QCD at nonzero chemical potential and the sign problem

INT lectures 2012

III: imaginary chemical potential

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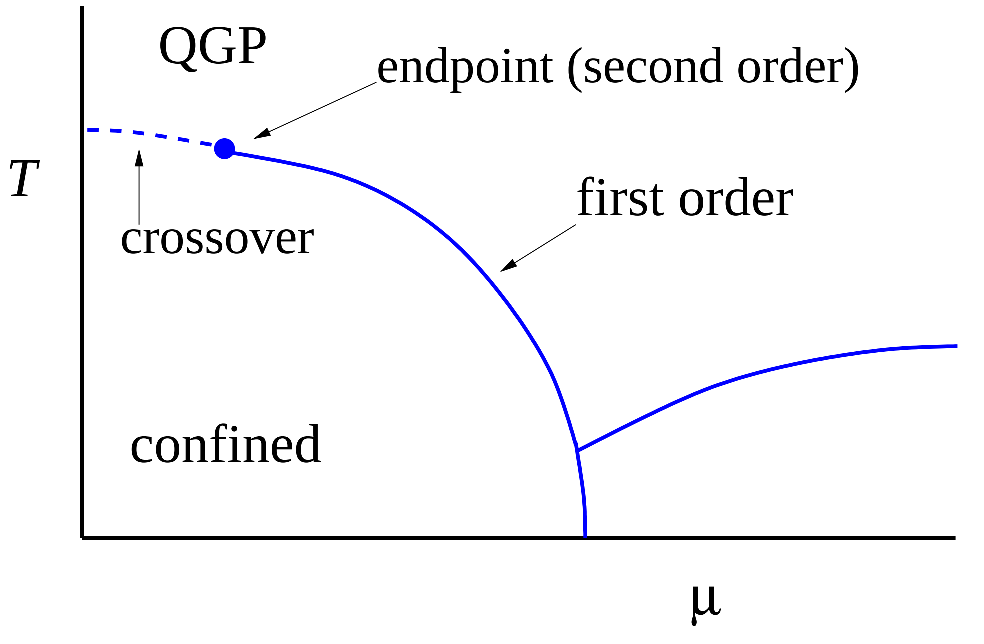
Reminder: physics goal

determine

- phase boundary between confined and deconfined phase at small μ
- critical endpoint (if it exists)



"standard conjectured" phase diagram

Imaginary μ

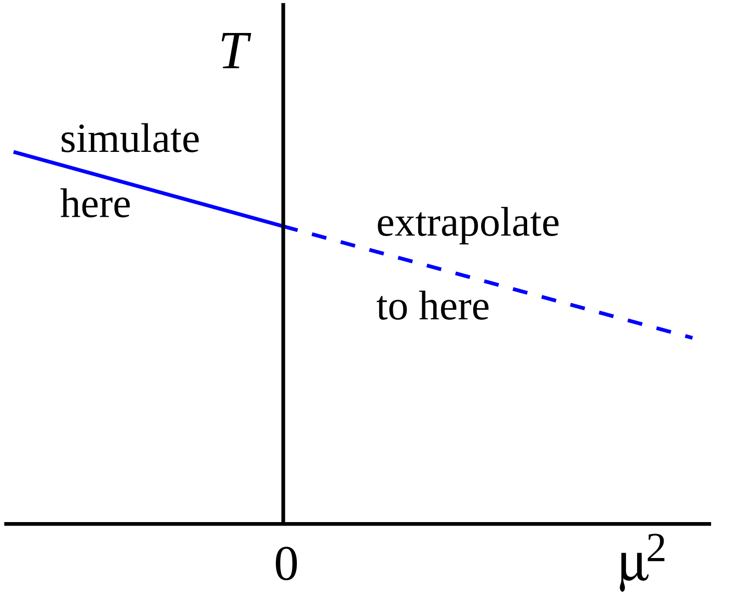
recall: $D^\dagger(\mu) = \gamma_5 D(-\mu^*)\gamma_5$

- if $\mu = i\mu_\mathrm{I}$, $\det D(i\mu_\mathrm{I})$ is real: importance sampling ok

$\Rightarrow$ perform ordinary simulations

- analytical continuation to real μ: $+\mu_\mathrm{I}^2 \to -\mu^2$

- determine phase boundary at $\mu^2 < 0$
- fit $T_c(-\mu^2)$
- obtain phase boundary at $\mu^2 > 0$



Imaginary μ

in fact: much richer than just analytical continuation

intricate phase structure

interplay with centre symmetry

- quark mass dependence of phase transition
  Columbia plot
- pure gauge theory
- with quarks and chemical potential
- Roberge-Weiss symmetry
- 3D Columbia plot
- critical properties

