm pq} = \frac{Z(a,b)}{Z(a,0)} = e^{-\frac{1}{2}b^2/a}$$

  sign problem only bad when b gets large

- for N degrees of freedom x_j, $j = 1, \ldots, N$

$$\langle e^{-ib\sum_j x_j} \rangle_{\rm pq} = e^{-\frac{1}{2}Nb^2/a}$$

  limits $b \to 0$, $N \to \infty$ do not commute

  severe sign problem for all $b \neq 0$ in $N \to \infty$ limit

  mimicks nonzero μ problem

Gaussian integral

$$Z(a,b) = \int dx\, e^{-\frac{1}{2}ax^2 - ibx} \qquad \langle x^2 \rangle = -2\frac{\partial \ln Z}{\partial a} = \frac{a - b^2}{a^2}$$

goal: compute numerically without importance sampling

first take $b = 0$:

- use analogy with Brownian motion

Parisi & Wu 81

particle in a fluid: friction (a) and kicks (η)

- Langevin equation

$$\frac{d}{dt}x(t) = -ax(t) + \eta(t) \qquad \langle \eta(t)\eta(t') \rangle = 2\delta(t - t')$$

Gaussian integral

- Langevin equation $\dot{x}(t) = -ax(t) + \eta(t)$
- analytical solution

$$x(t) = e^{-at}x(0) + \int_0^t ds\, \eta(s)e^{-a(t-s)}$$

- correlator [take $x(0) = 0$, no i.c. dependence]

$$\langle x^2(t)\rangle = \int_0^t ds \int_0^t ds'\, \langle \eta(s)\eta(s')\rangle e^{-a(2t-s-s')}$$

- noise averaged correlator, use $\langle \eta(s)\eta(s')\rangle = 2\delta(s-s')$

$$\lim_{t\to\infty} \langle x^2(t)\rangle = \frac{1}{a}$$

- no importance sampling, solution of stochastic process

Fokker-Planck equation

- associated distribution $\rho(x,t)$

$$\langle O(x(t)\rangle_\eta = \int dx\, \rho(x,t) O(x)$$

noise average   distribution average

- Langevin eq for $x(t)$ $\Leftrightarrow$ Fokker-Planck eq for $\rho(x,t)$

$$\dot{\rho}(x,t) = \partial_x \left(\partial_x + S'(x)\right) \rho(x,t)$$

- stationary solution: $\rho(x) \sim e^{-S(x)}$

review: Damgaard & Hüffel 87

Fokker-Planck equation

- stationary solution typically reached exponentially fast

$$\dot\rho(x,t) = \partial_x \left(\partial_x + S'(x)\right) \rho(x,t)$$

- write $\rho(x,t) = \psi(x,t) e^{-\frac{1}{2}S(x)}$

$$\dot\psi(x,t) = -H_{\rm FP}\psi(x,t)$$

- Fokker-Planck hamiltonian:

$$H_{\rm FP} = Q^\dagger Q = \left[-\partial_x + \frac{1}{2}S'(x)\right]\left[\partial_x + \frac{1}{2}S'(x)\right] \geq 0$$

$$Q\psi(x) = 0 \qquad \Leftrightarrow \qquad \psi(x) \sim e^{-\frac{1}{2}S(x)}$$

$$\psi(x,t) = c_0 e^{-\frac{1}{2}S(x)} + \sum_{\lambda>0} c_\lambda e^{-\lambda t} \to c_0 e^{-\frac{1}{2}S(x)}$$

Complex Gaussian integral

$$Z(a,b) = \int dx\, e^{-S(x)} \qquad S(x) = \frac{1}{2}ax^2 + ibx$$

$b \neq 0$:

- analytically: complete the square

  shift in the complex plane $x \to x + i\frac{b}{a}$
- achieve the same with Langevin equation

  "complexify" $x \to z = x + iy$

$$\dot{x} = -\mathrm{Re}\,\partial_z S(z) + \eta = -ax + \eta$$
$$\dot{y} = -\mathrm{Im}\,\partial_z S(z) \qquad = -ay - b$$

with $S(z) = S(x + iy)$

Complex Gaussian integral

- solution:
$$x(t) = x(0)e^{-at} + \int_0^t ds\, e^{-a(t-s)}\eta(s)$$
$$y(t) = [y(0) + b/a]e^{-at} - b/a$$

- correlators:
$$\begin{aligned}
\langle x^2(t)\rangle &= x^2(0)e^{-2at} + \left(1 - e^{-2at}\right)/a \to 1/a \\
\langle x(t)y(t)\rangle &= x(0)e^{-at}\left([y(0) + b/a]e^{-at} - b/a\right) \to 0 \\
\langle y^2(t)\rangle &= \left([y(0) + b/a]e^{-at} - b/a\right)^2 \to b^2/a^2
\end{aligned}$$

- combination $x \to x + iy$:
$$\lim_{t\to\infty}\langle [x(t) + iy(t)]^2\rangle = \langle x^2 - y^2 + 2ixy\rangle = \frac{1}{a} - \frac{b^2}{a^2} = \frac{a - b^2}{a^2}$$

correct!

Distribution

associated distribution $P(x,y;t)$ in complex plane

- real and positive distribution (if it exists)

$$\langle O(x+iy)(t)\rangle \quad = \quad \int dxdy\, P(x,y;t)O(x+iy)$$

Langevin eq for $x(t)$ and $y(t)$	Fokker-Planck eq for $P(x,y;t)$

- Fokker-Planck equation:

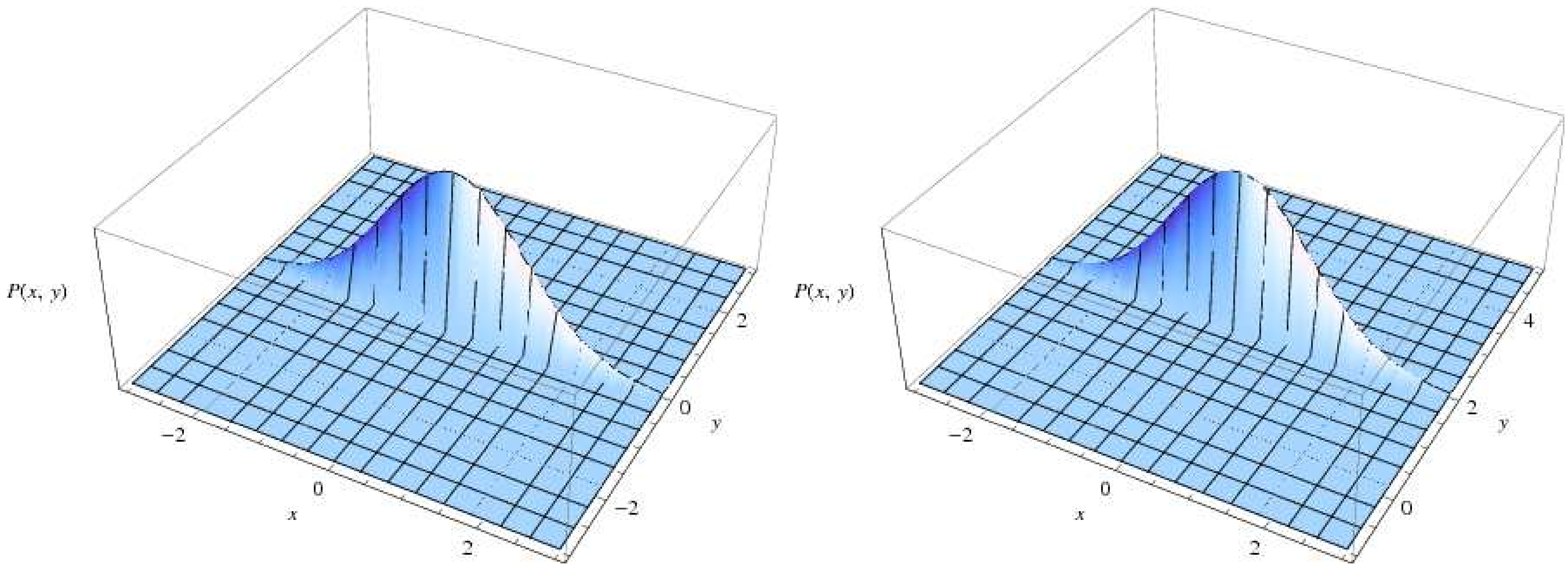
$$\dot{P}(x,y;t) = \left[\partial_x\left(\partial_x + \mathrm{Re}\,\partial_z S\right) + \partial_y \mathrm{Im}\,\partial_z S\right] P(x,y;t)$$

- solvable in Gaussian models (like here)
- no generic solutions known
  no semi-positive Fokker-Planck hamiltonian
  (in contrast to real Langevin/action)

Distribution

distribution $P(x,y)$ in the complex plane

$b = 0$ $\qquad\qquad\qquad\qquad$ $b = -2$



shift in the complex plane: $y \to -b/a$

Langevin process "finds" distribution:

$$P(x,y) \sim e^{-ax^2/2}\delta(y + b/a)$$

More interesting Gaussian integral

final Gaussian example:

- $S = \frac{1}{2}(a+ib)x^2$ $\qquad\qquad$ $\langle x^2 \rangle = \frac{1}{a+ib}$
- coupled Langevin equations

$$\dot{x} = -ax + by + \eta \qquad\qquad \dot{y} = -ay - bx$$

- solve and find correlators when $t \to \infty$

$$\langle x^2 \rangle = \frac{1}{2a}\frac{2a^2+b^2}{a^2+b^2} \qquad \langle y^2 \rangle = \frac{1}{2a}\frac{b^2}{a^2+b^2} \qquad \langle xy \rangle = -\frac{1}{2}\frac{b}{a^2+b^2}$$

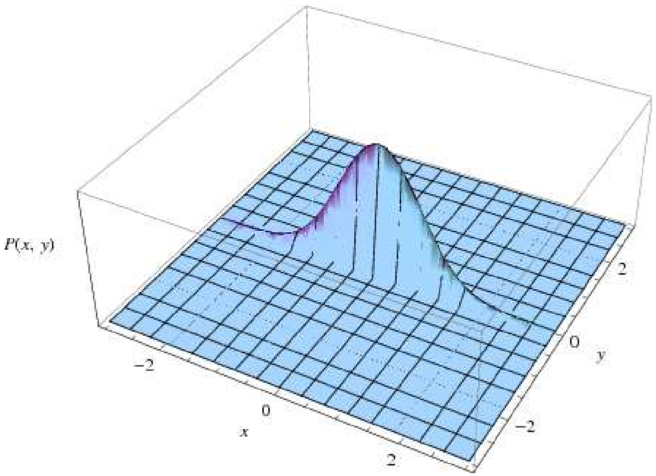
- correlator $\langle z^2 \rangle = \langle x^2 - y^2 + 2ixy \rangle = \frac{a-ib}{a^2+b^2} = \frac{1}{a+ib}$

correct!

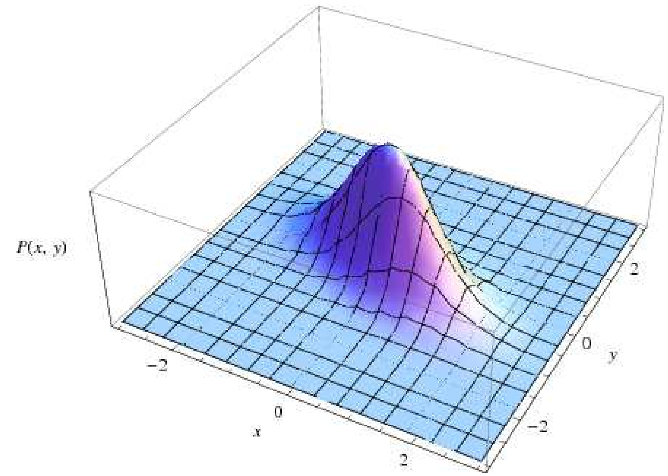
More interesting Gaussian integral

distribution $P(x, y)$ in the complex plane

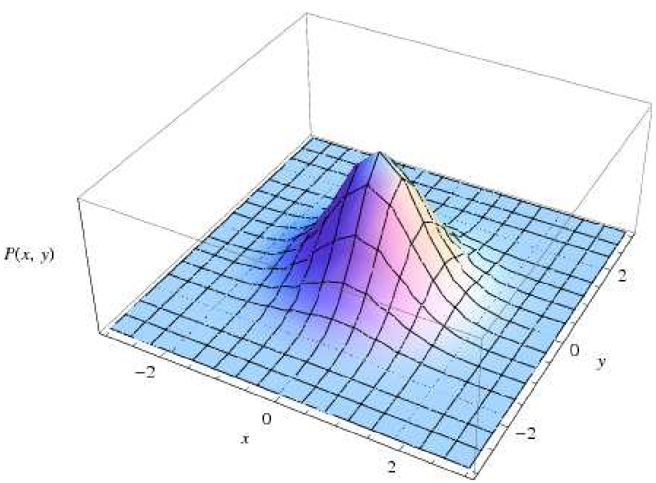
$b = 0.01$



$b = 1$



$b = 10$



Langevin process "finds" this distribution

original weight e^{-S} is complex

this distribution is real and positive

Equilibrium distributions

complex weight $\rho(x)$ real weight $P(x,y)$

- main premise:

$$\int dx\,\rho(x)O(x) = \int dxdy\,P(x,y)O(x+iy)$$

- if equilibrium distribution $P(x,y)$ is known analytically: shift variables

$$\int dxdy\,P(x,y)O(x+iy) = \int dx\,O(x)\int dy\,P(x-iy,y)$$

$$\Rightarrow \rho(x) = \int dy\,P(x-iy,y)$$

- correct in Gaussian examples
- hard to verify in numerical studies!

