Quantum anomalies in hydrodynamics

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Plan

- Relativistic hydrodynamics
- Triangle anomaly
- anomalies in hydrodynamics: insights from gauge/gravity duality
- What can we learn without gauge/gravity duality

A low-energy effective theory

Consider a thermal system: $T \neq 0$

Finite mean free path $\lambda_{\rm mfp}$

Dynamics at large distances $\ell \gg \lambda_{\rm mfp}$

is simple: most degrees of freedom do not matter

Degrees of freedom in hydrodynamics

D.o.f. that relax arbitrarily slowly in the long-wavelength limit:

- Conserved densities
- Goldstone modes (superfluids)
- Massless U(1) gauge field (magnetohydrodynamics)

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Equations of hydrodynamics can usually be written down from general principles: symmetries, conservation laws

Relativistic hydrodynamics

Conservation laws: $\partial_\mu T^{\mu\nu} = 0$

$\partial_\mu j^\mu = 0$ (one conserved charge)

Constitutive equations: local thermal equilibrium

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

$$j^\mu = n u^\mu$$

Total: 5 equations, 5 unknowns

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$$j^\mu = n u^\mu + \nu^\mu$$

Total: 5 equations, 5 unknowns

Dissipative terms, in local fluid rest frame:

$$\tau^{ij} = -\eta(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\vec{\nabla}\cdot\vec{u}) - \zeta\delta^{ij}\vec{\nabla}\cdot\vec{u} \qquad \nu^i = -\sigma T \partial^i \left(\frac{\mu}{T}\right)$$

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shear viscosity

bulk viscosity

conductivity
(diffusion)

Parity-odd effects?

- What happens if the conserved current is axial?
  - example: QCD with massless quarks: axial currents conserved in absence of external EM fields
- Parity invariance does not forbid

$$j^{5\mu} = n^5 u^\mu + \xi(T,\mu)\omega^\mu$$

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$$ vorticity

- The same order in derivatives as dissipative terms (viscosity, diffusion)

Landau-Lifshitz frame

- We can also have correction to the stress-energy tensor

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \xi'(u^\mu \omega^\nu + \omega^\mu u^\nu)$$

  - Can be eliminated by redefinition of u^μ

$$u^\mu \to u^\mu - \frac{\xi'}{\epsilon + P}\omega^\mu$$

Only a linear combination $\xi - \frac{n}{\epsilon + P}\xi'$ has physical meaning

Let us set $\xi' = 0$

New effect: chiral separation

- Rotating piece of quark matter
- Initially only vector charge density $\neq 0$
- Rotation: lead to j^5: chiral charge density develops
- Can be thought of as chiral separation: left- and right-handed quarks move differently in rotation fluid
- Similar effect in nonrelativistic fluids?

Chiral separation by rotation

Chiral separation by rotation

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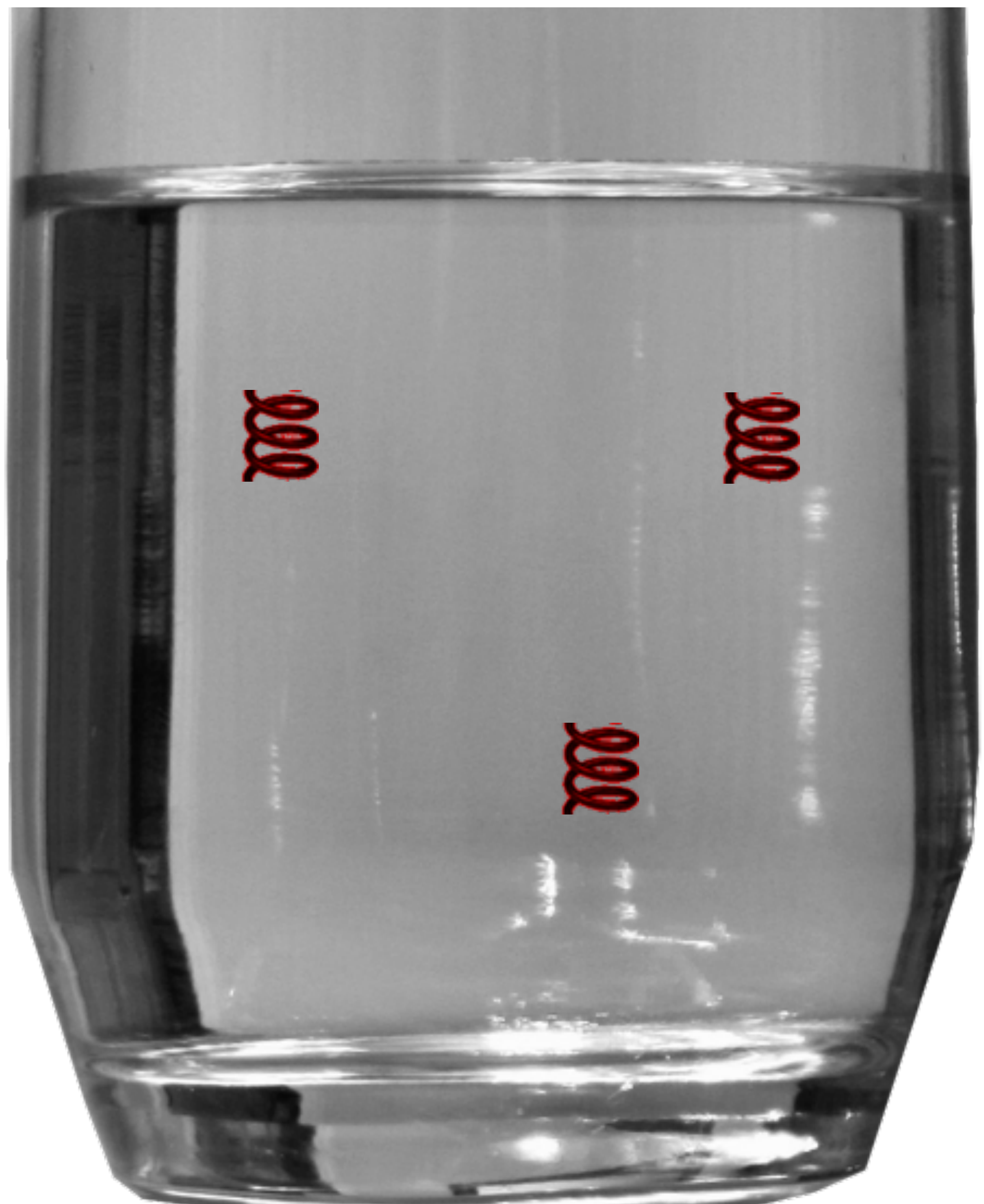
Chiral separation by rotation