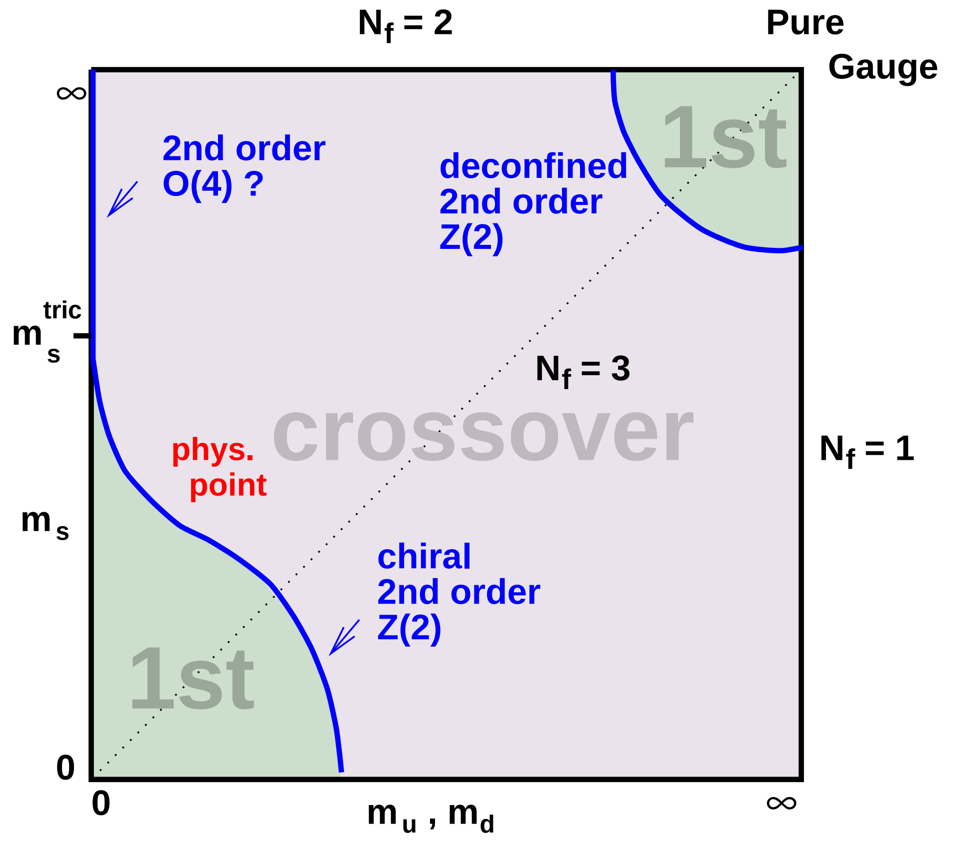
de Forcrand & Philipsen 02-now

d'Elia & Lombardo 02

d'Elia et al 02-now

Columbia plot

quark mass dependence of thermal deconfinement transition



massless quarks: chiral symmetry ⇒ chiral condensate

pure gauge: centre symmetry ⇒ Polyakov loop

Pure gauge: centre symmetry

- multiply each temporal link in a fixed time slice with phase factor z^k

$$U_4(\tau, \mathbf{x}) \quad \rightarrow \quad z^k U_4(\tau, \mathbf{x})$$

$$z^k = e^{2\pi i k/N} \qquad (k = 0, \ldots, N-1)$$

- $z^k \mathbb{1} \in Z_N$, element of the centre of SU(N): $\det(z^k \mathbb{1}) = 1$
- transformation leaves action and measure invariant: symmetry of pure gauge theory

gauge invariant order parameter: Polyakov loop

$$P(\mathbf{x}) = \mathrm{tr} \prod_{\tau=0}^{N_\tau - 1} U_4(\tau, \mathbf{x})$$

under transformation:

$$P(\mathbf{x}) \rightarrow z^k P(\mathbf{x})$$

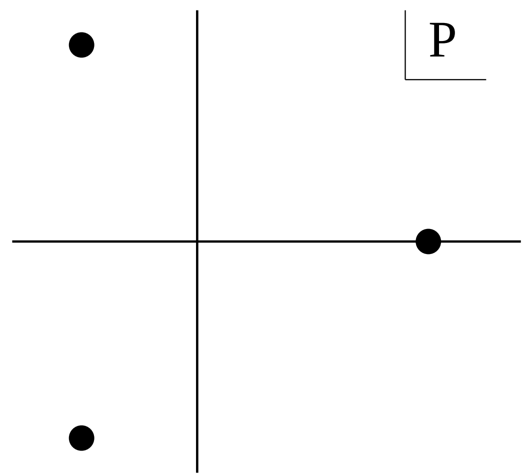
Pure gauge: centre symmetry

under transformation: $P(\mathbf{x}) \to zP(\mathbf{x})$

- if $\langle P \rangle = 0$: centre symmetry unbroken
- if $\langle P \rangle \neq 0$: centre symmetry broken: N equivalent vacua

$N = 3$:

$z = 1 \quad e^{\pm 2\pi i/3}$



Pure gauge: centre symmetry

interpretation:

- $\langle P \rangle = 0$: confined phase
- $\langle P \rangle \neq 0$: deconfined phase

why? P worldline of a massive (static) quark

free energy between static quark/anti-quark pair: $F_{q\bar{q}}(r)$

$$\langle P(\mathbf{x}) P^\dagger(\mathbf{y}) \rangle = e^{-F_{q\bar{q}}(r)/T} \qquad r = |\mathbf{x} - \mathbf{y}|$$

$$\lim_{r \to \infty} \langle P(\mathbf{x}) P^\dagger(\mathbf{y}) \rangle = \langle P(\mathbf{x}) \rangle \langle P^\dagger(\mathbf{y}) \rangle = |\langle P \rangle|^2$$

confined: $F_{q\bar{q}}(\infty) \to \infty \quad \Leftrightarrow \quad \langle P \rangle = 0$

deconfined: $F_{q\bar{q}}(\infty)$ finite $\quad \Leftrightarrow \quad \langle P \rangle \neq 0$

$\Rightarrow$ order parameter for (de)confinement

Centre symmetry with quarks

centre symmetry explicitly broken: $\bar{\psi}_x U_4 \psi_{x+4}$ not invariant

- pure gauge: all $Z(N)$ vacua equivalent
- with quarks: trivial vacuum preferred $\langle P \rangle \sim 1$
  'external magnetization'

add imaginary chemical potential: $U_4 \to e^{i\mu_I} U_4$

centre transformation can be undone by shift in μ_I!

