QCD at nonzero chemical potential and the sign problem

INT lectures 2012

II: standard approaches

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Where are we?

complex weight:

- straightforward importance sampling not possible
- overlap problem

various possibilities:

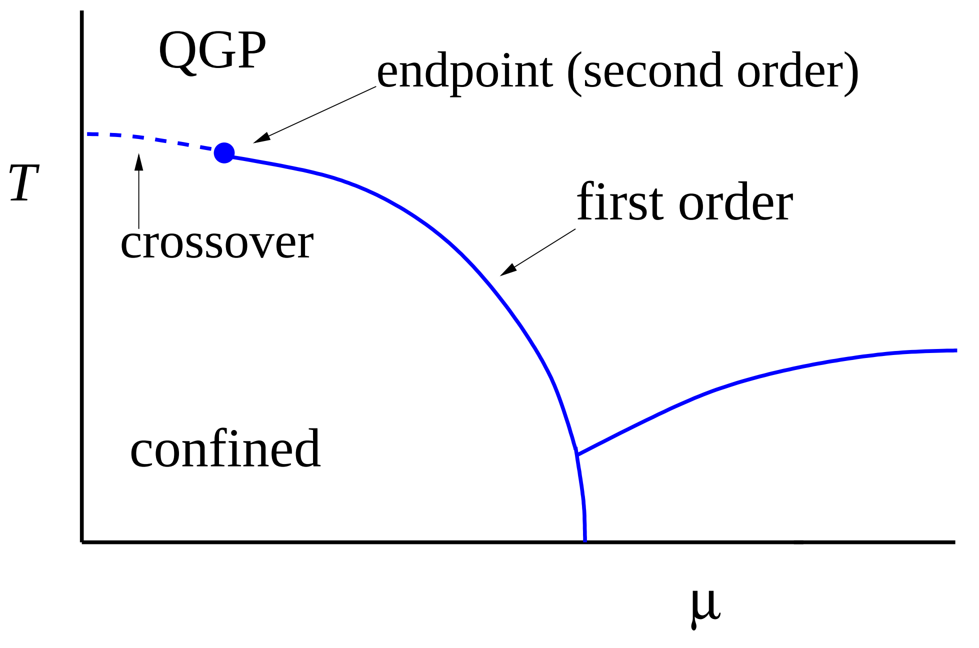
- preserve overlap as best as possible
- use approximate methods at small μ
- do something radical:
  - rewrite partition function in other dof
  - explore field space in a different way
  - . . .

discuss first two approaches

Reminder: physics goal

determine

- phase boundary between confined and deconfined phase at small μ
- critical endpoint (if it exists)



"standard conjectured" phase diagram

Reminder: physics goal

phase boundary at small μ:

- determine curvature of the phase boundary

$$\frac{T_c(\mu)}{T_c(0)} = 1 + \# \left(\frac{\mu}{T_c(0)}\right)^2 + \# \left(\frac{\mu}{T_c(0)}\right)^4 + \ldots$$

- (if crossover: this may depend on observable)

determine critical endpoint

- from this expansion
- directly

experimental search for critical endpoint is planned at FAIR (GSI, Darmstadt, Germany) in coming years

Method I: Reweighting

general strategy: $Z_w = \int DU\, w(U)$ $\qquad w(U) \in \mathbf{C}$

observable: $\langle O \rangle_w = \dfrac{\int DU\, O(U) w(U)}{\int DU\, w(U)}$

introduce new weight $r(U)$ (r for 'reweighting' or 'real'), chosen at will

$$\langle O \rangle_w = \frac{\int DU\, O(U) \frac{w(U)}{r(U)} r(U)}{\int DU\, \frac{w(U)}{r(U)} r(U)} = \frac{\langle O \frac{w}{r} \rangle_r}{\langle \frac{w}{r} \rangle_r}$$

