High Temperature and Density in Lattice QCD: Chiral symmetry restoration

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Chiral symmetry

- Continuum Euclidean fermion action for N_f flavors:

$$S_F = \sum_{f=1}^{N_f} \int d^4x [\bar{\psi}_f(x)\gamma_\mu(\partial_\mu + ieA_\mu^a\lambda^a/2)\psi_f(x) + m_f\bar{\psi}_f(x)\psi_f(x)]$$

- Degenerate masses: $SU(N_f) \times U(1)$ symmetry:

$$\delta\psi(x) = \left(1 + i\theta^0/2 + i\theta^k\tau^k/2\right)\psi(x)$$

where τ^k are generators of $SU(N_f)$.

- Zero masses: $SU(N_f)_L \times SU(N_f)_R \times U(1) \times U_A(1)$

$$\begin{aligned}
\delta\psi(x) &= \left(1 + i\theta^0/2 + i\theta^k\tau^k/2 + i\phi^0\gamma_5/2 + i\phi^k\tau^k\gamma_5/2\right)\psi(x) \\
\delta\bar{\psi}(x) &= \bar{\psi}(x)\left(1 - i\theta^0/2 - i\theta^k\tau^k/2 + i\phi^0\gamma_5/2 + i\phi^k\tau^k\gamma_5/2\right)
\end{aligned}$$

Chiral symmetry

- At zero temperature the $U_A(1)$ symmetry (ϕ^0 term) is broken by the gauge anomaly.
- The axial chiral symmetry (ϕ^k terms) is broken spontaneously at zero temperature
$$\langle \bar{\psi}_f \psi_f \rangle \neq 0.$$
- We have $N_f^2 - 1$ Goldstone bosons.

Transformation of quark bilinears ($N_f = 2$ case)

- Linear sigma model fields ($f_0 \equiv \sigma$. $a_0 \equiv \delta$.)

$$\pi^k = \bar\psi\tau^k\gamma_5\psi \quad f_0 = \bar\psi\psi$$
$$a_0^k = \bar\psi\tau^k\psi \quad \eta = \bar\psi\gamma_5\psi.$$

- Under an $SU(2)$ axial transformation

$$\delta\pi^k = i\phi^k f_0 \quad \delta f_0 = i\phi^k\pi^k$$
$$\delta a_0^k = i\phi^k\eta \quad \delta\eta = i\phi^k a_0^k$$

- Under a $U_A(1)$ (axial) transformation

$$\delta\pi^k = i\phi^0 a_0^k \quad \delta f_0 = i\phi^0\eta$$
$$\delta a_0^k = i\phi^0\pi^k \quad \delta\eta = i\phi^0 f_0$$

	$SU(2)_L \times SU(2)_R$		
$U_A(1)$	$\pi : \bar\psi\tau\gamma_5\psi$	$\leftrightarrow$	$f_0 : \bar\psi\psi$
	$\updownarrow$		$\updownarrow$
	$a_0 : \bar\psi\tau\psi$	$\leftrightarrow$	$\eta : \bar\psi\gamma_5\psi$

Chiral effective theory

- The chiral theory is a low-energy effective theory for QCD, based on $N_f^2 - 1$ Goldstone bosons, π^k. We define (nonlinear sigma model)

$$U = \exp\left(\frac{i}{f}\hat{\pi}_k \tau_k\right)$$

- f is a low energy constant (closely related to the pion decay constant).
- Chiral effective Lagrange density

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) + f^2 B \,\mathrm{Re}\,\mathrm{Tr}(MU)$$

- $M = \mathrm{diag}\{m_1, m_2, \ldots\}$ contains the quark masses. B is another low-energy constant.
- When $M = 0$ the Lagrange density is invariant under the chiral transformation