Chiral separation by rotation

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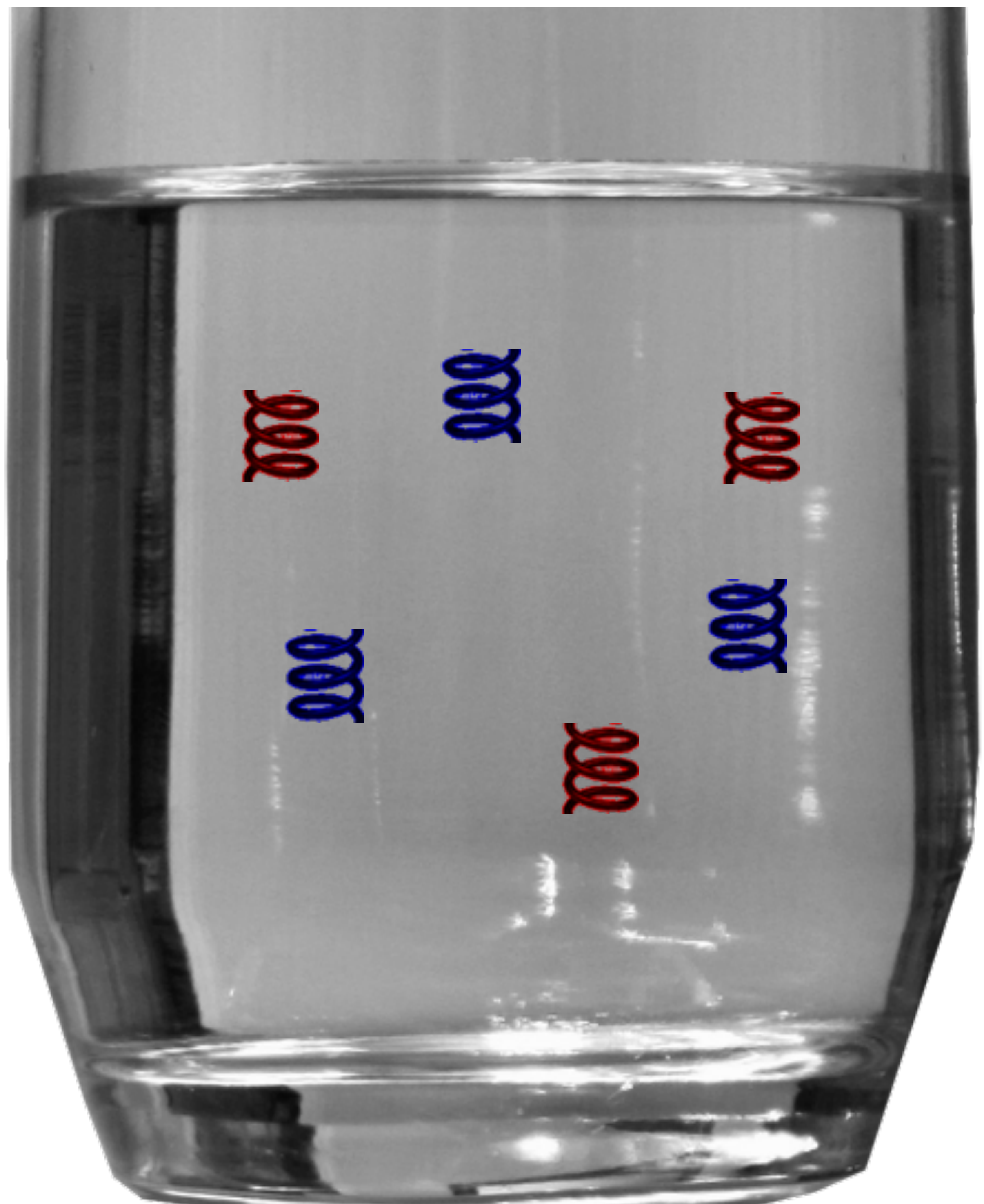

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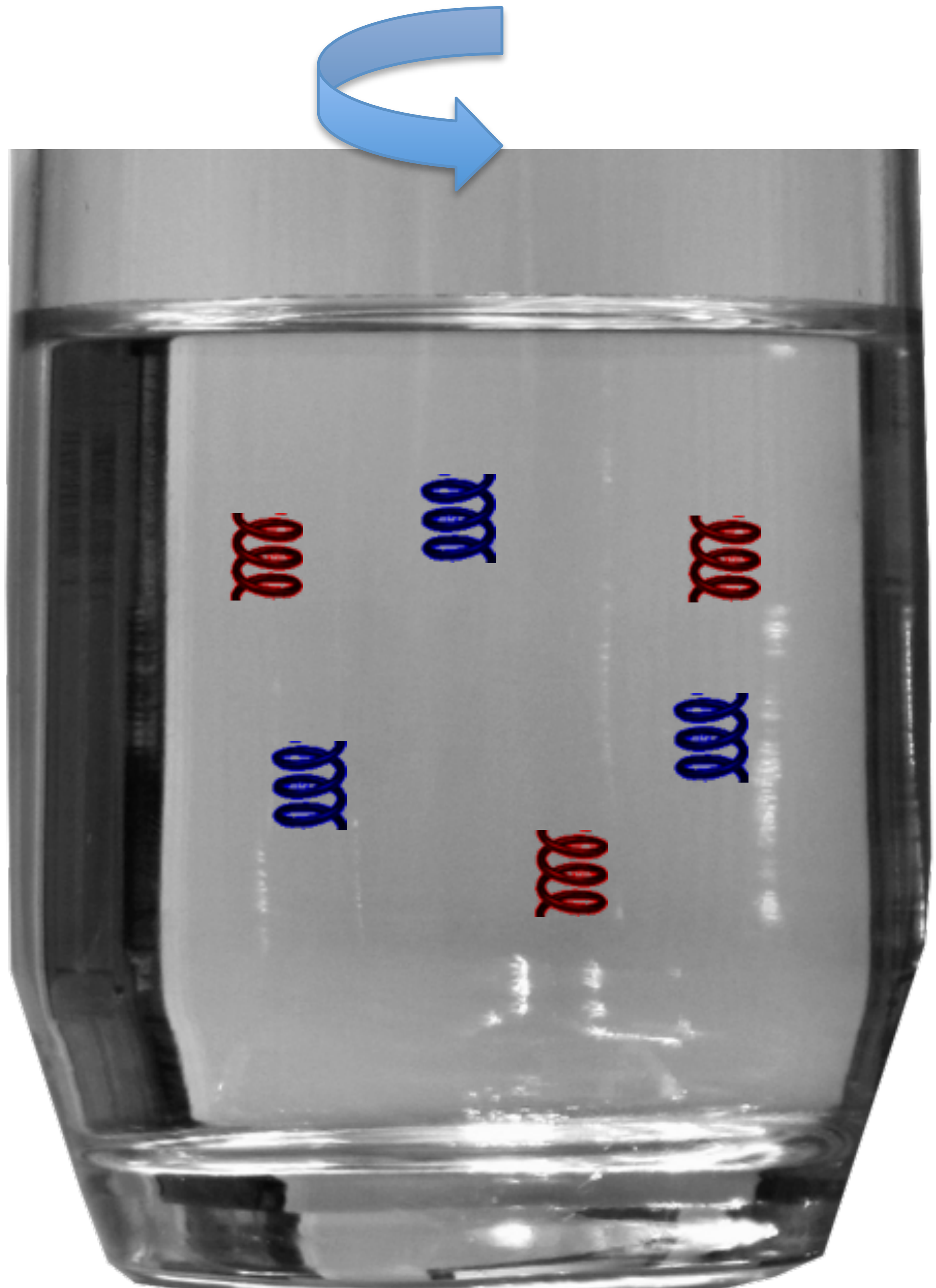