- move all μ_I dependence to final time slice: $e^{i\mu_I/T}$
- perform $Z(N)$ transformation on final time slice
- combination

$$z^k e^{i\mu_I/T} = \exp i \left(\frac{\mu_I}{T} + \frac{2\pi k}{N} \right)$$

Roberge-Weiss symmetry

shift in μ_I can be undone by centre transformation

new symmetry: Roberge & Weiss 86

$$Z\left(\frac{\mu}{T}\right) = Z\left(\frac{\mu}{T} + \frac{2\pi i k}{N}\right) \qquad Z(\mu) = Z(-\mu)$$

periodicity in the μ_I direction with period $2\pi/N \times T$

$\Rightarrow$ range of μ_I/T limited by π/N

Polyakov loop not invariant $\Rightarrow$ preferred vacuum:

- at $\mu_I/T \sim 0$: trivial vacuum $\langle P \rangle \sim 1$
- at $\mu_I/T \sim 2\pi/N$: rotated vacuum $\langle P \rangle \sim z$
- at $\mu_I/T \sim 4\pi/N$: rotated vacuum $\langle P \rangle \sim z^2$
- $\ldots$

Periodicity in μ_I

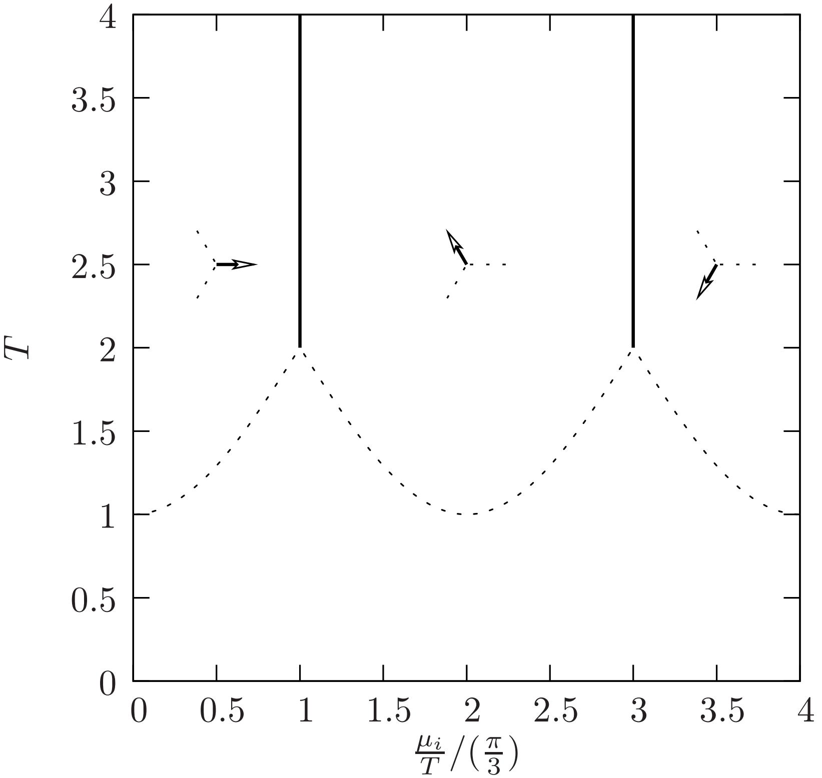
$$Z\left(\frac{\mu}{T}\right) = Z\left(\frac{\mu}{T} + \frac{2\pi i k}{N}\right)$$

- low T: confinement $\langle P \rangle = 0 \Rightarrow$ smooth transition
- high T: deconfinement $\langle P \rangle \neq 0 \Rightarrow$ sharp transitions

in deconfined phase:

1^{st} order phase transitions at $\mu_I/T = \pi/N$

Polyakov loop is proper order parameter



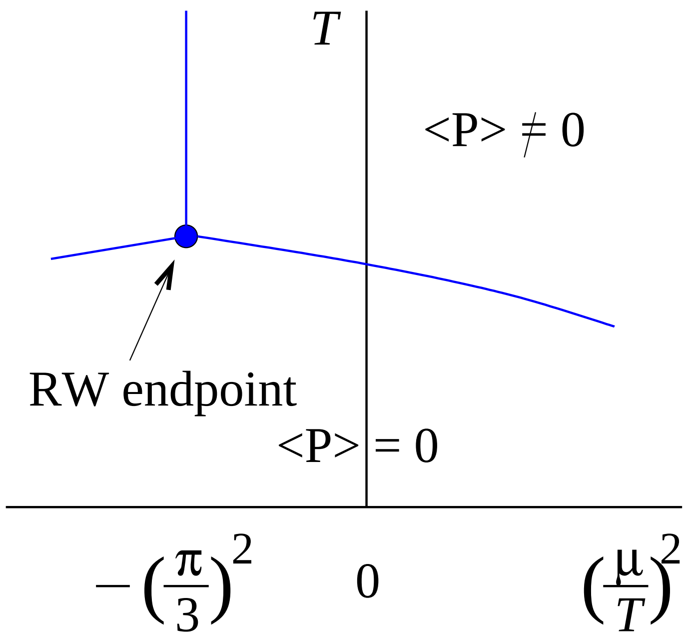
Roberge & Weiss 86

de Forcrand & Philipsen 10

Phase structure at imaginary μ

- high T: $1^{\rm st}$ order Roberge-Weiss lines at $\mu_I/T = (2r+1)\pi/N$
- include thermal (de)confinement line