Discretization

most cases not analytically solvable

numerical solution of Langevin equation

- discretize stochastic equation (Itō calculus)

$$x_{n+1} = x_n + \epsilon K_n^{\mathrm{R}} + \sqrt{\epsilon}\eta_n$$
$$y_{n+1} = y_n + \epsilon K_n^{\mathrm{I}}$$

- drift terms

$$K_n^{\mathrm{R}} = -\mathrm{Re}\,\frac{\partial S}{\partial z} \qquad\qquad K_n^{\mathrm{I}} = -\mathrm{Im}\,\frac{\partial S}{\partial z}$$

- noise

$$\langle \eta_n \eta_{n'} \rangle = \delta_{nn'}$$

- use adaptive stepsize if necessary

Stochastic quantizaton

adapt to field theory Parisi & Wu 81, Parisi, Klauder 83

- path integral $Z = \int D\phi \, e^{-S}$
- Langevin dynamics in "fifth" time direction

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{\delta S[\phi]}{\delta \phi(x,t)} + \eta(x,t)$$

- Gaussian noise

$$\langle \eta(x,t) \rangle = 0 \qquad \langle \eta(x,t)\eta(x',t') \rangle = 2\delta(x-x')\delta(t-t')$$

- compute expectation values $\langle \phi(x,t)\phi(x',t) \rangle$, etc
- study converge as $t \to \infty$

Phase transitions and the Silver Blaze

can complex Langevin dynamics handle:

- a severe sign problem?
- the thermodynamic limit?
- phase transitions?
- the Silver Blaze problem? Cohen 03
- . . .

study in a model with a phase diagram with similar features as QCD at low temperature

$\Rightarrow$ relativistic Bose gas at nonzero μ

GA 08-09

Relativistic Bose gas at nonzero μ

- scalar O(2) model with global symmetry
- continuum action

$$S = \int d^4x \Big[\, |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2$$

$$+\mu \left(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi\right) + \lambda |\phi|^4 \Big]$$

- complex scalar field, $d = 4$, $m^2 > 0$
- $S^*(\mu) = S(-\mu^*)$ as in QCD

Relativistic Bose gas at nonzero μ

- scalar O(2) model with global symmetry
- lattice action

$$S = \sum_x \Big[\left(2d + m^2\right) \phi_x^* \phi_x + \lambda \left(\phi_x^* \phi_x\right)^2 - \sum_{\nu=1}^{4} \left(\phi_x^* e^{-\mu\delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^* e^{\mu\delta_{\nu,4}} \phi_x \right) \Big]$$

- complex scalar field, $d = 4$, $m^2 > 0$
- $S^*(\mu) = S(-\mu^*)$ as in QCD

Relativistic Bose gas at nonzero μ

tree level potential in the continuum

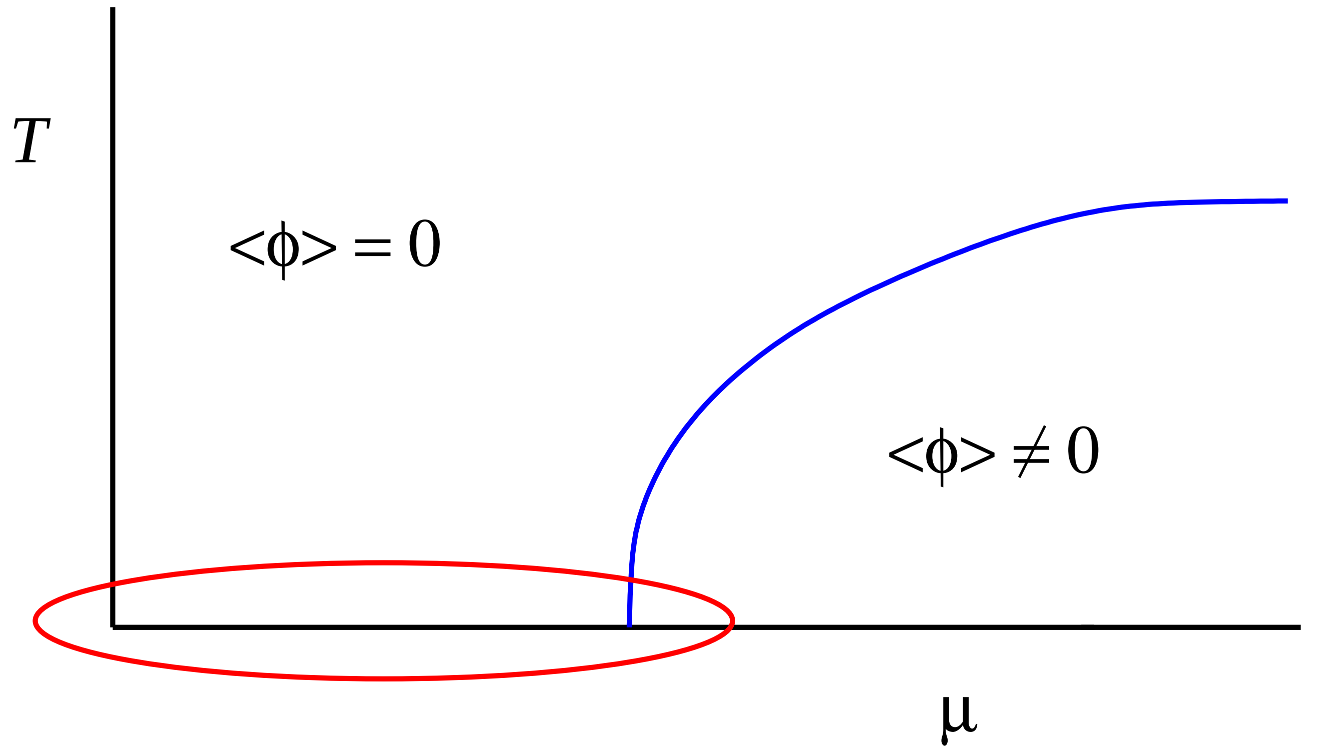
$$V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

condensation when $\mu^2 > m^2$, SSB

when $T = 0$
and $\mu < \mu_c$:

μ independence

Silver Blaze
problem



Relativistic Bose gas at nonzero μ

- write $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \;\;\Rightarrow\;\; \phi_a \;\; (a = 1, 2)$
- complexification $\phi_a \;\;\rightarrow\;\; \phi_a^{\rm R} + i\phi_a^{\rm I}$
- complex Langevin equations

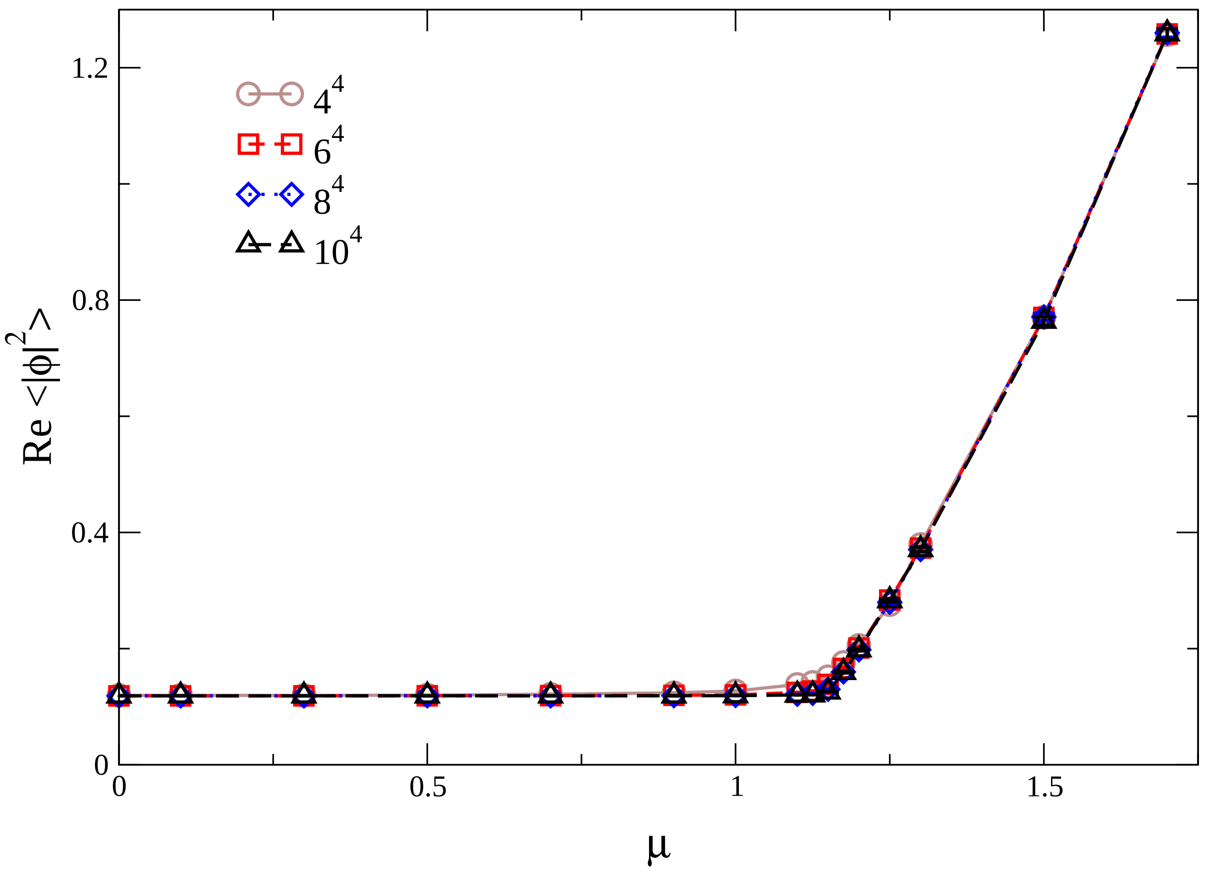
$$\frac{\partial \phi_a^{\rm R}}{\partial t} = -{\rm Re}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a\to\phi_a^{\rm R}+i\phi_a^{\rm I}} + \eta_a$$

$$\frac{\partial \phi_a^{\rm I}}{\partial t} = -{\rm Im}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a\to\phi_a^{\rm R}+i\phi^{\rm I}}$$

- straightforward to solve numerically, $m = \lambda = 1$
- lattices of size N^4, with $N = 4, 6, 8, 10$