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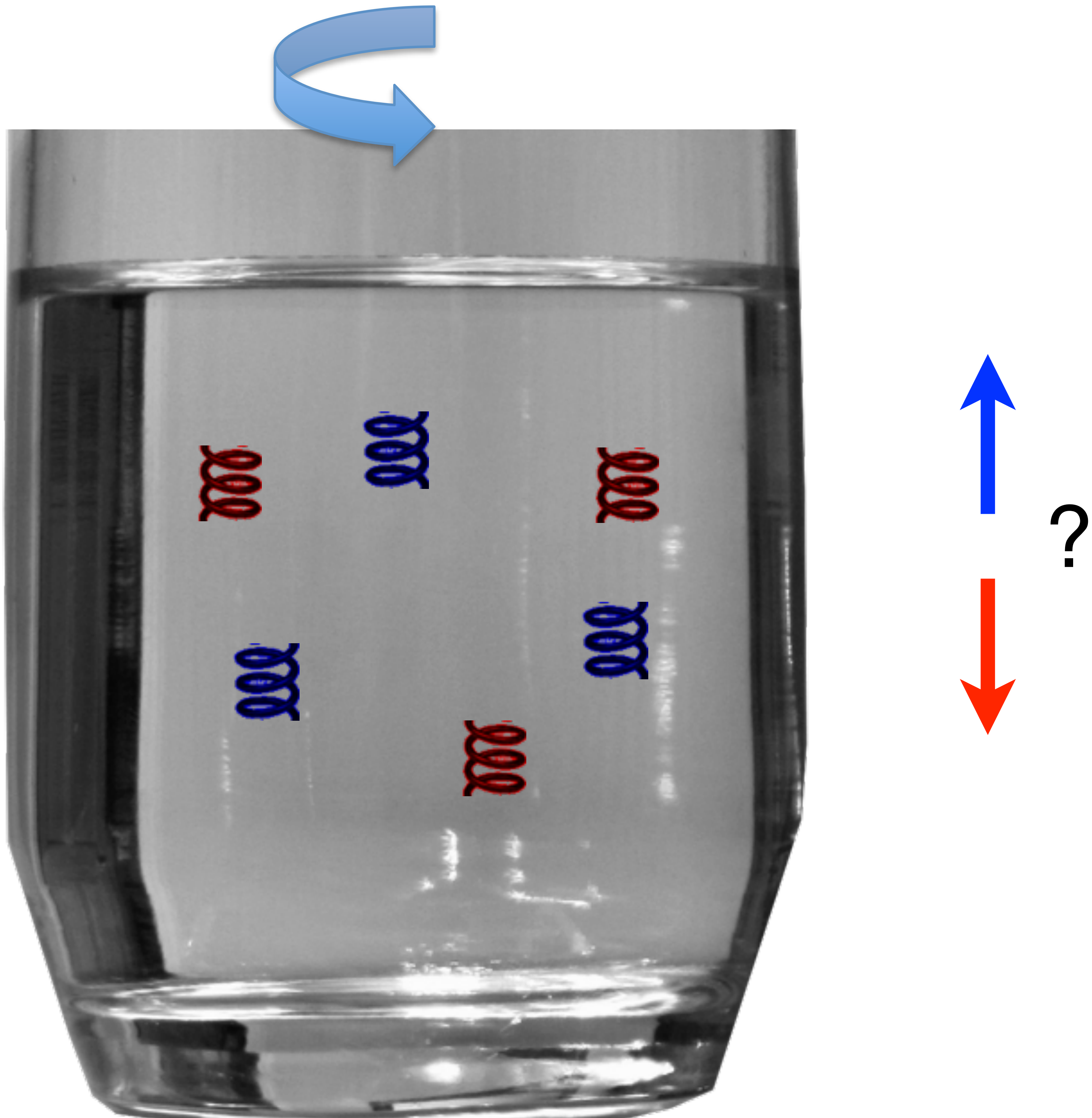


Can chiral separation occur in rigid rotation?

- If a chiral molecule rotates with respect to the liquid, it will moves
- In rigid rotation, molecules rotate with liquid
- $\Rightarrow$ no current in rigid rotation.

Relativistic theories are different

- There can be current ~ vorticity
- It is related to triangle anomalies

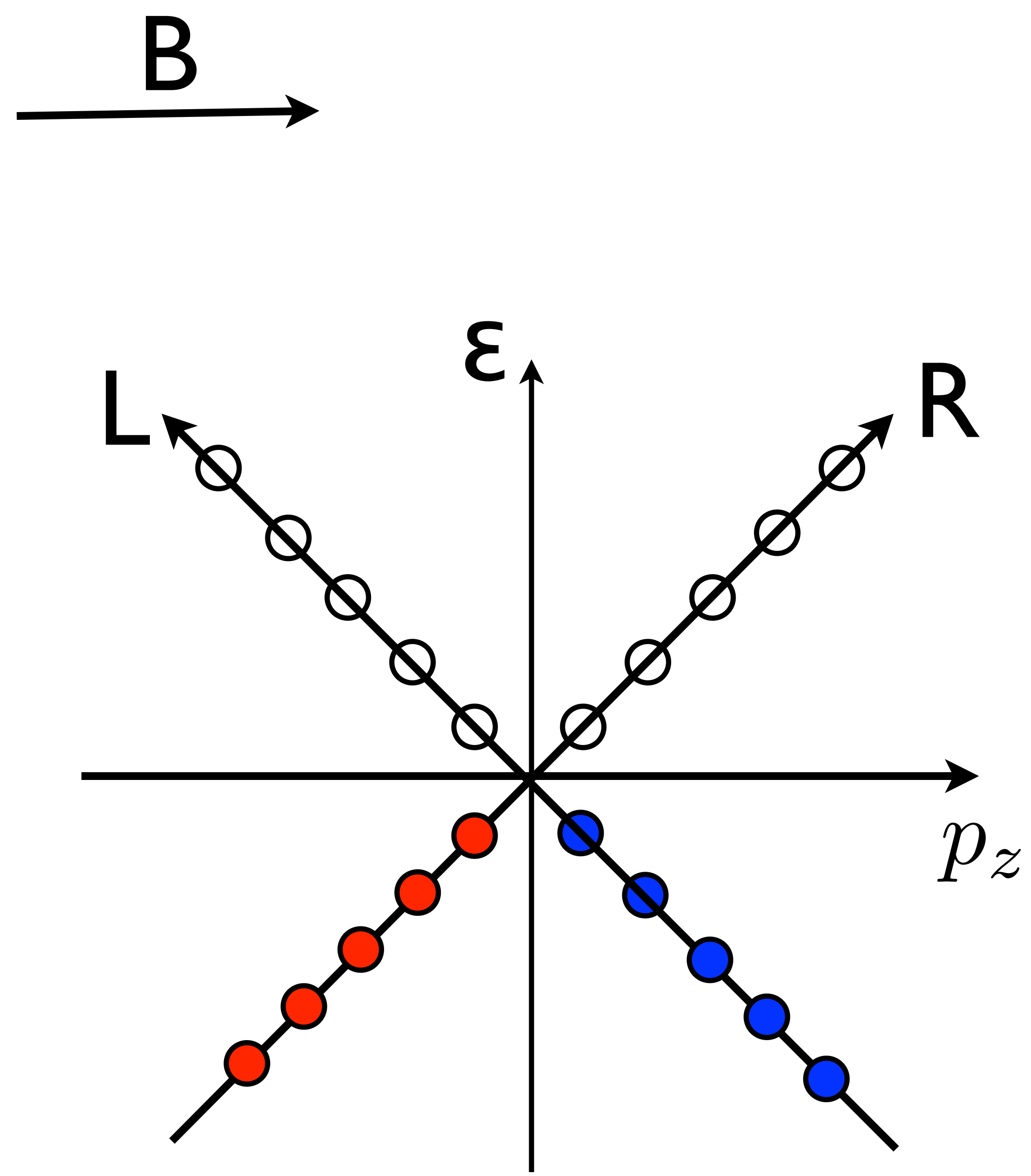
$$\partial_\mu j^{5\mu} = \# E \cdot B$$

  but the effect is there even in the absence of external field

- The kinetic coefficient ξ is determined (almost) completely by anomalies and equation of state

