Gauge/gravity duality and its applications

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Plan of of this lecture

- AdS/CFT correspondence at zero temperature.
- Thermodynamics of strongly coupled plasmas ($\mathcal{N} = 4$ supersymmetric Yang-Mills theory)
- Viscosity at strong coupling

Next lecture:

- Quantum anomalies in hydrodynamics

Motivation

- Strong coupling regime of QCD is difficult
- There exist gauge theories where the strong coupling regime can be studied analytically using AdS/CFT correspondence
- Hope: learn more about QCD from models

Some literature

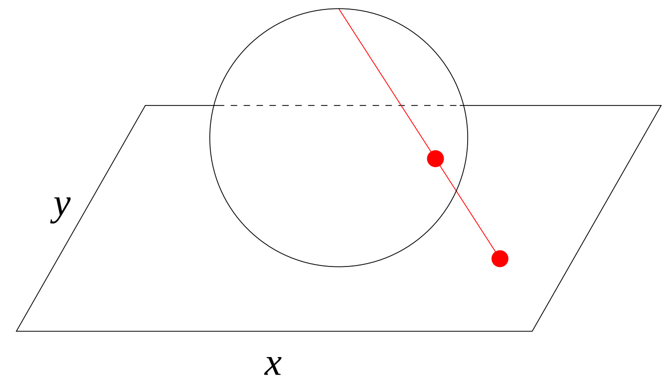
- Horowitz and Polchinski, Gauge/gravity duality, gr-qc/0602037
  general, philosophical, basic ideas
- Klebanov's TASI lectures Introduction to AdS/CFT correspondence, hep-th/0009139
  early and very readable introduction
- McGreevy Holographic duality with a view toward many-body physics, hep-th/0909.0518
  emphasizes recent "AdS/CMT" applications

Zero Temperature AdS/CFT

AdS space: 2D illustration

Sphere in projective coordinates:

$$ds^2 = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}$$

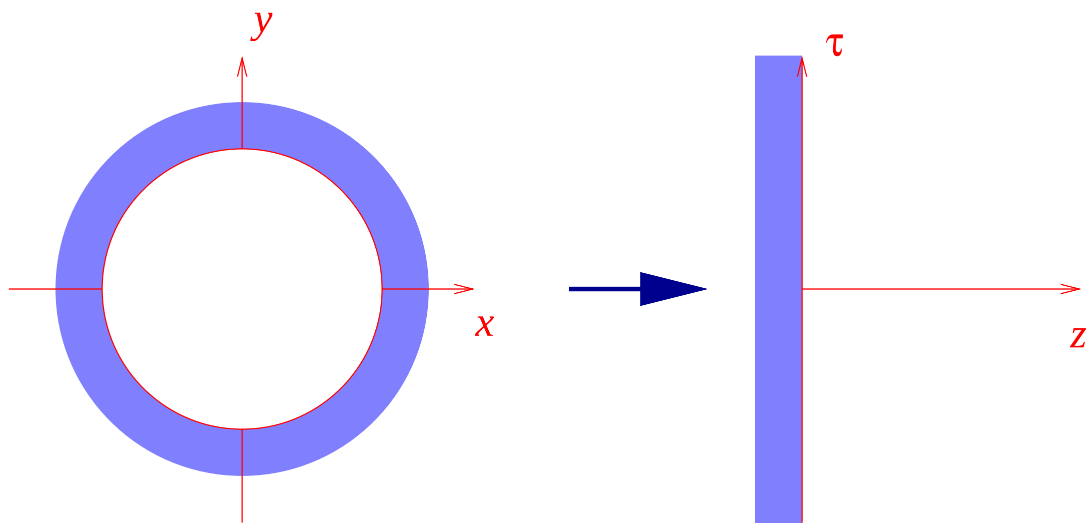


This is a space with constant positive curvature.
To make a space with constant negative curvature, we change signs in the denominator.
The result is Euclidean AdS_2 space:

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

AdS space (continued)

Now we perform conformal transformation:



$$z + i\tau = \frac{1 - x - iy}{1 + x + iy} \Rightarrow ds^2 = \frac{dz^2 + d\tau^2}{z^2}$$

In more than two dimensions, Minkowski signature:

$$ds^2 = \underbrace{\frac{R^2}{z^2}}_{\text{warp factor}} \left(-dt^2 + d\vec{x}^2 + dz^2\right)$$

