High Temperature and Density in Lattice QCD: Strong-coupling, high temperature limit and the Potts model paradigm

http://www.physics.utah.edu/~detar/int12/

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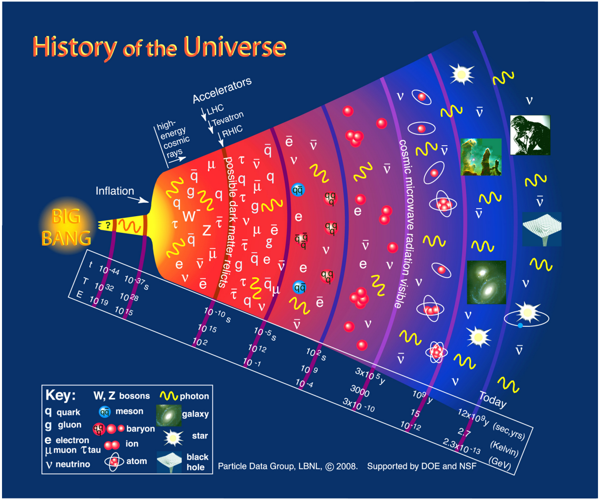
August 6-10, 2012

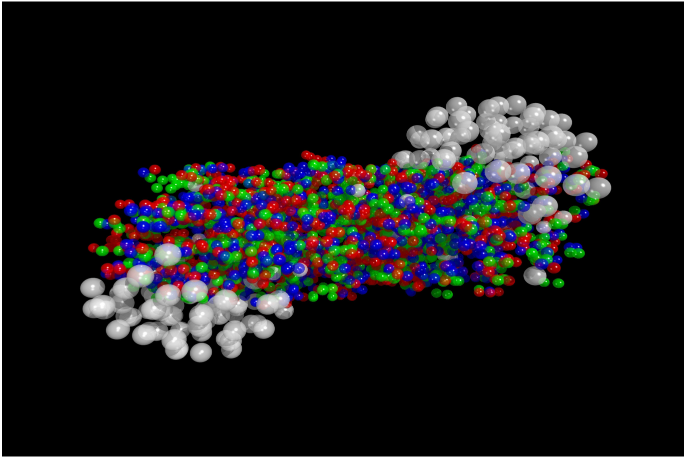
Why study high T and high density QCD?

- Early universe
- Dense stars



Why study high T and high density QCD?

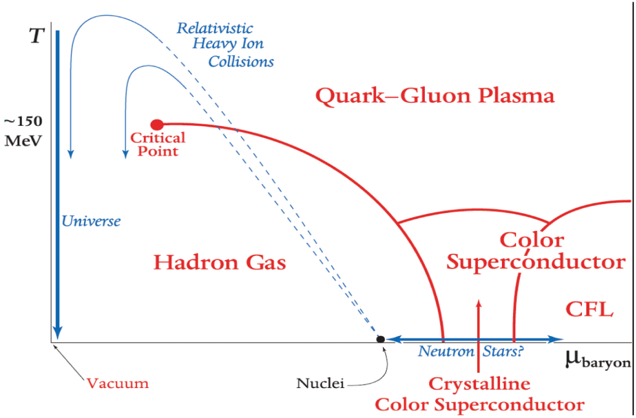
- Heavy ion collisions
- Intrinsic field theory interest



[Thanks CERN image]

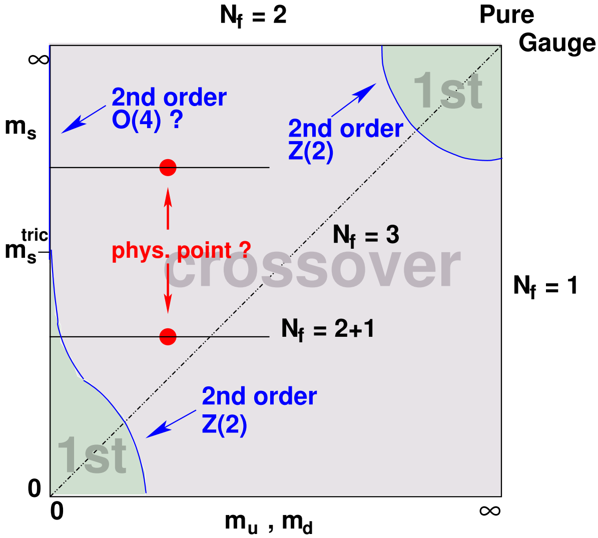
Phase structure in T and μ

- Confinement lost at high temperature or density
- "Quark-gluon plasma"
- Phase transition or crossover?
- Speculative phase diagram



