Lattice Methods for Hadron Spectroscopy: lattice symmetries and classifying states

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INT, Seattle, 8th August 2012

Plan

- A short recap of all-to-all propagators
- A little more on distillation details
- Classifying states by symmetry

recap from yesterday

- Calculating quark propagators is difficult. Solve $M\Psi(y) = \eta(x)$ for M^{-1} where M is a very large matrix.
- Techniques to calculate all to all propagators exist.

recap from yesterday

- Calculating quark propagators is difficult. Solve $M\Psi(y) = \eta(x)$ for M^{-1} where M is a very large matrix.
- Techniques to calculate all to all propagators exist.

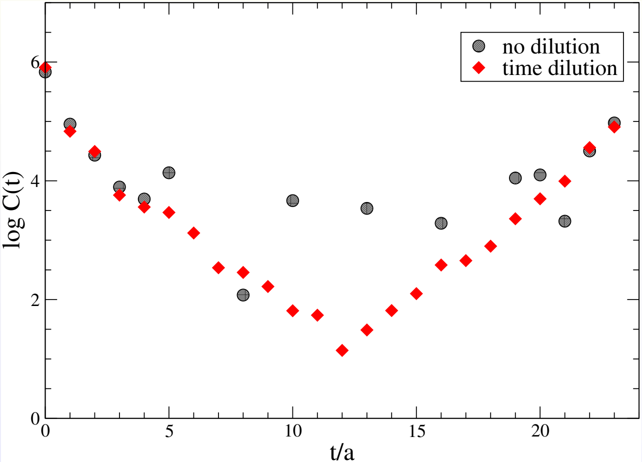
Stochastic estimation

- Stochastic estimation: fill the source vector η with noise. Then 1 inversion gives a stochastic estimate of M^{-1}

$$M^{-1} = \langle\langle \psi_{[r]} \otimes \eta^{\dagger}_{[r]} \rangle\rangle = \frac{1}{N_r} \sum_r \Psi_{[r]}(y) \eta^{\dagger}_{[r]}(x), \;\; N_r \to \infty$$

- Repeated for N_r noise vectors to reduce the error on variance.
- Improve this statistical reduction by dilution $\Rightarrow$ for 1 noise vector do more inversions but gain in an average over diluted vectors.
- Can combine exact and stochastic estimations.

Noise and dilution



- pseudoscalar (pion) propagator on $12^3 \times 24$ lattice
- black: 24 noise sources, no dilution: 24 inversions of M
- red: single noise source + time dilution: 24 inversions of M

Distillation

- Distillation: a redefinition of smearing as **explicitly** a low-rank operator.
- Effect: project out eigenmodes that do not contribute to hadronic physics.
- In the low-rank space M^{-1} can be calculated exactly.

- Consider an isovector meson two-point function:

$$C_M(t_1 - t_0) = \langle\langle \bar{u}(t_1)\Box_{t_1}\Gamma_{t_1}\Box_{t_1}d(t_1) \quad \bar{d}(t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0}u(t_0)\rangle\rangle$$

- Integrating over quark fields yields

$$C_M(t_1 - t_0) = \langle \mathrm{Tr}_{\{\underline{s},\sigma,c\}} \left(\Box_{t_1}\Gamma_{t_1}\Box_{t_1}M^{-1}(t_1,t_0)\Box_{t_0}\Gamma_{t_0}\Box_{t_0}M^{-1}(t_0,t_1)\right)\rangle$$

- Substituting the low-rank distillation operator $\Box$ reduces this to a **much smaller** trace:

$$C_M(t_1 - t_0) = \langle \mathrm{Tr}_{\{\sigma,\mathcal{D}\}} \left[\Phi(t_1)\tau(t_1,t_0)\Phi(t_0)\tau(t_0,t_1)\right]\rangle$$