Relativistic Bose gas

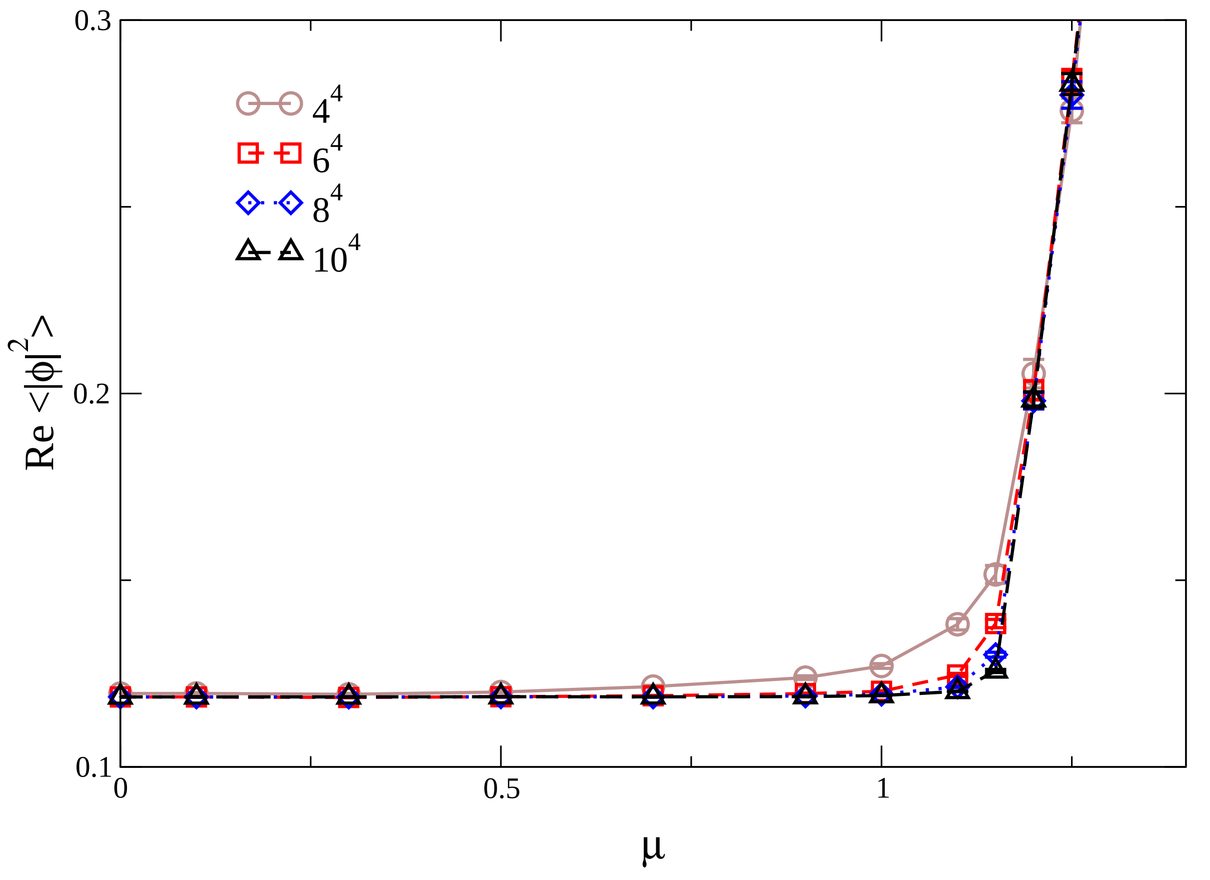
field modulus squared $$|\phi|^2 \to \frac{1}{2}\left({\phi_a^{\rm R}}^2 - {\phi_a^{\rm I}}^2\right) + i\phi_a^{\rm R}\phi_a^{\rm I}$$



Silver Blaze!

Relativistic Bose gas

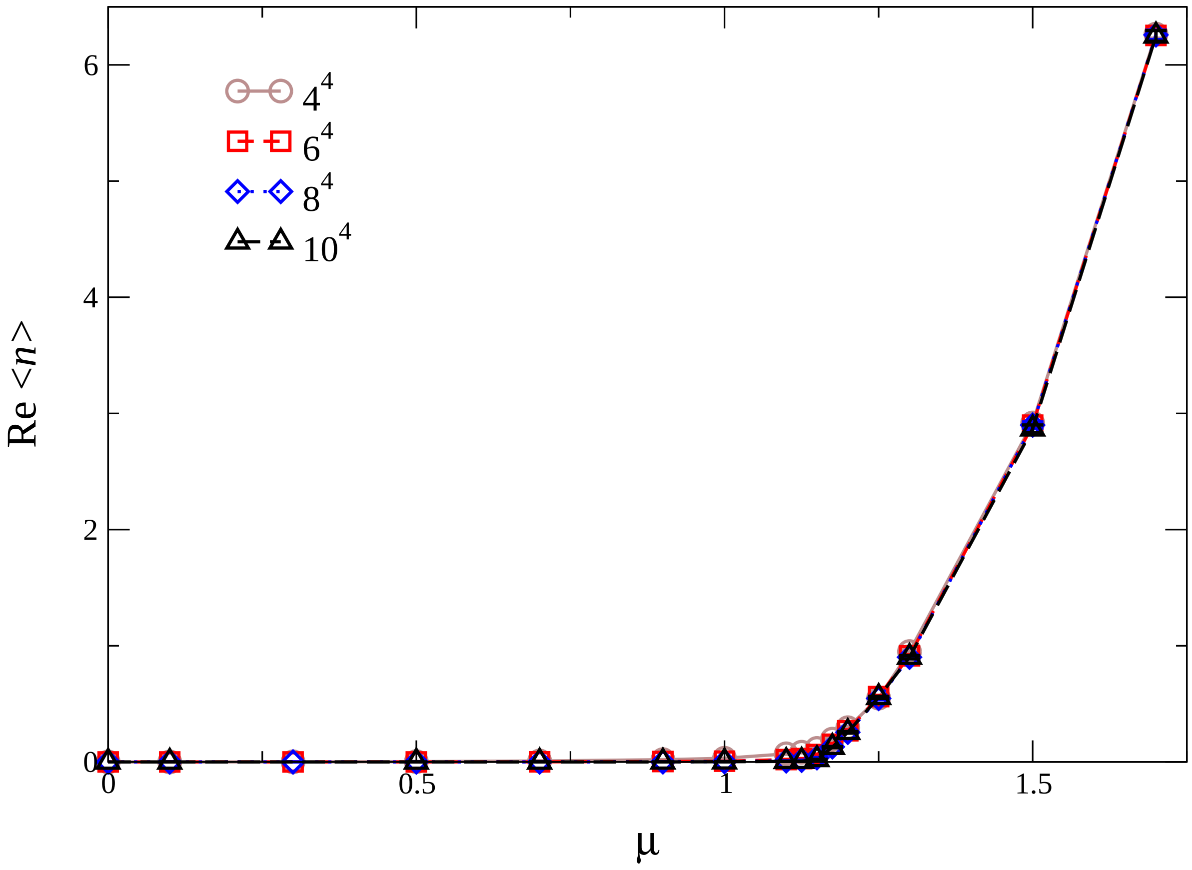
field modulus squared $$|\phi|^2 \to \frac{1}{2}\left({\phi_a^{\rm R}}^2 - {\phi_a^{\rm I}}^2\right) + i\phi_a^{\rm R}\phi_a^{\rm I}$$



second order phase transition in thermodynamic limit

Relativistic Bose gas

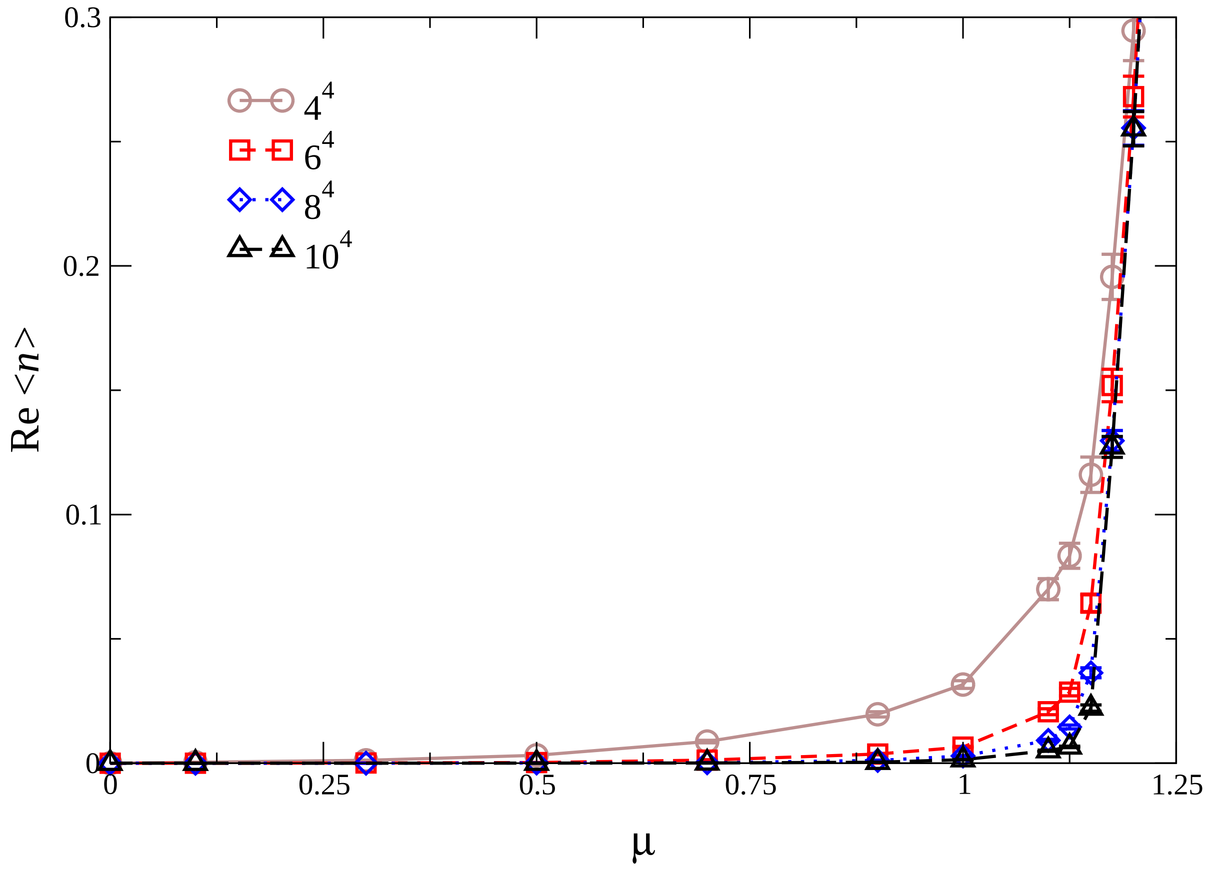
density $\langle n \rangle = (1/\Omega)\partial \ln Z/\partial\mu$



Silver Blaze

Relativistic Bose gas

density $\langle n \rangle = (1/\Omega)\partial \ln Z/\partial\mu$



second order phase transition in thermodynamic limit

Silver Blaze and the sign problem

Silver Blaze and sign problems are intimately related

- phase quenched theory $Z_{\rm pq} = \int D\phi |e^{-S}|$

physics of phase quenched theory:

- chemical potential appears only in mass parameter (in continuum notation)

$$V = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

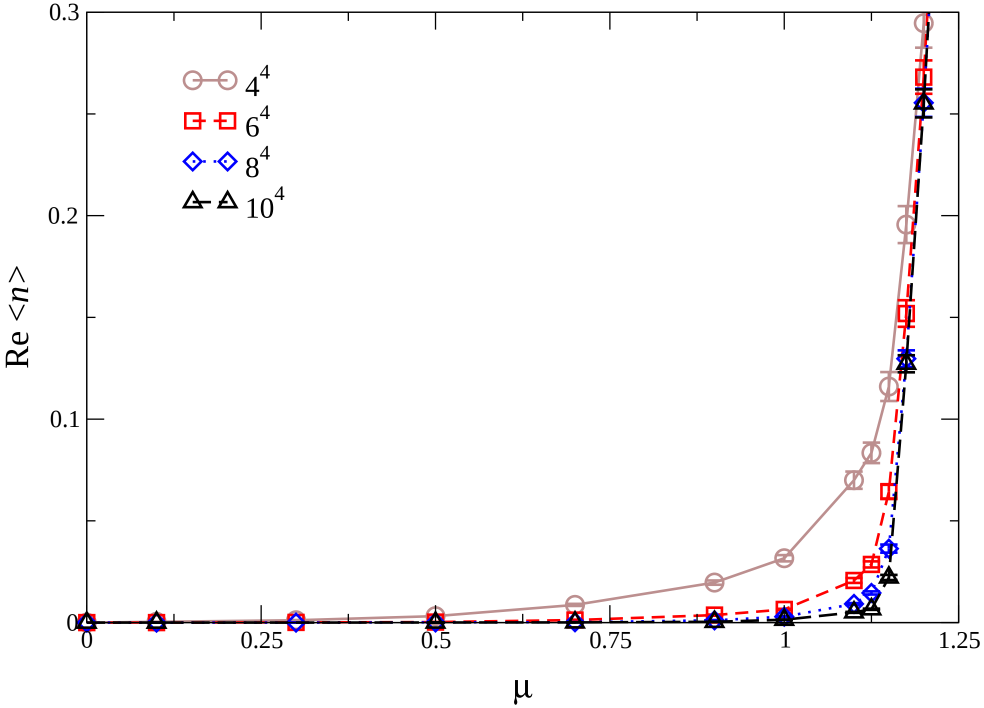
- dynamics of symmetry breaking, no Silver Blaze

