Introduction to Lattice QCD
Lecture 3

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Notes: http://www-hep.colorado.edu/~anna/INT_school

The continuum limit

The lattice is just a regulator, we want to take the $a \to 0$ continuum limit.

But the lattice action does not depend on a.

$$S[\psi] = \sum_n \frac{\beta}{6} \sum_p \mathrm{Tr}(\mathrm{U}_\Box + \mathrm{U}_\Box^\dagger)$$

Or even when it does, it just carries the dimension, it can be absorbed in the fields

$$S[\psi] = \sum_n \frac{\beta}{6} \sum_p \mathrm{Tr}(\mathrm{U}_\Box + \mathrm{U}_\Box^\dagger) + a^3(\bar{\psi}_n \gamma_\mu U_{n,\mu} \psi_{n+\mu} + \bar{\psi}_n \gamma_\mu U^\dagger_{n-\mu,\mu} \psi_{n-\mu} + am\bar{\psi}_n \psi_n)$$

The lattice spacing emerges through the coupling $\beta = 6/g^2$.

The continuum limit - massless case

On the lattice it is natural to work with bare quantities: $\beta = 6/g^2$ is the coupling.

Consider a physical quantity m_p at some coupling β :

$m_{\rm latt}(g(a)) = a m_p$

$$a\frac{d}{da}m_p = 0 = a\frac{dg(a)}{da} \times \frac{dm_{\rm latt}}{dg} - m_{\rm latt}$$

$$\beta(g) = a\frac{dg(a)}{da} = -\beta_0 g^3 - \beta_1 g^5 \ldots$$

is the RG β function ($\beta_0 > 0!$)

The continuum limit - massless case

Integrate the equation (lowest order for now):

$$\begin{aligned}\int \frac{dg}{\beta(g)} &= \int \frac{dm_{\mathrm{latt}}}{m_{\mathrm{latt}}} \\ \frac{-1}{2\beta_0 g^2} &= \ln \frac{\mathrm{m}_{\mathrm{latt}}}{\Lambda} \\ m_{\mathrm{latt}} &= \Omega e^{-1/(2\beta_0 g^2)} = a m_p\end{aligned}$$

where Ω is just an integration constant.

The continuum limit - massless case

$$a = \frac{1}{\Lambda} e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2} (1 + \mathcal{O}(g^2))$$

the relation between a and β

An example : Pure gauge Wilson action, $\beta = 6.0$; The string tension from the static potential is $a\sqrt{\sigma} = 0.220$. Since $\sqrt{\sigma} = 460 MeV$,

$$V(r) = V_0 + \frac{e}{r} + \sigma r$$

$$a(\beta = 6.0) = 0.095\mathrm{fm}$$

Now predict Λ and $a(g)$ everywhere else.

When this works, the system is in the asymptotic scaling regime.

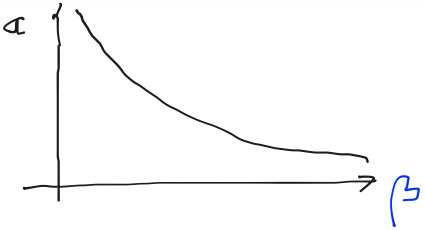
Dimensional transmutation

Magic happened: We started with a system that had no dimensional parameter and through (necessary) regularization we generated a mass scale.

$$\Lambda = \frac{1}{a} e^{-1/2\beta_0 g(a)^2} g(a)^{-\beta_1/\beta_0^2}$$

is the "Lambda parameter".

The continuum limit - massless case



$a \to 0$ means $g^2 \to 0$

At the same time

$$m_{\mathrm{latt}} = a m_{\mathrm{phys}} \longrightarrow 0 \quad \text{or} \quad \xi \sim \mathrm{m}_{\mathrm{latt}}^{-1} \to \infty$$

the bare coupling has to be tuned to a critical point with infinite correlation length.

The continuum limit - massless case

The general approach did not rely on perturbation theory.

Very general, very important:

A continuum limit can be defined by tuning the bare parameter to the critical surface of a lattice system. As long as the critical behavior is universal the lattice continuum limit is universal as well. Give up one physical quantity to set the physical scale - everything else is a prediction.