$$\frac{T_c(\mu)}{T_c(0)} = 1 - \# \left(\frac{\mu}{T_c(0)}\right)^2 + \ldots$$



determine deconfinement line at $\mu^2 > 0$ from phase structure at $\mu^2 < 0$

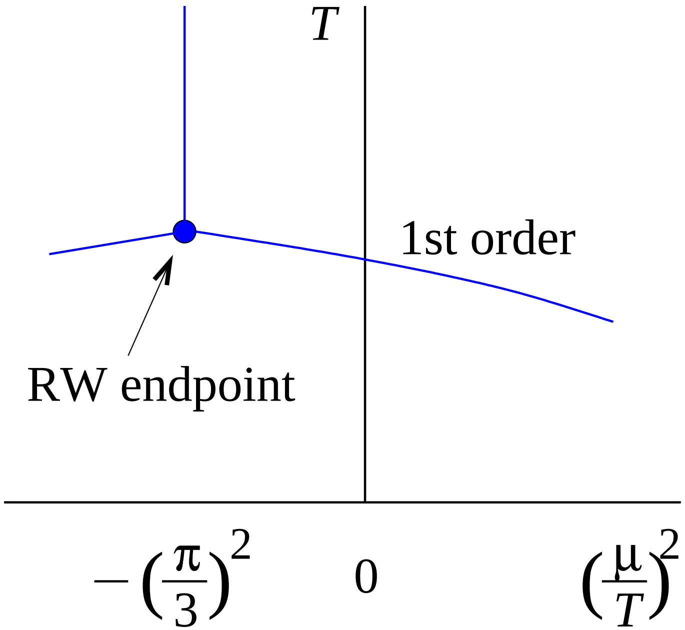
rich phase structure: strong quark mass dependence

Phase structure at imaginary μ

recall Columbia plot: depending on quark mass

- thermal transition $1^{\rm st}$ order or crossover at $\mu = 0$

heavy or light quarks: $1^{\rm st}$ order at $\mu = 0$ and all imaginary μ



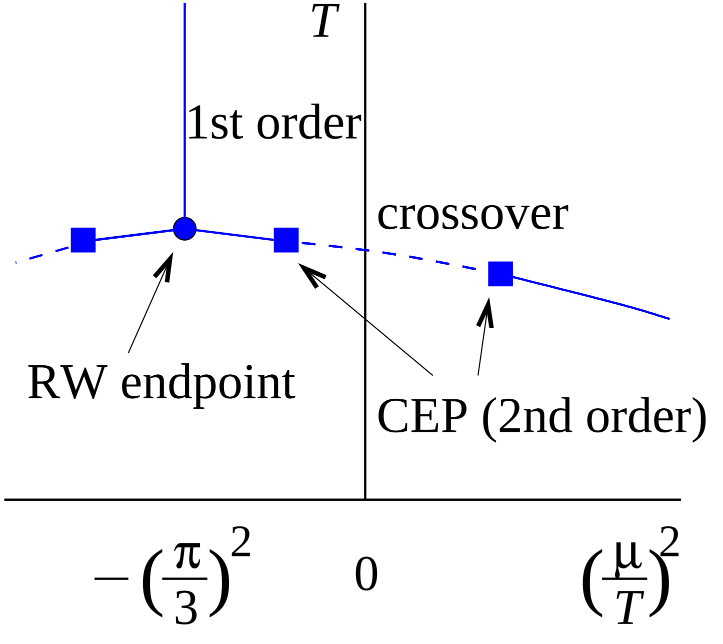
Roberge-Weiss endpoint: triple point

Phase structure at imaginary μ

recall Columbia plot: depending on quark mass

- thermal transition $1^{\rm st}$ order or crossover at $\mu = 0$

intermediate quarks: crossover at $\mu = 0$



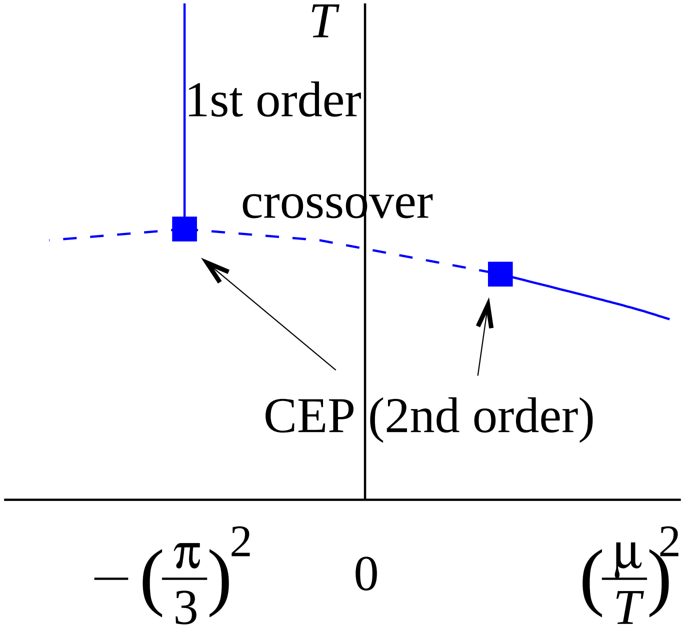
$2^{\rm nd}$ order critical endpoints at $\mu > 0$ and $\mu_I > 0$

Phase structure at imaginary μ

recall Columbia plot: depending on quark mass

- thermal transition 1^{st} order or crossover at $\mu = 0$

change quark mass even more



Roberge-Weiss endpoint $=$ critical endpoint!