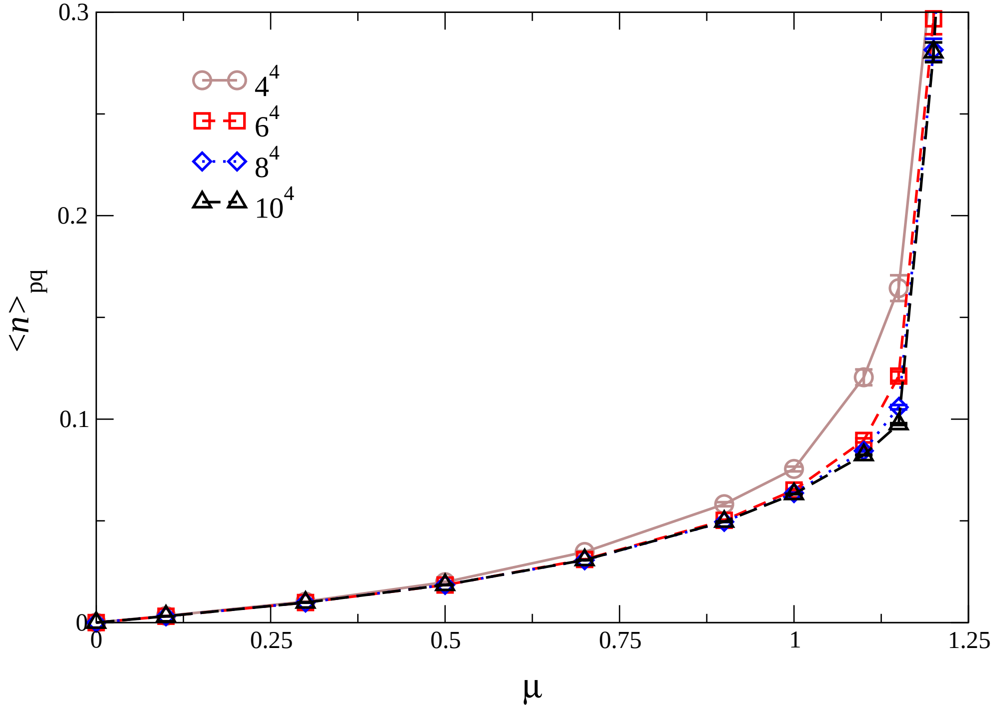
in QCD: phase quenched = finite isospin

onset at $\mu = m_\pi/2$ instead of $m_B/3$

Silver Blaze and the sign problem

density





complex | phase quenched

phase $e^{i\varphi} = e^{-S}/|e^{-S}|$ does precisely what is expected

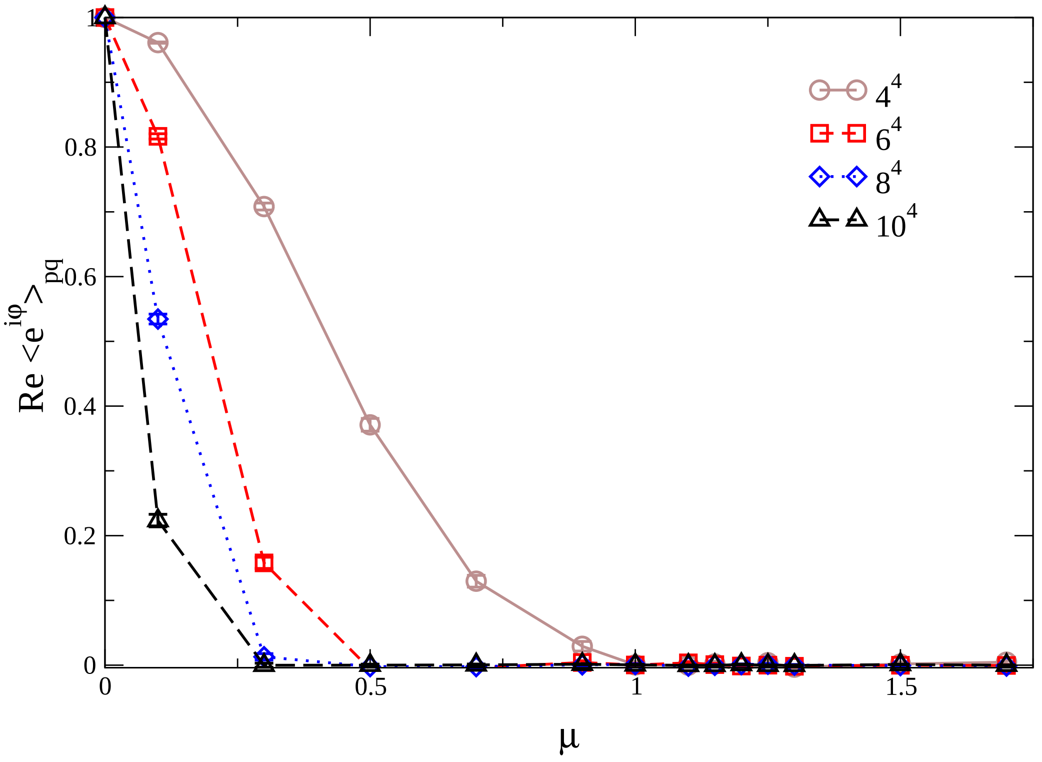
How severe is the sign problem?

- complex action $e^{-S} = |e^{-S}|e^{i\varphi}$
- average phase factor in phase quenched theory

$$\langle e^{i\varphi}\rangle_{\rm pq} = \frac{Z_{\rm full}}{Z_{\rm pq}}$$

$$= e^{-\Omega\Delta f} \to 0$$

as $\Omega \to \infty$



- exponentially hard in thermodynamic limit

Lattice gauge theory

- partition function

$$Z = \int DU\, e^{-S_B} \det M$$

- M is the fermion matrix
- fermion determinant is complex

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

SU(3) lattice gauge theory

Langevin update for link variables $U_{x\nu}$:

$$U_{x\nu}(t+\epsilon) = R_{x\nu}(t)\, U_{x\nu}(t) \qquad R_{x\nu} = \exp\left[i\lambda_a \left(\epsilon K_{x\nu a} + \sqrt{\epsilon}\eta_{x\nu a}\right)\right]$$

Gell-mann matrices λ_a ($a = 1, \ldots 8$)

- drift term

$$K_{x\nu a} = -D_{x\nu a} S_{\text{eff}}[U] \qquad S_{\text{eff}} = S_B + S_F \qquad S_F = -\ln\det M$$

- noise

$$\langle \eta_{x\nu a} \rangle = 0 \qquad \langle \eta_{x\nu a} \eta_{x'\nu' a} \rangle = 2\delta_{xx'}\delta_{\nu\nu'}\delta_{aa'}$$

real action: $\Rightarrow K^\dagger = K \Leftrightarrow R^\dagger R = 1 \Leftrightarrow U \in$ SU(3)

complex action: $\Rightarrow K^\dagger \neq K \Leftrightarrow R^\dagger R \neq 1 \Leftrightarrow U \in$ SL$(3, \mathbb{C})$

Heavy dense QCD

- bosonic action: standard SU(3) Wilson action

$$S_B = -\beta \sum_P \left(\frac{1}{6} \left[\mathrm{Tr}\, U_P + \mathrm{Tr}\, U_P^{-1} \right] - 1 \right)$$

- determinant $\det M$ for Wilson fermions

  fermion matrix:

$$M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left(e^{\mu} \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_{-4} \right)$$

Heavy dense QCD

- hopping expansion:

$$\det M \approx \det\left[1-\kappa\left(e^{\mu}\Gamma_{+4}U_{x,4}T_4 + e^{-\mu}\Gamma_{-4}U^{-1}_{x,4}T_{-4}\right)\right]$$

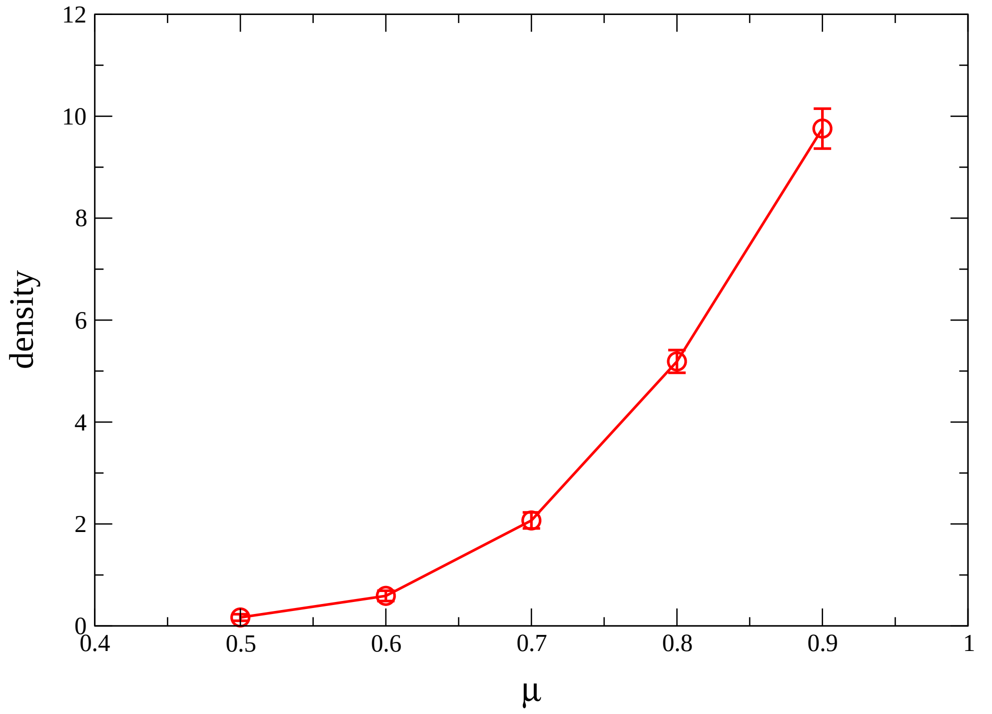
$$= \prod_{\mathbf{x}} \det\left(1 + he^{\mu/T}\mathcal{P}_{\mathbf{x}}\right)^2 \det\left(1 + he^{-\mu/T}\mathcal{P}^{-1}_{\mathbf{x}}\right)^2$$

  with $h = (2\kappa)^{N_\tau}$ and (conjugate) Polyakov loops $\mathcal{P}^{(-1)}_{\mathbf{x}}$

- static quarks propagate in temporal direction only: Polyakov loops
- full gauge dynamics included

GA & Stamatescu 08

Density

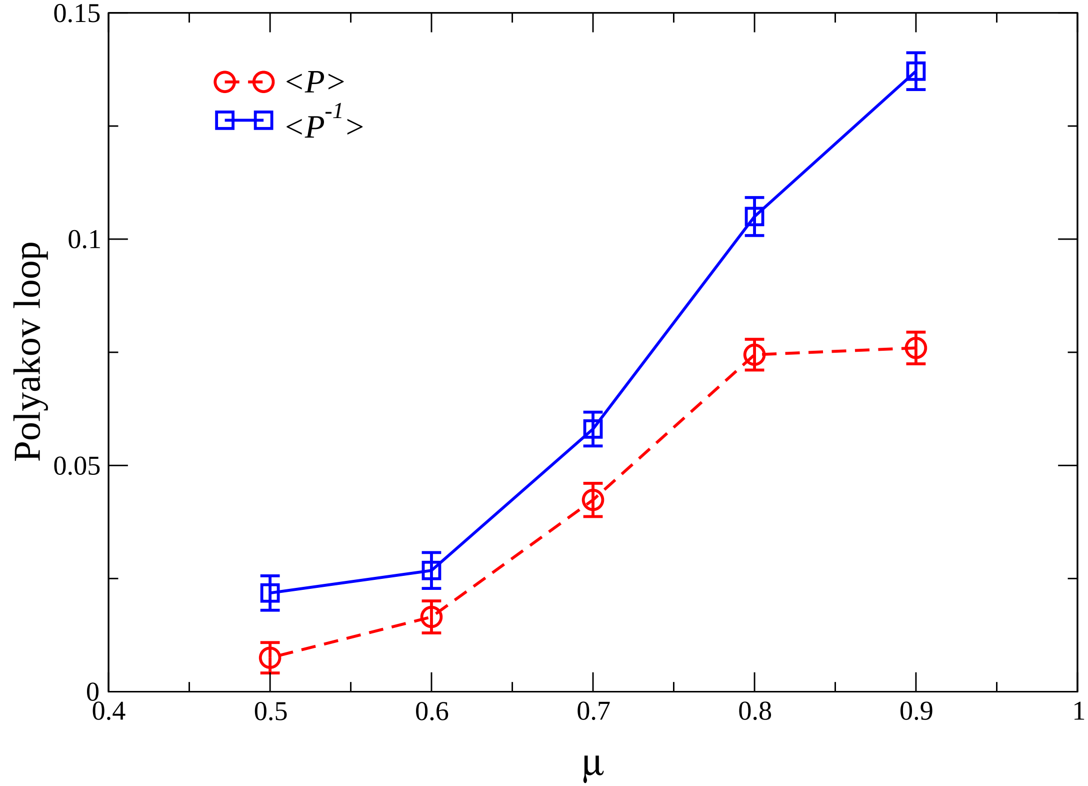


first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density phase $\Rightarrow$ high-density phase

(conjugate) Polyakov loops

results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$



low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase

SU(3) $\to$ SL(3,$\mathbb{C}$)