Anomalies

Massless fermions: lowest Landau level is chiral



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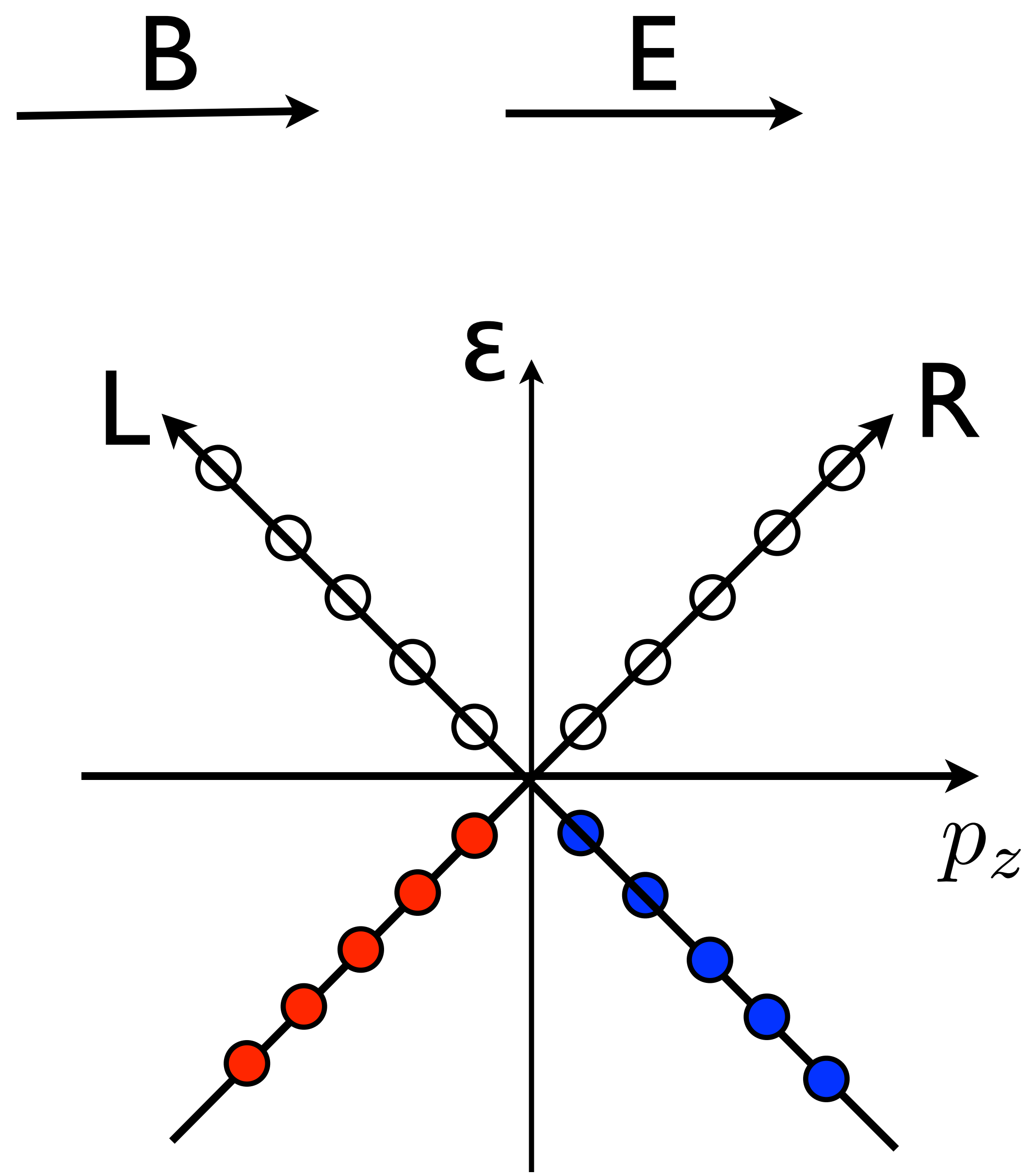

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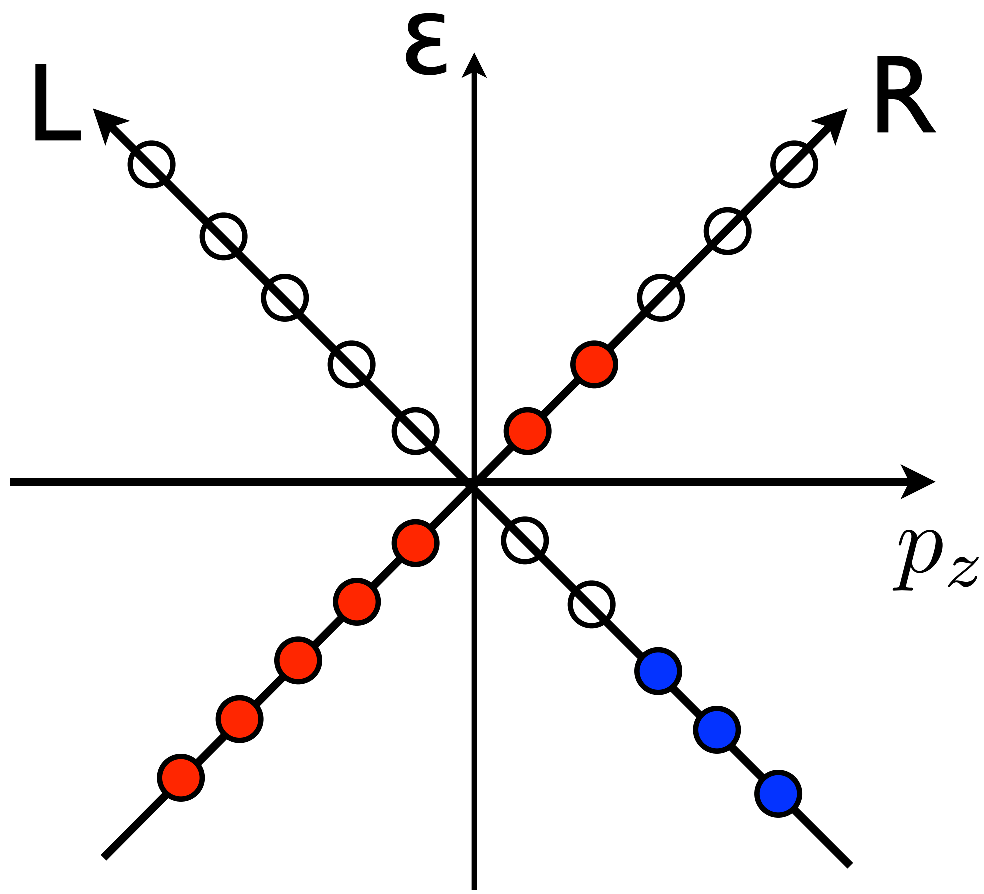

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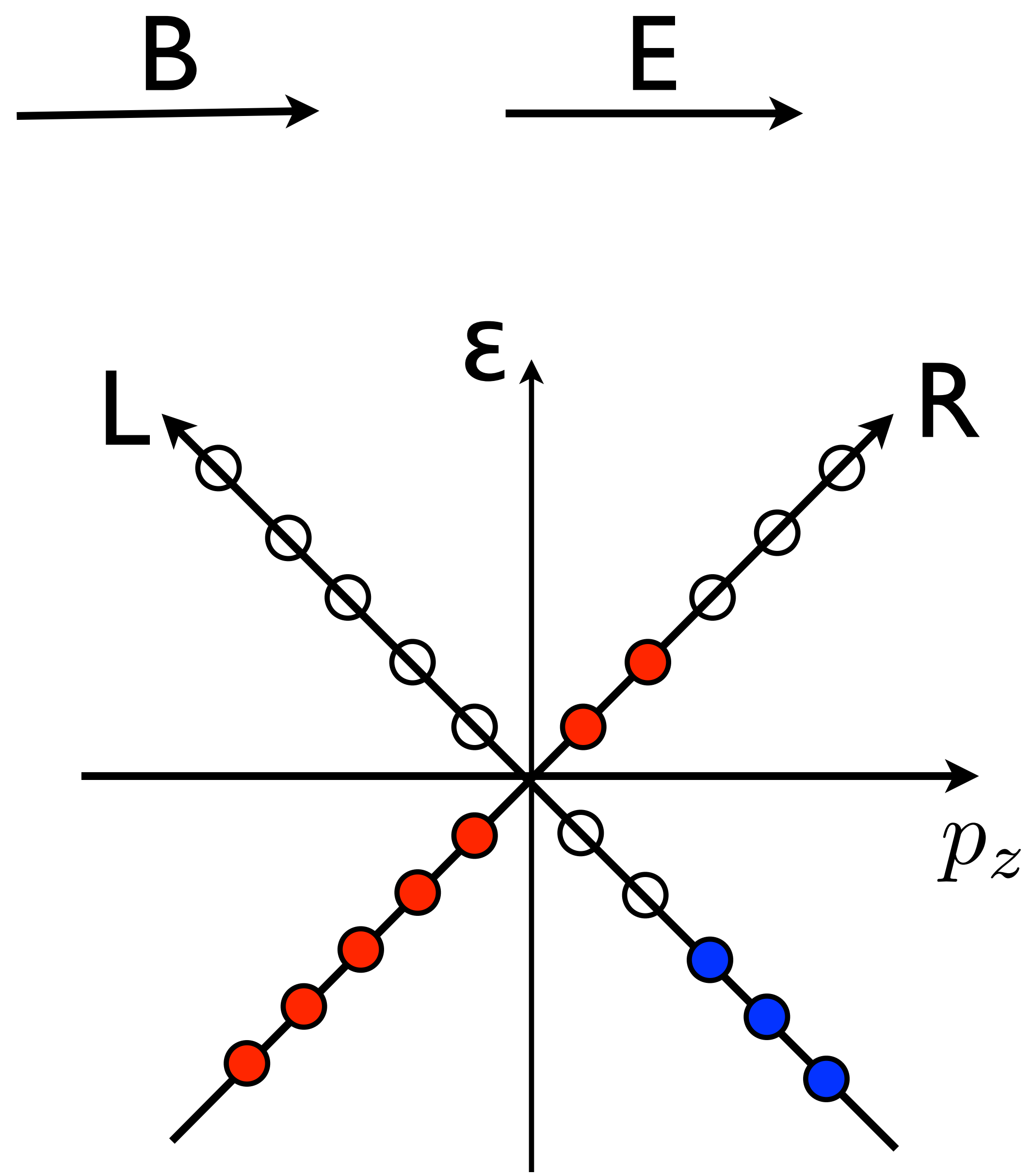

