Introduction to Lattice QCD
Lecture 2

Anna Hasenfratz
University of Colorado

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Fourier transform

Standard definitions:

$$f(p) = a^4 \sum_n e^{-ipna} f_n, \quad f_n = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} e^{ipna} f(p).$$

$f(p)$ is periodic within the Brillouin zone

$$f(p) = f(p + \frac{2\pi}{a} m_\mu), \quad m_\mu \in Z,$$

i.e. the lattice momentum is restricted to $|p_\mu| \leq \pi/a = \Lambda_{\mathrm{cutoff}}$.

Notation: $k = ap$ dimensionless momentum;

$\int_p = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4}$

A useful formula

$$\sum_{n_\mu} e^{ink} = (2\pi)^4 \delta(k)$$

Fourier transform

The derivatives

$$\sum_n \phi_n(\Delta_\mu \Delta^*_\mu)\phi_n = -\int_p \frac{4}{a^2}\sin^2(\frac{\mathrm{ap}_\mu}{2})\phi(-\mathrm{p})\phi(\mathrm{p})$$

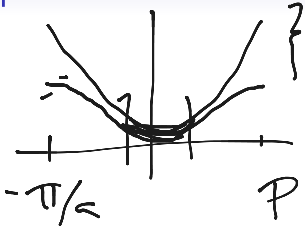
$$\frac{1}{2}\sum_n \bar{\psi}_n(\Delta_\mu + \Delta^*_\mu)\psi_n = \int_p \frac{i}{a}\sin(\mathrm{ap}_\mu)\bar{\psi}(-\mathrm{p})\psi(\mathrm{p})$$

Notation: $\hat{p}_\mu = \frac{2}{a}\sin\frac{\mathrm{ap}_\mu}{2} = \mathrm{p}_\mu + \mathcal{O}(\mathrm{a}^2)$

$2(\cos p - 1)$

$\phi_{n+\mu} - 2\phi_n + \phi_{n-\mu}$

Free scalars - again



The free scalar lattice action:

$$S[\psi] = a^4 \sum_n \left(\frac{1}{2}(\Delta_\mu \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 \right)$$

$$= \frac{a^4}{2} \int_p (\hat{p}^2 + m^2)\phi(p)\phi(-p), \qquad \hat{p}_\mu = \frac{2}{a}\sin(\frac{\mathrm{ap}_\mu}{2})$$

$\hat{p} \to p$ substitution relates continuum & lattice. The continuum and lattice descriptions agree for small p.

$$\hat{p} = p + O(a^2)$$

Fermion doubling

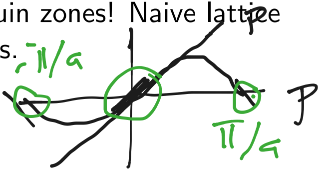
The naive fermion lattice action:

$$S[\psi] = a^4 \sum_{n,\mu} (\bar{\psi}_n \gamma_\mu (\Delta^*_\mu + \Delta_\mu)\psi_n + m\bar{\psi}_n \psi_n)$$

$$= a^4 \int_p \bar{\psi}(-p)[\frac{i}{a}\mathrm{sin}(\mathrm{p}_\mu \mathrm{a})\gamma_\mu + \mathrm{m}]\psi(\mathrm{p})$$

Now the continuum to lattice replacement is $p_\mu \to \mathrm{sin}(\mathrm{p}_\mu \mathrm{a})/\mathrm{a}$ very different at the edge of the Brillouin zones! Naive lattice fermions describe 16 continuum species.

$\sin(p.a) \sim p$



Homework

Derive the energy-momentum relation both for scalar and naive fermions. That's the reliable way to understand the particle content of the systems.

1) the 2-point function is

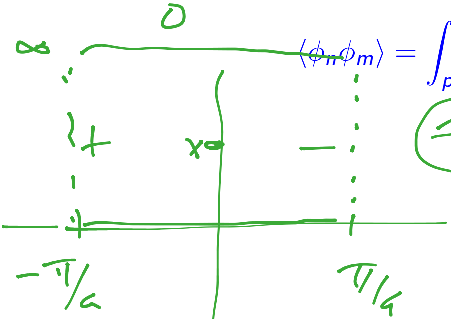
$$\langle \phi_n \phi_m \rangle = \int_p \frac{e^{-iap(n-m)}}{\hat{p}^2 + m^2}$$

2) Integrate out p_0. This is a complex contour integral, so find the poles $\omega(p) = \pm i p_0$.