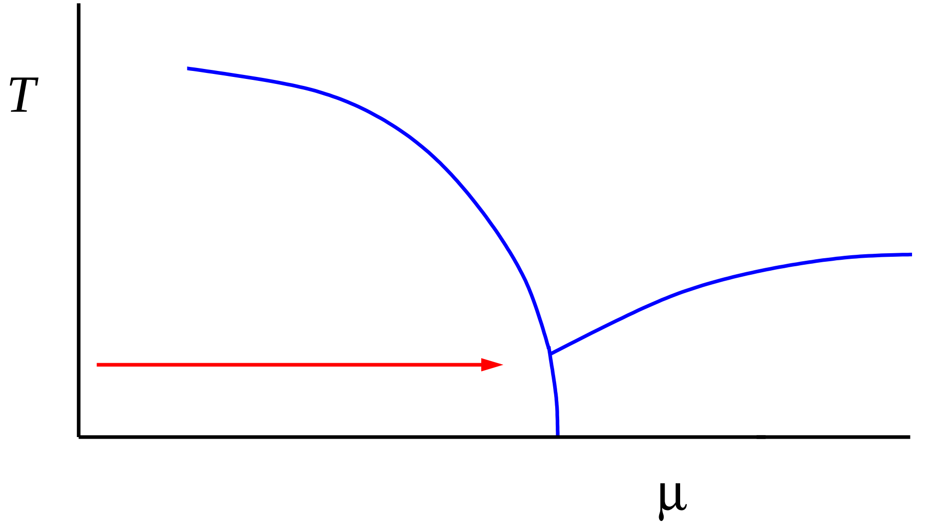
reweighting factor, average sign:

$$\left\langle \frac{w}{r} \right\rangle_r = \frac{Z_w}{Z_r} = e^{-\Omega \Delta f} \qquad \Delta f = f_w - f_r \geq 0$$

Method I: Reweighting

choose weight r to adapt to problem:

- Glasgow reweighting: fix β (or T)



$$\frac{w}{r} \sim \frac{\det M(\mu)}{\det M(0)}$$

severe overlap problem

probe high-density phase with $\mu = 0$ hadronic physics!

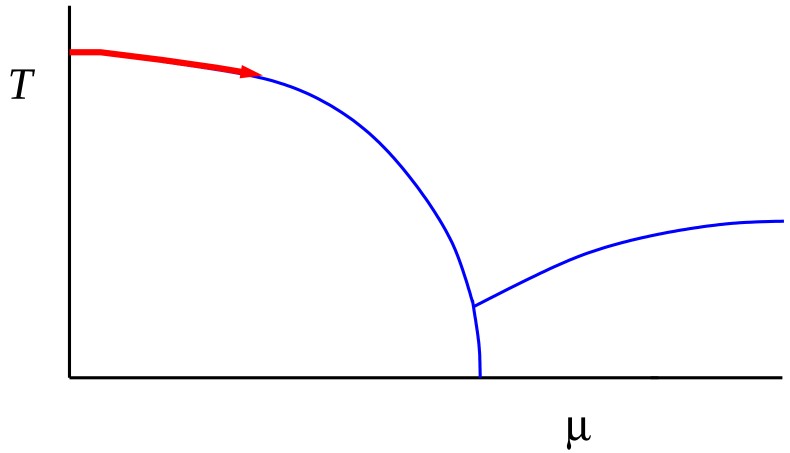
doomed to fail ...

Method I: Reweighting

choose weight r to adapt to problem:

- Fodor-Katz reweighting
  or multi-parameter/overlap preserving reweighting

$\Rightarrow$ adapt β as well



$$\frac{w}{r} \sim \frac{\det M(\mu)}{\det M(0)} e^{-\Delta\beta S_{\mathrm{YM}}}$$