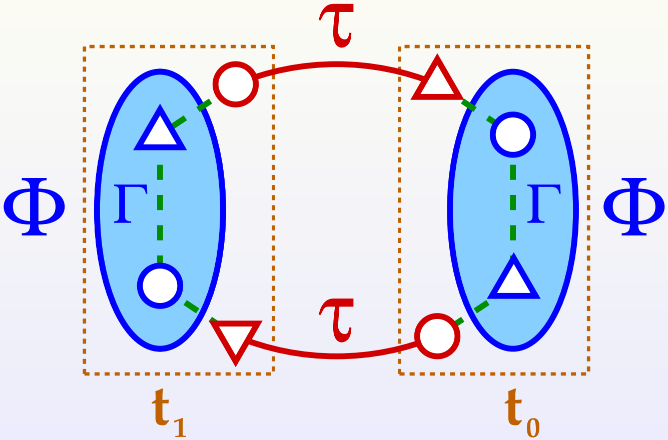
- $\Phi^{\alpha,a}_{\beta,b}$ and $\tau^{\alpha,a}_{\beta,b}$ are $N_{\mathcal{D}} \times N_\sigma$ square matrices.

$$\Phi(t) = V^\dagger(t)\Gamma_t V(t)$$

$$\tau(t,t') = V^\dagger(t)M^{-1}(t,t')V(t')$$

The "perambulator"

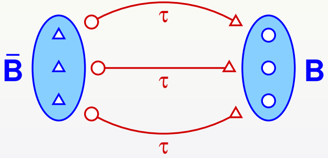
Meson two-point function



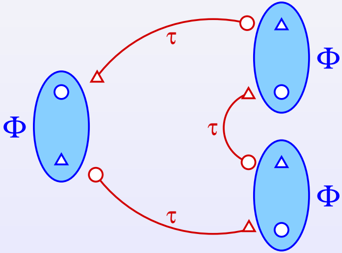
Distilled meson two-point correlation function

$$C_M(t_1 - t_0) = \mathrm{Tr}_{\{\sigma,D\}} \left[\Phi(t_1)\ \tau(t_1, t_0)\ \Phi(t_0)\ \tau(t_0, t_1)\right]$$

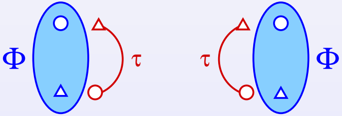
More diagrams: baryons, multihadrons, disconnected



$\bar{B}_{abc}\tau_{aa'}\tau_{bb'}\tau_{cc'}B_{a'b'c'}$

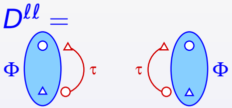


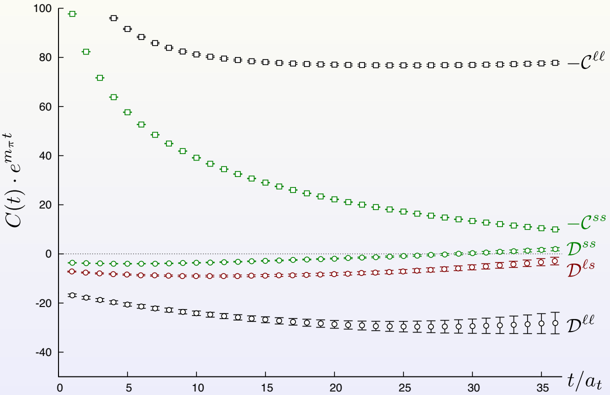
$\mathrm{Tr}\left[\Phi\tau\Phi\tau\Phi\tau\right]$



$\mathrm{Tr}\left[\Phi\tau\right]\ \mathrm{Tr}\left[\Phi\tau\right]$

Good news: precision spectroscopy (2)





- Correlation functions for $\bar{\psi}\gamma_5\psi$ operator, with different flavour content (s, l).
- 16^3 lattice (about 2 fm). [arXiv:1102.4299]

Stochastic estimation and distillation

- Construct a **stochastic identity matrix** in $\mathcal{D}$: introduce a vector η with $N_{\mathcal{D}}$ elements and with

$$E[\eta_i] = 0 \text{ and } E[\eta_i \eta_j^*] = \delta_{ij}$$

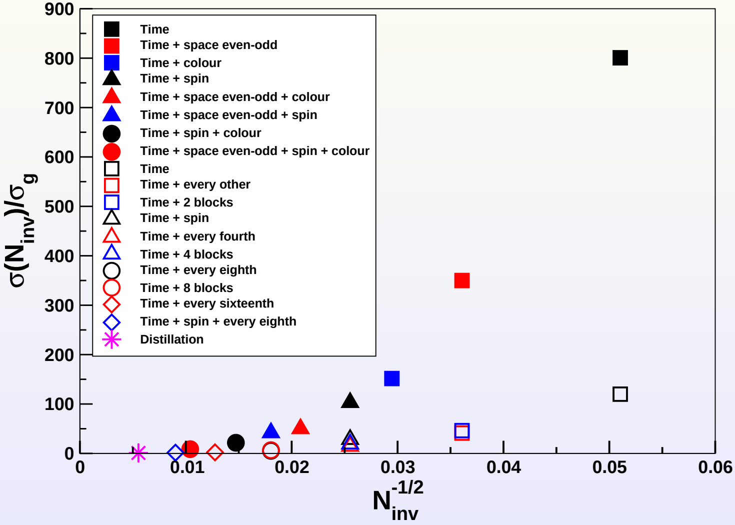
  [$E =$expectation value]
- Now the distillation operator is written