Phase structure in m_{ud} and m_s

- Whether there is a phase transition depends on quark masses
- Speculative phase diagram



Open questions

- What happens at high density?
- Does the critical end-point exist?
- Is it experimentally accessible?
- Is there a first order phase transition at very small nonzero m_{ud}?

Lattice QCD to the rescue!

- These are nonperturbative questions
- LQCD is the only nonperturbative *ab initio* method we have.
- BUT: Lattice QCD is based on equilibrium thermodynamics
- Heavy ion collisions are dynamical.
- Relevant only where thermal equilibrium is a good approximation.
- Good for dense stars, early universe, and some stages in heavy ion collisions.

Outline of lectures

1. Strong-coupling, high temperature limit and the Potts model paradigm.
2. Deconfining transition in QCD
3. Chiral symmetry restoration in QCD
4. Connection with phenomenology

Classic Wilson action

- Path integral: $T = 1/(N_\tau a)$

$$Z_W = \mathrm{Tr}\exp(-H/T) = \int [dU][d\psi d\psi]\exp(-S)$$

- $S = S_G + S_F$

$$\begin{aligned} S_G &= \frac{6}{g^2}\sum_{x,\mu<\nu}[1 - \mathrm{Re}\,\mathrm{Tr}\,U_P(x;\mu,\nu)/3] \\ S_F &= \sum_x \bar{\psi}(x)\psi(x) \\ &- \kappa\sum_{x,\mu}[\bar{\psi}(x)(1+\gamma_\mu)U_\mu(x)\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})(1-\gamma_\mu)U_\mu^\dagger(x)\psi(x)] \end{aligned}$$

- Conserved (Noether) current

$$J_\mu(x) = \kappa[\bar{\psi}(x)(1+\gamma_\mu)U_\mu(x)\psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu})(1-\gamma_\mu)U_\mu^\dagger(x)\psi(x)]$$

External point-charge source

- Introduce an external point charge g in the fundamental representation, moving along the world line C.
- Continuum representation

$$\delta S = -ig \oint_C \lambda^a A_a^\mu dx_\mu$$

- For gauge invariance C must be closed.
- Lattice representation ($\gamma_{-\mu} = -\gamma_\mu$; $U_{-\mu}(x) = U_\mu^\dagger(x - \hat{\mu})$)

$$\begin{aligned} Z &= \int [dU][d\psi d\psi] \exp(-S) L_C \\ L_C &= \mathrm{Tr} \prod_{x,\mu \in C} (1 + \gamma_\mu) U_{x,\mu} \end{aligned}$$

Static charge

- The static charge worldline C is fixed at $\mathbf{x}$, moving forward only in τ.
- Product of time-like links, closing by periodicity in imaginary time:

$$L_C \propto \mathrm{Tr} \prod_\tau U_{\mathbf{x},\tau;0}$$

- Called a "Polyakov loop" (also, sometimes "Wilson line").

Gauge field at strong coupling, high T

References: L.G. Yaffe and B. Svetitsky, Phys. Rev. D 26, 963 (1982); T.A. DeGrand and C.D., Nucl. Phys. B225[FS9], 590 (1983)

- Taking soluble limits often provides insight into the workings of a theory.
- Anisotropic lattice ($a_t \neq a_s$)

$$S_G = \frac{6a_s}{a_t g^2} \sum_{x,i} [1 - \text{Tr } U_P(x;0,i)/3] + \frac{6a_t}{a_s g^2} \sum_{x,i>j} [1 - \text{Tr } U_P(x;i,j)/3]$$