Problem 1

Find a coordinate transformation that maps

$$ds^2 = \frac{dy_1^2 + dy_2^2 + \cdots dy_n^2}{(1 - y_1^2 - y_2^2 - \cdots y_n^2)^2}$$

to

$$ds^2 = \frac{1}{z^2}(dz^2 + dx_1^2 + dx_2^2 + \cdots dx_{n-1}^2)$$

Verify that the Ricci tensor for these metrics satisfies

$$R_{\mu\nu} = -\Lambda g_{\mu\nu}$$

where Λ is some constant

Original AdS/CFT correspondence

Maldacena; Gubser, Klebanov, Polyakov; Witten

between $N = 4$ supersymmetric Yang-Mills theory
and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$ds^2 = \frac{R^2}{z^2}(d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2$$

This is a solution to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$(T_{\mu\nu} = F_\mu{}^{\alpha\beta\gamma\delta} F_{\nu\alpha\beta\gamma\delta})$

Large 't Hooft limit in gauge theory $\Leftrightarrow$ small curvature limit in string theory

$$g^2 N_c \gg 1 \Leftrightarrow R/l_s = \sqrt{\alpha'}\, R \gg 1$$

Correlation function are computable at large 't Hooft coupling, where string theory $\rightarrow$ supergravity.

The dictionary of gauge/gravity duality

gauge theory	gravity
operator $\hat{O}$ energy-momentum tensor $T_{\mu\nu}$ dimension of operator	field ϕ graviton $h_{\mu\nu}$ mass of field
globar symmetry conserved current anomaly ...	gauge symmetry gauge field Chern-Simon term ...

$$\int e^{iS_{\rm 4D}+\phi_0 O} = \int e^{iS_{\rm 5D}}$$

where $S_{\rm 5D}$ is computed with nontrivial boundary condition

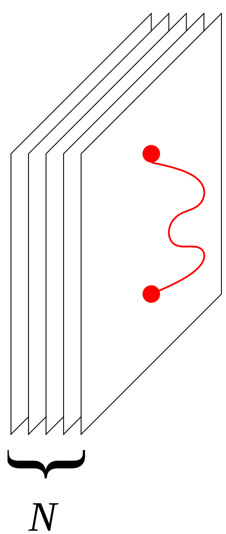
$$\lim_{z\to 0}\phi(\vec{x},z) = \phi_0(\vec{x})$$

Origin of the idea

Consider type IIB string theory in (9+1)D

contains: strings (massless string modes = graviton, dilaton, etc.)
Dp-branes, $p = 1, 3, 5, 7$

Stack N D3 branes on top of each other:



Fluctuations of the branes are described by
$N = 4$ super Yang-Mills theory
field = open strings

$N \gg 1$: space time is curved
closed strings in a curved background

Duality

Hypothesis: the two pictures are two different descriptions of the same object.

Type IIB string theory on AdS$_5 \times$ S^5 $\Leftrightarrow$ $\mathcal{N}=4$ super Yang-Mills in flat space time a conformal field theory

This is supported by many checks:

- Symmmetries: conformal symmetry $\leftrightarrow$ isometry of AdS$_5$, R-symmetry $\leftrightarrow$ SO(6) symmmetry of S^5.
- Correlation functions: some can be computed exactly in field theory and checked with AdS/CFT calculations
- ...

Computing correlators

In the limit $N_c \to \infty$, $g^2 N_c \to \infty$, calculation of correlators reduces to solving classical e.o.m:

$$Z[J] = \int D\phi \, e^{iS[\phi] + i \int JO} = e^{iW[J]}$$

$$W[J] = S_{\text{cl}}[\varphi_{\text{cl}}] \text{ : classical action}$$

φ_{cl} solves e.o.m., $\varphi_{\text{cl}}|_{z\to 0} \to J(x)$.

Example: correlator of R-charge currents

- R-charge current in 4D corresponds to gauge field in 5D
- Field equation for transverse components of gauge fields is

$$\partial_\mu (\sqrt{-g}\, g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}) = 0$$

- In the gauge $A_z = 0$, equation for transverse and longitudinal parts of A_μ decouple. The equation for the transverse part is

$$\partial_z \left(\frac{1}{z} \partial_z A_\perp(z,q) \right) - \frac{q^2}{z} A_\perp = 0 \Rightarrow A_\perp(z,q) = QzK_1(Qz) A_\perp(0,q)$$

Computing correlators (continued)