$$U \to V_R U V_L^\dagger$$

Symmetry restoration

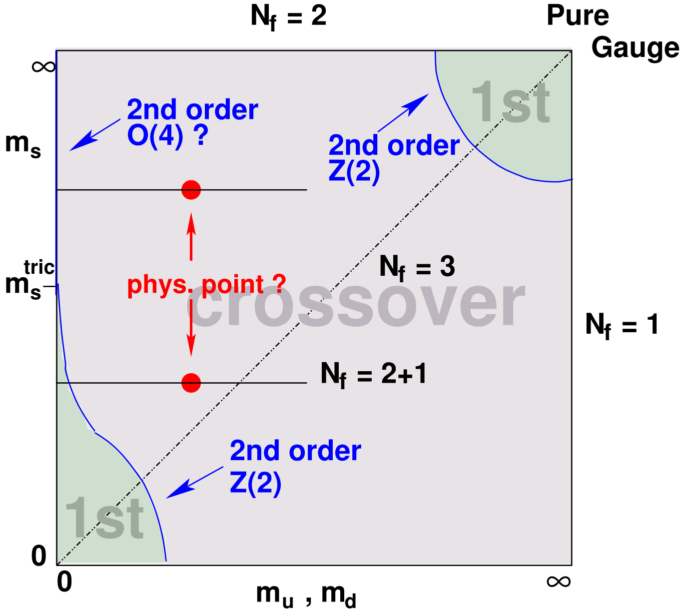
- The chiral model behaves like a ferromagnetic spin system. For $N_f = 2$ it is equivalent to $O(4)$.
- Quark masses play the role of a magnetic field. $\mathrm{Re}\,\mathrm{Tr}\,U$ plays the role of magnetization. It is the analog of $\langle \bar{\psi}\psi \rangle$.
- At low temperatures we expect spontaneous symmetry breaking and at high temperatures we expect symmetry restoration.
- Restoration of $SU(N_f)_L \times SU(N_f)_R$ at high T in QCD, therefore seems certain.
- At sufficiently high mass we expect no phase transition.
- Whether the $U_A(1)$ symmetry is restored depends on the fate of the anomaly at high T.

Chiral symmetry

Reference: R. Pisarski and F. Wilczek, Phys. Rev. **D29**, 338 (1984).

- The nature of the chiral phase transition depends on N_f.
- For $N_f \geq 3$ the phase transition is first order.
- The $U_A(1)$ symmetry should be restored at least asymptotically at high T but its restoration needn't occur at the same temperature as that of $SU(N_f)_L \times SU(N_f)_R$.
- For $N_f = 2$, the nature of the phase transition depends on what happens with $U_A(1)$.
- If $U_A(1)$ is effectively restored at the same temperature as $SU(N_f)_L \times SU(N_f)_R$, the transition can be a fluctuation-driven first-order transition.
- Otherwise, it is continuous (second order).

Suggested phase diagram



Question to be answered in lattice simulations

- Is there a phase transition at the physical (nonzero) values of the quark masses?
- A first order transition is resistant to small changes, so if the transition is first order when $m_u = m_d = m_s = 0$, how large can the masses be before the phase transition is lost?
- At what temperature is $U_A(1)$ (at least effectively) restored?

Signals of chiral symmetry restoration

- At zero mass the order parameter $\langle \bar{\psi}_f \psi_f \rangle$ should vanish.
- If quark masses are not zero, we can still use $\langle \bar{\psi}_f \psi_f \rangle$ as an indicator.
- The chiral susceptibility $\chi_f = \frac{\partial}{\partial m_f} \langle \bar{\psi}_f \psi_f \rangle$ should peak at the transition (or crossover) temperature.
- Hadron correlators (implies masses) become equal. For $SU(2)_L \times SU(2)_R$ we have $C_\pi = C_{f_0}$.

$$\begin{aligned} C_{f_0}(x) &= \langle f_0(x) f_0(0) \rangle \\ C_\pi(x)\delta_{k,k'} &= \left\langle \pi^k(x) \pi^{k'}(0) \right\rangle \end{aligned}$$

- Similarly $C_\eta = C_{a_0}$.
- With restoration of $U_A(1)$ we have $C_\pi = C_{a_0}$ and $C_{f_0} = C_\eta$.

Lattice treatment of chiral symmetry

The lattice implementation of chiral symmetry depends on the fermion formulation

- Wilson/clover fermions break chiral symmetry explicitly
- Staggered (asqtad, HISQ) fermions preserve a remnant of chiral symmetry.
- Overlap and domain wall fermions aim to treat chiral symmetry exactly.

Chiral observables

- Regulating the ultraviolet divergence

$$\langle \bar\psi_f \psi_f \rangle = m_f/a^2 + \ldots .$$

- So subtract the light quark ($m_u = m_d$) and strange quark condensates:

$$D_{ud,s}(T) = [\langle \bar\psi\psi \rangle_{ud} - m_{ud}/m_s \langle \bar\psi\psi \rangle_s]$$

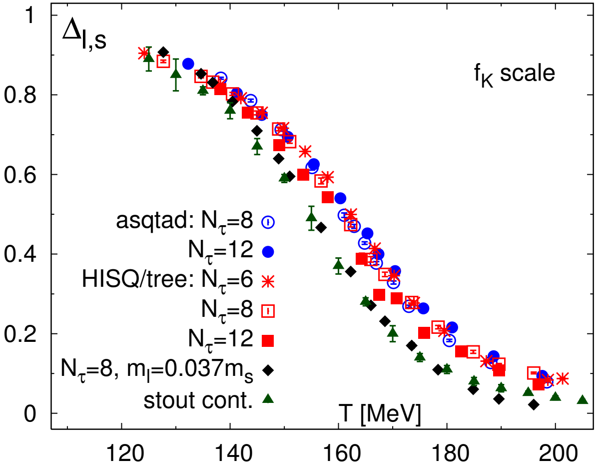
- Also subject to multiplicative renormalization (independent of T).

$$Z_S \bar\psi\psi$$

- So take the ratio before comparing results from different calculations.