$$\square = E[V\eta\eta^\dagger V^\dagger] = E[WW^\dagger]$$

- Introduces noise into computations
- **Dilution:** "thin out" the stochastic noise.
- Use N_η orthogonal projectors to make a variance-reduced estimator of $E[WW^\dagger] = \sum_{k=1}^{N_\eta} E[V\mathcal{P}_k\eta\eta^\dagger\mathcal{P}_k V^\dagger]$, with $W_k = V\mathcal{P}_k\eta$, a $N_\eta \times (N_s \times N_c)$ matrix
- Remember V was size $N_{\mathcal{D}} \times (N_s \times N_c)$ and $N_{\mathcal{D}}$ scales like V^2.

[arXiv:1104.3870]

Stochastic estimation: baryon correlator



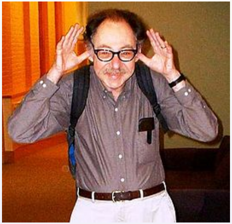
- Convergence faster for noise in distillation space

[arXiv:1011.0481]

Lattice symmetries and classifying spin

Plan

"A man who is tired of group theory is a man who is tired of life." - Sidney Coleman

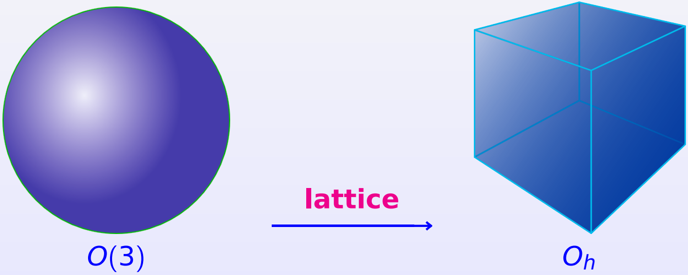


- Continuum and Lattice symmetry groups
- Classifying states by irreducible representations (irreps)
- From lattice to continuum irreps
- [The group theory of two-particle states]

Classifying States

Continuum QCD:

- angular momentum and parity, J^P correspond to irreducible representations (irreps) of improper rotation group, $O(3)$.
- irreps include bosonic (single-valued) and fermionic (double-valued) representations.
- the projection of angular momentun onto some axis, J_z labels rows of the representation.



Symmetry and lattice QCD

- a spatially isotropic lattice breaks $O(3) \to O_h$, the cubic point group
- eigenstates of the lattice Hamiltonian transform under irreps of O_h.
- lattice states are classified by a "quantum letter" Λ^P, the irreps of O_h and not by J^P.
- continuum states with same J^P quantum numbers but different J_z values are in general separated acorss lattice irreps
- need operators which couple strongly to lattice eigenstates , ie project into the irreps of O_h.

Group theory primer

Representations

- a d-dimensional representation Γ of a group G: a set of $d \times d$ matrices each acting on $g_i \in G$ such that $\Gamma(g_1 g_2) = \Gamma(g_1)\Gamma(g_2)$.
- A group of matrices satisfying the same multiplication as the elements of the group is a representation.

- A representation is reducible iff it is possible to perform the same similarity transform on all matrices in the rep and reduce them to block diagonal form.
- Otherwise it is irreducible.

what does this mean?

- say $\Gamma^{(1)}(g)$ and $\Gamma^{(2)}(g)$ representations of the same group. Then

$$\Gamma(g) = \begin{pmatrix} \Gamma^{(1)}(g) & 0 \\ 0 & \Gamma^{(2)}(g) \end{pmatrix}$$

also a representation. Write $\Gamma(g) = \Gamma^{(1)}(g) \oplus \Gamma^{(2)}(g)$.