Two-point correlator = second derivative of classical action over boundary values of fields

$$\langle j^\mu j^\nu \rangle \sim \frac{\delta^2 S_{\text{Maxwell}}}{\delta A_\perp(0)^2} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q}^2 \right) \frac{1}{z} \frac{\partial}{\partial z} [\underbrace{QzK_1(Qz)}_{1+Q^2 z^2 \ln(Qz)}]$$

$$= \#(g^{\mu\nu} q^2 - q^\mu q^\nu) \ln Q^2$$

This structure is the consequence of conformal symmetry

Problem 2

It is known that the operator $O = F_{\mu\nu}^2$ in 4D field theory corresponds to the dilaton field ϕ in 5D. The dilaton couples minimally to the metric:

$$S = -\frac{1}{2}\int d^5x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

- Find the general solution to the equation of motion for ϕ in AdS space

$$ds^2 = \frac{R^2}{z^2}(dz^2 + d\vec{x}^2)$$

  with the boundary condition $\phi(z=0,\vec{x}) = \phi_0(\vec{x})$, $\phi(z=\infty,\vec{x}) = 0$.
- Find the classical action S as a functional of $\phi_0(\vec{x})$,
- Find the correlation function of O,

$$\langle O(\vec{x})O(\vec{y}) = \frac{\delta^2 S}{\delta\phi(\vec{x})\delta\phi(\vec{y})}$$

Ref.: Gubser, Klebanov, Polyakov 1998

Plasma Thermodynamics and Black Holes

Reminder of GR and black holes

Metric: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Einstein equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

where $R_{\mu\nu}$ is the Ricci tensor ($\sim$ curvature, $\partial^2 g_{\mu\nu}$)
$T_{\mu\nu}$ is the stress-energy tensor of matter

Example: Schwarzschild black hole is a solution with $T_{\mu\nu} = 0$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{d\Omega_2^2}$$

Properties

- $r \to \infty$: flat space
- $r = r_0 = 2GM$: metric is singular, but curvature is finite (coordinate singularity)
  this is the black hole horizon

Behavior near horizon

Near $r = r_0$, $ds^2 = \#(r - r_0)dt^2 + \#dr^2/(r - r_0) + \cdots$

For r near r_0, we introduce new coordinate ρ:

$$r - r_0 = \frac{\rho^2}{4r_0} \Longrightarrow ds^2 = -\frac{\rho^2}{4r_0^2}dt^2 + d\rho^2 + r_0^2 d\Omega_2^2$$

This is simply a Minkowski version of polar coordinates:

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2, \qquad \varphi = \frac{it}{2r_0^2}$$

and has no curvature singularity at $\rho = 0$, if φ is a periodic variable

Hawking temperature: $\varphi \sim \varphi + 2\pi$

corresponding to periodic Euclidean time: $it \sim it + \underbrace{4\pi r_0}_{\beta=1/T}$

$$T_H = \frac{1}{4\pi r_0}$$

Black hole entropy

We inteprete the black hole as a thermodynamical system with energy

$$E = M = \frac{r_0}{2G}$$

and temperture $T = T_H$.

$$dS = \frac{dE}{T} = 4\pi r_0 \frac{dr_0}{2G}$$

$$S = \frac{\pi r_0^2}{G} = \frac{A}{4G}$$

$A = 4\pi r_0^2$ area of horizon

Finite-temperature AdS/CFT correspondence

Black 3-brane solution:

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + d\vec{x}^2] + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \qquad f(r) = 1 - \frac{r_0^4}{r^4}$$

- $r_0 = 0$, $f(r) = 1$: is AdS$_5 \times$ S^5, $r = R^2/z$.
- $r_0 \neq 0$: corresponds to $\mathcal{N} = 4$ SYM at temperture

$$T = T_H = \frac{r_0}{\pi R^2}$$

Entropy density

Entropy $= A/4G$

A is the area of the event horizon
G is the 10D Newton constant.

$$A = \int dx\, dy\, dz\, \sqrt{g_{xx} g_{yy} g_{zz}} \times \underbrace{\pi^3 R^5}_{\text{area of S}^5} = V_{3D} \pi^3 r_0^3 R^2 = \pi^6 V_{3D} R^8 T^3$$

On the other hand, from AdS/CFT dictionary

$$R^4 = \frac{\sqrt{8\pi G}}{2\pi^{5/2}} N_c$$

(for derivation see e.g., Klebanov hep-th/0009139).
Therefore

$$S = \frac{\pi^2}{2} N_c^2 T^3 V_{\text{3D}}$$

This formula has the same N^2 behavior as at zero 't Hooft coupling $g^2 N_c = 0$ but the numerical coefficient is 3/4 times smaller.