$$\frac{d}{dt}(N_R - N_L) \sim E \cdot B$$

Forbidden?

- Terms with epsilon tensor do not appear in the standard (e.g., Landau-Lifshitz) treatments of hydrodynamics
- Usual argument: 2nd law of thermodynamics:
- additional requirement beside symmetries, conservations law:

  hydrodynamic equations must be consistent with the existence of a non-decreasing entropy

Dissipative terms

Standard textbook manipulations (single U(1) charge)

$$\partial_\mu[(\epsilon + P)u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0$$

$$\partial_\mu(n u^\mu) + \partial_\mu \nu^\mu = 0$$

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$$\partial_\mu(su^\mu) = \frac{\mu}{T}\partial_\mu\nu^\mu+\frac{1}{T}\quad u_\nu\,\partial_\mu\tau^{\mu\nu}$$

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$$\partial_\mu\left(su^\mu-\frac{\mu}{T}\nu^\mu\right)=\qquad\frac{\mu}{T}\partial_\mu\nu^\mu+\frac{1}{T}\quad u_\nu\,\partial_\mu\tau^{\mu\nu}$$

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entropy current s^μ

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$$\partial_\mu\Big(\underbrace{su^\mu-\frac{\mu}{T}\nu^\mu}_{\text{entropy current } s^\mu}\Big)=-\partial_\mu\frac{\mu}{T}\quad\nu^\mu-\frac{1}{T}\partial_\mu u_\nu\quad\tau^{\mu\nu}$$

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Positivity of entropy production constrains the dissipation terms: only three kinetic coefficients η, ζ, and σ (right hand side positive-definite)

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Is there a place for a new kinetic coefficient?

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) = -\frac{1}{T} \tau^{\mu\nu} \partial_\mu u_\nu - \nu^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

Consider a theory with a single conserved chiral charge

Can we add to the current: $\nu^\mu = \cdots + \xi \omega^\mu$?

Problem: Extra term in current would lead to

$$\partial_\mu s^\mu = \cdots - \xi \omega^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$ not manifestly zero

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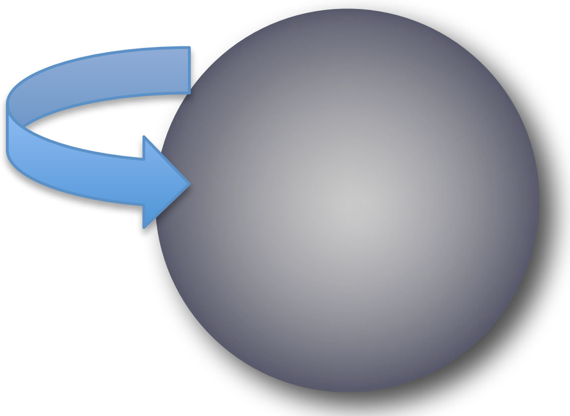
Forbidden by 2nd law of thermodynamics?

Holography

The first indication that standard hydrodynamic equations are not complete comes from considering



rotating 3-sphere of N=4 SYM plasma ↔ rotating BH

If the sphere is large: hydrodynamics should work

no shear flow: corrections ~ 1/R^2

Instead: corrections ~ 1/R   Bhattacharyya, Lahiri, Loganayagam, Minwalla

Holography (II)

Erdmenger et al. arXiv:0809.2488

Banerjee et al. arXiv:0809.2596

considered N=4 super Yang Mills at strong coupling
finite T and μ

should be described by a hydrodynamic theory

discovered that there is a current ~ vorticity

Found the kinetic coefficient ξ(T,μ)

$$\xi = \frac{N^2}{4\sqrt{3}\pi^2}\mu^2 \left(\sqrt{1+\frac{2}{3}\frac{\mu^2}{\pi^2 T^2}}+1\right)\left(3\sqrt{1+\frac{2}{3}\frac{\mu^2}{\pi^2 T^2}}-1\right)^{-1}$$

Fluid-gravity correspondence

- Long-distance dynamics of black-brane horizons (in AdS) are described by hydrodynamic equations
    - finite-T field theory ↔ AdS black holes

  described by hydrodynamics

- Charged black branes in Einstein-Maxwell theory: hydrodynamics with conserved charges
- Anomalies: Chern-Simons term in 5D action of gauge fields

A holographic fluid

$$S = \frac{1}{8\pi G}\int d^5x\,\sqrt{-g}\left(R - 12 - \frac{1}{4}F_{AB}^2 + \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB} F_{CD}\right)$$

encodes anomalies

Black brane solution (Eddington coordinates)

$$ds^2 = 2dvdr - r^2 f(r,m,q)dv^2 + r^2 d\vec{x}^2 \qquad f(m,q,r) = 1 - \frac{m^4}{r^4} + \frac{q^2}{r^6}$$