Renormalization group

We are doing the bare coupling analogue of the continuum (Callan-Symanzik) RG treatment:

$g(\mu)$: renormalized coupling at scale μ, m_p some physical quantity

$$\mu \frac{d}{d\mu} m_p(g(\mu), \mu) = 0 = \mu \frac{dg(\mu)}{d\mu} \times \frac{\partial m_p}{\partial g} + m_p$$

has the same solution. The renormalized β function

$$\beta(g(\mu)) = \mu \frac{dg(\mu)}{d\mu} = -\beta_0 g^3 - \beta_1 g^5 \ldots$$

$$m_p = \Lambda_r e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2}(1 + \mathcal{O}(g^2))$$

But are the bare and renormalzied β functions the same? How about the Λ parameters?

Homework

Show that the RG β functions corresponding to different definitions of the renormalized or bare couplings are identical to two loop level, i.e. β_0 and β_1 are universal. Use

$$g_1 = g_2 + c g_2^3$$

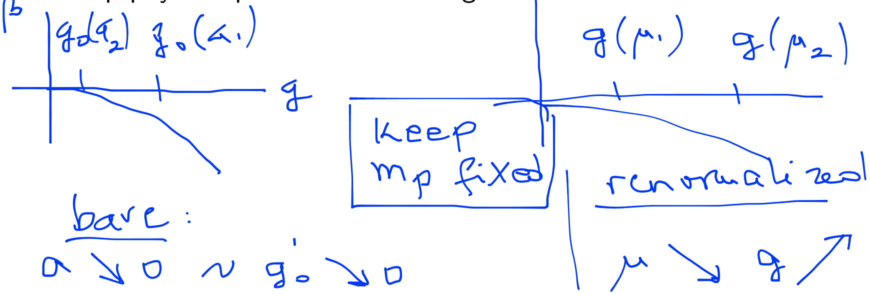
The Λ parameters are different between different schemes but can be connected at the 1-loop level, i.e knowing c is sufficient.

Renormalization group

Tuning

- Bare parameters: tune the *bare* couplings as a changes
- Continuum: tune the *renormalized* couplings as μ changes

to keep physical predictions unchanged



Renormalized coupling

There are many possible definitions:

- Perturbative, based on some subtraction scheme ($\overline{\mathrm{MS}}$ or MOM, etc)
- Non-perturbative:
  - Coulomb term of the static potential
  - Schrodinger functional coupling
  - Wilson flow coupling

They can be connected perturbatively.

Including the quark mass

There are two lattice parameters, β and m

$$a\frac{dg}{da} \equiv \beta(g) = \beta_0 g^3 + \beta_1 g^5 + \dots$$
$$a\frac{dm}{da} \equiv m\gamma(g) = m(\gamma_0 g^2 + \gamma_1 g^4 \dots)$$

Solve the first equation as before

$$a = \frac{1}{\Lambda} e^{-1/2\beta_0 g^2} \quad \text{- - - -}$$

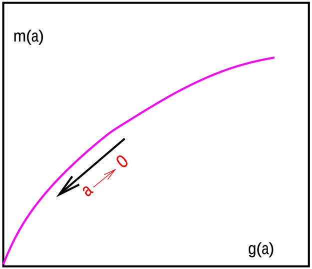
and the second

$$\int \frac{dm}{m} = \int \frac{da}{a}\gamma(g) = \int \frac{dg}{\beta(g)}\gamma(g)$$

Including the quark mass

$$m = Mg^{\gamma_0/\beta_0} \sim M \ln(1/a\Lambda)^{-\gamma_0/\beta_0}$$

M is physical (fixed), $m(a)$ is now predicted



Need two physical quantities to set Λ and m

Tune both β and m to the Gaussian FP

β function

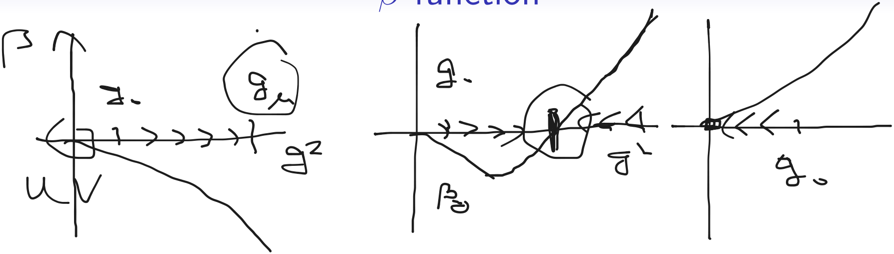
N_f fermions in the fundamental representation, gauge group SU(N)