Thermodynamics

$$S = f(g^2 N_c) N_c^2 T^3 V_{3D}$$

where the function f interpolates between weak-coupling and strong-coupling values, which differ by a factor of 3/4.

Problem 3

The $\mathcal{N} = 4$ SYM theory contains: one gauge boson, 4 Weyl fermions and 6 real scalar fields. Each field is in the adjoint representation of the gauge group SU(N_c). Find the entropy density at zero 't Hooft coupling and finite temperature, and check that it is 4/3 times larger than quoted value found from AdS/CFT correspondence.

Hydrodynamics from Black Hole Physics

Hydrodynamics

- is the effective theory describing the long-distance, low-frequency behavior of interacting finite-temperature systems. Hydrodynamic regime
- Valid at distances $\gg$ mean free path, time $\gg$ mean free time.
- At these length/time scales: local thermal equilibrium: T, μ vary slowly in space.
- Simplest example of a hydrodynamic theory: the Navier-Stokes equations

$$\partial_t \rho + \nabla \dot{(} \rho \mathbf{v}) = 0$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \eta \nabla^2 \mathbf{v}$$

- The quark-gluon plasma can be described by similar equations.
- All microscopic physics reduces to a small number of *kinetic coefficients* (shear viscosity η, bulk viscosity, diffussion coeffecients).

Idea

The idea is to use AdS/CFT correspondence to explore the hydrodynamic regime of thermal gauge theory.

- Finite-T QFT $\Leftrightarrow$ black hole with translationally invariant horizon

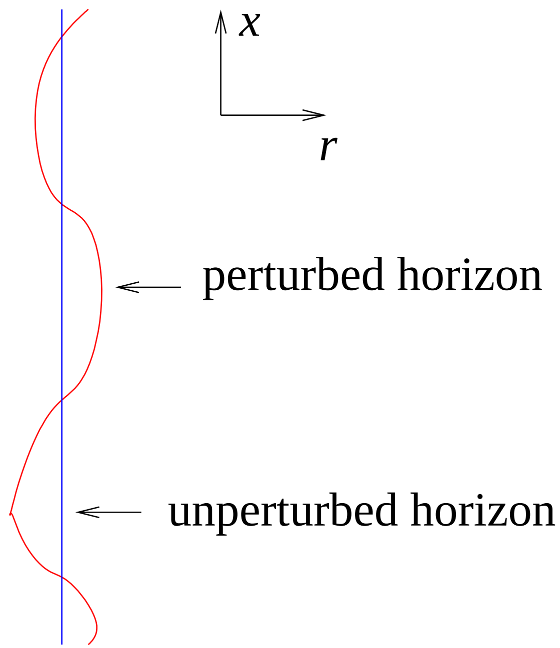
$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}(\frac{dr^2}{f} + r^2 \partial\Omega_5^2)$$

  horizon: $r = r_0$, $\vec{x}$ arbitrary.

- Local thermal equilibrium $\Leftrightarrow$ parameters of metric (e.g., r_0) slowly vary with $\vec{x}$.

  remember that $T \sim r_0/R^2$.

Dynamics of the horizon



$T \sim r_0 = r_0(\vec{x})$

Generalizing black hole thermodynamics M, Q,... to black brane hydrodynamics
$T = T_H(\vec{x})$, $\mu = \mu(\vec{x})$

While we know $S = A/(4G)$, what is the viscosity from the point of view of gravity?

Kubo's formula: physical meaning

Viscosities can be expressed in terms of Green's functions

- Hydrodynamics = effective theory describing response of a system to external long-distance perturbations.
- Example of such perturbation: gravitational waves

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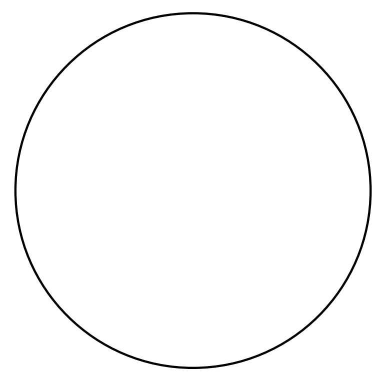