- High temperature: $N_t = 1$ so $a_t = 1/T$ and $a_t/a_s \ll 1$. Drop the space-space term.
- We have only $U(\mathbf{x},0)$ and $U(\mathbf{x},i)$

$$\text{Tr } U_P(x;0,i) = \text{Tr } U_0(\mathbf{x},0) U_i(\mathbf{x}) U_0^\dagger(\mathbf{x}+\hat{\mathbf{i}}) U_i^\dagger(\mathbf{x})$$

- The trace takes its maximum value of 3 when $U_0(\mathbf{x},0) = z(\mathbf{x}) I \in Z(3)$, the center of $SU(3)$: $\{1, \exp(\pm 2\pi i/3)\}$.

Gauge field at strong coupling, high T

- Appoximate the integral over the gauge fields by a sum over $Z(3)$

$$Z = \int \prod_{x,\mu} [dU_\mu(x)] \exp(S_G) \to \sum_{z_{\mathbf{x}}} \exp\left[\frac{6a_s}{g^2 a_t} \sum_{\mathbf{x},\mathbf{i}} \mathrm{Re}\left(z_{\mathbf{x}}^* z_{\mathbf{x}+\hat{i}} \right) \right]$$

- The theory becomes the three-state, 3D Potts model.
- Global Z(3) Symmetry: $z_{\mathbf{x}} \to Y z_{\mathbf{x}}$ for $Y \in Z(3)$.

Gauge field at strong coupling, high T

- Again
$$Z = \sum_{z_{\mathbf{x}}} \exp\left[\frac{6a_s}{g^2 a_t}\sum_{\mathbf{x},\mathbf{i}} \mathrm{Re}\left(z^*_{\mathbf{x}} z_{\mathbf{x}+\hat{i}}\right)\right]$$
- In a spin system, we would replace $6a_s/(g^2 a_t) = J/T_{\rm Potts}$
- So $T_{\rm Potts} \propto g^2$ at fixed a_t/a_s.
- At fixed a_t/a_s we vary $a_t = 1/T_{\rm QCD}$ by varying g^2.
- As $g^2 \to 0$ asymptotic freedom says $a_t \to 0$ so T_{QCD} increases while $T_{\rm Potts}$ decreases.
- At low $T_{\rm Potts}$ the spin system is in a ferromagnetic state.
- The order parameter is the magnetization $\langle z\rangle$.
- There is a first order magnetization phase transition. The ordered phase corresponds to high $T_{\rm QCD}$.
- The order parameter corresponds to $\mathrm{Tr}\, U_0(\mathbf{x})$.
- More generally, it is the "Polyakov loop".
$$L(\mathbf{x}) = P\exp\left[\int igA_0(\mathbf{x},\tau)d\tau\right]$$

Chemical potential

- Conserved charges Q_f are flavor number (or baryon number).
- Grand canonical ensemble

$$Z_W = \mathrm{Tr} \exp \left(-H/T + \sum_f \mu_f Q_f / T \right)$$

Chemical potential

- Conserved charge density

$$\rho_f(x) = \kappa[\bar{\psi}_f(x)(1+\gamma_0)U_0(x)\psi_f(x+\hat{0}) - \bar{\psi}_f(x+\hat{0})(1-\gamma_0)U_0^\dagger(x)\psi_f(x)]$$

- So we add to the action

$$\mu_f Q_f / T = \mu_f \int d\tau Q_f = \int d^4x \mu_f \rho_f(x)$$

- Note this term is just like the time-like kinetic term in the action except for a sign. We get a factor $(1 + a\mu)$ for forward hopping and $(1 - a\mu)$ for backward. It is more natural to use $e^{\pm\mu a}$.
- So we replace

$$\begin{aligned}
\bar{\psi}(x)(1+\gamma_0)U_0(x)\psi(x+\hat{0}) &\to \bar{\psi}(x)(1+\gamma_0)U_0(x)\psi(x+\hat{0})e^{\mu a} \\
\bar{\psi}(x+\hat{0})(1-\gamma_0)U_0^\dagger(x)\psi(x) &\to \bar{\psi}(x+\hat{0})(1-\gamma_0)U_0^\dagger(x)\psi(x)e^{-\mu a}
\end{aligned}$$

- Note the the fermion determinant $\det M(\mu)$ is not real now.