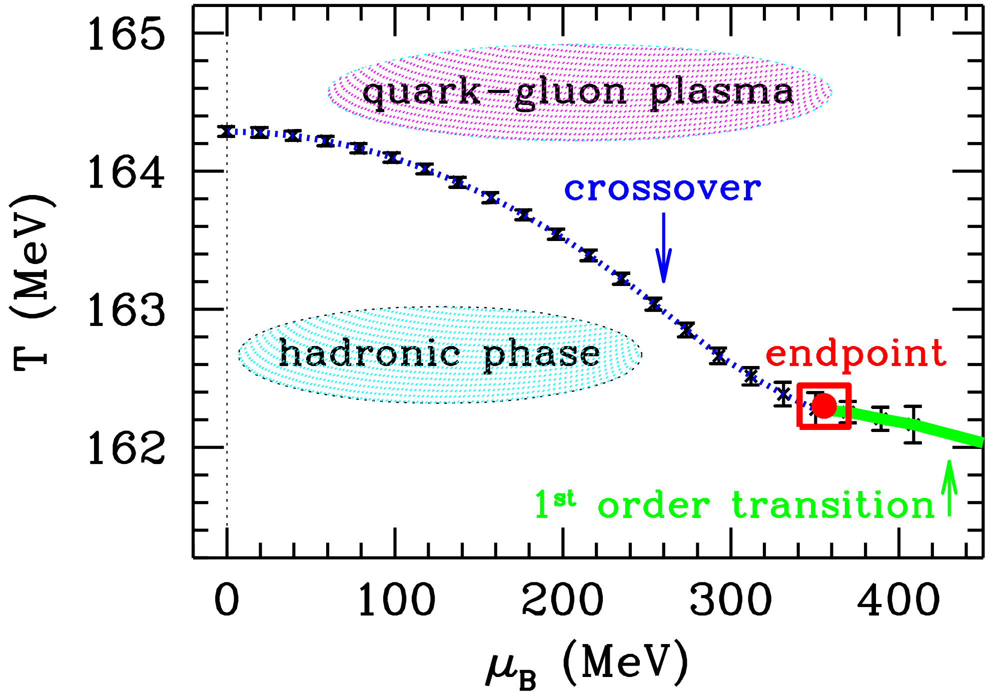
stay on
pseudo-critical line $T_c(\mu)$

improved (ensured?) overlap: sample from both phases

Method I: Reweighting

Fodor-Katz reweighting: multi-parameter/overlap preserving

Fodor & Katz 02/04



locate
critical endpoint:

$\mu_E^q = 120(3)$ MeV
$T_E = 162(2)$ MeV

physical m_q

$N_\tau = 4$

never repeated

breakdown of method, (un)expected role of pions?

Splittorff 07

Method II: Taylor series

- $Z(\mu)$ is even in μ (charge conjugation invariance)
- $\langle n(\mu)\rangle \sim \frac{\partial}{\partial\mu} \ln Z$ is odd in μ

$\Rightarrow$ Taylor series around $\mu = 0$

Bielefeld-Swansea, Gavai-Gupta 02/05

MILC, hotQCD 10

grand-canonical ensemble $\quad p = \frac{T}{V} \ln Z$

$$\Delta p(\mu) = p(\mu) - p(0) = \frac{\mu^2}{2!} \frac{\partial^2 p}{\partial \mu^2}\Big|_{\mu=0} + \frac{\mu^4}{4!} \frac{\partial^4 p}{\partial \mu^4}\Big|_{\mu=0} + \ldots$$

$$\frac{\Delta p(\mu)}{T^4} = \sum_{n=1}^{\infty} c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n}$$

determine coefficients c_{2n}

Method II: Taylor series

explicit expressions:

$$Z = \int DU \, (\det M)^{N_f} \, e^{-S_{\rm YM}} = \int DU \, e^{-S_{\rm YM} + N_f \ln \det M(\mu)}$$

straightforward:

$$\frac{\partial \ln Z}{\partial \mu} = \left\langle N_f \frac{\partial}{\partial \mu} \ln \det M \right\rangle$$

$$\frac{\partial^2 \ln Z}{\partial \mu^2} = \left\langle N_f \frac{\partial^2}{\partial \mu^2} \ln \det M \right\rangle + \left\langle \left(N_f \frac{\partial}{\partial \mu} \ln \det M \right)^2 \right\rangle - \left\langle N_f \frac{\partial}{\partial \mu} \ln \det M \right\rangle^2$$

etc.

Method II: Taylor series

explicit expressions: $\ln \det M = \mathrm{Tr}\, \ln M$

$$\frac{\partial}{\partial \mu} \ln \det M = \mathrm{Tr}\, M^{-1} \frac{\partial M}{\partial \mu}$$

$$\frac{\partial^2}{\partial \mu^2} \ln \det M = \mathrm{Tr}\, M^{-1} \frac{\partial^2 M}{\partial \mu^2} - \mathrm{Tr}\, M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu}$$

etc.
straightforward to work out to higher order, but:

- number of terms increases rapidly, $c_n \sim 6^n$ terms
- huge cancelations required: p is intensive, c_n are finite, but individual contributions may scale differently (generalized susceptibilities)