Properties and rules for irreps

- vectors from matrices of different reps are orthogonal

$$\sum_g \Gamma_i(g)_{mn}\Gamma_j(g)_{mn} = 0, \;\; i \neq j$$

- vectors from same rep but differnet matrix elements are orthogonal

$$\sum_g \Gamma_i(g)_{mn}\Gamma_j(g)_{m'n'} = 0 \;\; m \neq m' \;\text{ or }\; n \neq n'.$$

- vectors from the same rep and same matrix elements have magnitude h/ℓ_i.

$$\sum_g \Gamma_i(g)_{mn}\Gamma_i(g)_{mn} = h/\ell_i$$

where h is the order of the group and ℓ_i the dimension of Γ_i

Useful rules: for irreps and their characters

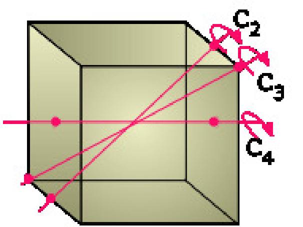
Recall that the character χ of representation $\Gamma(g)$ is

$$\chi(g) = \sum_j \Gamma_{jj}(g), \quad \text{for each } \ g \in G.$$

- $\sum_i \ell_i^2 = h$
- $\sum_g [\chi_i(g)]^2 = h$ a simple test of irreducibility
- $\sum_g \chi_i(g)\chi_j(g) = h\delta_{ij}$
- In a given rep or irrep the characters of all matrices belonging to the same class are identical.
- In a group, number of irreps = number of classes.

Symmetry group of the cube

- O: the symmetry group of the octahedron (dual to a cube)
- 24 rotational (orientation-preserving/proper) symmetries
- 48 including combinations of reflection and rotation:
- cubic point group: $O_h = O \otimes \{\mathbb{I}, \mathbb{I}_s\}$



Operation	No.	Label
identity	1	$\mathbb{I}$
90° about axes through centres of opposite faces	6	C_2
180° about the same axes	3	C_2
120° about diagonals connecting opposite vertices	8	C_3
180° about axes through centers of opposite edges	6	C_4

A soothing exercise in group theory

Think about the symmetries of the cube that we have described. Construct matrices forming an irreducible representation.
The identity operation

$$E\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotation of $\pm\frac{\pi}{2}$ about x, y, z axes gives

$$C_x(1) \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ z \\ -y \end{pmatrix}, \quad C_x(-1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ -z \\ y \end{pmatrix},$$

$$C_y(1) \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} -z \\ y \\ x \end{pmatrix} \quad C_y(-1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

$$C_z(1) \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} y \\ -x \\ z \end{pmatrix} \quad C_z(-1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

More on symmetry

- O has 5 conjugacy classes (O_h has 10)
- number of conjugacy classes = number of irreps
- Schur: for G is a group and Γ_i an irrep of G.

$$|G| = \sum_i \dim(\Gamma_i)^2$$

- So for O we get: $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$.
- irreps called: A_1, A_2, E, T_1, T_2.

Character tables

- A character table is the tabulation by class of the irreps of a group.
- The entries consist of characters, the trace of the matrices representing group elements of the column's class in the given row's group representation.

Character table for O

O	$\mathbb{I}$	$8C_3$	$6C_2$	$6C_4$	$3C_2(=(C_4)^2)$
A_1	+1	+1	+1	+1	+1
A_2	+1	+1	-1	-1	+1
E	+2	-1	0	0	+2
T_1	+3	0	-1	+1	-1
T_2	+3	0	+1	-1	-1

The cubic point group: O_h

- Note: the extension to O_h includes the 24 improper rotations (spatial inversions) of O such that

$$\mathbb{I}_s \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

- The number of group elements is now 48 with 10 irreps labelled $A_{1g}, A_{1u}, A_{2g}, A_{2u}, E_g, E_u, T_{1g}, T_{1u}, T_{2g}, T_{2u}$.
- (g, u) label the even ($g$erade) and odd ($u$ngerade) behaviour under spatial inversion.