Chemical potentials and Monte Carlo

- At nonzero chemical potential $\det[M(\mu)]$ is complex.
- Can't be used as a Monte Carlo probability weight.
- Phase oscillations grow with the volume of the system V.
- Can't take the thermodynamic limit $V \to \infty$.

Fermions at strong coupling, large mass, high T

- Wilson action. Anisotropic ($a_t \neq a_s$). Chemical potential μ.

$$\begin{aligned} S_F &= \sum_x \bar\psi(x)\psi(x) \\ &- \kappa \sum_x [\bar\psi(x)(1+\gamma_0)U_0(x)e^{-\mu a_t}\psi(x+\hat 0) + \bar\psi(x+\hat 0)(1-\gamma_0)U_0^\dagger(x)e^{\mu a_t}\psi(x)] \\ &- \frac{\kappa a_t}{a_s}\sum_{x,i}[\bar\psi(x)(1+\gamma_i)U_i(x)\psi(x+\hat i) + \bar\psi(x+\hat i)(1-\gamma_i)U_i^\dagger(x)\psi(x)] \end{aligned}$$

- $1/\kappa = 6a_t/a_s + 2 + 2Ma_t$
- High temperature: $N_t = 1$ and $a_t/a_s = 1/(a_s T) \to 0$. Drop the space-like term.
- The fermion matrix is diagonal in space-time with values on each spatial site

$$1 - \kappa(1+\gamma_0)ze^{-\mu a_t} - \kappa(1-\gamma_0)z^* e^{\mu a_t}$$

Fermions at strong coupling, large mass, high T

- For large mass $\Rightarrow$ small κ the fermion determinant becomes

$$\exp\left[h_0(\kappa,\mu) + h(\kappa,\mu)\sum_{\mathbf{x}} \mathrm{Re} z_{\mathbf{x}} + i h'(\kappa,\mu)\sum_{\mathbf{x}} \mathrm{Im} z_{\mathbf{x}}\right]$$

- For small κ we have

$$h(\kappa,\mu) \approx 24\kappa\cosh(a_t\mu) \quad h'(\kappa,\mu) \approx 24\kappa\sinh(a_t\mu)$$

Fermions at strong coupling, large mass, high T

- The quark mass corresponds to an external real magnetic field. The chemical potential introduces an imaginary magnetic field

$$H = -J \sum_{\mathbf{x},\mathbf{i}} \mathrm{Re}\left(z_{\mathbf{x}}^* z_{\mathbf{x}+\hat{i}}\right) - \sum_{\mathbf{x}} [h\,\mathrm{Re} z_{\mathbf{x}} - ih'\,\mathrm{Im} z_{\mathbf{x}}]$$

  for $h \approx 24\kappa\cosh(a_t\mu)$ and $h' \approx 24\kappa\sinh(a_t\mu)$
- The real external field weakens the phase transition and eventually destroys it.
- The imaginary field further weakens it.

Exercise

In mean field theory we consider the statistical mechanics of a single site, assuming that the neighbors of the site take on the same mean value. So for the Potts model we have a single-site partition function

$$Z(\bar{z}) = \sum_{z} \exp[-H(z, \bar{z})/T_{\mathrm{Potts}}]$$

where the single-site $H(z, \bar{z})$ *is obtained from the full* H *by setting all spins to the mean value* $\bar{z}$*, except for one site, which carries variable spin* z*.*