3) How many poles are there? At what momenta?

4) Now repeat for the fermions.

Homework - Spectral representation



$$\langle \phi_n \phi_m \rangle = \int_p \frac{e^{-iap(n-m)}}{\hat{p}^2 + m^2} = 2\pi i \, \mathrm{Res}(\text{pole})$$

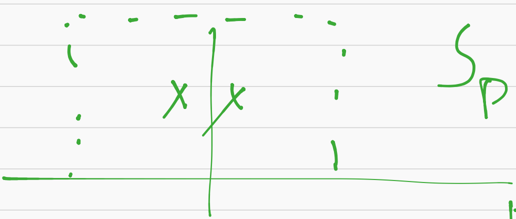
p_0

$a \to 0$

$$i p_0 = \omega_p \sim \sqrt{p^2 + m^2}$$

$$\langle \phi \phi \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} e^{i(\;)}$$

Fermions



$S_p \longrightarrow S + S$

$\omega_{p_1} \longrightarrow \gamma$

$\omega_{p_2} \longrightarrow \gamma$

Left handed?

$\frac{(1-\gamma_5)}{2}$

1 Left $\longrightarrow$ $\gamma R + \gamma L$

Fermion doubling

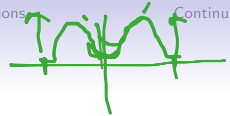
Nilsen-Ninomiya no-go theorem:

It is not possible to construct a lattice fermion action that is

- Ultra local
- Chirally symmetric
- Has the correct continuum limit
- Undoubled

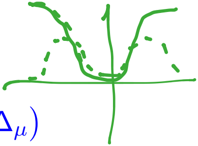
Solutions:

- Wilson fermions - break chiral symmetry
- Staggered fermion - brake taste symmetry
- Ginsparg-Wilson fermions - extend chiral symmetry; local but not ultra local actions (still universal).
- Twisted mass fermions
- Other variants



Wilson fermions

Modify the fermion action so the modes at $p_\mu \sim \pi$ become heavy:



$$
\begin{aligned}
S_W &= a^4 \sum_n \bar{\psi}_n (D_W + m)\psi_n \\
D_W &= \frac{1}{2}\left(\gamma_\mu(\Delta^*_\mu + \Delta_\mu) - ar\Delta^*_\mu \Delta_\mu\right)
\end{aligned}
$$

The Wilson term is quadratic in Δ_μ like the scalar kinetic term. It

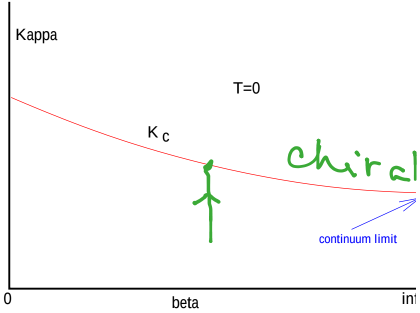
- $\mathcal{O}(a)$ irrelevant operator
- Lifts the doublers
- Breaks chiral symmetry even in the chiral limit (it's a mass term)

Wilson fermions - hopping form

The Wilson action after rescaling $\psi \to \sqrt{2\kappa}\psi$, $\kappa = \frac{1}{2ma+8r}$

$$S_W = a^4 \sum_n \left(\bar{\psi}_n \psi_n - \kappa(\bar{\psi}_n (r-\gamma_\mu) U_{n,\mu}\psi_{n+\mu} + \bar{\psi}_n (r+\gamma_\mu) U_{n,\mu}\psi_{n-\mu})\right)$$

κ is the "hopping parameter", $r = 1$ standard choice



$\frac{1}{8}$ ✓

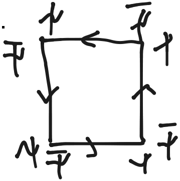
Critical surface $\kappa_{cr}(\beta)$
Deviation from $1/8$ is the measure of chiral symmetry breaking.