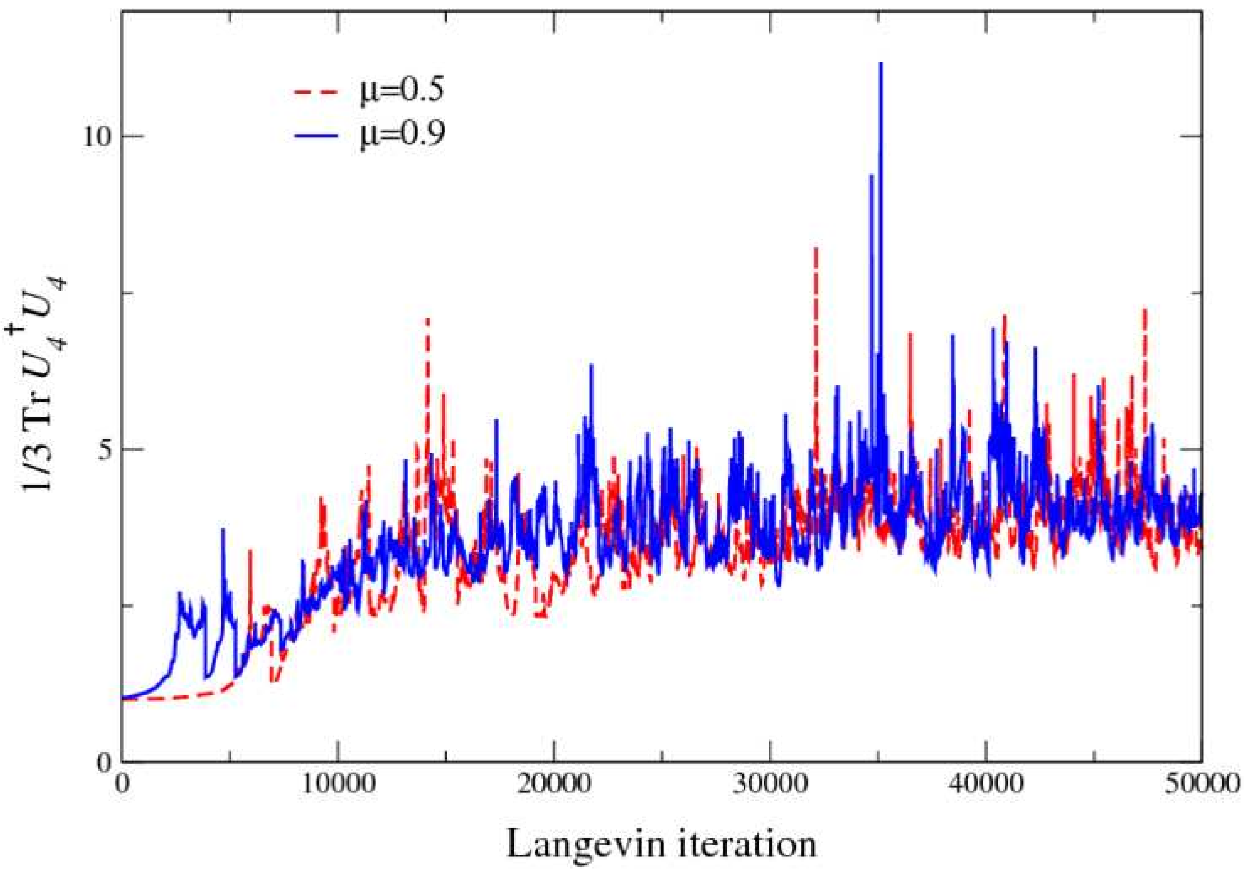
- complex Langevin dynamics: no longer in SU(3)
- instead $U \in$ SL(3,$\mathbb{C}$)
- in terms of gauge potentials $U = e^{i\lambda_a A_a/2}$
  A_a is now complex
- how far from SU(3)?

consider

$$\frac{1}{N}\mathrm{Tr}\, U^\dagger U \begin{cases} = 1 \ \text{ if } \ U \in \text{SU}(N) \\ \\ \geq 1 \ \text{ if } \ U \in \text{SL}(N,\mathbb{C}) \end{cases}$$

SU(3) $\to$ SL(3,$\mathbb{C}$)

$$\frac{1}{3}\mathrm{Tr}\ U^\dagger U \geq 1 \qquad\qquad = 1 \ \text{ if } \ U \in \text{SU(3)}$$



One-dimensional QCD

- exactly solvable Gibbs 86, Bilic & Demeterfi 88
- phase quenched: transition at $\mu = \mu_c$, full: no transition

severe sign problem when $|\mu| > |\mu_c|$

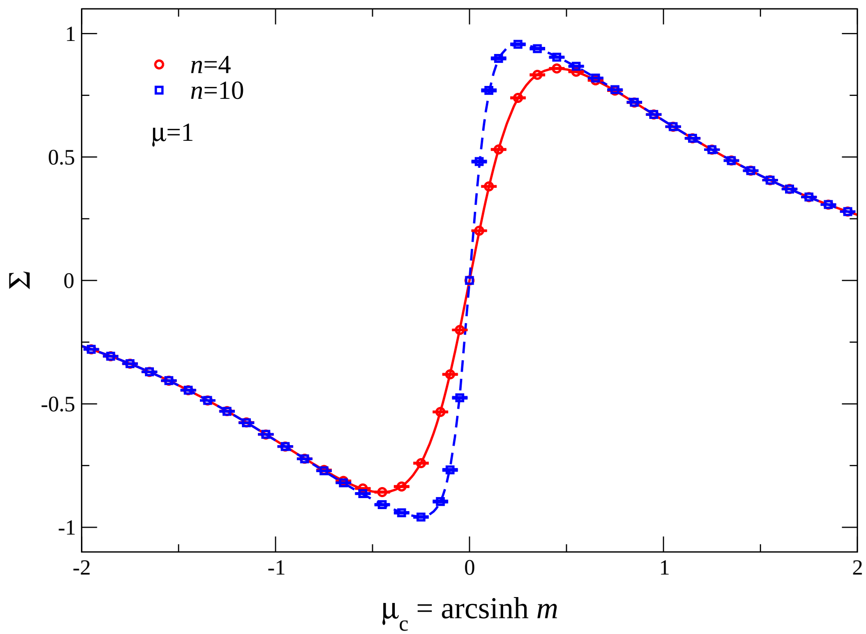
- chiral condensate:
  write as integral over spectral density

$$\Sigma = \int d^2 z \, \frac{\rho(z;\mu)}{z+m} \qquad\qquad \mu_c = \text{arcsinh}\, m$$

- $\rho(z;\mu)$ complex and oscillatory Ravagli & Verbaarschot 07
- condensate independent of μ: Silver Blaze
- solve with complex Langevin GA & Splittorff 10

One-dimensional QCD

- exact results reproduced
- discontinuity at $\mu_c = 0$ in thermodynamic limit $n \to \infty$



- sign problem severe when $|\mu_c| < |\mu|$
- condensate independent of μ: Silver Blaze

One-dimensional QCD

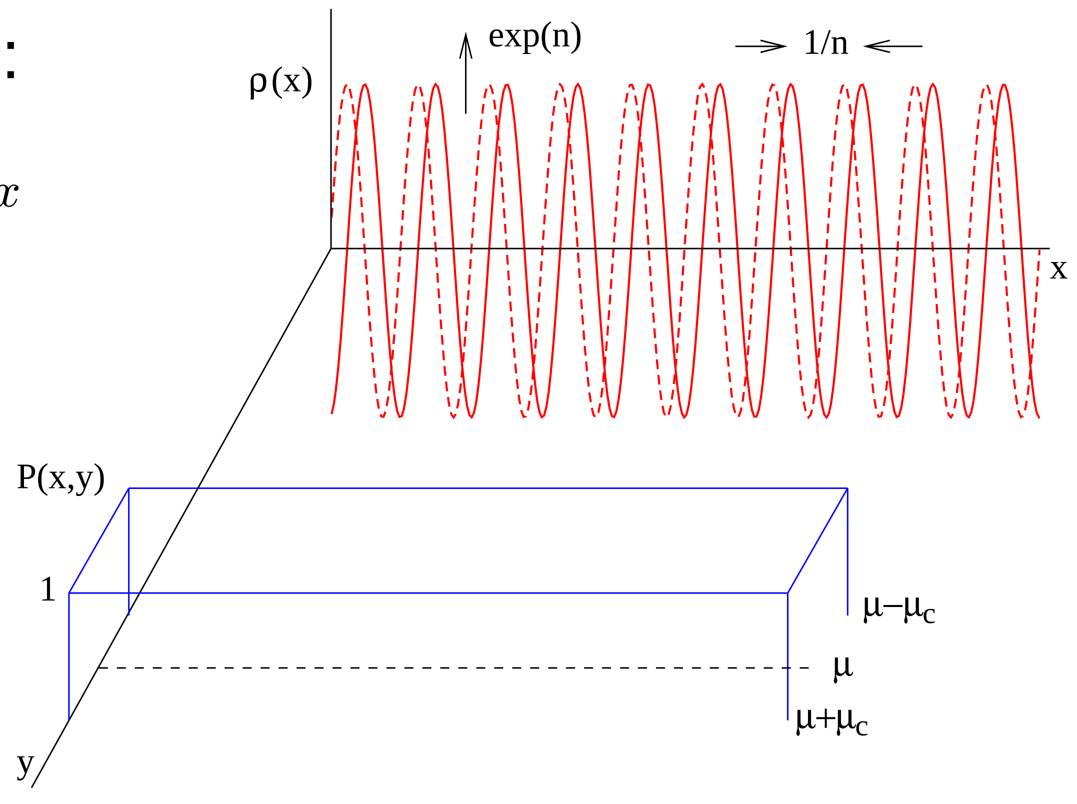
elegant analytical solution in thermodynamic limit:

- original distribution:

$$\rho(x) \sim e^{n(\mu-\mu_c)} e^{inx}$$

  when $n \to \infty$

- real distribution sampled by complex Langevin:



$$P(x,y) = \begin{cases} 1 & \mu - \mu_c < y < \mu + \mu_c \\ 0 & \text{elsewhere} \end{cases}$$

Troubled past

1. numerical problems: runaways, instabilities

   $\Rightarrow$ adaptive stepsize

   no instabilities observed, works for SU(3) gauge theory

   `GA, James, Seiler & Stamatescu 09`

   `a la Ambjorn et al 86`

2. theoretical status unclear

   $\Rightarrow$ detailed analyis, identified necessary conditions

   `GA, FJ, ES & IOS 09-12`

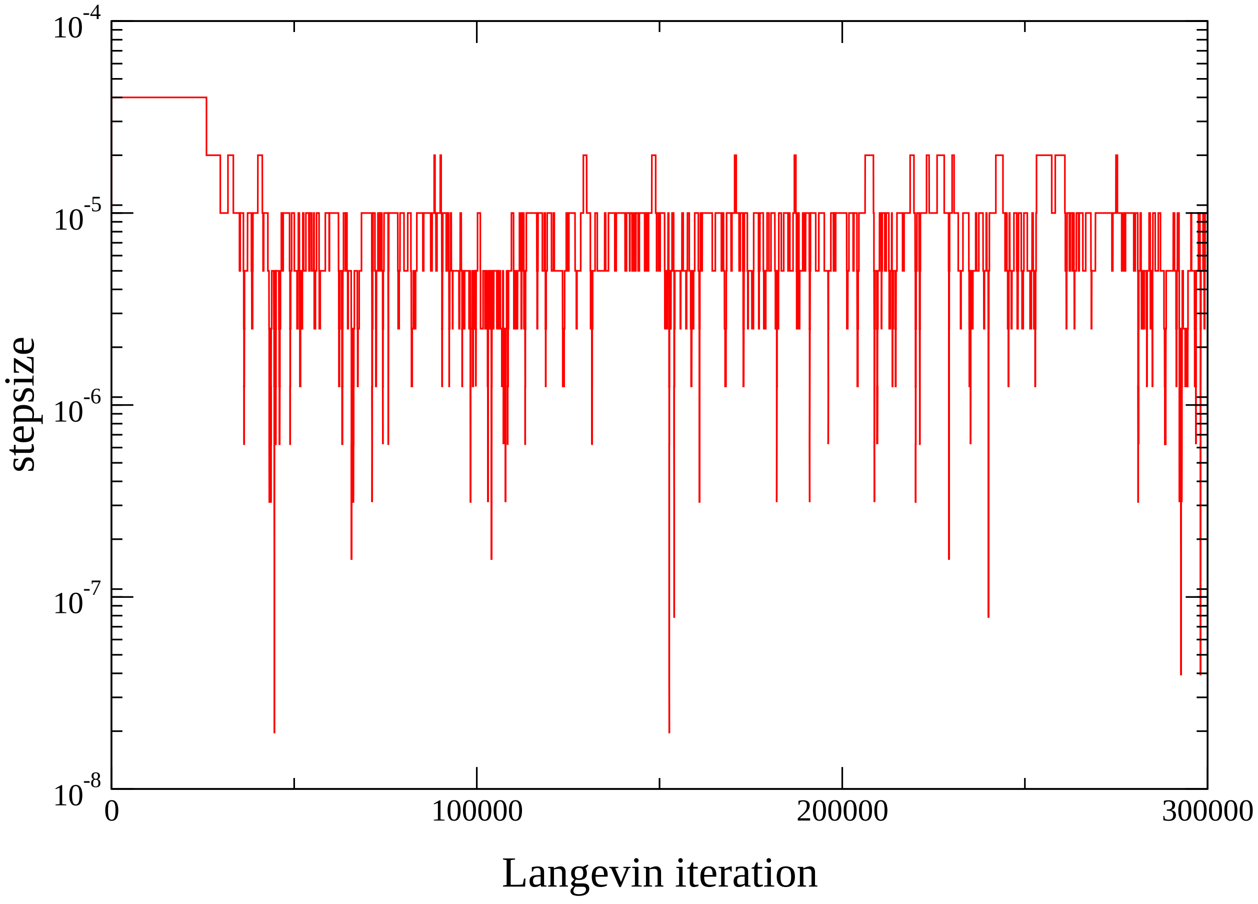
3. convergence to wrong limit

   $\Rightarrow$ better understood but not yet resolved

   `in progress`

Instabilities: heavy dense QCD

adaptive time step during the evolution



occasionally *very* small stepsize required

can go to longer Langevin times without problems

Analytical understanding

consider expectation values and Fokker-Planck equations

one degree of freedom x, complex action $S(x)$, $\rho(x) \sim e^{-S(x)}$

- wanted: $\langle O(x,t)\rangle_\rho = \int dx\ \rho(x,t)O(x)$

$$\partial_t \rho(x,t) = \partial_x \left(\partial_x + S'(x)\right) \rho(x,t)$$

- solved with CLE:

$$\langle O(x+iy,t)\rangle_P = \int dxdy\ P(x,y;t)O(x+iy)$$

$$\partial_t P(x,y;t) = \left[\partial_x \left(\partial_x - K_x\right) - \partial_y K_y\right] P(x,y;t)$$

with $K_x = -\mathrm{Re}S'$, $K_y = -\mathrm{Im}S'$

- question: $\langle O(x+iy,t)\rangle_P = \langle O(x,t)\rangle_\rho$?

Analytical understanding

question: $\langle O(x+iy,t)\rangle_P = \langle O(x,t)\rangle_\rho$ as $t \to \infty$?

answer: yes, provided some conditions are met:

- distribution $P(x,y)$ should drop off fast enough in y direction
- partial integration without boundary terms possible
- actually $O(x+iy)P(x,y)$ for large enough set $O(x)$

$\Rightarrow$ distribution should be sufficiently localized

- can be tested numerically via criteria for correctness

$$\langle LO(x+iy)\rangle = 0$$

with L Langevin operator 0912.3360, 1101.3270

SU(3) spin model

apply these ideas to 3D SU(3) spin model GA & James 11

- earlier solved with complex Langevin Karsch & Wyld 85
  Bilic, Gausterer & Sanielevici 88
- however, no detailed tests performed

$\Rightarrow$ test reliability of complex Langevin using developed tools

- analyticity in μ^2:
  - from imaginary to real μ
  - Taylor series
- criteria for correctness
- comparison with flux formulation Gattringer & Mercado 12

SU(3) spin model

3-dimensional SU(3) spin model: $S = S_B + S_F$

$$S_B = -\beta \sum_{<xy>} \left[P_x P_y^* + P_x^* P_y \right]$$

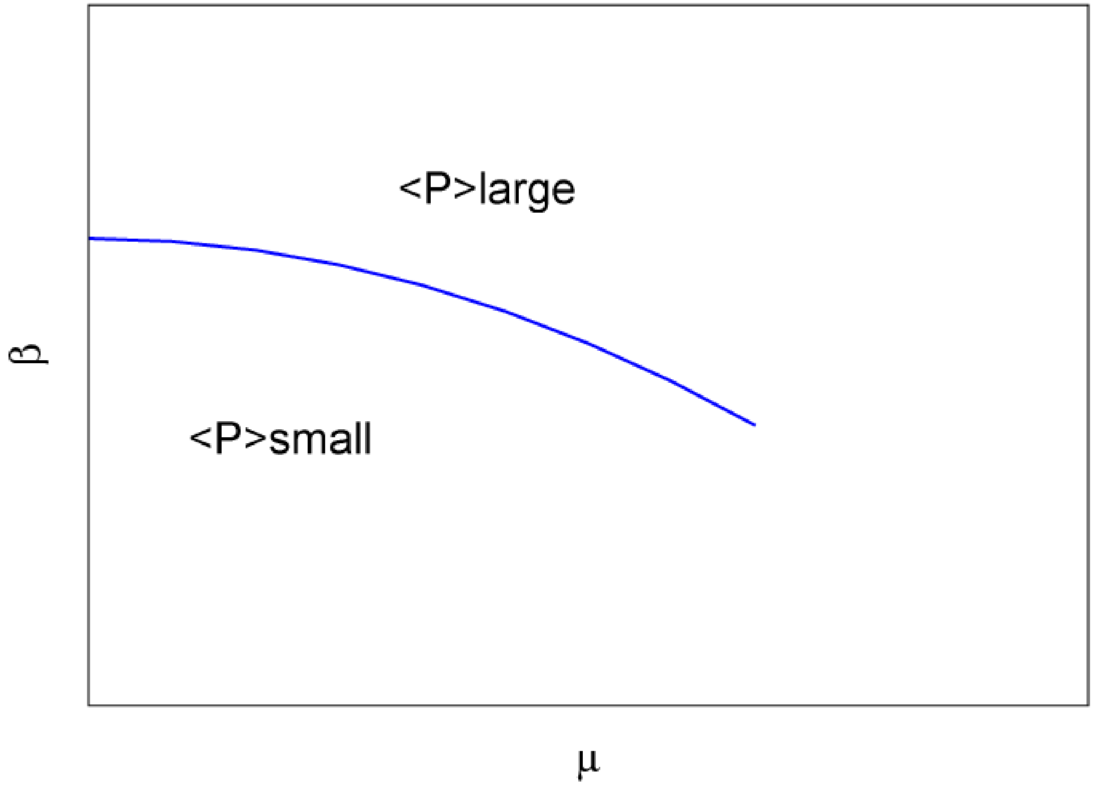
$$S_F = -h \sum_x \left[e^{\mu} P_x + e^{-\mu} P_x^* \right]$$

- SU(3) matrices: $P_x = \mathrm{Tr}\, U_x$
- gauge action: nearest neighbour Polyakov loops
- (static) quarks represented by Polyakov loops
- complex action $S_F^*(\mu) = S_F(-\mu^*)$

effective model for QCD with static quarks, centre symmetry

SU(3) spin model

- phase structure

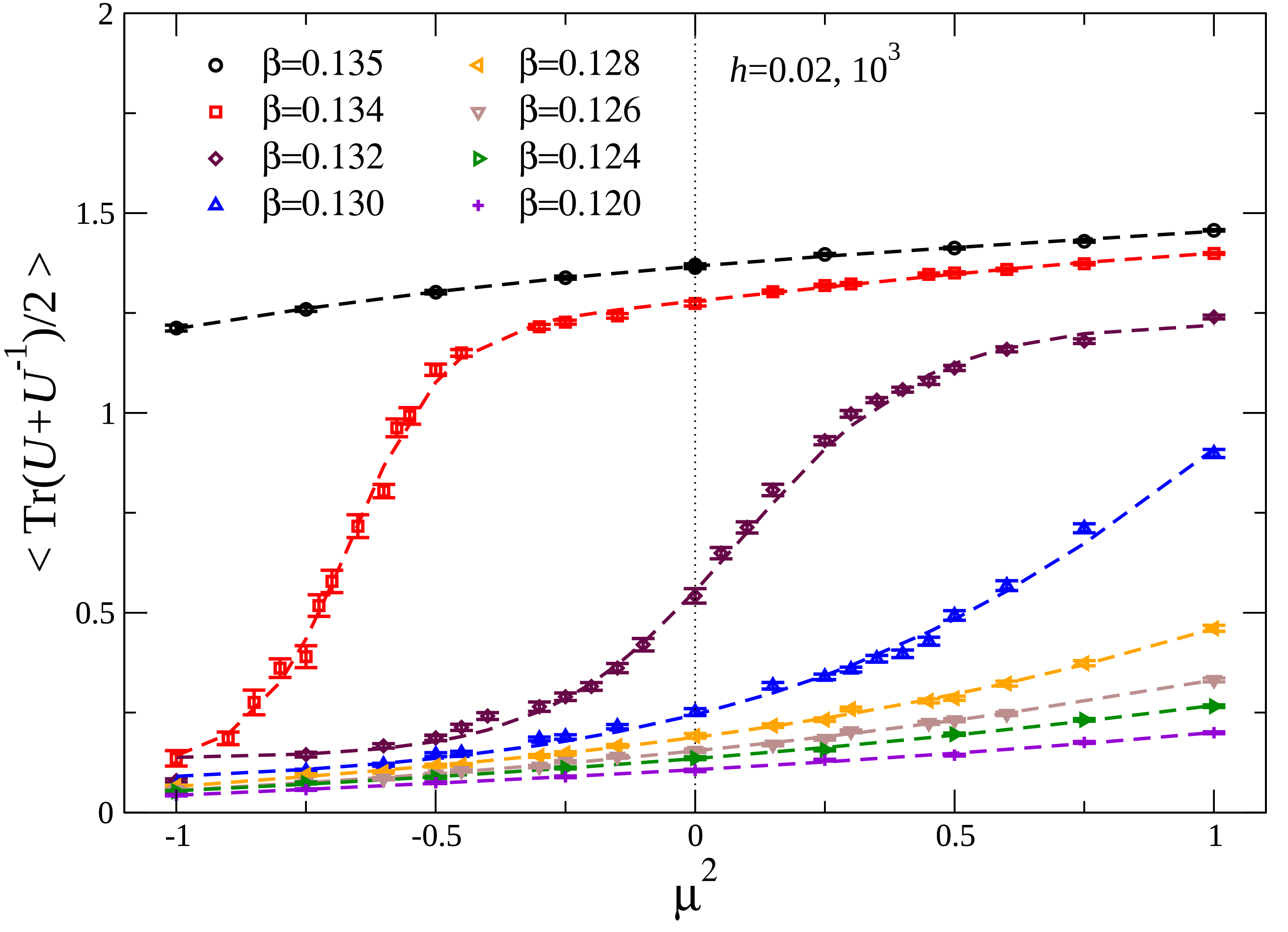


- effective model for QCD with static quarks

SU(3) spin model

real and imaginary potential:

first-order transition in $\beta - \mu^2$ plane, $\langle P + P^* \rangle / 2$

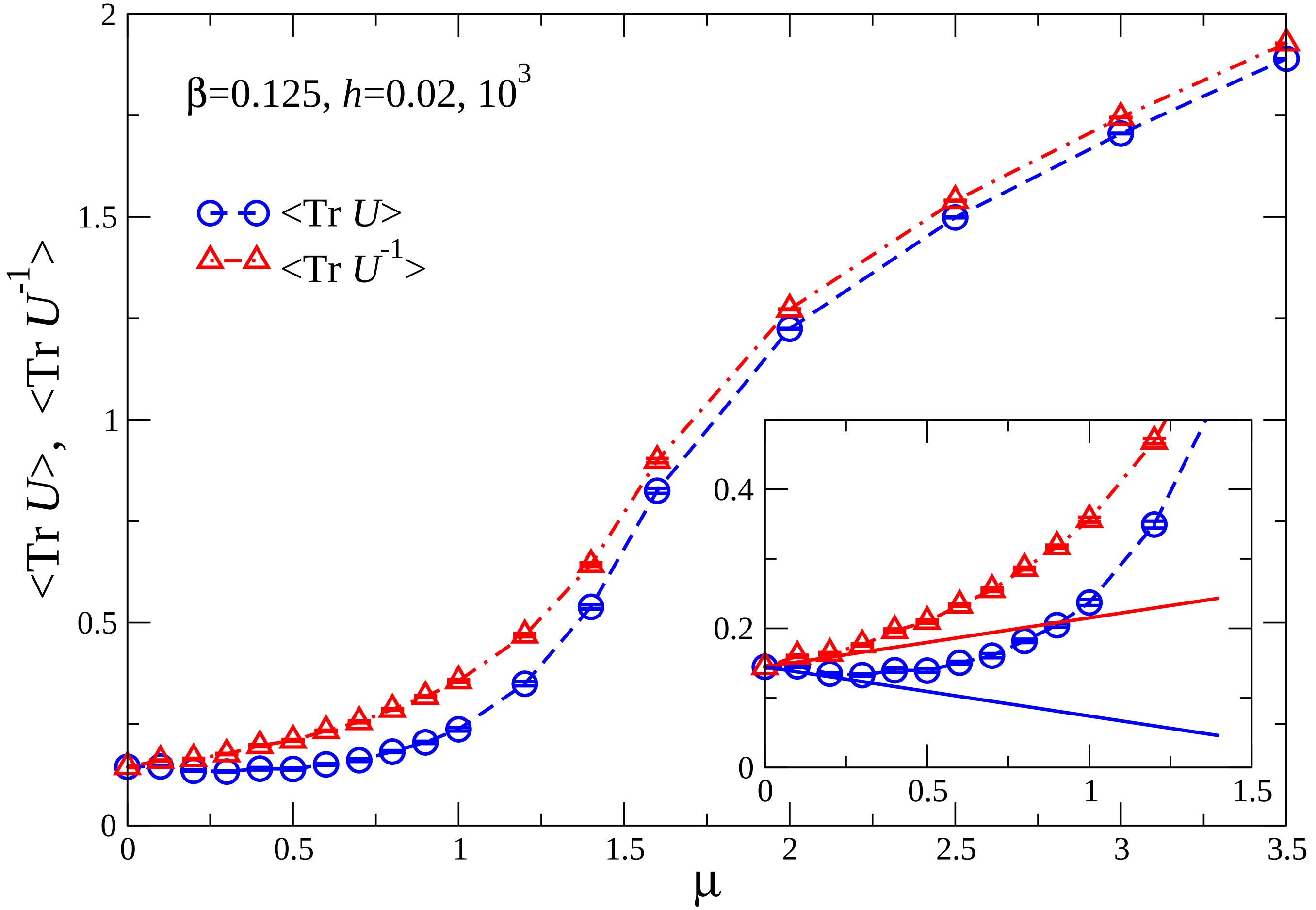


negative μ^2: real Langevin — positive μ^2: complex Langevin

SU(3) spin model

real chemical potential

immediate splitting between $\langle P \rangle$ and $\langle P^* \rangle$: no Silver Blaze



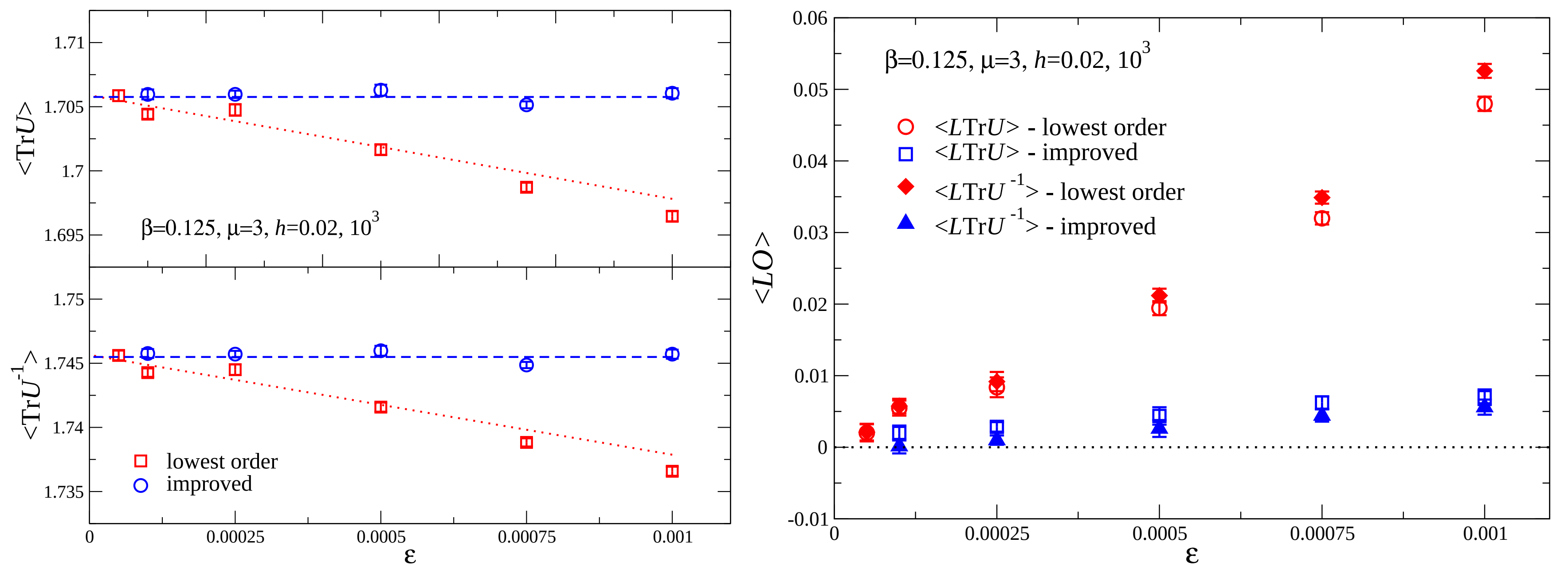
inset: lines from first-order Taylor expansion

SU(3) spin model

stepsize dependence

left: $\langle P \rangle$ (top) and $\langle P^* \rangle$ (bottom) at $\mu = 3$

right: criteria for correctness $\langle LO \rangle = 0$

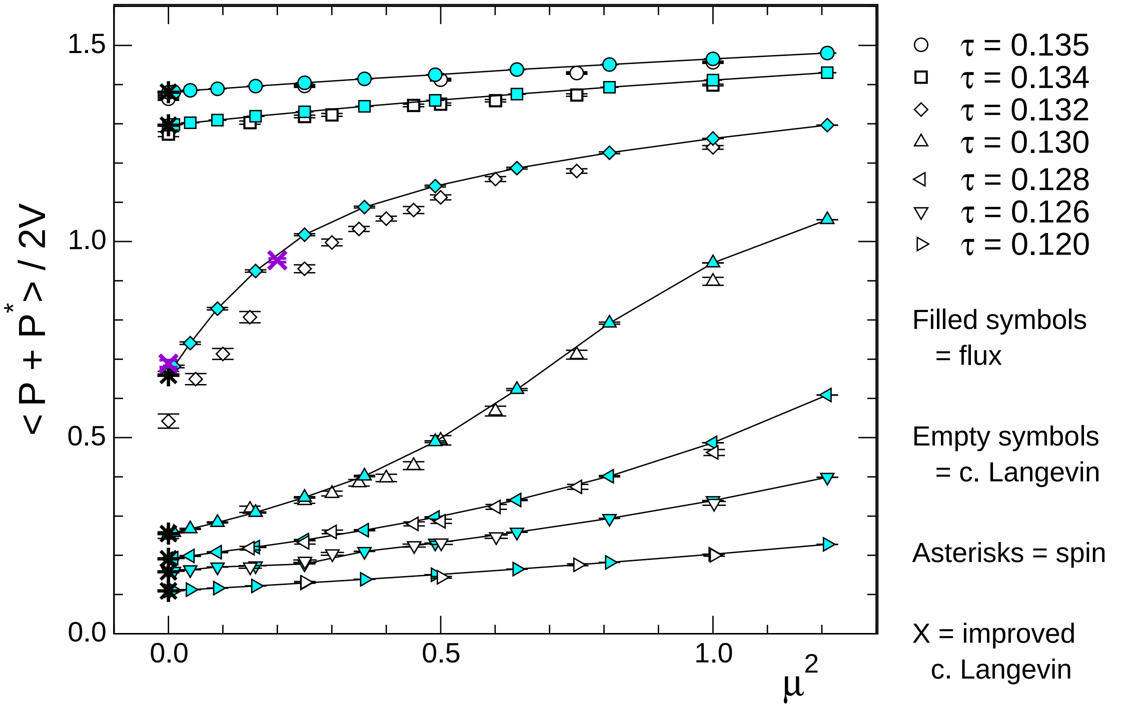


improved stepsize algorithm to eliminate linear dependence

criteria satisfied as stepsize $\epsilon \to 0$

SU(3) spin model

comparison with result obtained using flux representation



- CL: finite stepsize errors in lowest-order algorithm
- improved algorithm removes discrepancy in critical region

SU(3) spin model

complex Langevin passes all the tests: why?

- localized distribution: fast decay in imaginary direction
- real manifold is stable under small fluctuations
- Haar measure plays essential role

$\Rightarrow$ Haar measure contribution to complex drift restoring

Stabilizing drift

- Haar measure contribution to complex drift restoring
- controlled exploration of the complex field space

employ this: generate Jacobian by field redefinition

$$Z = \int dx\, e^{-S(x)} \qquad x = x(u) \qquad J(u) = \frac{\partial x(u)}{\partial u}$$

$$= \int du\, e^{-S_{\rm eff}(u)} \qquad S_{\rm eff}(u) = S(u) - \ln J(u)$$

drift: $$K(u) = -S'_{\rm eff}(u) = -S'(u) + J'(u)/J(u)$$

which field redefinition?

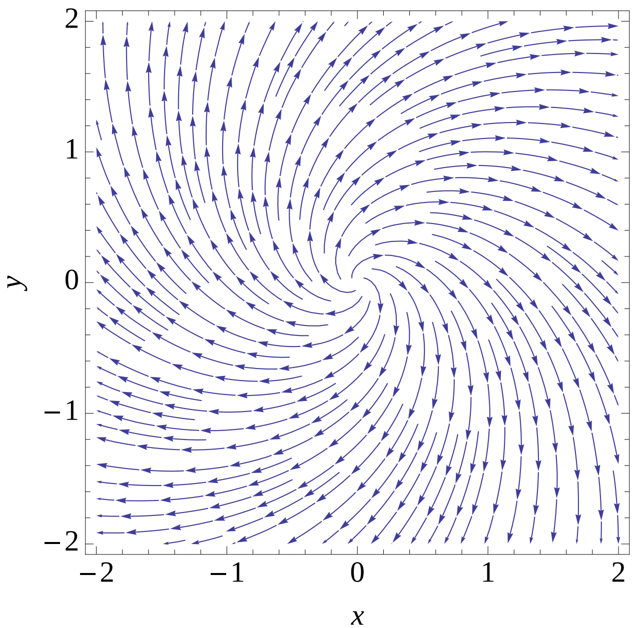
singular at $J(u) = 0$ but restoring in complex plane

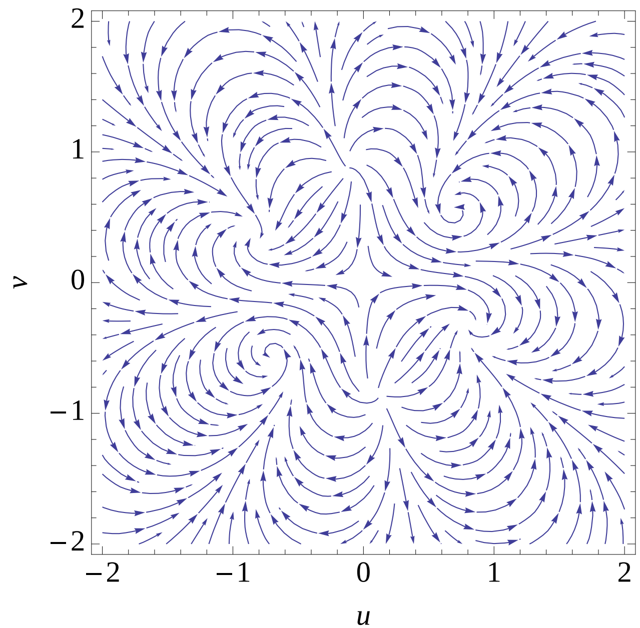
Fun with complex Langevin

Gaussian example: defined when $\mathrm{Re}(\sigma) = a > 0$

$$Z = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\sigma x^2} \qquad \sigma = a + ib \qquad \langle x^2 \rangle = \frac{1}{\sigma}$$

what if $a < 0$? flow in complex space for $a = -1, b = 1$:



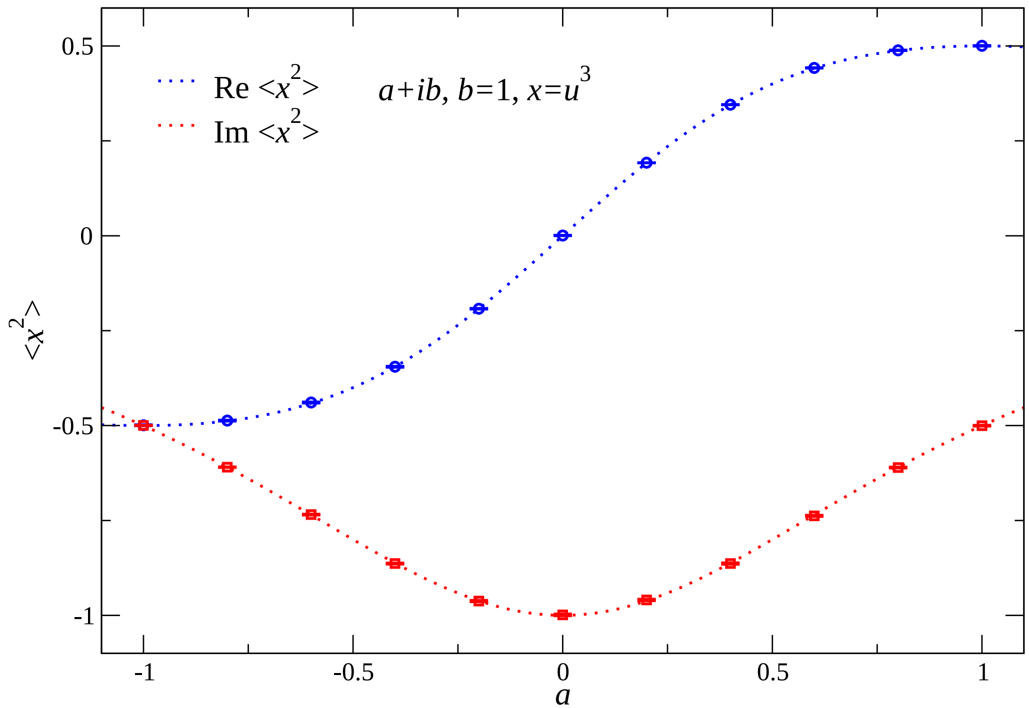


left: highly unstable

right: after transformation $x(u) = u^3$
attractive fixed points

Fun with complex Langevin

do CLE in the u formulation and compute $\langle x^2\rangle = \langle u^6\rangle$



$$\langle x^2\rangle = \frac{1}{\sigma} = \frac{a-ib}{a^2+b^2}$$

take also negative a

CLE finds the analytically continued answer to negative a!

clearly needs more exploration $-$ potential for stabilization $-$ affects convergence

XY model

three-dimensional XY model at nonzero μ

$$S = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0}\right)$$

- μ couples to the conserved Noether charge
- symmetry $S^*(\mu) = S(-\mu^*)$

unexpectedly difficult to simulate with complex Langevin!

numerics shares many features with heavy dense QCD

GA & James 10

also studied by Banerjee & Chandrasekharan using worldline formulation
hep-lat/1001.3648

Convergence: XY model

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_{\rm I}$

real μ, complex action:

$$S = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} - i\mu\delta_{\nu,0}\right)$$

imaginary $\mu = i\mu_{\rm I}$, real action:

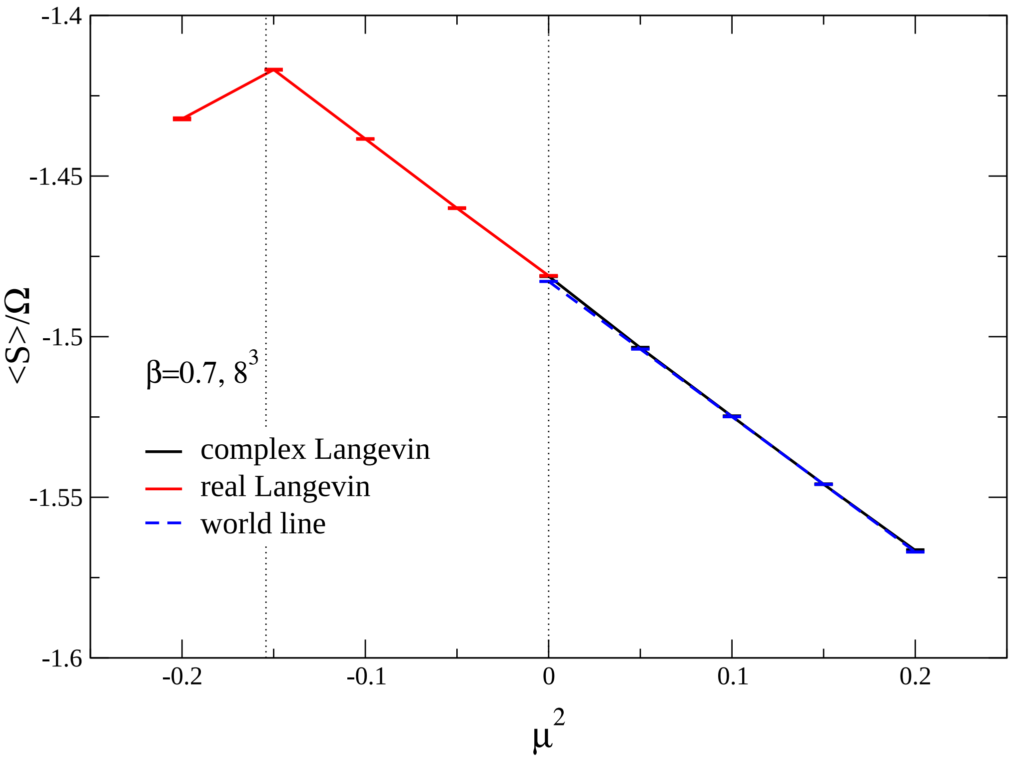
$$S_{\rm I} = -\beta \sum_x \sum_{\nu=0}^{2} \cos\left(\phi_x - \phi_{x+\hat{\nu}} + \mu_{\rm I}\delta_{\nu,0}\right)$$

- real and imag μ results analytic in μ^2

GA & James 10

Convergence: XY model

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_{\rm I}$



action density versus μ^2

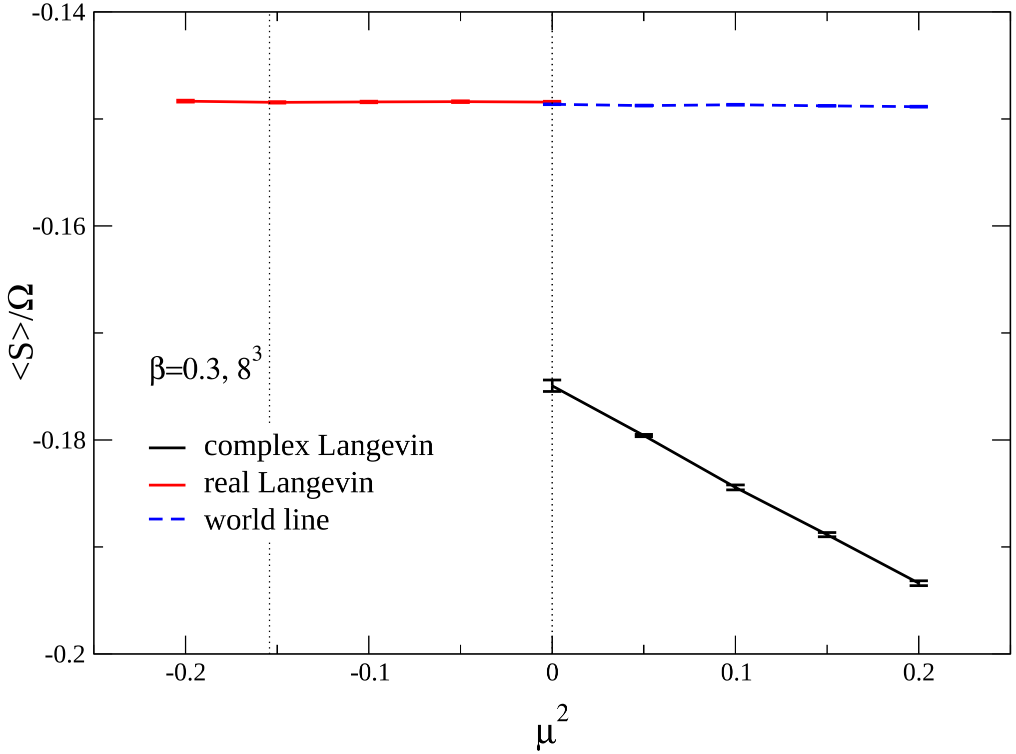
$\beta = 0.7$

ordered phase

- "Roberge-Weiss" transition at $\mu_{\rm I} = \pi/N_\tau$

Convergence: XY model

- comparison with known result (world line formulation)
- analytic continuation from imaginary $\mu = i\mu_{\rm I}$



action density versus μ^2

$\beta = 0.3$

disordered phase

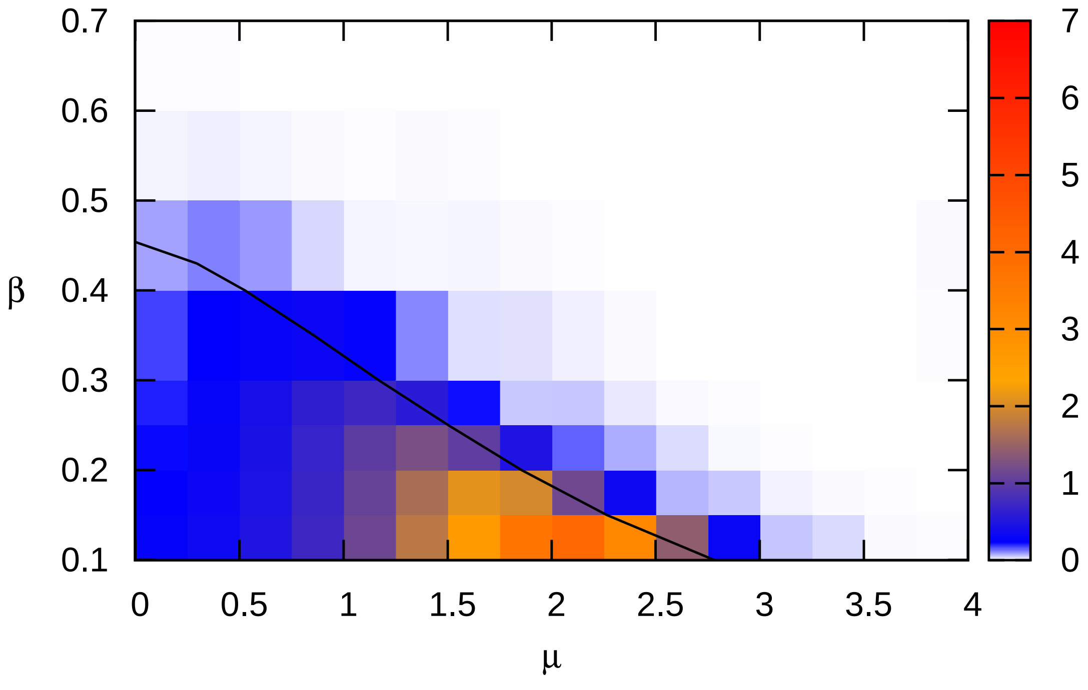
- Silver Blaze feature at small β and μ

Convergence: XY model

XY MODEL

- comparison with known result (world line formulation)

phase diagram:



relative deviation:

$$\Delta S = \frac{\langle S \rangle_{\mathrm{cl}} - \langle S \rangle_{\mathrm{wl}}}{\langle S \rangle_{\mathrm{wl}}}$$

high β: ordered

low β: disordered

- phase boundary from Banerjee & Chandrasekharan
- highly correlated with ordered/disordered phase

Convergence: XY model

- apparent correct results in the ordered phase
- incorrect result in the disordered/transition region

diagnostics:

- distribution $P[\phi_{\rm R}, \phi_{\rm I}]$ qualitatively different
- classical force distribution qualitatively different
- complexified dynamics $\neq$ real dynamics when $\mu = 0$

but:

- independent of strength of the sign problem

conclusion: failure not due to sign problem

Summary

many stimulating results: examples where complex Langevin can handle

- sign problem
- Silver Blaze problem
- phase transition
- thermodynamic limit

problems from the 80s:

- instabilities and runaways $\rightarrow$ adaptive stepsize
- convergence: correct result not guaranteed

resolution in progress, important:

- failure does not depend on strength of sign problem
- distinct from all other approaches

Outlook

QCD at nonzero μ $\Leftrightarrow$ sign problem

relevant for QCD phase diagram, heavy-ion collisions, dense objects, ...

- sign problem has been studied from many perspectives
- 'well understood' (overlap, Silver Blaze, ...)
- no solution for QCD (yet ...)

sign problem appears not only in QCD

also in many (lower-dimensional, condensed matter) theories

$\Rightarrow$ learn from those models as well

Outlook

some approaches with limited applicability in full QCD:

- overlap preserving reweighting
- Taylor series
- imaginary μ and analytical continuation
- ...

partial or full solutions in not quite QCD:

- strong coupling QCD
- flux representations in spin models (not discussed)
- complex Langevin
- ...

Outlook

to do (possible):

determine QCD phase diagram for

- imaginary chemical potential
- isospin chemical potential

$\Rightarrow$ no sign problem

$\Rightarrow$ large-scale numerical project

$\Rightarrow$ intricate phase structure depending on quark masses

Outlook

to do (possible?):

solve sign problem

don’t give up!