Verify that the extended character table is

$\mathbf{O_h}$	$\mathbb{I}$	$8C_3$	$6C_2$	$6C_4$	$3C_2(=(C_4)^2)$	$\mathbb{I}_s$	$6S_4$	$8S_6$	$3\sigma_h$	$6\sigma_d$
A_{1g}	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	-1	-1	1	1	-1	1	1	-1
E_g	2	-1	0	0	2	2	0	-1	2	0
T_{1g}	3	0	-1	1	-1	3	1	0	-1	-1
T_{2g}	3	0	1	-1	-1	3	-1	0	-1	1
A_{1u}	1	1	1	1	1	-1	-1	-1	-1	-1
A_{2u}	1	1	-1	-1	1	-1	1	-1	-1	1
E_u	2	-1	0	0	2	-2	0	1	-2	0
T_{1u}	3	0	-1	1	-1	-3	-1	0	1	1
T_{2u}	3	0	1	-1	-1	-3	1	0	1	-1

Group theory for baryons

- To determine fermionic reps of O its double group O^D must be used.
- 48-element group obtained from O by adding a negative identity: rotations through 2π. The group of rotations for which you recover the identity after a rotation of 4π.

O^D

- 8 single-valued irreps
- In 5, rotation by 2π represented by the identity matrix - coincide with irreps of O.
- 3 new irreps: G_1, G_2, H and $24 = \sum_i l_i^2 = 2^2 + 2^2 + 4^2$ (irrep dimensions 2,2 and 4)

Connecting lattice and continuum groups

- Considering O for clarity.
- There are an infinite number of irreps (J values) in the continuum but just 5 on the lattice
- To identify which continuum states can occur in a particular irrep note that O is a subgroup of $SO(3)$
- Restricting the irreps of $SO(3)$ labelled by J to rotations allowed by the lattice generates representations that are reducible ie J is reducible under O or O_h
- Subduction is the method for generating these representations
- Using

$$n_J^{(\alpha)} = \frac{1}{N_G} \sum_k n_k \chi_k^{(\alpha)} \chi_k^{(J)}$$

it is possible to find the multiplicity of the irreps of $SO(3)$ in O

Connecting lattice and continuum groups (2)

	A_1	A_2	E	T_1	T_2
$J=0$	1				
$J=1$				1	
$J=2$			1		1
$J=3$		1		1	1
$J=4$	1		1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Connecting lattice and continuum groups (2)

	A_1	A_2	E	T_1	T_2
$J = 0$	1				
$J = 1$				1	
$J = 2$			1		1
$J = 3$		1		1	1
$J = 4$	1		1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- In principle then to identify a $J = 2$ state, results from E and T_2 at finite a should extrapolate to the same result.
- an expensive business(!)
- Even then, is this enough information to disentangle high-spin states eg $4 = 0 \oplus 1 \oplus 2$?
- In charmonium a radial excitation of the near-degenerate $(0^{++}, 1^{++}, 2^{++})$ could be close in energy to the 4^{++} ground state.

Summary

- States (hadrons) on the lattice are classified the symmetries of the cubic point group O_h
- States are labelled by irreps of this group
- The relationship between continuum J^P states and lattice states is made using group theory: subduction
- For moving hadrons or two-particle states the set of symmetries is further reduced $\Rightarrow$ more group theory!

Lattice irrep, Λ	Dimension	Continuum irreps, J
A_1	1	0, 4, ...
A_2	1	3, 5, ...
E	2	2, 4, ...
T_1	3	1, 3, ...
T_2	4	2, 3, ...
G_1	3	1/2, 7/2, ...
G_2	3	5/2, 7/2, ...
H	4	3/2, 5/2, ...

Group theory of two particles in a box

- Consider two identical particles, with momentum p and $-p$ (so zero total momentum).
- $\Omega(p)$, set of all momentum directions related by rotations in O_h
- Can make a set of operators, $\{\phi(p)\}$ from Ω and these form a (reducible) representation of O_h.
- Example: $\Phi = \{\phi(1,0,0), \phi(0,1,0), \phi(0,0,1)\}$ contains the A_1 and E irreps
- Different particles: $+p$ and $-p$ are not equivalent

p	irreducible content
(0,0,0)	A_1^g
(1,0,0)	$A_1^g \oplus E^g$
(1,1,0)	$A_1^g \oplus E^g \oplus T_2^g$
(1,1,1)	$A_1^g \oplus T_2^g$

References

- John F. Cornwell *Group Theory in Physics, Vols 1 and 2* Academic Press
- H. F. Jones *Groups, Representations and Physics* (Hilger, Bristol 1990)
- Robert Gilmore *Lie Groups, Lie Algebras, and Some of Their Applications* (Dover Publ.:2006)