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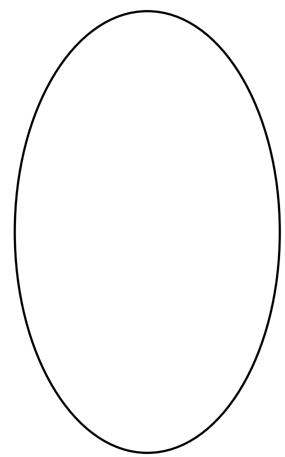

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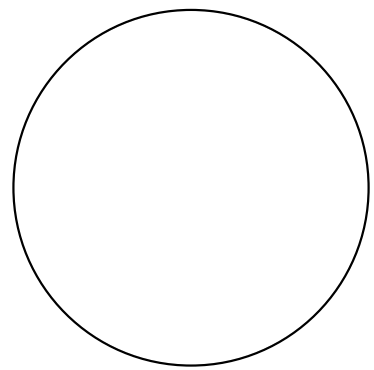

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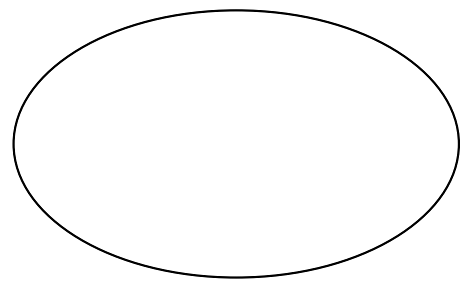

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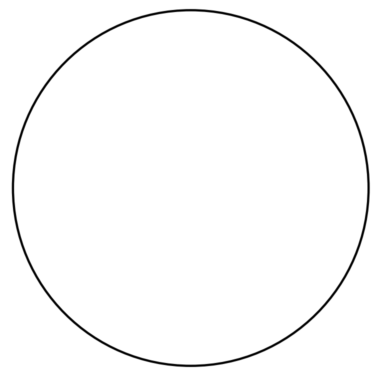

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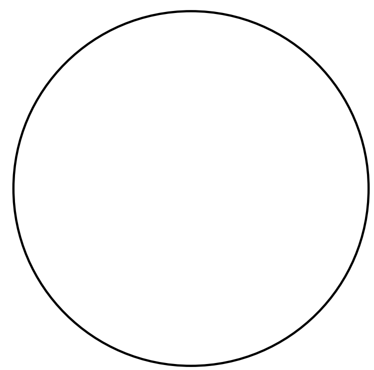

Long-wavelength ravitational waves induce hydrodynamic perturbations

Kubo's formula

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega}\int dt\, d\vec{x}\, e^{i\omega t}\langle[T_{xy}(t,\vec{x}),\, T_{xy}(0,0)]\rangle$$
$$= \lim_{\omega\to 0}\lim_{\vec{q}\to 0}\operatorname{Im} G^R_{xy,xy}(\omega,\vec{q})$$

imaginary part of the retarded correlator of T_{xy}.

Similar relations exist for other kinetic coefficients.

Gravity counterpart of Kubo's formula

Klebanov: $\mathrm{Im} G^R$ is proportional to the absorption cross section by the black hole.

$$\sigma_{\mathrm{abs}} = -\frac{16\pi G}{\omega} \mathrm{Im}\, G^R(\omega)$$

That means viscosity = absorption cross section for low-energy gravitons

$$\eta = \frac{\sigma_{\mathrm{abs}}(0)}{16\pi G}$$

The absorption cross section can be found classically.
There is a theorem that the cross section at $\omega = 0$ is equal to the area of the horizon.
But the entropy is also proportional to the area of the horizon

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Viscosity/entropy ratio and uncertainty principle

Estimate of viscosity from kinetic theory

$$\eta \sim \rho v \ell, \qquad s \sim n = \frac{\rho}{m}$$

$$\frac{\eta}{s} \sim m v \ell \sim \hbar \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

Quasiparticles: de Broglie wavelength $\lesssim$ mean free path

Therefore $\eta/s \gtrsim \hbar$

- Weakly interacting systems have $\eta/s \gg \hbar$.
- Theories with gravity duals have universal η/s, but we don't know how to derive the constancy of η/s without AdS/CFT.
- Computing the viscosity of finite-temperature QCD is a challenging problem!

Further references

- Viscosity from AdS/CFT correspondence: DTS, Starinets 0704.0240
- Determination of viscosity from RHIC data: Luzum, Romatschke 0804.4015
- Attempts to determine viscosity on the lattice: H. Meyer