Introduction to Lattice QCD
Lecture 3

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Improved actions

The goal is to remove lattice artifacts
both from the action and the operators!

Two approaches

- Systematic: usually perturbative, order by order
- Intuitive : like smearing

But

- Improved actions are usually more expensive to simulate (but not always)
- Sometimes improvement hurts more than helps

Best approach : combine intuitive (usually works) with perturbative improvement so you can put your action up on a pedestal (control the continuum extrapolation).
(Is it worth the cost?)

Lattice artifacts

Free scalar lattice action

$$S[\phi] = a^4 \sum_n \left(\frac{1}{2} (\Delta_\mu \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 \right)$$
$$= a^4 \int_p (\hat{p}^2 + m^2) \phi^*(-p) \phi(p)$$

The 2-point function

$$\Gamma_2(p) = \hat{p}^2 + m^2 = \frac{4}{a^2} \sin^2 \frac{\mathrm{p}_\mu \mathrm{a}}{2} + \mathrm{m}^2$$
$$= p^2 + m^2 - \frac{1}{12} a^2 \sum p_\mu^4 + \dots$$

Can we modify the action to cancel p_μ^4?

Lattice artifacts

We can add dimension 6 operators to the action:

$$
\begin{aligned}
(\Delta_\mu^2 \phi)^2 \\
(\Delta_\mu \phi)(\Delta_\mu \Delta_\nu^2 \phi) \\
(\Delta_\mu \Delta_\nu \phi)^2
\end{aligned}
$$

Improved scalar action

$$S[\phi] = a^4 \sum_n \left(\frac{1}{2}(\Delta_\mu \phi_n)^2 + \frac{1}{12}(\Delta_\mu^2 \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 \right)$$

is $\mathcal{O}(a^2)$ improved action.
Do this systematically : tree level improved action
Add loop corrections : 1-loop, etc improved action
This is the Symanzik improvement program (and can be done systematically)

Homework

Calculate the $\mathcal{O}(a^4)$ improved scalar action. (That might be more than what you want to do tonight.)

Symanzik improved gauge action

$$\mathrm{Tr}(\mathrm{U}_{\Box} + \mathrm{U}_{\Box}^{\dagger}) = 2\mathrm{Tr}\mathbf{1} + \mathrm{a}^4\mathrm{Tr}(\mathrm{F}_{\mu\nu})^2 + \mathcal{O}(\mathrm{a}^6)$$

There are 3 dimension-6 operators in the continuum:
On the lattice we can use the 6-link operators to generate them:

Symanzik improved gauge action

Improvement conditions:

- Normalization: $c_0 + 8c_1 + 8c_2 + 16c_3 = 0$
- Improvement: $c_2 = 0$
- Improvement: $c_3 + c_1 = 1/12$

The usual choice: $c_3 = 0$, $c1 = -1/12$, $c_0 = 3/4$ Luscher-Weisz tree level action

Symanzik improved gauge action - 1-loop

the 1-loop correction changes the coefficients as

$$c_i = c_{i0} + g^2 c_{i1}$$

Using $g^2 = 6/\beta$ the 1-loop correction does not help much;
Non-perturbative improvement is needed:
Tadpole improved 1-loop Symanzik action is the MILC collaboration's choice

Tadpole improvement

Heuristic but pretty good:
Lattice perturbation theory converges faster if we sum up a set of lattice-induced diagrams, tadpoles:

In practice: rescale

$$U_{n,mu} \longrightarrow u_0 U_{n,\mu}$$

with

$$u_0 = (\frac{1}{N}\langle \mathrm{Tr} \mathrm{U}_{\Box} \rangle)^{1/4}$$

$$S_{TP} = \frac{\beta}{6u_0^4} \sum_p \mathrm{Tr}(\mathrm{U}_{\Box} + \mathrm{U}_{\Box}^{\dagger}) + \ldots$$

Increases the weight of the longer loop terms.

Clover term for Wilson fermions

The Wilson term adds an $\mathcal{O}(a)$ error. How can we remove that?

There are 5 basic dimension-5 operators:

$$
\begin{array}{ll}
O_1 & \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \\
O_2 & \bar{\psi}D_\mu D_\mu\psi \\
O_3 & mg_0^2\mathrm{Tr}(\mathrm{F}_{\mu\nu}^2) \\
O_4 & m\bar{\psi}(D_\mu - D_\mu^*)\psi \\
O_5 & m^2\bar{\psi}\psi
\end{array}
$$

Only the first one is needed

$$S_{CW} = S_W + a^5 c_{SW}\frac{i}{4}\sum \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$$

$c_{SW} = 1 + \mathcal{O}(g^2)$.

Homework

Show that the clover term indeed gives $F_{\mu\nu}$. (Not as hard as it sounds.)

Non-perturbative improvement -I

It has been observed that the 1-loop perturbative improvement (especially for c_{SW}) is not enough.
Two approaches are used to improve the convergence

- Non-perturbative calculation of g^2 coefficient - Schrodinger functional method : systematic calculation
- Heuristic (and quick) tadpole improvement:
  "guesses" (intelligently!) a redefinition of the gauge coupling to improve the convergence of PT

.

Many actions use tadpole improved coefficients both in gauge and fermion terms (Asqtad, HISQ)

Smearing

What causes the most trouble?
Large fluctuations at the plaquette scale
$\rightarrow$ smear the short scale without changing the IR physics.

$$U_{n,\mu} \longrightarrow W_{n,\mu}$$

in a gauge invariant way.

Smearing

Smeared links can be used in gauge operators but most useful when coupled to fermions:

$$
\begin{aligned}
S &= S_g[U] + \\
&a^4 \sum_n \left(\bar{\psi}_n \psi_n - \kappa(\bar{\psi}_n(1-\gamma_\mu) W_{n,\mu} \psi_{n+\mu} + \bar{\psi}_n(1+\gamma_\mu) W_{n,\mu} \psi_{n-\mu})\right)
\end{aligned}
$$

APE smearing

First smearing transformation proposed

Used for operator improvement

How does it help? Reduces scaling violations, improves chiral symmetry, reduces taste breaking.

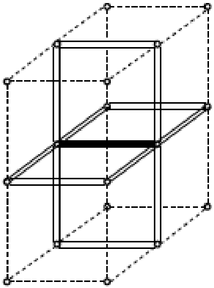
Can it hurt? If repeated too many times, it smears out short scale physical properties.

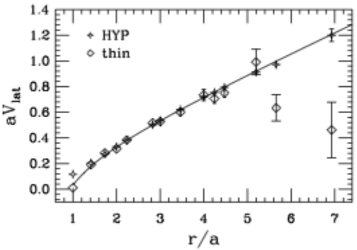
APE smearing

Main problem: projected link is not differentiable, cannot be used in dynamical simulations.
Do we need the projection?
Yes.
The Asqtad action containes up to 7- link loops in smearing but does not project - has considerably worst taste breaking than projected APE.

Smearing

HYP smearing : 3 sets of APE smearing that stays within a hypercube: very compact yet effective. (A.H., F. Knechtli, 1999)





HYP smearing makes an improved Wilson loop operator

Main problem: not differentiable, cannot be used in dynamical simulations.

Stout smearing

First differentiable smeared link, useble in dynamical simulations (Morningstar, Peardon, 2002)
Main idea: $U_{n,\mu} \in SU(3)$ is the link, $\Sigma_{n,\mu} \in G(3,3)$ is the APE staple sum.

$$Q = \frac{i}{2}(\Sigma U^\dagger - U\Sigma^\dagger) - \frac{i}{2N}\mathrm{Tr}(\Sigma \mathrm{U}^\dagger - \mathrm{U}\Sigma^\dagger)$$

is Hermitian, so

$$W = e^{i\rho Q} U_{n,\mu} \in SU(3)$$

Is it smearing? Kind of. Certainly on smoother lattices; Used extensively.

Stout smearing

How it smears

$$Q = \frac{i}{2}(\Sigma U^\dagger - U\Sigma^\dagger) - \frac{i}{2N}\mathrm{Tr}(\Sigma \mathrm{U}^\dagger - \mathrm{U}\Sigma^\dagger)$$

$$W = e^{i\rho Q} U_{n,\mu} = (1 + iQ \ldots) U$$

U(3) projection

An even simpler projection:

$$Q = (1-\alpha)U_{n,\mu} + \frac{\alpha}{6}\Sigma$$

$$W = Q(QQ^{\dagger})^{-1/2} \in U(3)$$

is sufficient projection and it is differentiable.

Smeared actions

Stout (2-6 times) is used with Wilson fermions (Wuppertal, Jlab, etc)
HYP combined with stout (HEX) is used in Wuppertal
U(3) projected HYP is used in BSM (Colorado/Tel Aviv)
U(3) projection is part of HISQ
HISQ(Highly improved staggered quarks) : smeared + perturbatively improved: all the bells& whistles you can imagine

Smearing has helped every fermion action it has been tried with.
Smeared action simulations are frequently faster than unsmeared ones.

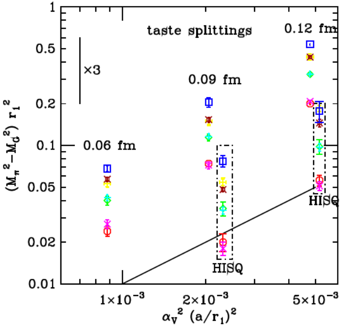
Taste breaking

Staggered actions brake "taste":

- The Dirac components of the 4 species (tastes) are distributed in a hypercube
- The 16 pseudo scalars get their components from different sites $\rightarrow$ have different masses
- There is only one Goldstone pion, the rest are heavy (but will become massless in the continuum limit)

On the lattice coarse gauge fields make taste break strongly.

Taste breaking in HISQ action



Taste breaking of the HISQ and Asqtad actions

$$(M_\pi^2 - M_G^2)r_1^2 \quad r_1^2 \approx 0.45 GeV^{-2}$$

Smeared actions

If you (or your adviser) are still using an unsmeared action, you are most likely

- Wasting computer time
- Live with large lattice artifacts

Improved actions are not necessary . . . but extremely useful.