Method II: Taylor series

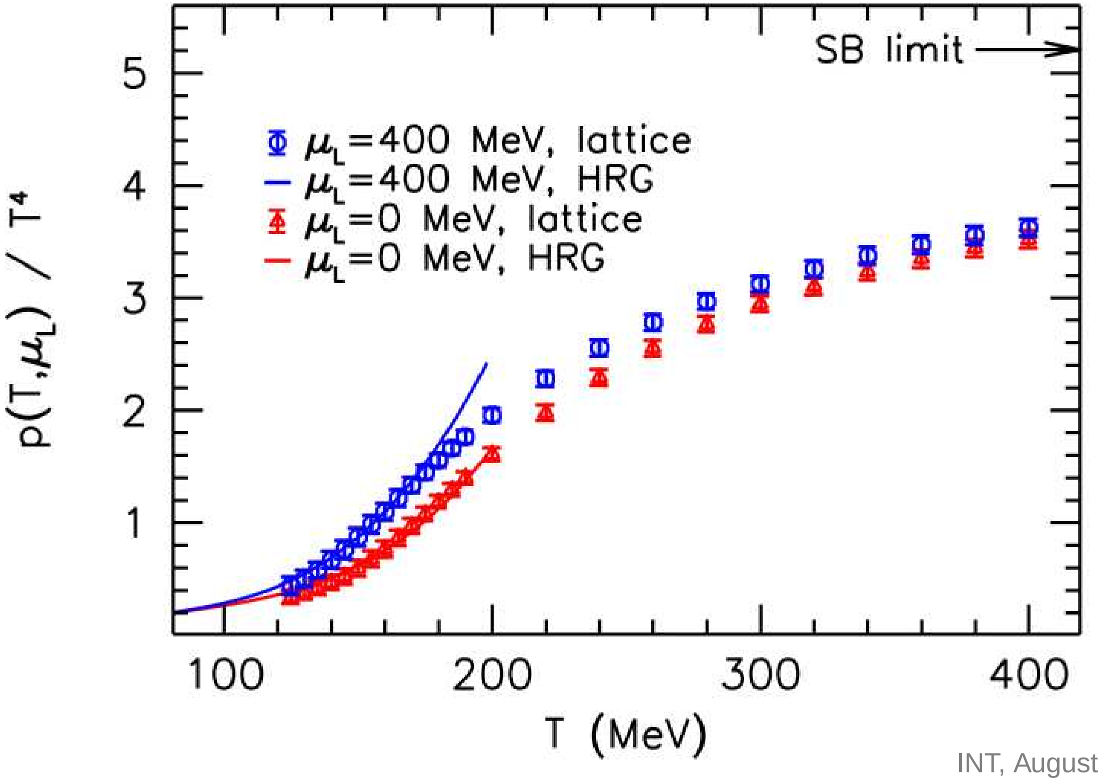
current standard:

- most groups: $\#\mu^2 + \#\mu^4 + \#\mu^6$
- $\ldots + \#\mu^8$ Gavai-Gupta 08
- coarse lattices: $N_\tau = 4, 6$

only continuum extrapolated result:

equation of state to $\mathcal{O}(\mu^2)$

Borsanyi, Fodor & Katz et al 12

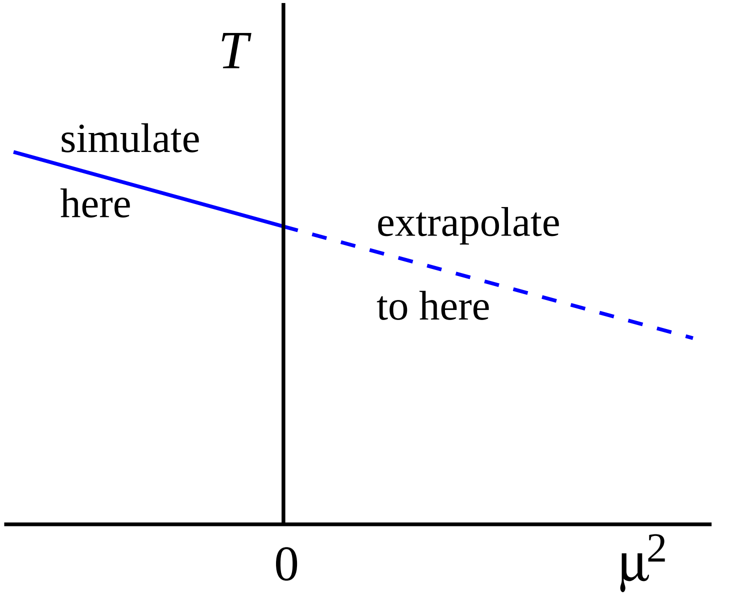


Method III: imaginary μ

recall: $D^\dagger(\mu) = \gamma_5 D(-\mu^*)\gamma_5$

- if $\mu = i\mu_\mathrm{I}$, $\det D(i\mu_\mathrm{I})$ is real: perform ordinary simulations
- analytical continuation to real μ: $+\mu_\mathrm{I}^2 \to -\mu^2$

- determine phase boundary at $\mu^2 < 0$
- fit $T_c(-\mu^2)$
- obtain phase boundary at $\mu^2 > 0$



de Forcrand & Philipsen 02-now
d'Elia & Lombardo 02

d'Elia et al 02-now

Method III: imaginary μ

- in fact: much richer than just analytical continuation
- intricate phase structure at imaginary μ
- see below

other methods (not discussed here):

- canonical ensemble
- histograms

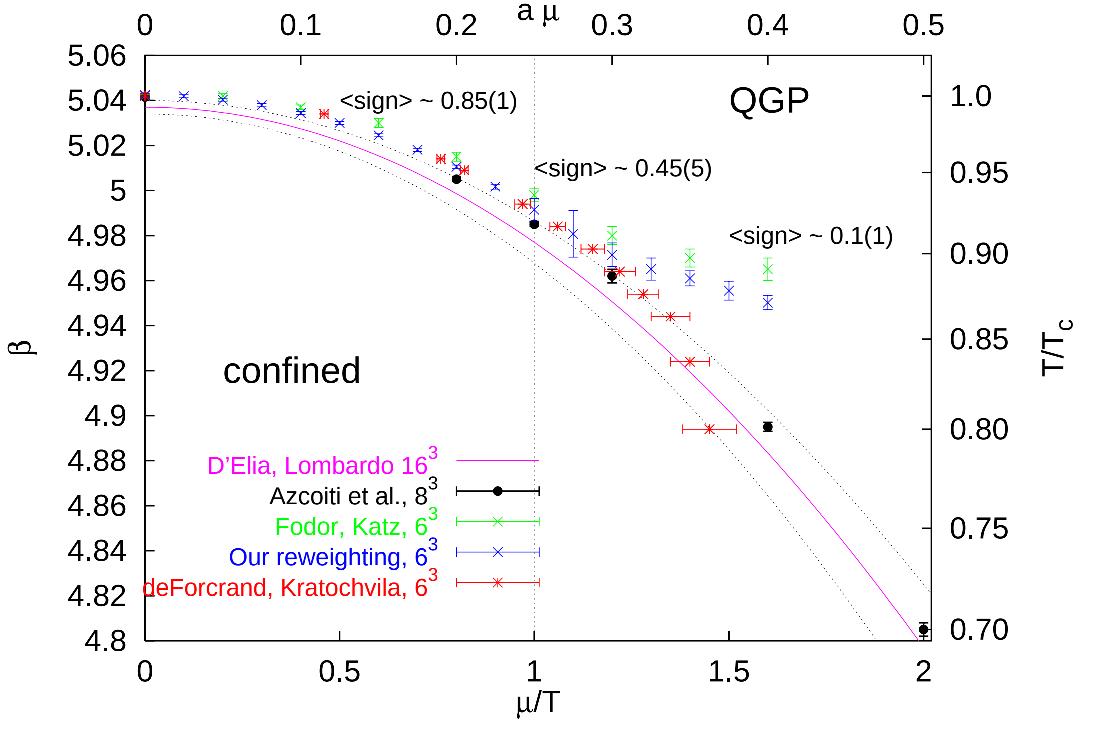
Summary

- agreement between methods at small $\mu/T \lesssim 1$

phase boundary:

de Forcrand LAT09

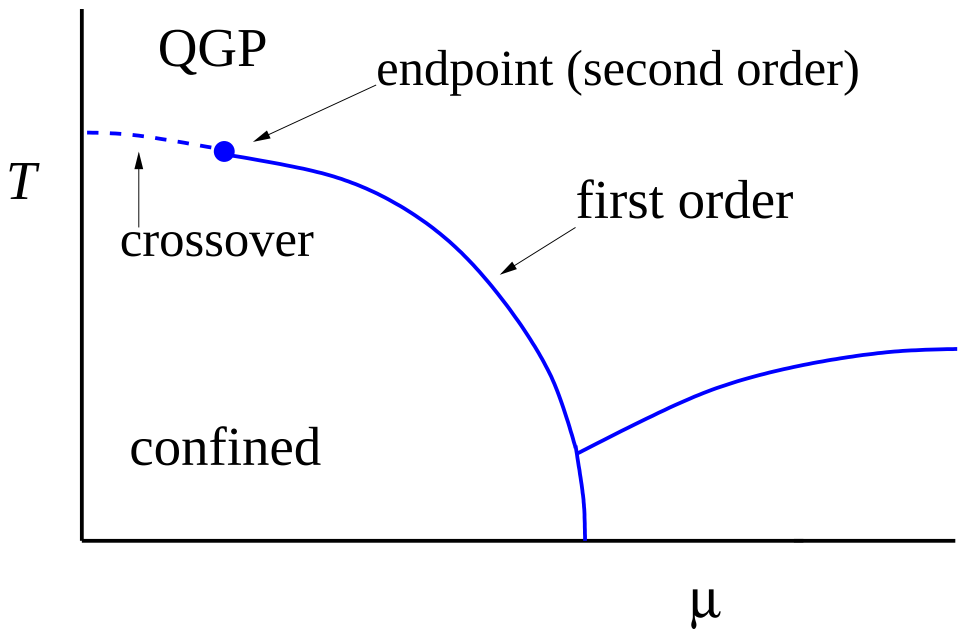
imaginary μ
2 parameter imag. μ
double reweighting
(Lee-Yang zeroes)
double reweighting
(susceptibilities)
canonical



- sign problem under control, fixed $N_\tau = 4$

Critical endpoint

indications for existence of critical endpoint?



- imaginary chemical potential: not obvious (see below)
- Taylor series: number of terms is really small
  estimate radius of convergence?

Critical endpoint

crossover at small μ:

- transition temperature not uniquely defined
- depends on observable
- no non-analyticity

study transition using

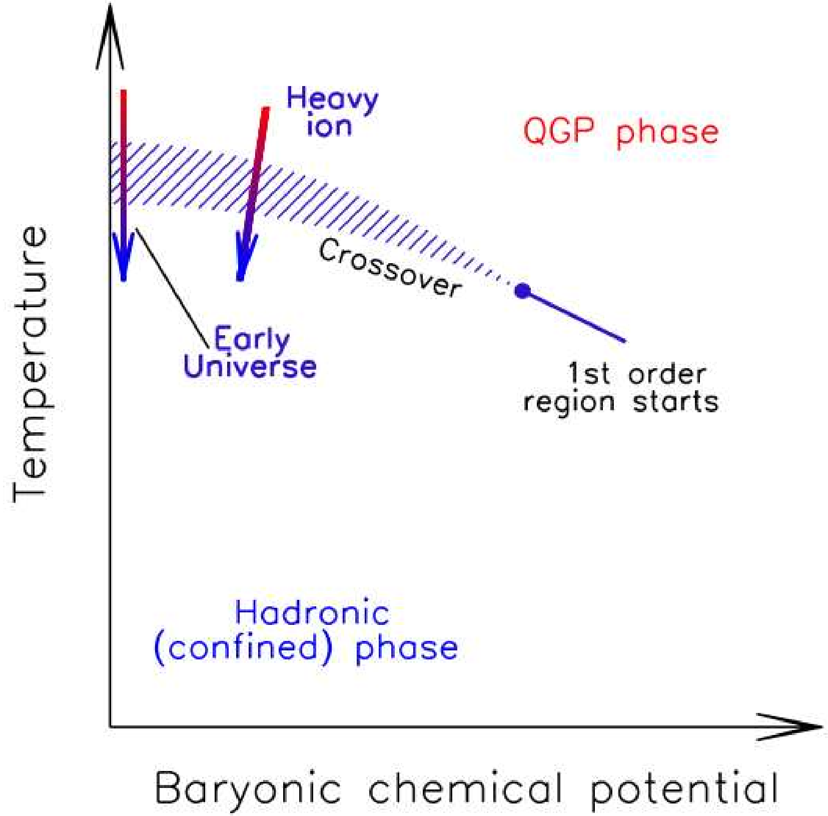
- Polyakov loop susceptibility
- chiral condensate
- strange quark number susceptibility
- ...