Phase structure at imaginary μ

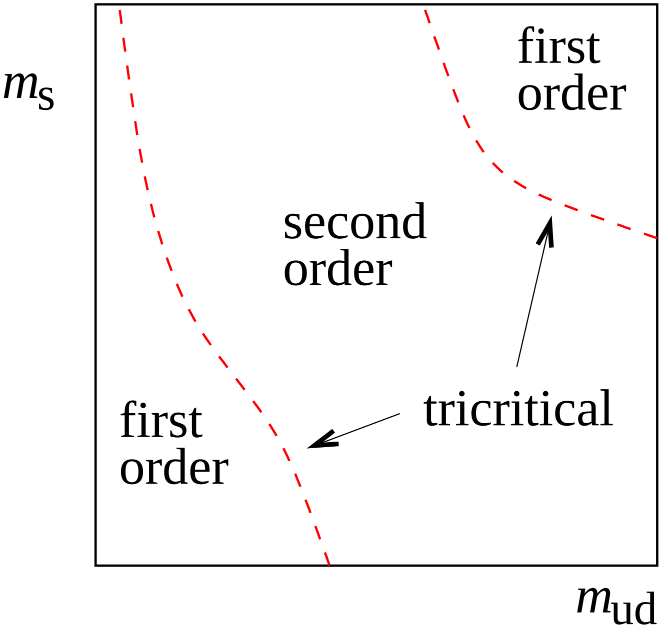
Roberge-Weiss endpoint is either

- first order triple point (heavy and light quarks)
- 2^{nd} order critical endpoint (intermediate mass)

redraw Columbia plot
at $\mu_I/T = \pi/N$

whole plane is critical

tricritical lines when quark mass is varied

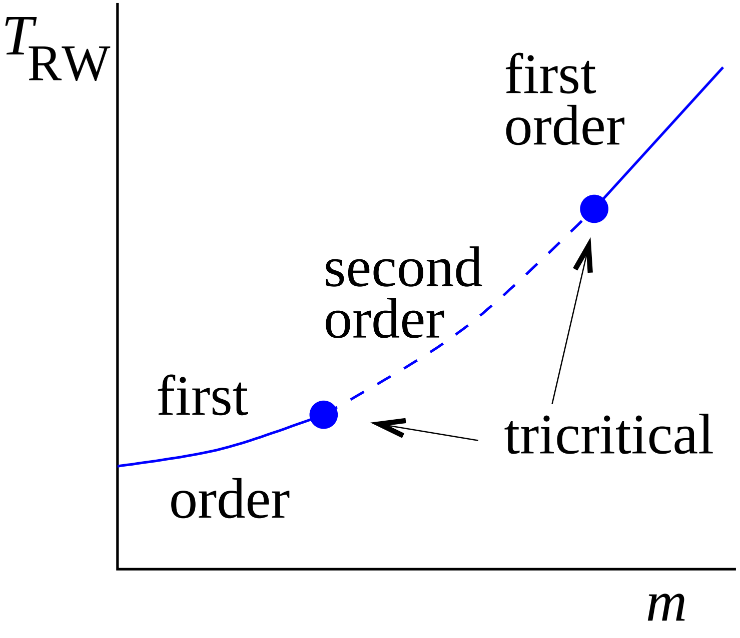


no sign problem: can be determined numerically

Phase structure at imaginary μ

detailed study of properties of RW endpoint

(at fixed $\mu_I/T = \pi/3$)



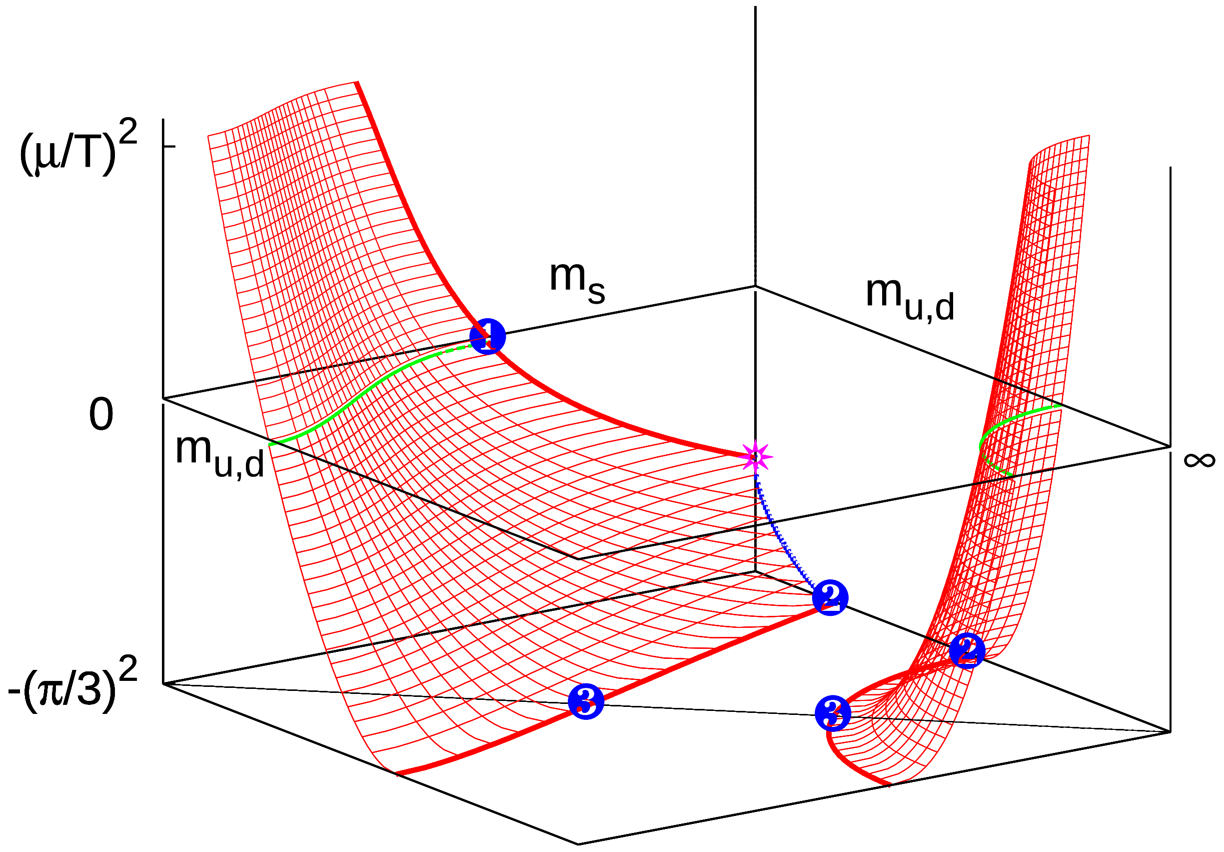
d'Elia & Sanfilippo 09 ($N_f = 2$)

de Forcrand & Philipsen 10 ($N_f = 3$)