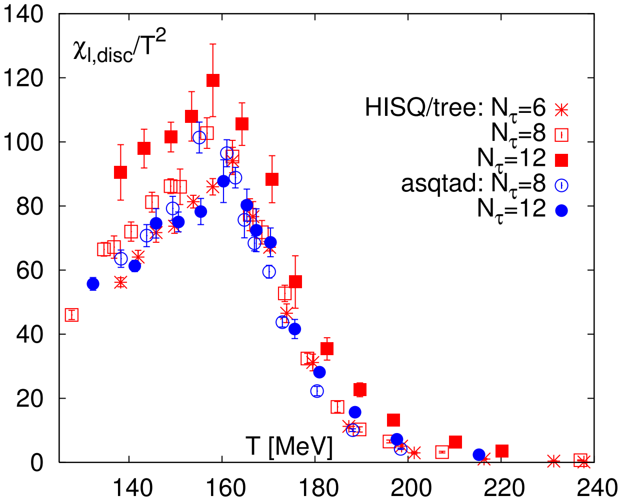
$$\Delta_{ud,s}(T) = D_{ud,s}(T)/D_{ud,s}(T = 0)$$

Numerical results



[HotQCD Phys. Rev. D85 (2012) 054503]

Chiral susceptibility



[HotQCD Phys. Rev. D85 (2012) 054503]

Banks-Casher relations

[Ohno, Heller, Karsch, Mukherjee, PoS LATTICE2011 (2011) 210]

- Susceptibilities

$$\chi_S = \left\langle \int C_{f_0}(x) \right\rangle \quad \chi_P = \left\langle \int C_\eta(x) \right\rangle .$$

- Banks-Casher (extended)

$$\begin{aligned} \langle \bar\psi\psi \rangle &= m \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda)}{\lambda^2 + m^2} \\ \chi_P - \chi_S &= \int_{-\infty}^{\infty} d\lambda \frac{2m^2 \rho(\lambda)}{(\lambda^2 + m^2)^2} \end{aligned}$$

- As $m \to 0$ $\langle \bar\psi\psi \rangle = \pi\rho(0)$.
- If $SU(2)_R \times SU(2)_L$ is restored we get $\chi_P - \chi_S = 0$.
- This happens if $\rho(\lambda)$ opens up a gap, for example.

Exercise

In terms of the Euclidean Dirac matrix $M = m + \not{D}$, the chiral condensate is

$$\langle \bar{\psi}\psi \rangle = \mathrm{Tr}\, M^{-1}$$

The eigenvalues and eigenvectors of $\not{D}$ satisfy $M u_n = i\lambda_n u_n$. Assume that the antihermitian $\not{D}$ operator also satisfies the anticommutation relation $\{\not{D}, \gamma_5\} = 0$.

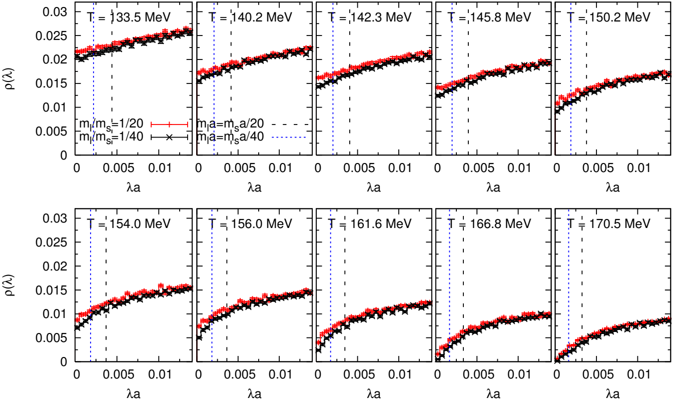
Prove the Banks-Casher relation

$$\langle \bar{\psi}\psi \rangle = m \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda)}{\lambda^2 + m^2}.$$

where the spectral density is constructed from $1/V \sum_n \to \int d\lambda \rho(\lambda)$.

Then show that at zero mass $\langle \bar{\psi}\psi \rangle = \pi \rho(0)$.