$$A_0(r) = \#\frac{q}{r^2}$$

Boosted black brane: also a solution

$$ds^2 = -2u_\mu dx^\mu dr + r^2(P_{\mu\nu} - f u_\mu u_\nu)dx^\mu dx^\nu$$

$$A_\mu(r) = -u_\mu \# \frac{q}{r^2}$$

Promoting parameters into variables

$$u_\mu \to u_\mu(x) \qquad m \to m(x) \qquad q \to q(x)$$

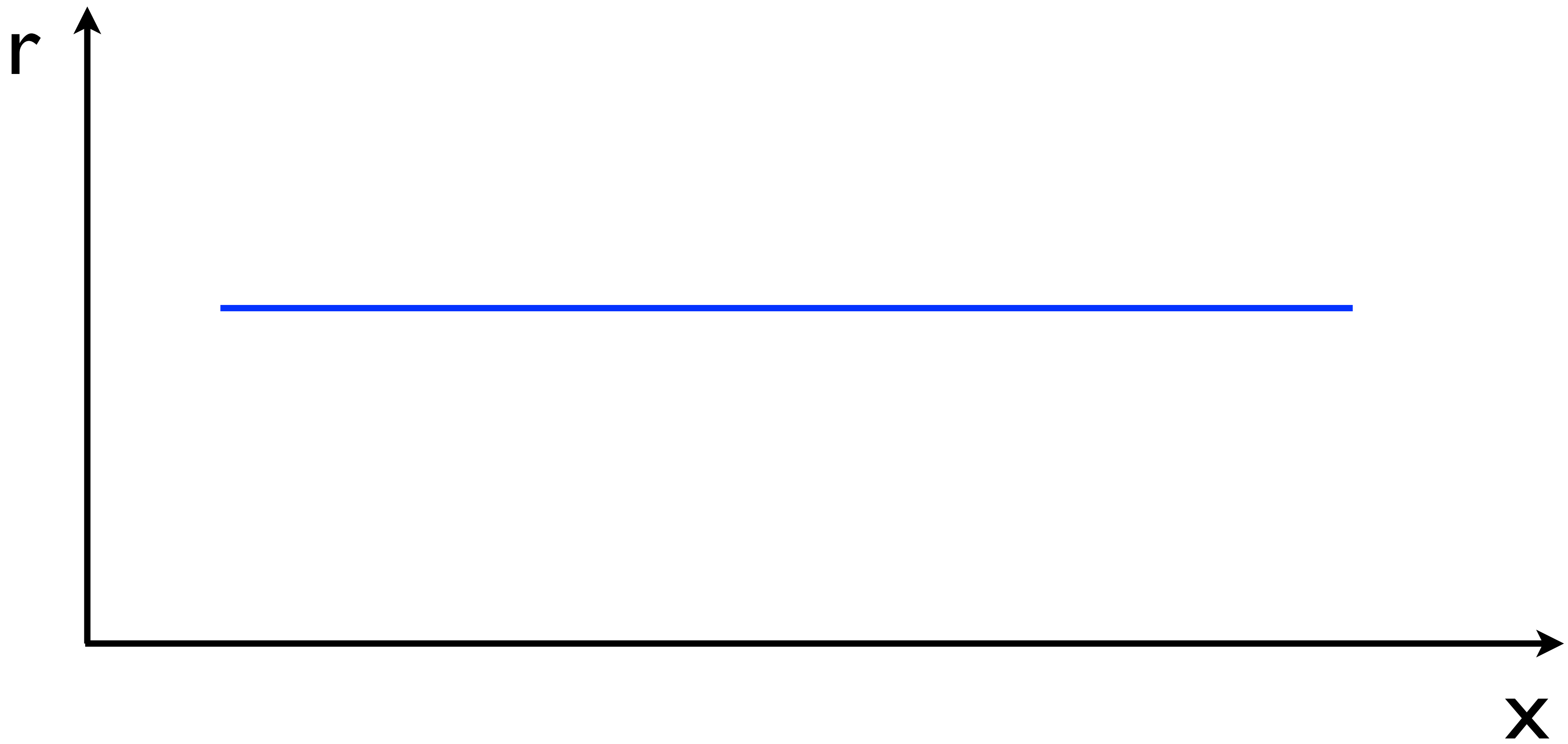
$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(m, q, u) + g^{1}_{\mu\nu}$$

proportional to ∇m, ∇q, ∇u

Solve for g^1 perturbatively in derivaties

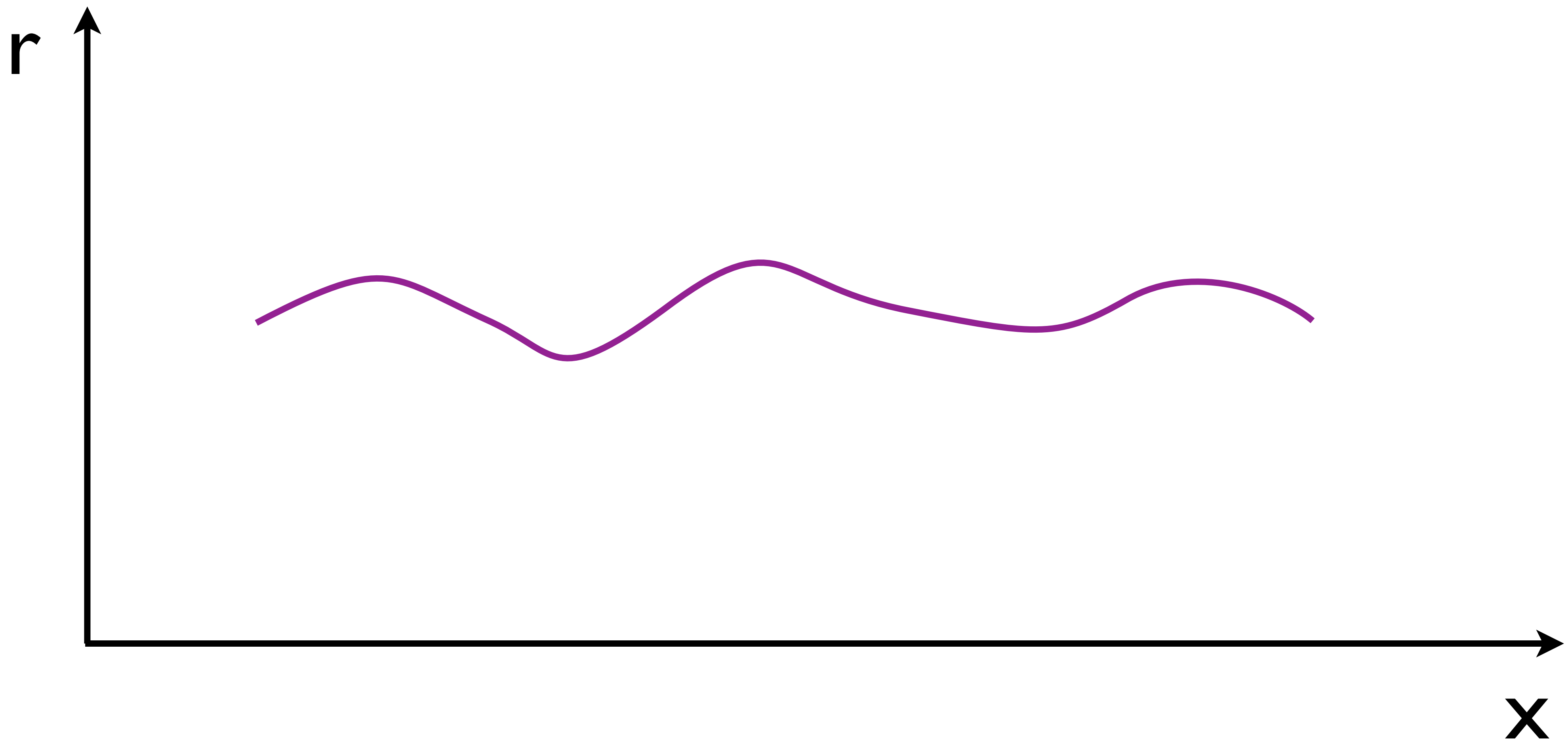
Condition: no singularity outside the horizon

In picture



BH horizon in equilibrium

In picture



BH horizon out of equilibrium

Learning from holography

- Chern-Simons term enters the equation of motion

$$\Box A^{\mu} \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta}$$

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$$\Box \underset{\substack{\uparrow\\ i}}{A^{\mu}} \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\underset{\substack{\uparrow\\ 0}}{\nu}\underset{\substack{\uparrow\\ r}}{\lambda}} F_{\underset{\substack{\uparrow\\ j}}{\alpha}\underset{\substack{\uparrow\\ k}}{\beta}}$$

Learning from holography

- Chern-Simons term enters the equation of motion

$$\square A^{\mu} \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta}$$

i 0 r j k

$A_i \sim u_i$

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$$A_i \sim u_i$$

- This lead to correction to the gauge field
    - $\delta A_i \sim \varepsilon_{ijk} \partial_j u_k$
- Current is read out from asymptotics of A near the boundary: $j \sim \omega$

Back to hydrodynamics

- How can the argument based on 2nd law of thermodynamics fail?
  - 2nd law not valid? unlikely...
  - Maybe we were not careful enough?

$$\partial_\mu s^\mu = \cdots - \xi \omega^\mu \partial_\mu \left(\frac{\mu}{T} \right)$$

Can this be a total derivative?