Enlarged Columbia plot

extend Columbia plot with third direction

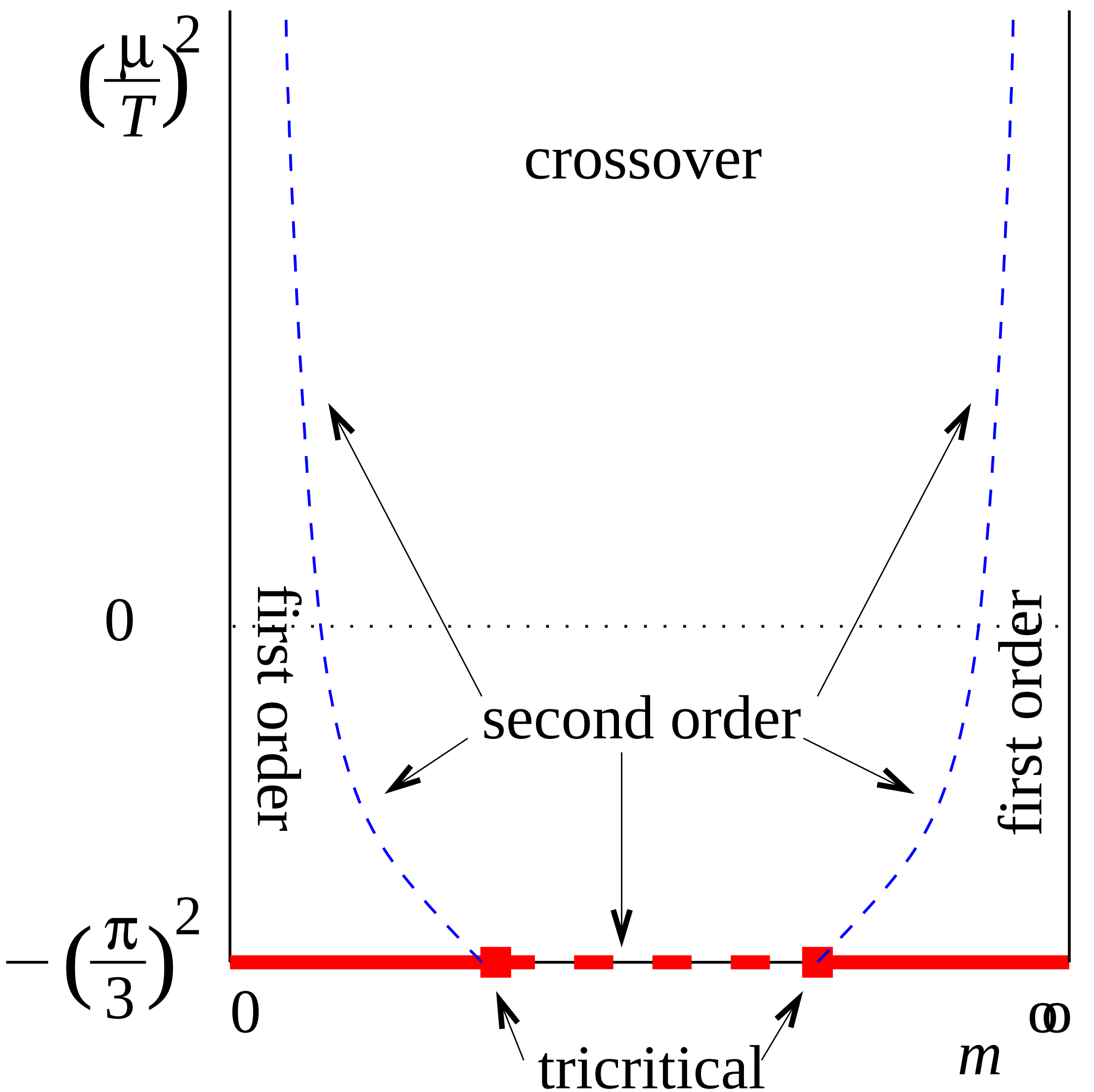
$$-(\pi/3)^2 \leq (\mu/T)^2 < \infty$$



Bonati, de Forcrand, d'Elia & Philipsen 12

Enlarged Columbia plot

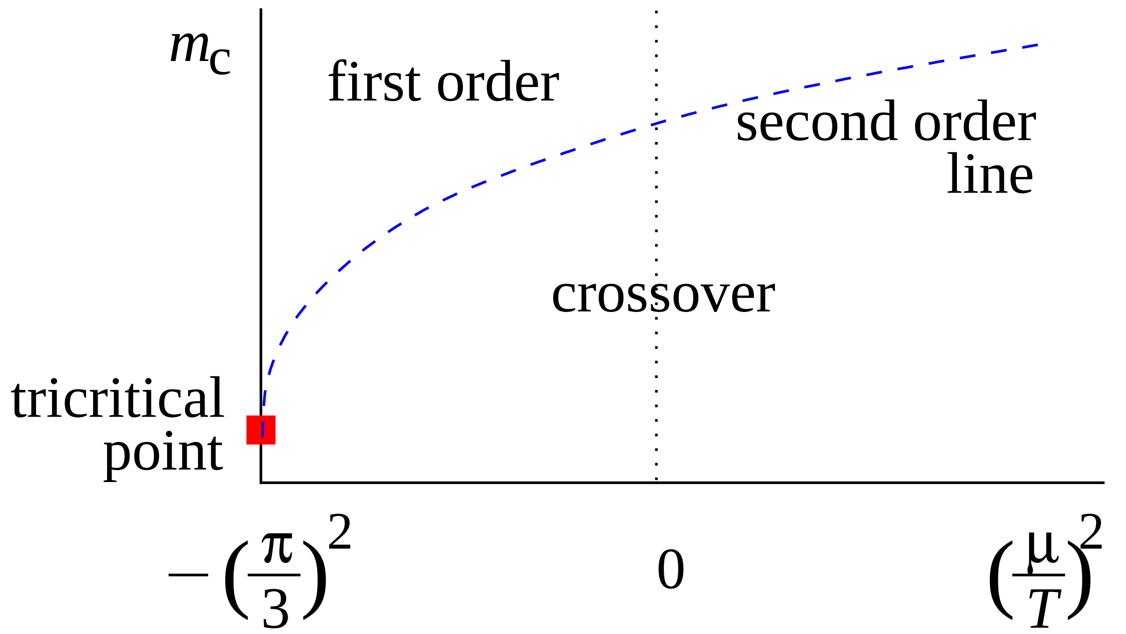
fix to $N_f = 3$ flavours: cut through 3D Columbia plot