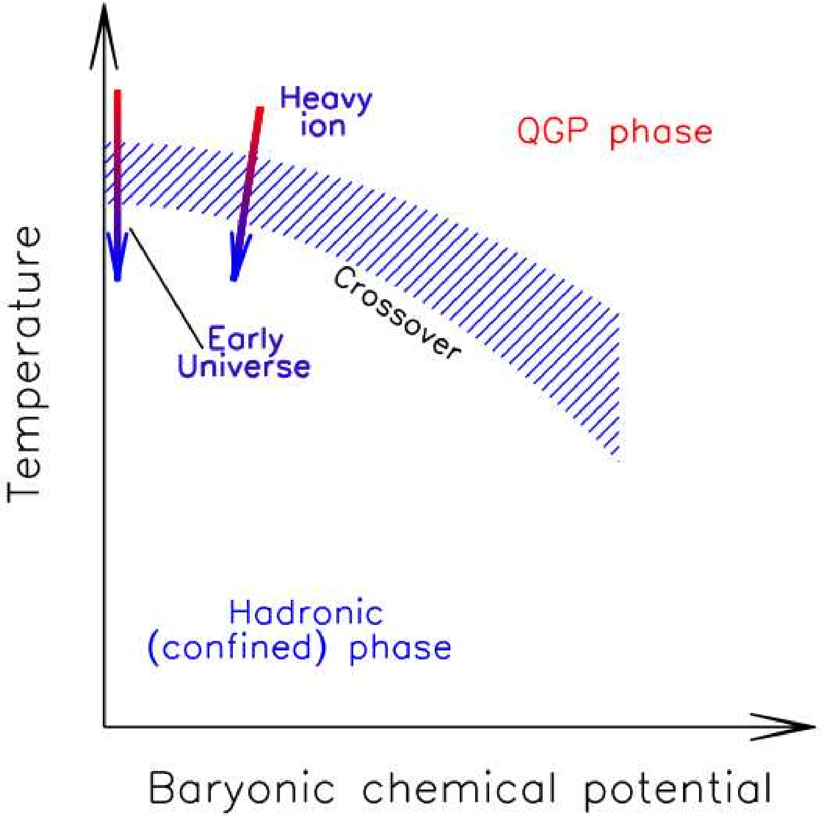
do not have to agree: crossover region loosely defined

Critical endpoint

two scenarios

- if crossover region shrinks with increasing μ: CEP
- if crossover region extends with increasing μ: no CEP





Endrodi, Fodor, Katz & Szabo 11

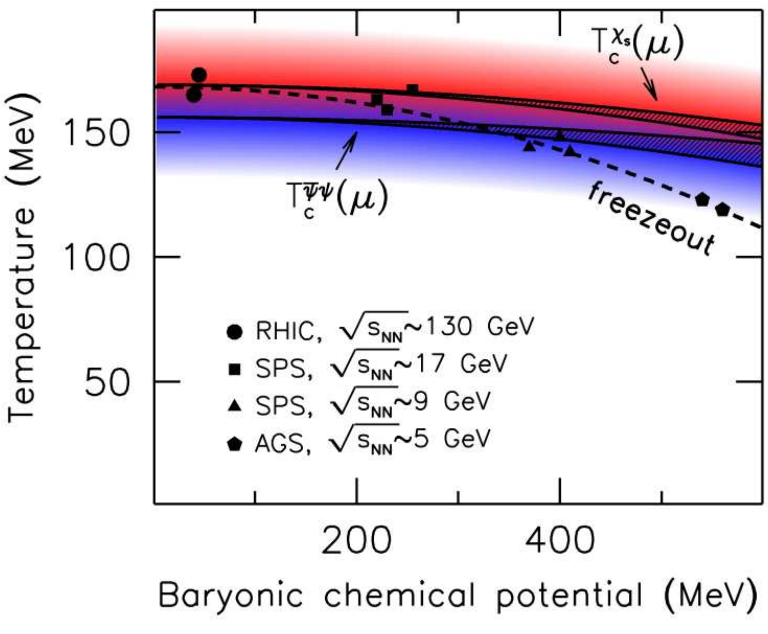
Critical endpoint

study crossover region using

- chiral condensate $\langle\bar{\psi}\psi\rangle$
- strange quark number susceptibility χ_s

using Taylor series expansion

Endrodi et al 11



conclusion: no indication for scenario I

Summary

standard approaches . . .

- . . . can be used for some questions
- are limited in applicability
- do not solve the sign problem . . .