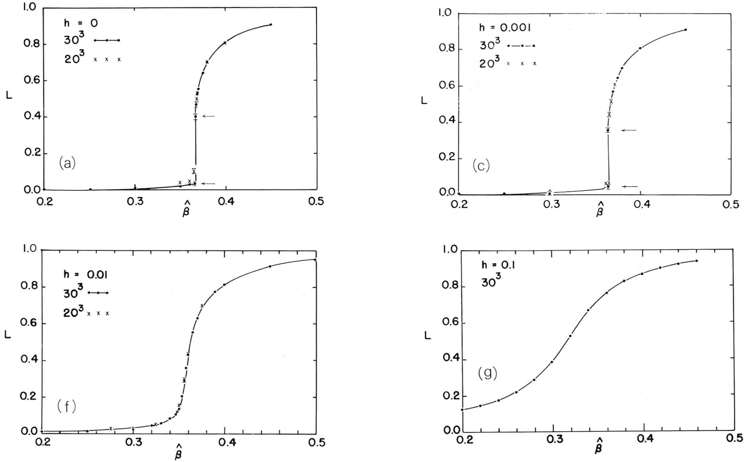
We then impose self-consistency by calculating the output mean value of the spin on the single site and requiring that it equal the input mean value.

Do this for the 3D 3-state Potts model with $h = h' = 0$*, and show that there is one real solution for low* J/T *and three real nonzero solutions for sufficiently high* J/T*. (In the latter case, the middle one happens to be unstable.) Then show that the transition is first order.*

Maple, Mathematica, or gnuplot can help with the numerics here.

3 state 3 D Potts model

T.A. DeGrand and C.D., Nucl. Phys. B225[FS9], 590 (1983)



- Magnetization *vs.* inverse Potts temperature with external field h.
- The first order phase transition disappears with increasing field.

3D flux tube model of QCD

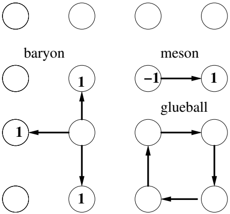
A. Patel, Nucl. Phys. B 243, 411 (1984); Phys. Lett. 139B, 394 (1984);

- Sites: $Z(3)$ charges $n_{\mathbf{x}} \in \{0, 1, -1\}$
- $+1 =$ quark; $-1 =$ antiquark.
- Links: $Z(3)$ flux $\ell_{\mathbf{x},i} \in \{0, 1, -1\}$
- Gauss' Law

$$\sum_i (\ell_{\mathbf{x},i} - \ell_{\mathbf{x},-i}) \mod 3 = n_{\mathbf{x}}$$

- Hamiltonian

$$H = \sigma \sum_{\mathbf{x},\mathbf{i}} |\ell_{\mathbf{x}}, i| + m \sum_{\mathbf{x}} |n_{\mathbf{x}}|.$$



Flux tube model at nonzero chemical potential

J. Condella and C. DeTar, Phys. Rev. **D61**, 074023 (2000).

- Grand canonical partition function ($N = \sum_{\mathbf{x}} n_{\mathbf{x}}$)

$$Z = \sum_{n_{\mathbf{x}}, \ell_{\mathbf{x},\mathbf{i}}} \exp[-(H - \mu N)/T]$$

- There is no complex phase problem at nonzero density here.
- The fluxtube model is equivalent to the $Z(3)$ Potts model
- Not difficult to show. Use the $Z(3)$ identity

$$\frac{1}{3} \sum_{z} z^{\ell} = \delta_{\ell,0}$$

  to replace the Gauss' Law Kronecker delta in the partition function.
- Lesson: changing from a field basis to a color singlet basis cures the complex phase problem in the high-T strong-coupling limit.

Flux tube model at nonzero chemical potential

- In the field basis the complex phase comes from the imbalance between forward time-like and backward time-like hopping, combined with the presence of complex time-like gauge links.

$$\bar{\psi}(x)(1+\gamma_0)U_0(x)e^{-\mu a_t}\psi(x+\hat{0})+\bar{\psi}(x+\hat{0})(1-\gamma_0)U_0^\dagger(x)e^{\mu a_t}\psi(x)$$

- Integration over the time-like links enforces Gauss' Law at each lattice site.
- Changing from the field basis to the hadron basis eliminates the complex phase.
- With $SU(3)$, it is much more difficult to formulate the path integral with a basis change because of the infinite number of $SU(3)$ irreps.
- Moreover, there will still be a fermion sign problem, just as with electrons in condensed matter physics.
- Finally, strong coupling, large mass doesn't capture chiral symmetry.