Critical properties

in all known cases:
the first order regions shrink as $(\mu/T)^2$ is increased

made very precise for heavy quarks: rotate graph



tricritial point determines scaling of $2^{\rm nd}$ order line!

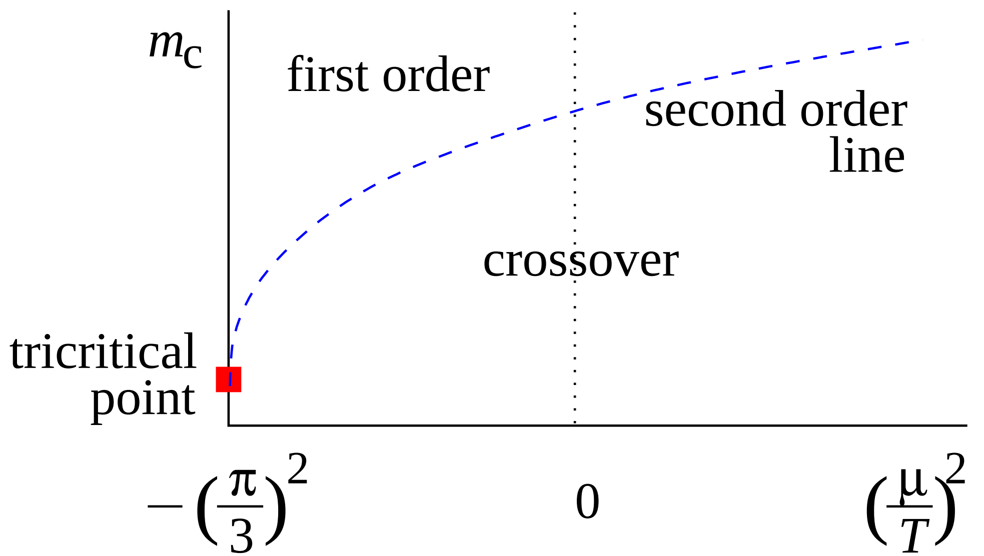
Critical properties

tricritical scaling:

- denote $x = (\mu/T)^2$ $x_* = -(\pi/3)^2$
- then

$$m_c(x) = m_c(x_*) + K\,(x - x_*)^{2/5}$$

- K is free parameter, exponent $2/5$ fixed by universality

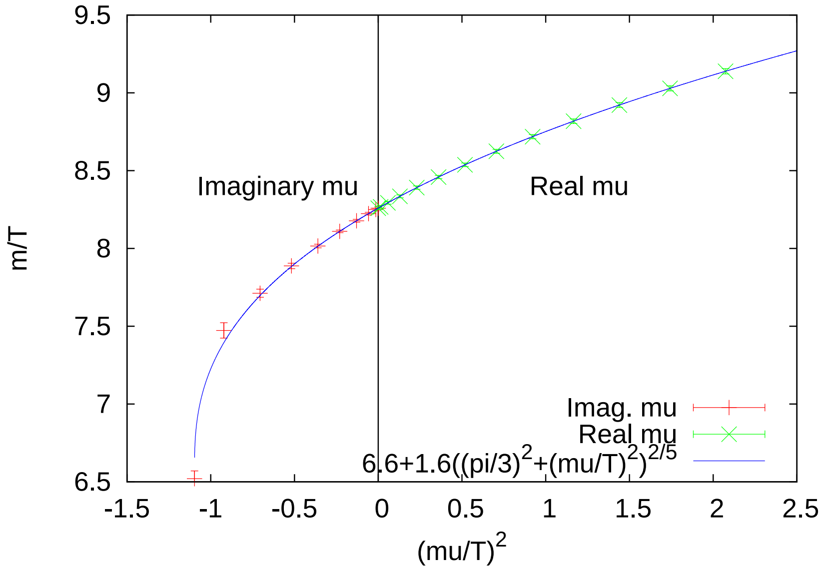


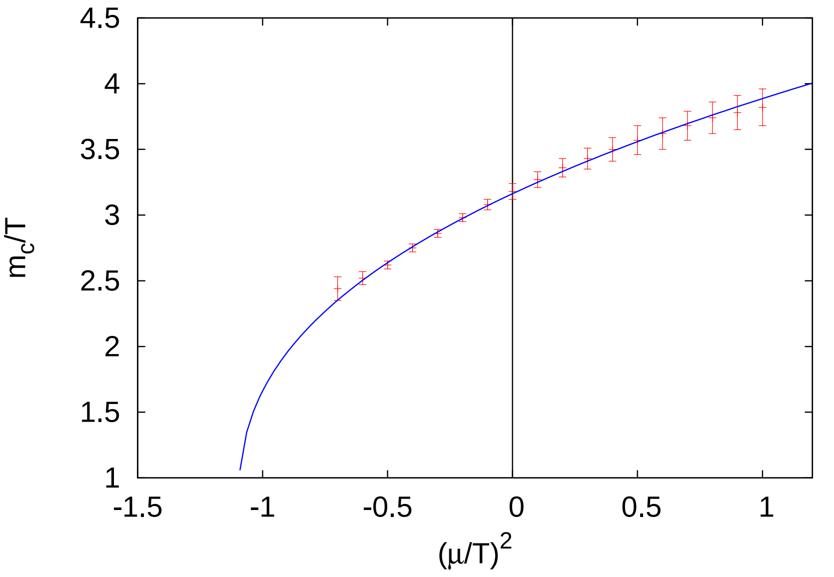
de Forcrand & Philipsen 10

Critical properties

how well does it work? (how large is the scaling region?)

- test in models where the sign problem is mild:





3D Potts model strong coupling QCD

- works much better than (naively) expected
- scaling region extends well into $\mu^2 > 0$ domain

Tricritical scaling

conclusion

- Roberge-Weiss endpoint is triple or critical endpoint
- quark mass dependence: tricritical point
- tricritical scaling in scaling region

for heavy quarks

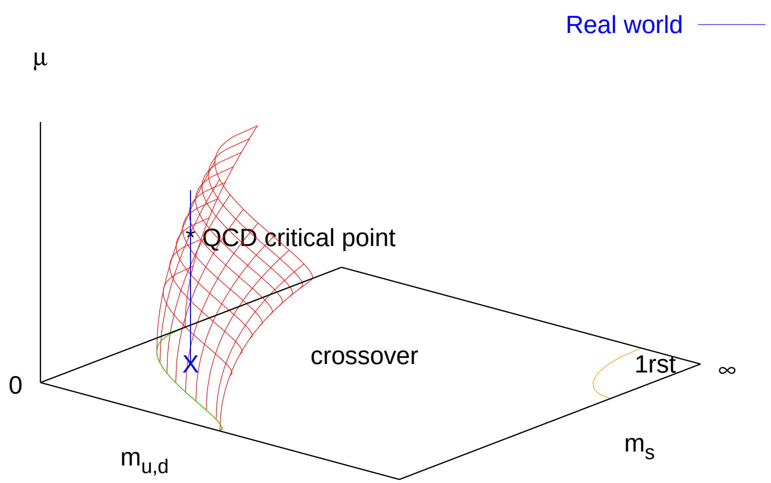
- presence of RW endpoint at imaginary μ determines phase structure at real μ
- goes beyond Taylor series and analytical continuation

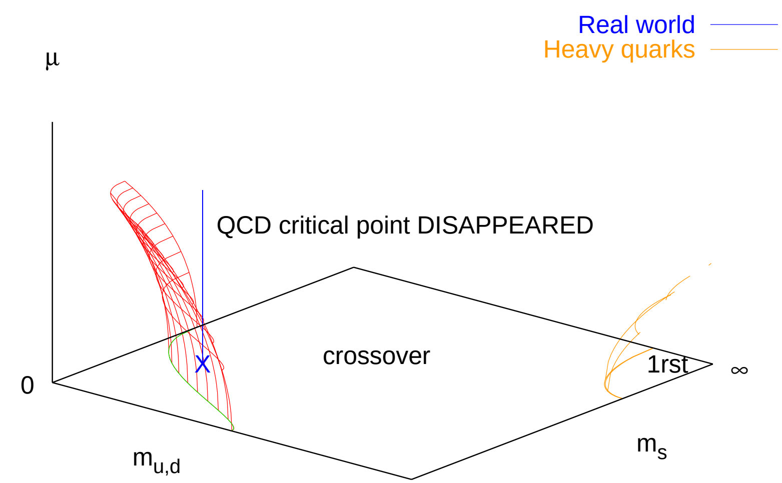
open questions

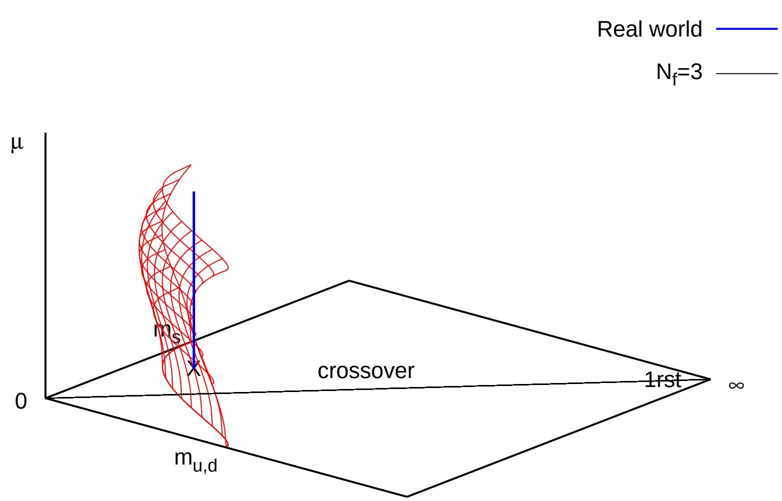
- continuum limit?
- light quarks?

Critical endpoint

first order regions shrink?







- nontrivial curvature?
- caveat: most studies so far on coarse lattices $N_\tau = 4$

Summary

- imaginary chemical potential: no sign problem
- intricate phase structure: 3D Columbia plot
- can/should be determined (so far $N_\tau = 4$ only)
- implications for real μ