If yes, then all we need to to is to modify s^μ

$$s^\mu \to s^\mu + D(T,\mu)\omega^\mu$$

so our task is to find D so that

$$\partial_\mu [D(T,\mu)\omega^\mu] = \xi(T,\mu)\omega^\mu \partial_\mu \left(\frac{\mu}{T}\right)$$

for all solutions to hydrodynamic equations

This is possible for a special class of $\xi(T,\mu)$ (expressible in terms of a function of 1 variable: μ/T

but we are still not able to relate ξ to anomalies

Turning on external fields

- To see where anomalies enter, we turn on external background U(1) field A_μ
- Theory still makes sense if A_μ is non dynamical
- Now the energy-momentum and charge are not conserved

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda$$

$$\partial_\mu j^\mu = -\frac{C}{8} \epsilon^{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho}$$

- Power counting: A~1, F~O(p): right hand side has to be taken into account

Anomalous hydrodynamics

- These equations have to be supplemented by the constitutive relations:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$ +viscosities

$$j^\mu = n u^\mu + \xi \omega^\mu + \xi_B B^\mu \qquad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$$

+diffusion+Ohmic current

- We demand that there exist an entropy current with positive derivative: $\partial_\mu s_\mu \geq 0$
- The most general entropy current is

$$s^\mu = s u^\mu - \frac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu$$

Entropy production

- Positivity of entropy production almost completely fixes all functions ξ, ξ_B, D, D_B

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right) + C_1 T^2\left(1 - \frac{2n\mu}{\epsilon + P}\right)$$

anomaly coefficient

not fixed (grav. anomaly)

$$\xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right)$$

$$j^\mu = \cdots + \xi\omega^\mu + \xi_B B^\mu$$

These expressions have been checked for N=4 SYM

A more convenient frame

$$u^\mu \to u^\mu + \frac{1}{\epsilon + P}[(\tfrac{2}{3}C\mu^3 + 2C_1\mu T^2)\omega^\mu + \tfrac{1}{2}(C\mu^2 + C_1 T^2)B^\mu]$$

$$j^\mu = nu^\mu + (C\mu^2 + C_1 T^2)\omega^\mu + C\mu B^\mu$$

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + (u^\mu q^\nu + q^\mu u^\nu)$$

"heat flux"

$$q^\mu = (\tfrac{2}{3}C\mu^3 + 2C_1\mu T^2) + \tfrac{1}{2}(C\mu^2 + C_1 T^2)B^\mu$$

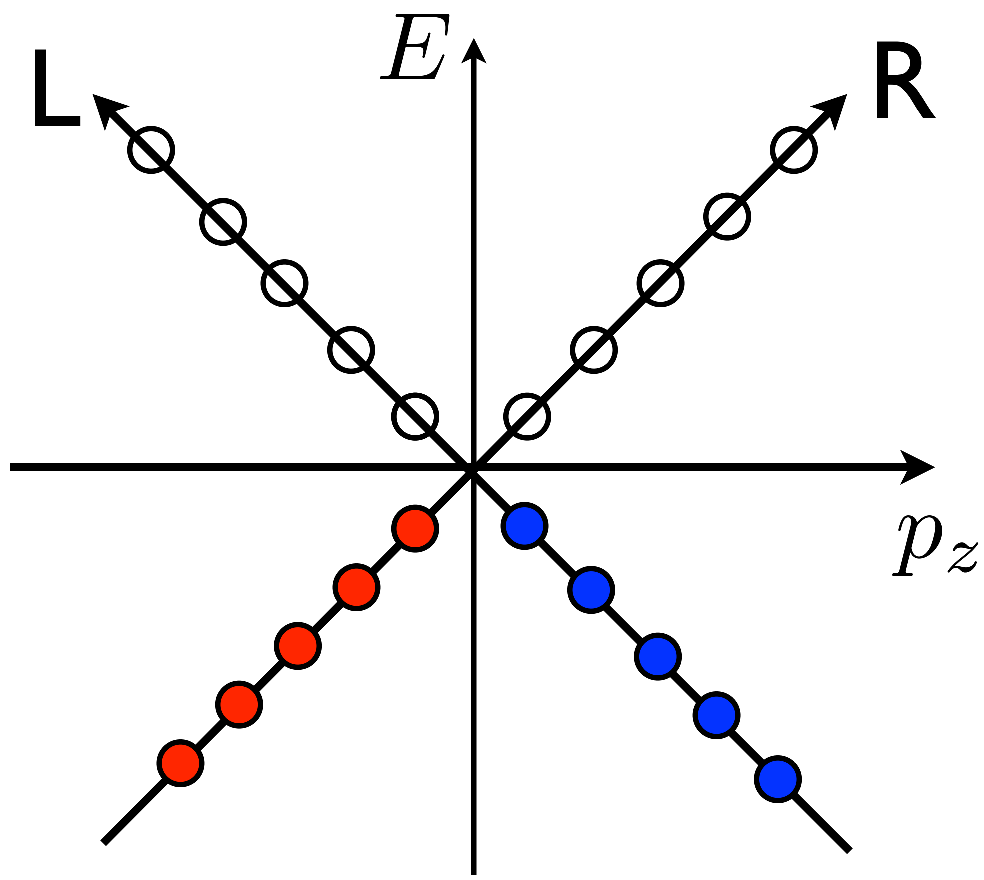
anomalous terms are have simpler forms

Current induced by magnetic field

Spectrum of Dirac operator:

$$E^2 = 2nB + p_z^2$$

All states LR degenerate except for n=0



$$j_{\mathrm{L}} \sim -C\mu B$$
$$j_{\mathrm{R}} \sim C\mu B$$

$$j_5 = j_R - j_L \sim C\mu B$$

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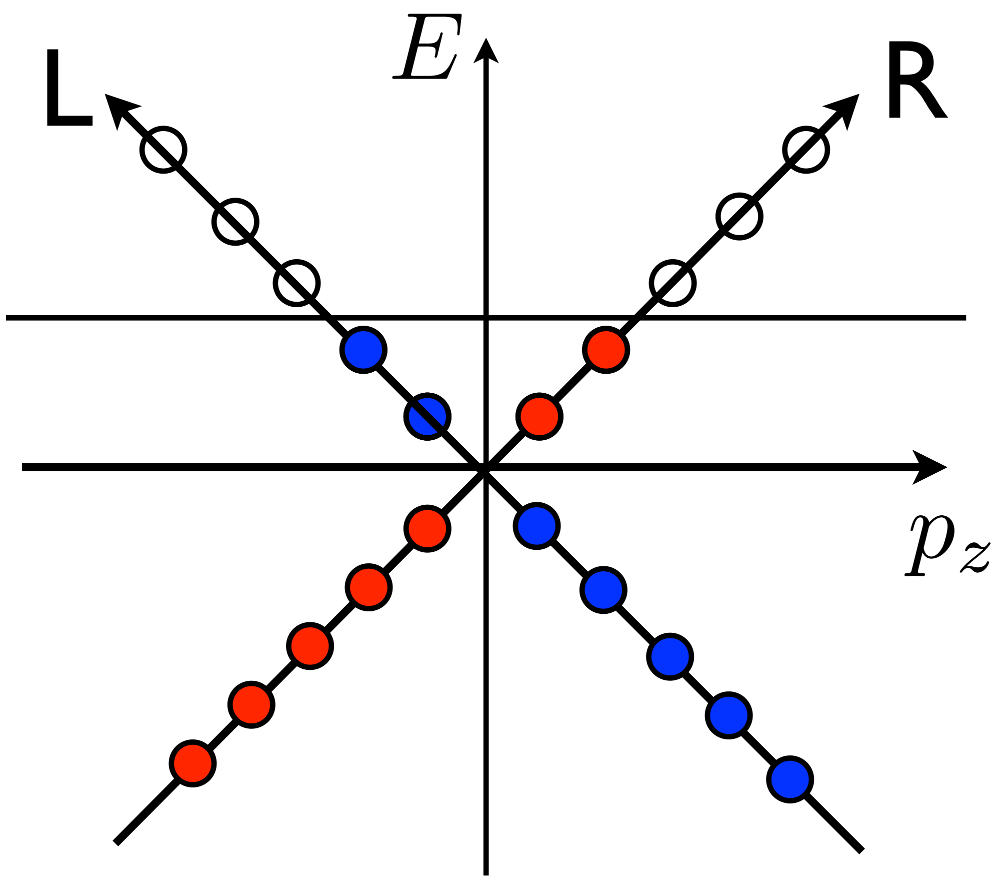


$$j_{\mathrm{L}} \sim -C\mu B$$
$$j_{\mathrm{R}} \sim C\mu B$$

$$j_5 = j_R - j_L \sim C\mu B$$

If there is only right-handed fermions:

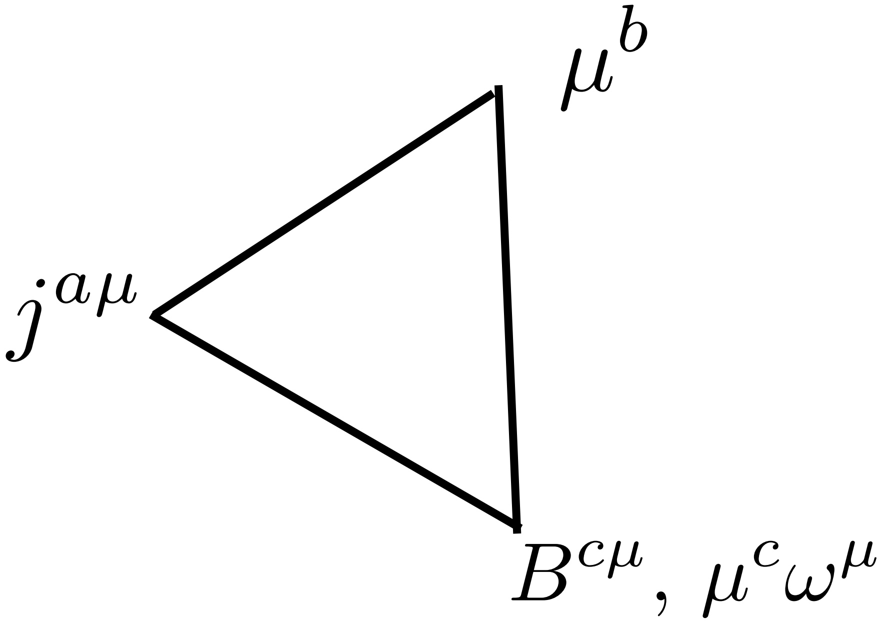
$$j^\mu = nu^\mu + C\mu B^\mu$$
$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + \frac{C}{2}\mu^2(u^\mu B^\nu + u^\nu B^\mu)$$

going to the Landau-Lifshitz frame gives the correct ξ_B

No similar picture for vorticity induced current

Multiple charges

In the case when there are multiple conserved charges: anomalous contribution to each current



$$\sim C^{abc}$$

$$j^{a\mu} = \cdots + \# C^{abc}\mu^b\mu^c\omega^\mu + \# C^{abc}\mu^b B^{c\mu}$$

(these are gauge invariant, non-conserved currents)

For U(1)A currents : $j^{5\mu} = \cdots + C' T^2 \omega^\mu$

Multiple charges (II)

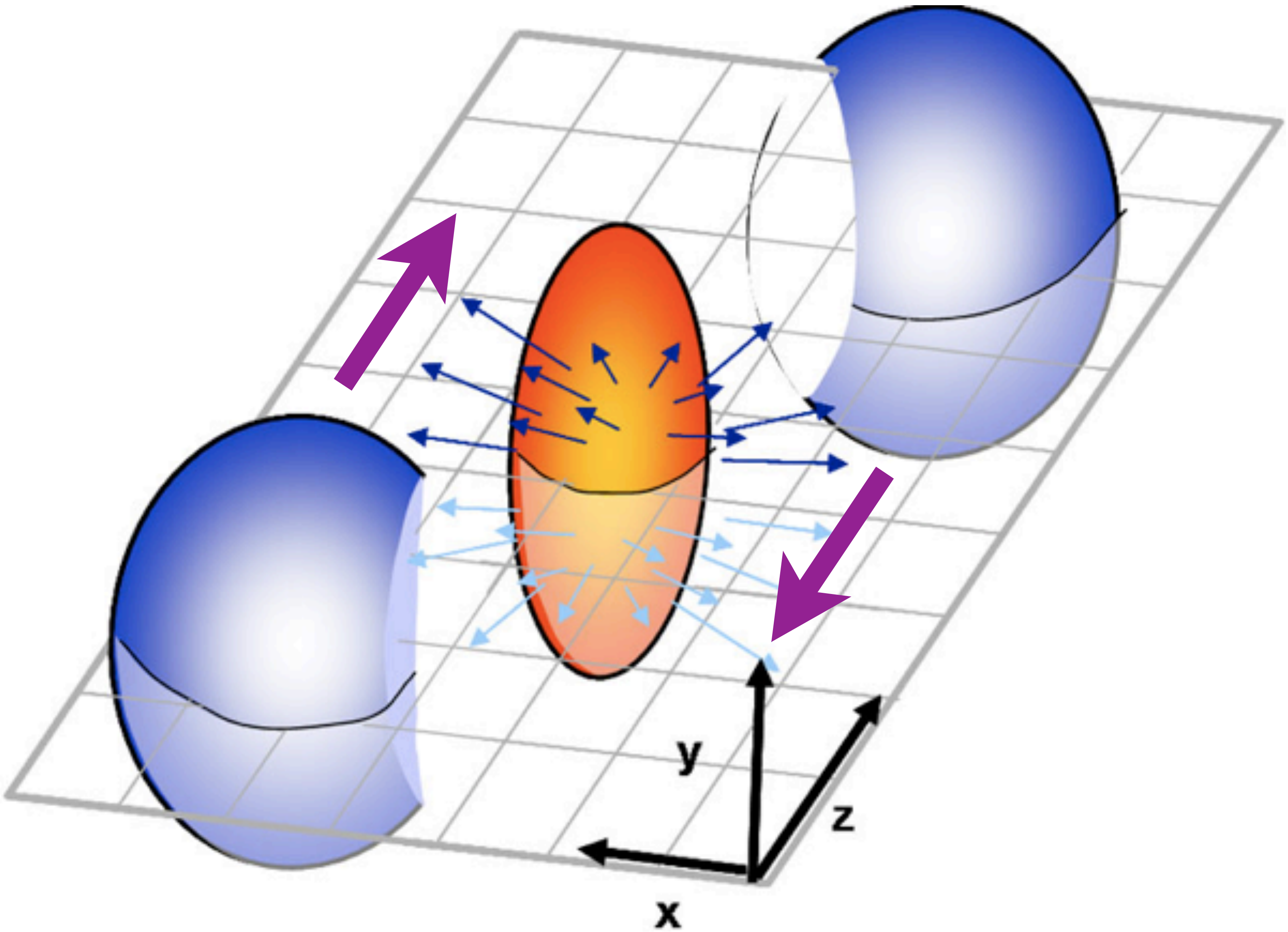
Example: theory with one massless Dirac fermion

$$j^\mu = \frac{1}{2\pi^2}(2\mu\mu_5\omega^\mu + \mu_5 B^\mu + \mu B_5^\mu)$$

$$j_5^\mu = \frac{1}{2\pi^2}((\mu^2 + \mu_5^2)\omega^\mu + \mu B^\mu + \mu_5 B_5^\mu)$$

$$+C' T^2 \omega^\mu$$

Observable effect on heavy-ion collsions?



Chiral charges accumulate at the poles: what happens when they decay?

“Chiral magnetic effect”

- Large axial chemical potential μ_5 for some reason
- Leads to a vector current: charge separation
- π^+ and π^- would have anticorrelation in momenta
- Some experimental signal?
- Attempts to explain the signal by $j \sim \mu_5 B$ Kharzeev et al

Further developments

- Anomalies in kinetic theory: effect of "Berry curvature" on the Fermi surface (DTS, Yamamoto, 1203.2697)
- Static (Euclidean) view on the anomalous kinetic coefficients: Jensen, Loganayagam, Yarom 1207.5824; Golkar & DTS 1207.5806

Conclusions

- Anomalies affect hydrodynamic behavior of relativistic fluids
- First seen in holographic models, but can be found by other methods
- New terms in hydrodynamics completely fixed
- Interplay between the quantum and classical theories