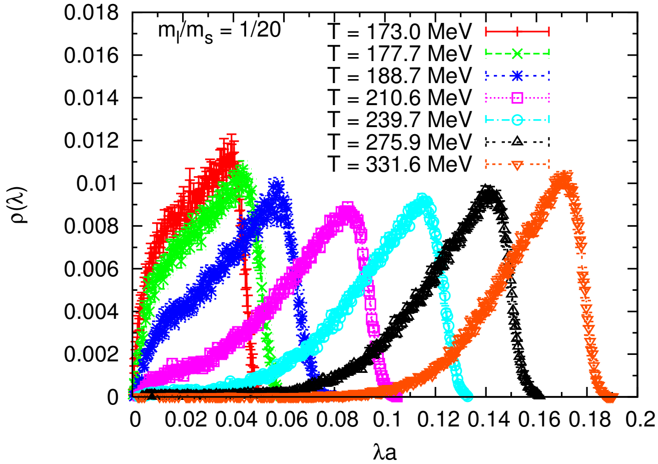
$SU(2) \times SU(2)$ restoration



[H. Ohno, U. M. Heller, F. Karsch, S. Mukherjee, PoS LATTICE2011 (2011) 210.]

- $N_\tau = 8$ and light quark masses.
- There appears to be a zero for $T > 168$ MeV.

$U_A(1)$ restoration



[H. Ohno, U. M. Heller, F. Karsch, S. Mukherjee, PoS LATTICE2011 (2011) 210.]

- There appears to be a gap for $T > 240$ MeV, approximately.

Universality and critical behavior

- Assume the phase transition happens at $T = T_c$ and $m_{ud} = 0$.
- Rescale T and m_{ud} to give t and h:

$$
\begin{aligned}
t &= \frac{1}{t_0} \frac{T - T_c^0}{T_c^0} \\
h &= \frac{1}{h_0} H \quad \text{for } H = \frac{m_l}{m_s}
\end{aligned}
$$

- Free energy density as a function of quark masses and temperature in the vicinity of a critical point

$$
f = -\frac{T}{V} \log Z \equiv f_{\rm sing}(t, h) + f_{\rm reg}(T, m_{ud}, m_s) \ .
$$

- The singular part is singular at the phase transition.
- Up to a rescaling of the variables, the singular part is universal, depending only on the symmetries and dimensionality of the system! Here it is $O(4)$ in 3D.

Universality and critical behavior

- Repeating,

$$f = -\frac{T}{V} \log Z \equiv f_{\mathrm{sing}}(t, h) + f_{\mathrm{reg}}(T, m_{ud}, m_s) \ .$$

- Define $z = t/h^{1/\beta\delta}$ for universal critical exponents δ and β. Then we have, further

$$f_{\mathrm{sing}}(t, h) = h^{1/\delta} f_s(z)$$

- Define $M_b \equiv \frac{m_s \langle \bar{\psi}\psi \rangle_{ud}}{T^4}$ where $\langle \bar{\psi}\psi \rangle_{ud} = T/V \partial \log Z / \partial m_{ud}$.
- Then

$$M_b(T, H) = h^{1/\delta} f_G(t/h^{1/\beta\delta}) + f_{M,\mathrm{reg}}(T, H)$$

- The function f_G is universal.

Universality and critical behavior

- The chiral susceptibility is the derivative $\chi_{ud} = \frac{\partial}{\partial m_{ud}} \left\langle \bar{\psi}\psi \right\rangle_{ud}$
- We get a scaling expression for it by differentiation

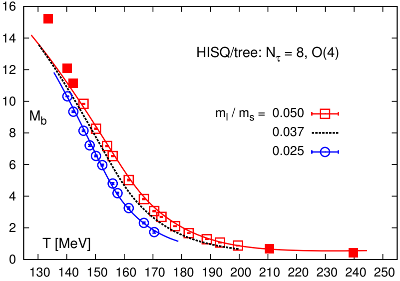
$$\frac{\chi_{ud}}{T^2} = \frac{T^2}{m_s^2} \left(\frac{1}{h_0} h^{1/\delta - 1} f_\chi(z) + \frac{\partial f_{M,reg}(T, H)}{\partial H} \right)$$

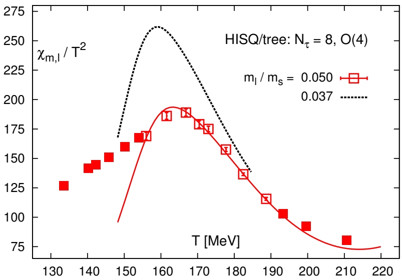
where

$$f_\chi(z) = \frac{1}{\delta} \left[f_G(z) - \frac{z}{\beta} f_G'(z) \right] .$$

- So the behavior of $\left\langle \bar{\psi}\psi \right\rangle_{ud}$ and χ_{ud} is governed by the same singular function.

Universality and critical behavior





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- Used only a leading order Taylor expansion for the regular part
- Such considerations lead to $T_c = 154(9) MeV$.