Hopping parameter expansion

Hopping parameter expansion: expand in terms of κ (valid for small κ or large mass)

$$Z = \int \mathcal{D}[U\bar{\psi}\psi] e^{(-S_g - \bar{\psi}\psi)} \prod_{\text{link}} (1 - \kappa\bar{\psi}_n (r - \gamma_\mu) U_{n,\mu} \psi_{n+\mu} - \kappa\bar{\psi}_n (r + \gamma_\mu) U_{n,\mu} \psi_{n-\mu})$$

Only terms with 1-1 $\bar{\psi}$ and ψ contribute $\rightarrow$ leads to closed gauge loops.



$$\mathrm{Tr}\,(1+\gamma_\mu)(1+\gamma_\nu)(1-\gamma_\mu)(1-\gamma_\nu) \cdot \kappa^4 \cdot U_\Box$$

Effective gauge action

$$Z = \int \mathcal{D}[U] e^{-S_g} \sum_{\text{closed loops}} \kappa^l c_l \mathrm{ReTr} \prod_{\mathcal{C}} \mathrm{U}$$

Re-exponentiate to find the effective gauge action from the fermions

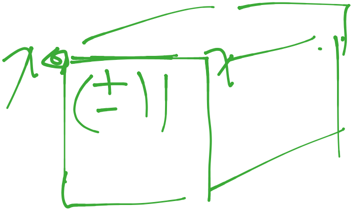
$$S_{\text{eff}} = 16\kappa^4 N_f \sum_p (3 - \mathrm{ReTrU}_\Box) + \dots \quad (\mathrm{r} = 1)$$

Fermions always introduce an effective positive gauge coupling - true for other formulations, large κ, etc.

Staggered fermions

Simple idea: distribute the 4 components of the Dirac spinor to different lattice sites.

Counting: 1 component per site x 16 fold doubling = 4 species or tastes



Staggered fermions

Formalism:

Unitary transformation $\psi_n \to \Omega_n \psi'_n$, $\Omega_n = \gamma_0^{n_0}\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}$ makes the naive fermion action spin-diagonal:

$$S = \frac{1}{2a}\sum_n \bar{\psi}'_n \alpha_\mu(n)[U_{n,\mu}\psi_{n+\hat{\mu}} - U^\dagger_{n-\hat{\mu},\mu}\psi_{n-\hat{\mu}}] + m\sum_n \psi'_n\psi'_n$$

$\alpha_\mu(n) = (-1)^{n_0+\cdots+n_{\mu-1}}$ is a phase factor, ± 1

Drop 3 of the four components of $\psi \to \chi$ and we have a 4-taste staggered action

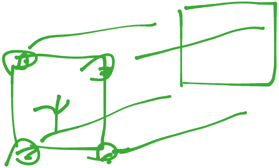
$$S = \frac{1}{2a}\sum_n \bar{\chi}_n \alpha_\mu(n)[U_{n,\mu}\chi_{n+\hat{\mu}} - U^\dagger_{n-\hat{\mu},\mu}\chi_{n-\hat{\mu}}] + m\sum_n \chi_n\chi_n$$

Staggered fermions

How do we recover a 4-component Dirac spinor?

$$\Psi_n^{\alpha a} = \frac{1}{8} \sum_{\eta} \Omega_\eta^{\alpha a} \chi_{2n+\eta}, \quad \eta_\mu = 0 \text{ or } 1$$

i.e. collect them from a hypercube. (a is taste, α is spinor)
Problems? the different elements of a Dirac spinor "see" different gauge fields, leads to taste breaking.
The coarser the gauge fields, the worst it is;



$$\bar{\Psi}_n \, U_{n\mu} \, \psi_{n+\mu}$$

Staggered fermions

Some staggered symmetries:

- Translation:

$$\begin{aligned} \chi(n) &\rightarrow \xi_\mu(n)\chi(n+\hat{\mu}), \quad \bar{\chi}(n) \rightarrow \xi_\mu(n)\bar{\chi}(n+\hat{\mu}), \\ U_{n,\mu} &\rightarrow U_{n+\hat{\mu},\mu}, \quad (\xi_\mu(n) = (-1)^{\sum_{\nu>\mu} n_\nu}) \end{aligned}$$

- Remnant U(1):

$$\chi_n \rightarrow e^{\pm i\theta}\chi_n, \quad n = \text{even or odd}$$

protects the quark mass but gives only 1 pion per 4 tastes

Staggered fermions

Rooting: In order to reduce the 4 tastes to 1 the 4th root of the staggered determinant is taken in simulations. The resulting lattice action is non-local but there is growing evidence that in the continuum limit it approaches the correct universality class.

$$Z = \int D\bar{\chi}\chi \, e^{-\bar{\chi}_i M \chi_i} = (\det M)^{N_f}$$

$$\rightarrow \det(M) \longrightarrow (\det M)^{1/4}$$