$$\beta_0 = \frac{1}{16\pi^2}(11N/3 - 2N_f/3)$$
$$\beta_1 = (\frac{1}{16\pi^2})^2(34N^2/3 - 10NN_f/3 - N_f(N^2-1)/N)$$
$$\gamma_0 = \frac{1}{16\pi^2}\frac{3(N^2-1)}{N}$$

- $\beta_0 = 0$ when $N_f = 16.5$ $(N = 3)$: loss of asymptotic freedom
- $\beta_1 = 0$ when $N_f = 8.05$ $(N = 3)$: second zero of the β function

What does that mean?

β function



$N_f < 8$ $\qquad$ $8 < N_f \leq 16$ $\qquad$ $N_f > 16.$

PT!

$$\beta = a \frac{dg}{da} \Longleftrightarrow \beta = \mu \frac{dg}{d\mu}$$

The continuum limit - ~~massless case~~

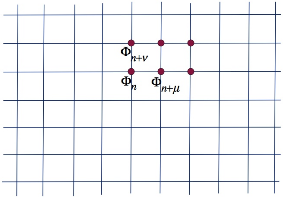
The general approach did not rely on perturbation theory.
Very general, very important:
A continuum limit can be defined by tuning the bare parameters to the critical surface of a lattice system. As long as the critical behavior is universal the lattice continuum limit is universal as well.

If the critical surface is not at $g^2 = 0$ the continuum theory is not going to be perturbative.

Perturb. $\left\{ \begin{array}{l} g^2 \to 0 \\ m \to 0 \end{array} \right.$ || non-perturb.

Wilson renormalization group

Take a system with lattice spacing a or momentum cutoff π/a.
Integrate out the short distance/ high momentum modes



$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi' e^{-S'[\phi']}$$

$$S[\phi] \longrightarrow S'[\phi]$$

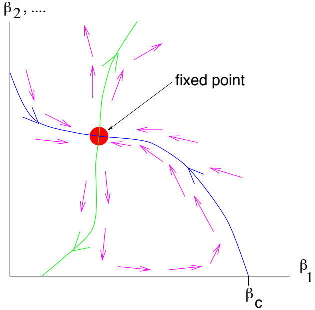
describes the flow in action space.
Long distance (infrared) physics is unchanged while

$$a \longrightarrow a' = sa \quad m_{\mathrm{latt}} \longrightarrow sm_{\mathrm{latt}}$$

Wilson renormalization group

Every allowed (by symmetries) coupling will be generated

$S'[\phi]$ is a complicated action, but with diminishing cutoff effects



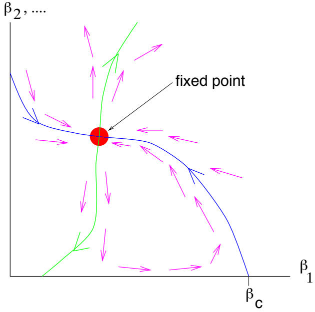
Flow lines follow the change of the action

$$S[\phi] \longrightarrow S'[\phi]$$

Fixed points are at $m_{\rm latt} = 0 \text{ or } \infty$

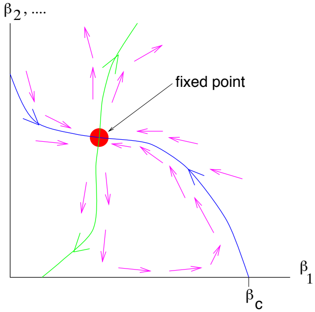
Operators either flow towards (irrelevant - blue) or away (relevant - green) the fixed point

Wilson renormalization group



Relevant operators must be tuned towards the FP to keep infrared physics unchanged.
In QCD, g^2 and m are relevant operators

Wilson renormalization group



Irrelevant directions will die out, but while present contribute as lattice artifacts.

In QCD all other (infinitely many) operators are irrelevant. But they can make a lattice action really good or pretty bad.

The fixed point structure of QCD

β - m plane: